Free Vibrations of Circular Cylindrical Shells

Tatsuzo Koga

e-mail: tatkoga@yahoo.co.jp

URL: http://www.geocities.co.jp/SiliconValley-Bay/1245

Abstract

The eigenvalue problem of the free vibrations of thin elastic circular cylindrical shells is one of the well-established classical topics in structural mechanics. All the characteristic values of interest can be calculated nowadays to a desired degree of accuracy as a routine work with the aid of digital computers. A number of analytical solutions have been proposed, which may help to gain a good insight into the physical nature underlying the flood of numerical data poured from the computers. This paper reviews the historical background and provides a unified view of the current state of the art through asymptotic solutions recently obtained by the author. Emphasis is placed on the effects of the boundary conditions.

1. Inextensional Vibrations

It was more than a hundred years ago that a valid solution for the free vibrations of thin elastic circular cylindrical shells was first given by Lord Rayleigh, with the birth name John William Strutt. In his famous treatise on "The Theory of Sound"(1) published in 1877, Lord Rayleigh derived an expression for the natural frequency of an infinitely long cylindrical shell, assuming that the midsurface of the shell admits no stretching and that the mode shape is cylindrical. Lord Rayleigh's solution for the natural frequency, f_R, is given by

$$f_R^2 = Eh^2n^2(n^2-1)^2/48\pi^2(1-\nu^2)\rho R^4(n^2+1) \qquad (1)$$

where h is the thickness, R the radius, E Young's modulus, ν Poisson's ratio, ρ mass per unit volume, and n the circumferential wave number. The mode shape is represented by the lateral deflection w such that

$$w = W_0 \cos n\theta \sin 2\pi f_R t \qquad (2)$$

where W_0 is an arbitrary constant, θ the circumferential coordinate, and t the time. Rayleigh's solution also holds for the inplane vibrations of a circular ring. Indeed, he showed that it is identical in form with the one derived by Hoppe for a ring in a memoir published in Crelle, Bd.63, 1871. In 1881, Lord Rayleigh presented a paper(2) in which he developed a theory of the inextensional vibrations for shells of revolution and applied it to shells of spherical, conical and cylindrical shape. His primary interest in this paper seems to have been in the calculation of natural frequencies of hemispherical shell as a model of a church bell, because a substantial part of the paper consists of the analysis of this problem.

Love presented a paper(3) in 1888 (received January 19,read February 9), in which he criticized Lord Rayleigh's theory as inexact in that it leads to an expression for the displacements which can not satisfy the boundary conditions at the free ends. He argued that the equations imposing the condition of inextensibility are, in the most general case, a system of the third order, while the boundary conditions are four in number, and hence these equations are not, in general, of a sufficiently high order to admit of solutions which satisfy the boundary conditions at the free ends. He explicitly showed that this is the case for a spherical shell. He has also derived a strain energy expression which consists of two parts; one due to membrane stretching and proportional to the thickness h and the other due to flexural bending proportional to h^3. He argues that the term proportional to h^3 is small for thin shells in comparison with the term proportional to h, and that the former instead of the latter should be omitted in the limit of $h \to 0$. He has thus formulated a theory for the extensional vibrations and applied it to spherical and cylindrical shells. Lord Rayleigh responded immediately to Love's criticism in a paper(4) presented in the same year (received December 1, read December 19). He states that it is a general mechanical principle that, if displacements be produced in a system by forces, the resulting deformation is determined by the condition that the potential energy of deformation shall be as small as possible, and that the large potential energy which would accompany any stretching will not occur. He gives an explanation to the violation of the boundary conditions at free ends by means of an example of a long cylinder subjected to a pair of concentrated normal forces at the extremities of a diameter of the cross section at the center of the shell. As the thickness reduces, the deformation assumes more and more the character of pure bending such that every normal cross section deforms into an identical configuration. If the thickness remains small but finite, a point will at last be attained when the energy can be made least by a sensible local stretching of the midsurface such as will dispense with the uniform bending otherwise necessary over so great a length.

The second edition of "The Theory of Sound" was published in 1894. All the corrections of importance and new matter added to the first edition are clearly indicated. A new chapter is incorporated devoted to shells, in which it is noted that any extension that may occur in the inextensional vibrations must be limited to a region of infinitely small area adjacent to the ends and affects neither the mode nor the frequency of the vibrations. The first edition of Love's treatise on "The Mathematical Theory of Elasticity"(5) was published in 1892 (volume 1) and 1893 (volume 2). It's fourth edition was published in 1927, whose American printing in 1959 is the one available to the author at the present time. Love seems to have acknowledged Rayleigh's theory, and he notes that the membrane strain which is necessary in order to secure the satisfaction of the boundary conditions is practically confined to so narrow a region near the ends that its effect in altering the total amount of the potential energy is negligible and a greater part of the shell vibrates without accompanying it. Love himself has derived a solution of the inextensional vibrations of a cylindrical shell of finite length having free ends. Love's solution differs from Rayleigh's in that the vibration mode is linear along the generator and asymmetric about the cross section bisecting the axis and also in that it contains a length parameter. The natural frequency of Love's solution, f_L, is given by

$$f_L^2 = f_R^2\{1+6(1-\nu)R^2/n^2L^2\}/\{1+3R^2/n^2(n^2+1)L^2\} \qquad (3)$$

where L is one half the length of the cylinder. It should be noted that f_L is always higher than f_R but the difference is very small when the shell length is greater than or about the same as the radius. The mode shape is represented by a linear function of the axial coordinate x.

$$w = W_0 x \cos n\theta \sin 2\pi f_L t \qquad (4)$$

The issue seems to have been settled. State of the art remains essentially the same for a century. It has been widely accepted that the inextensional vibrations occur either in Rayleigh's or Love's mode either when the cylinder is indefinitely long or when both ends are free for a finite cylinder. Many have reported that they have detected the inextensional vibrations and measured the resonance frequencies, but none of them seems to have identified the resonance modes.
In 1980, Koga[6] suggested a possibility of yet another mode of the inextensional vibrations when one end of the cylinder is free and the other is supported in such a manner that it can move freely in the axial direction. Subsequently, Koga and his collaborators[7,8] have shown by asymptotic method that the inextensional vibrations can occur in three distinct modes; Rayleigh's, Love's and a new mode, at a frequency approximately equal to f_R, and that the mode shape function of the new mode is represented by a linear combination of those of Rayleigh's and Love's, such that

$$W = W_0 (x+L) \cos n\theta \sin 2\pi f_R t \qquad (5)$$

where x= L corresponds to the free end and x= - L to the supported end. Typical mode shapes of these inextensional vibrations are depicted schematically in Fig.1.

Koga and his collaborators conducted an experiment and have proved the existence of these three modes by detecting and identifying their modes in resonance. In the experiment, the mode shapes were visualized by holographic interferometry. Typical fringe patterns of the resonance modes are shown in Fig.2. Both Rayleigh's and Love's modes were observed in a test specimen having completely free ends. The specimen was attached by a supporting column by a small screw at a point bisecting the axis of the cylinder. The new mode was observed in a test specimen which had a free end and a supported end. The supported end was fitted with a diaphragm made from a flexible thin annular plate whose inner circle was fastened to a supporting column by a pair of nuts. The cross sections of the test specimens are shown in Fig.3.

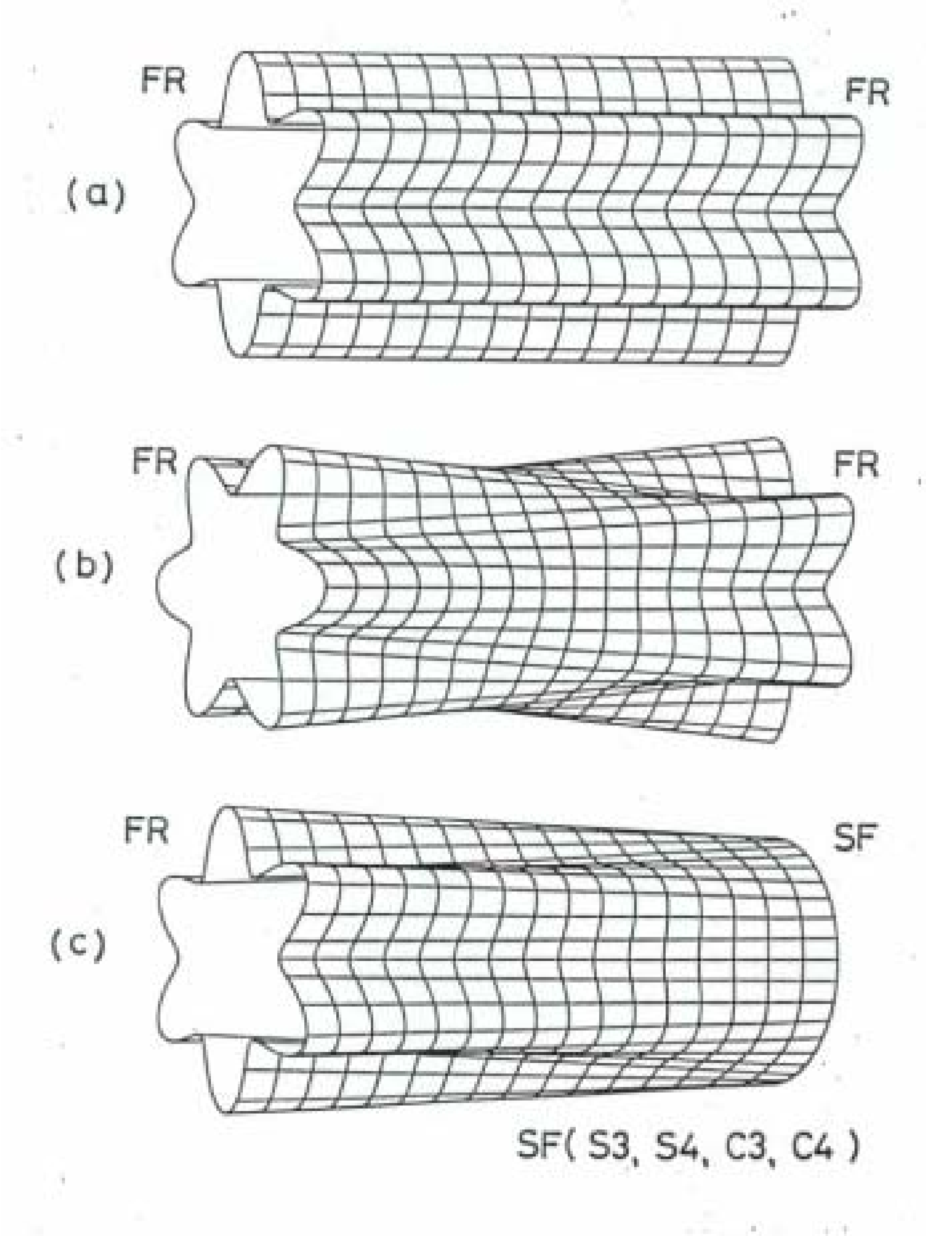

Fig.1. Free Vibration Modes: (a)Rayleigh mode, (b)Love mode, (c)new mode

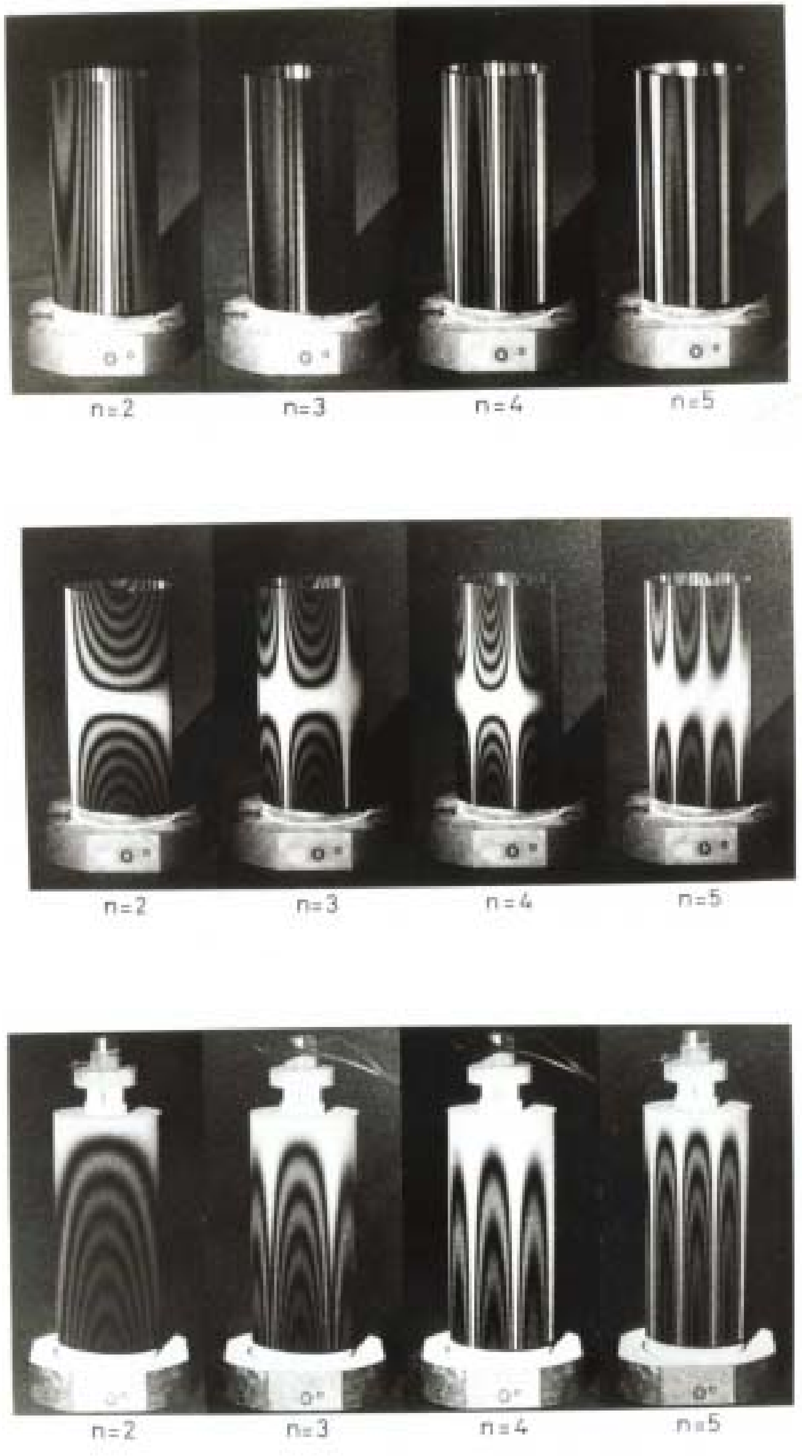

Fig.2. Fringe Patterns of Rayleigh, Love, and new modes

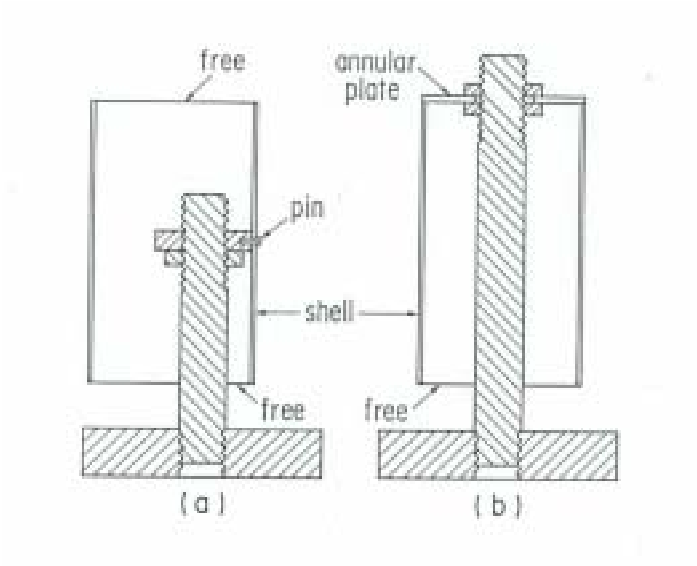

Fig.3. Cross Sections of Test Specimens: (a)FR-FR, (b)FR-SF

2. General Flexural Vibrations

Love suggested in Ref.5 a method for solving eigenvalue problems of free vibrations in general modes which may accompany a slight midsurface stretching, but he had made no attempt to apply it to any specific problem. Flugge published a book[9] in 1934, in which he has established a general method of solution for the free vibrations of circular cylindrical shells. Flugge's method may be summarized as follows:

Let the axial, circumferential and lateral displacements be denoted by u, v and w, and let them be assumed in the form

$$u = U_0 \exp(px) \cos n\theta \ \sin 2\pi ft$$

$$v = V_0 \exp(px) \cos n\theta \ \sin 2\pi ft \qquad (6)$$

$$w = W_0 \exp(px) \cos n\theta \ \sin 2\pi ft$$

where U_0, V_0 and W_0 are arbitrary constants. Substitution of Eqs.(6) into the equations of motion results in the auxiliary equation which is an algebraic equation of the fourth degree in p^2 and n^2 and of the third degree in f^2.

Thus, if values of n and f are assumed, eight roots for p may be determined. Once the values of p, n and f as well as the thickness-to-radius ratio and the material constants are known, the characteristic equations resulting from consideration of the boundary conditions may be solved for the length of the cylinder 2L. The minimum of them determines the length of the cylinder which vibrates at f in a mode having 2n of nodal lines along the generator and none around the circumference.

Calculation of numerical solutions by Flugge's method is tedious in general without the use of high speed digital computers. Flugge could only applied it to a special case where the ends are simply supported such that

$$w = M_x = N_x = v = 0 \; ; \text{ at } x = L \text{ and } x= -L \qquad (7)$$

where N_x is the axial stress resultant and M_x is the axial stress couple. In this special case, the exponential functions in Eqs.(6) can be represented by trigonometric functions such that

$$u = U_0 \sin \pi (x+L)/2L \cos n\theta \sin 2\pi ft$$

$$v = V_0 \cos \pi (x+L)/2L \sin n\theta \sin 2\pi ft \qquad (8)$$

$$w = W_0 \sin \pi (x+L)/2L \cos n\theta \sin 2\pi ft$$

It can be shown easily that Eqs.(8) satisfy Eqs.(7) exactly. The auxiliary equation can now be solved for f for given values of n.

Flugge calculated three roots of f for a specific numerical example. Comparing the amplitude ratios of the displacement components for each root, he has shown that the free vibrations corresponding to the lowest, intermediate and highest values of f are predominantly in lateral, axial and circumferential motion, respectively. He has derived a simple formula for the lowest natural frequencies neglecting in the auxiliary equation cubic and quadratic terms in f^2.

Much of the research effort thereafter and before the emergence of digital computers was directed toward obtaining approximate solutions and verifying them by experiment. Boundary conditions considered in those early days are those of the simply supported ends defined by Eqs.(7), the rigidly clamped ends defined by

$$w = w' = u = v = 0 \text{ ; at } x = L \text{ and } x = -L \qquad (9)$$

or the free ends defined by

$$Q_x = M_x = N_x = T_{x\theta} = 0 \text{ ; at } x = L \text{ and } x = -L \qquad (10)$$

where w' is the partial derivative of w with respect to x, and Q_x and $T_{x\theta}$ are the lateral and circumferential components of the equivalent edge-shear. Arnold and Warburton(10) calculated the natural frequencies of circular cylindrical shells of finite length, which are either simply supported or rigidly clamped at both ends. An equivalent wave length has been introduced in order to deal with cylinders having solidly built-in or flanged ends. In experiment, the modal characteristics were determined by detecting the sound intensity variations by stethoscope. Baron and Bleich(11) calculated the natural frequencies of an infinitely long cylindrical shells by Rayleigh's method. Yu(12) used Donnell type of approximate equations to calculate the natural frequencies of cylindrical shells which were either simply supported or rigidly clamped at both ends, or simply supported at one end and rigidly clamped at the other. Assuming that the shell is relatively long and vibrates in a mode characterized by a small number of the longitudinal waves and a large number of the circumferential waves, he has derived a simple formula for the natural frequency. The characteristic equations turned out to be identical with those of an elastic beam in lateral vibrations. Yu's method was used by Smith and Haft(13) and Vronay and Smith(14) to calculate the natural freqencies of cylindrical shells having rigidly clamped ends. The Donnell type of approximation was also used by Heki(15). From an analysis on the characteristic values of both the circumferentially closed and open shells, Heki has drawn the following conclusions: (i) The Donnell type of approximation gives accurate solutions for natural frequency provided

that the frequency is not very high and the shell is not extremely long. (ii) The edge-zone bending solutions which decay out rapidly as the distance from the ends increases do not have significant influence on the natural frequency. (iii) The natural frequency is governed predominantly by the global solutions which vary gradually over the entire surface of the shell and affected significantly by the inplane boundary conditions. Weingarten(16) made an analysis in the case where one end is rigidly clamped and the other is free and conducted an experiment on butt-welded steel test specimens using a microphone. Watkins and Clary(17) conducted an experiment on spotwelded stainless steel test specimens having the free-free and the clamped-free ends. Sewall and Naumann(18) conducted an analytical and experimental study on the free vibrations of unstiffened shells. The unstiffened test specimens were fabricated from aluminum alloy sheet. Four sets of boundary conditions were considered; the free-free, the simply supported-simply supported, the clamped-free, and the clamped-clamped ends. Nau and Simmonds(19) have derived a simple formula for the low natural frequency of a clamped-clamped shell by asymptotic method.
In the meantime, the development and rapid progress of digital computers enabled researchers to calculate more accurate solutions numerically. The accuracy and the range of validity of the approximate solutions have been examined by comparing with more accurate solutions of the classical shell theory or with elasticity solutions. Greenspon(20) calculated the natural frequency of the flexural vibrations of a hollow cylinder of finite length considering it as a three dimensional body. The results were compared with those of the Timoshenko beam theory in the case of n=1 and with those calculated by Arnold and Warburton(10) in the case of n=2. Gazis(21) made an analysis by elasticity theory on the plane-strain free vibrations of a thick-walled cylinder of finite length. He has shown that the extensional and shear modes can exist uncoupled in the axisymmetric vibrations and derived approximate expressions of the frequencies for these modes, which approach those of the simple thickness-stretch and thickness-shear modes of an infinite plate as the thickness-to radius ratio approaches zero. The minimum of them is given by f_s of the simple thickness-shear mode:

$$f_s^2 = E/8(1+\nu)\rho h^2 \qquad (11)$$

Armenakas[22] calculated the natural frequencies of a simply supported shell by the Flugge and the Donnell type equations and has established a range of validity of these equations on the basis of numerical comparison with the results of Refs.20 and 21. Sharma and Johns[23] calculated the natural frequencies of a finite shell whose ends are either rigidly clamped and free or rigidly clamped and ring-stiffened. The results show that there is no significant difference between the natural frequencies calculated by Flugge theory and those by Timoshenko's version of the Love theory. It is well anticipated, as noted by Kalnins[24], from the results of Gazis' analysis that the solutions for the natural frequency f calculated by the classical theory are valid in a spectrum well below f_s. Since the classical theory is established on the basis of the fundamental assumption

$$h/R \ll 1 \qquad (12)$$

it may be stated that it is valid when

$$f/f_s = O(h/R) \qquad (13)$$

Furthermore, Eqs.(1) and (2) imply that the natural frequency of highly flexural modes is as low as

$$f/f_s = O[(nh/R)^2] \qquad (14)$$

provided that nh/R is much smaller than unity. The last condition always holds for the classical theory of thin shells: Niordson[25] has proved that the fundamental assumptions underlying the classical theory are equivalent to stating that

$$(h/\lambda_m)^2 \ll 1 \qquad (15)$$

where λ_m is the minimum wave length of deformation or vibrations.

3. Effects of Boundary Conditions

Arnold and Warburton[10] made a systematic attempt to clarifying the effects of the boundary conditions by calculating approximate solutions under various boundary conditions of practical interest and by conducting an experiment. They were followed by many researchers, Refs.11-19, who calculated the natural frequencies under the boundary conditions prescribed by Eqs.(7), (9) or (10). As a result, it became clear that the natural frequency depends on these three sets of boundary conditions. The importance of the inplane boundary conditions was first noted by Heki[15] in his memorable but unduly overlooked paper. The significance of the effect of the inplane boundary conditions became more convincing by an extensive numerical analysis of Forsberg[26,27].
In principle, in the framework of the classical theory of thin shells, the boundary conditions at an end of a circular cylindrical shell can be imposed by an appropriate combination of the following four pairs:

$$\begin{aligned} &w = 0 \quad \text{or} \quad Q_x = 0 \\ &w' = 0 \quad \text{or} \quad M_x = 0 \\ &u = 0 \quad \text{or} \quad N_x = 0 \\ &v = 0 \quad \text{or} \quad T_{x\theta} = 0 \end{aligned} \tag{16}$$

There are sixteen possible combinations. Forsberg solved Flugge's equations numerically in all sixteen cases of the boundary conditions in a broad range of geometric parameters. He has presented the results only in ten representative cases including those specified with different boundary conditions at opposite ends. One of the most important conclusions of his analysis on the breathing vibrations ($n \geq 2$) is that the effect of the axial constraint is significant even for very long shells for all values of the thickness-to-radius ratio, so that the minimum frequencies differ by more than 50 per cent depending on whether $u=0$ or $N_x=0$ is imposed at both ends. As to the axisymmetric ($n=0$) and the beam-like bending ($n=1$) vibrations, he has shown that the vibration characteristics are strongly influenced by the circumferential as well as the axial constraints.

The simply supported and the rigidly clamped ends are often used in engineering practice as idealized mathematical models for the supported ends of real shell structures which fall in somewhere between the two models. There are four variations for each of these models depending on the inplane boundary conditions. They are designated and defined as follows:

For simply supported ends

S1: $w = M_x = u = v = 0$

S2: $w = M_x = u = T_{x\theta} = 0$

S3: $w = M_x = N_x = v = 0$ (17)

S4: $w = M_x = N_x = T_{x\theta} = 0$

For clamped ends

C1: $w = w' = u = v = 0$

C2: $u = w' = u = T_{x\theta} = 0$

C3: $u = w' = N_x = v = 0$ (18)

C4: $u = w' = N_x = T_{x\theta} = 0$

Yamaki[(28)] calculated numerical solutions for the eigenvalue problems under various combinations of the boundary conditions of Eqs.(17) and (18). His results confirm Forsberg's conclusions on the effects of the axial constraints.

More recently, Koga[(29)] have derived asymptotic solutions dealing with all possible combinations of the boundary conditions for the free ends, Eqs.(10), the simply supported ends, Eqs.(17), and the clamped ends, Eqs.(18). If the set of the boundary conditions defined by Eqs.(10) is designated by FR, the asymptotic solutions are valid for all forty-five possible combinations between the opposite ends of the nine sets consisting of S1, S2, S3, S4, C1, C2, C3, C4, and FR. The asymptotic solution for the natural frequency is given by

$$f^2 = f_R^2[\ 1 + 12(1-\nu^2)R^2\xi^4/h^2(n^2-1)^2\ ] \qquad (19)$$

where f_R is Rayleigh's solution given by Eq.(1), and ξ is the eigenvalue to be determined from consideration of boundary conditions. Equation (19) is identical in form with that derived by Nau and Simmonds[19] for the cases of C1-C1. The characteristic equations to be satisfied by ξ have been obtained for all forty-five combinations of the boundary conditions. It turned out that it suffices for deriving the characteristic equations to specify only three sets of representative boundary conditions defined and desiganated as

$$\begin{aligned} &SR: w = u = 0 \\ &SF: w = N_x = 0 \\ &FR: N_x = T_{x\theta} = 0 \end{aligned} \tag{20}$$

TYPE	CHARACTERISTIC EQUATIONS	COMBINATIONS OF B.Cs.	REPRESENTATIVE B.Cs.
I	$\cosh 2n\xi\ell \cos 2n\xi\ell - 1 = 0$	S1-S1, S1-S2, S1-C1, S1-C2 S2-S2, S2-C1, S2-C2, C1-C1 C1-C2, C2-C2, (FR-FR)	SR - SR
II	$\cosh 2n\xi\ell \sin 2n\xi\ell - \sinh 2n\xi\ell \cos 2n\xi\ell = 0$	S1-S3, S1-S4, S1-C3, S1-C4 S2-S3, S2-S4, S2-C3, S2-C4 C1-S3, C1-S4, C1-C3, C1-C4 C2-S3, C2-S4, C2-C3, C2-C4 (FR-SF)	SR - SF
III	$\sin 2n\xi\ell = 0$	S3-S3, S3-S4, S3-C3, S3-C4 S4-S4, S4-C3, S4-C4, C3-C3 C3-C4, C4-C4	SF - SF
IV	$\cosh 2n\xi\ell \cos 2n\xi\ell + 1 = 0$	FR-S1, FR-S2, FR-C1, FR-C2	FR - SR
V	$\xi^2 = 0$	FR-FR FR-S3, FR-S4, FR-C3, FR-C4	FR - FR FR - SF

Table 1. Characteristic Equations and Combinations of Boundary Conditions

There are six combinations of these sets between two ends; SR-SR, SR-SF, SF-SF, SR-FR, SF-FR, and FR-FR. The last two result in the solutions for the inextensional vibrations described in Section 1. The remaining four sets result in four mutually independent characteristic equations which are identical in form with those for the free vibrations of an elastic beam. The characteristic equations and the combinations of the boundary conditions are summarized in Table 1.

The representative boundary conditions, Eqs.(20), indicate clearly that the free vibration characteristics of a circular cylindrical shell depend on whether the ends are free or supported, and whether the supported ends are axially constrained.
The minimum roots of the characteristic equations are given by

$$
\begin{aligned}
2nL\xi/R &= 4.730 \text{ (Type I)}, \\
&= 3.927 \text{ (Type II)}, \\
&= 3.142 \text{ (Type III)}, \\
&= 1.875 \text{ (Type IV)}, \\
&= 0 \text{ (Type V)} \qquad (21)
\end{aligned}
$$

For given values of L/R and n, the values of ξ are determined from Eqs.(21), which are then substituted in Eq.(19) to calculate the natural frequencies. Results are shown accurate enough for engineering purpose on the basis of a comparison with more accurate numerical solutions. This has been also proved by an experiment. In the test specimens, the representative boundary conditions of SR were established by soldering an end to a metal block by a low melting point metal, those of SF by attaching a flexural annular plate at an end by adhesive resin, and those of FR by leaving an end completely free. The cross sections of the test specimens of Type I, II, III, and IV are depicted in Fig.4. Test results show a satisfactory agreement with the theoretical preditions by Eq.(19).

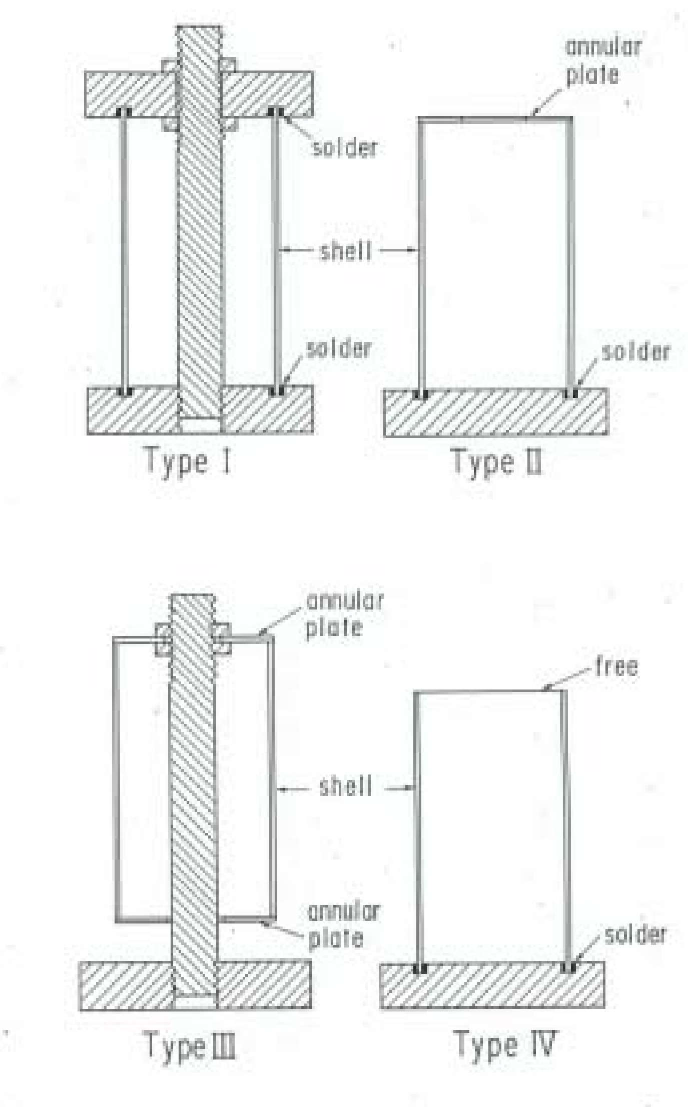

Fig.4. Cross Sections of Test Specimens of Type I, II, III, IV

4. Some Related Eigenvalue Problems

In the theory of thin-walled structures, surface tractions acting on the wall surfaces and body forces in the volume of mass points are treated alike by replacing them by the equivalent forces acting on the reference surfaces. It can be anticipated very well therefore that the analyses and conclusions of the preceding sections for the free vibrations can readily be extended to include the effect of pressurization. If the inertial force vanishes and only the external pressure is present, the analyses will provide solutions for the bifurcation buckling. When the cylindrical shell is filled with a liquid, the fructuating pressure exerted upon the shell from the oscillating liquid may be incorporated in the shell body as inertial force acting on virtual mass. Then, again, the asymptotic method and solutions of the preceding sections can readily be applied to this problem to obtain the natural frequency of a liquid-filled shell.

Koga and his collaborators[30,31,32,33] have obtained solutions for these problems.

Natural frequency under pressure p:

$$f^2 = f_R^2[\, 1 + 12(1-\nu^2)R^2\xi^4/h^2(n^2-1)^2] + n^2(n^2-1)p/4\pi^2\rho hR(n^2+1) \qquad (22)$$

where p is positive for internal pressure.

Buckling pressure:

$$p = -\,[\, Eh^3(n^2-1)/12(1-\nu^2)R^3\,]\{\, 1 + 12(1-\nu^2)R^2\xi^4/h^2(n^2-1)^2\,\} \qquad (23)$$

Natural frequency of a liquid-filled shell:

$$f^2 = \{\, Eh^2n^2(n^2-1)^2/48\pi^2(1-\nu^2)R^4[(\rho+\rho_v)n^2 + \rho\,]\,\}\{\, 1 + 12(1-\nu^2)R^2\xi^4/h^2(n^2-1)^2\,\} \qquad (24)$$

in which r_v is the virtual mass per unit volume and is given in terms of the liquid-mass per unit volume, r_L, approximately by

$$\rho_v = \rho_L R/nh \qquad (25)$$

References

(1) Lord Rayleigh, The Theory of Sound, 2nd. Ed. Dover Publications, 1945 (1st. Ed.1877, 2nd. Ed. 1894)

(2) Lord Rayleigh, On the Infinitesimal Bending of Surfaces of Revolution, Proc. of London Math. Soc., Vol.13, Nov. 1881, pp.4-17

(3) Love, A.E.H., The Small Free Vibrations of and Deformation of a Thin Elastic Shell, Phil. Trans. Roy. Soc. of London, Ser.A, Vol.179, Feb.1888, p.491-546

(4) Lord Rayleigh, On the Bending and Vibration of Thin Elastic Shells, Especially of Cylindrical Form, Proc. Of Roy. Soc. Of London, Vol.45, Dec. 1888, pp.105-123
(5) Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity," 4th. Ed., Cambridge Univ. Press, American printing by Macmillan Co., 1959 (1st. ed. 1892 and 1893, 2nd. Ed. 1906, 3rd. Ed. 1920, 4th. Ed. 1927)
(6) Koga, T., Minimum Natural Frequency and Vibration Characteristics of Circular Cylindrical Shells, Trans. Of Japan Soc. Of Mech. Eng., Vol.46, No.411, Nov. 1980, pp.1280-1289 (in Japanese)
(7) Koga, T. and Kimpara, Y. and Saito, A., Inextensional Free Vibrations of Circular Cylindrical Shells, Proc. Of 28th. JSASS/JSME Structures Conf., July 1986, pp.166-169 (in Japanese)
(8) Koga, T. and Saito, A., Inextensional Free Vibrations of Circular Cylindrical Shells, AIAA Journal, Vol.26, No.12, Dec. 1988, pp.1499-1505
(9) Flugge, W., Statik und Dynamik der Schalen, 3rd. ed. Springer-Verlag, 1962
(10) Arnold, R.N. and Warburton, G.B., Flexural Vibrations of the Walls of Thin Cylindrical Shells having Freely Supported Ends, Proc. of Royal Soc. of London, Series A, Vol.197, No.1049, June 1949, pp.283-256
(11) Baron, M.L. and Bleich, H.H., Tables for Frequencies and Modes of Free Vibration of Infinitely Long Thin Cylindrical Shells, Journal of Appl. Mech., Vol.21, No.2, June.1954, pp.178-184
(12) Yu, Y.-Y., Free Vibrations of Thin Cylindrical Shells Having Finite Lengths With Freely Supported and Clamped Edges, Journal of Appl. Mech., Vol.22, No.4, Dec. 1955, pp.547-552
(13) Smith, B.L. and Haft, E.E., Natural Frequencies of Clamped Cylindrical Shells, AIAA Journal, Vol.6, No.4, April 1968, pp.720-721
(14) Vronay, D.F. and Smith, B.L., Free Vibration of Circular Cylindrical Shells of Finite Length, AIAA Journal, Vol.8, No.3, March 1970, pp.601-603
(15) Heki, K., Vibration of Cylindrical Shells, Journal of Inst. of Polytechnics, Osaka City Univ., Series F, Vol.1, No.1, November 1957
(16) Weingarten, V.I., Free Vibration of Thin Cylindrical Shells, AIAA Journal, Vol.2, No.4, April 1964, pp.717-722
(17) Watkins, J.D. and Clary, R.R., Vibrational Characteristics of Some Thin-Walled Cylindrical and Conical Frustum Shells, NASA TN D-2729, March 1965

(18) Sewall, J.L. and Naumann, E.C., An Experimental and Analytical Vibration Study of Thin Cylindrical Shells With and Without Longitudinal Stiffeners, NASA TN D-4705, September 1968
(19) Nau, R.W. and Simmonds, J.G., Calculation of the Low Natural Frequencies of Clamped Cylindrical Shells by Asymptotic Methods, Int. Journal of Solid and Structures, Vol.9, No.3, May 1973, pp.591-605
(20) Greenspon, J.E., Flexural Vibrations of a Thin Walled Circular Cylinder, Proc. of 3rd. U.S. National Congress of Appl. Mech., June 1958, pp.163-173
(21) Gazis, D.C., Exact Analysis of Plane-Strain Vibrations of Thick-Walled Hollow Cylinders, Journal of Acoust. Soc. of America, Vol.30, No.8, August 1958, pp.786-794
(22) Armenakas, A.E., On the Accuracy of Some Dynamic Shell Theories, Journal of Eng. Mech. Div., Proc. of American Soc. of Civil Eng., Vol.93, EM-5, October 1967, pp.95-109
(23) Sharma, C.B. and Johns, D.J., Vibration Characteristics of a Clamped/Free and Clamped/Ring-Stiffened Circular Cylindrical Shell, Proc. of Symp. on Struct. Dynamics, Loughborough Univ. of Tech., May 1970, Paper No.B.7
(24) Kalnins, A., Dynamic Problems of Elastic Shells, Appl. Mech. Review, Vol.18, No.11, November 1965, pp.867-872
(25) Niordson, F.I., A Consistent Refined Shell Theory, Danish Center for Appl. Math. and Mech., Tech. Univ. of Denmark, Report No.104, May 1976
(26) Forsberg, K., Influence of Boundary Conditions on the Modal Characteristics of Thin Cylindrical Shells, AIAA Journal, Vol.2, No.12, December 1964, pp.2150-2157
(27) Forsberg, K., Axisymmetric and Beam-Type Vibrations of Thin Cylindrical Shells, AIAA Journal, Vol.7, No.2, February 1969, pp.221-227
(28) Yamaki, N., Free Bending Vibrations of Circular Cylindrical Shells, Memoirs of Inst. of High Speed Mech., Tohoku Univ., Vol.26, No.268, 1970 (in Japanese)
(29) Koga, T., Effects of Boundary Conditions on the Free Vibrations of Circular Cylindrical Shells, AIAA Journal, Vol.26, No.11, November 1988, pp.1387-1394
(30) Koga, T. and Lei, P.J.,Bifurcation Buckling and Free Vibrations of Cylindrical Shells Under Pressure, Proc. of ASME 1989 Press. Vessel and

Piping Conf., Hawaii, July 1989
(31) Koga, T. and Kodama, T., Bifurcation Buckling and Free Vibrations of Cylindrical Shells under Pressure, Int. Journal of Pressure Vessels and Piping
(32) Koga, T. and Morimatsu, S., Bifurcation Buckling of Circular Cylindrical Shells under Uniform External Pressure, AIAA Journal, Vol.27, No.2, February 1989, pp.242-248
(33) Koga, T. and Tsushima, M., Breathing Vibrations of a Liquid-Filled Circular Cylindrical Shell, Int. Journal of Solids and Structures
(34)Koga, T. Free Vibrations of Circular Cylindrical Shells, JSME International Journal, Japan Society for Mechanical Engineers, Series I, Vol.32, No.3, 1989, pp.311-319