Thesis
Black Hole Evaporation in Semi-classical Approach

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Abstract

We consider how the mass of the black hole decreases by the Hawking radiation in the Vaidya spacetime, using the concept of dynamical horizon equation, proposed by Ashtekar and Krishnan. Using the formula for the change of the dynamical horizon, we derive an equation for the mass incorporating the Hawking radiation. It is shown that the final state is the Minkowski spacetime in our particular model. We finally solved the equation which describes how black hole mass decreases. The back-reaction problem of the Hawking radiation has not been solved by the conventional method by solving the Einstein equation. While we can solve this problem using the following three ideas. First idea is to use the dynamical horizon equation which only needs information of the horizon surface. Then we calculate usual field equation as the integration equation. Second we taken negative energy into account near the black hole horizon. Using the negative energy we can enlarge the dynamical horizon to the time-like case. Third, we use the Vaidya metric. Usually the Einstein equation and the dynamical horizon equation are not compatible. However, using the Vaidya metric as a background we can use the dynamical horizon equation in place of the Einstein equation.

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Chapter 1

Introduction

The black hole is a peculiar phenomenon in general relativity. In the problem of the black hole there is singularity problem and information loss paradox. The singularity problem means the breakdown of classical general relativity so that to avoid this we should quantize the black hole. The information loss problem is related with the Hawking radiation. In the classical theory of black hole can only absorb and not emit particles. However, quantum mechanical effect causes black holes to create and emit particles. This thermal emission leads to a slow decrease in the mass of the black hole and eventually disappeared. Once the particle enters or information enters in black hole this information vanishes by the black hole evaporation. The Hawking radiation is the phenomenon that the particle is created by the coupling of background black hole spacetime. In the null infinity there are the thermal radiation by the Hawking effect. Relating to the information loss paradox there is the black hole evaporation problem. How the black hole evaporates or final state of the black hole by the Hawking radiation has been an open problem. This problem is known as the back reaction problem. We have solved this problem in this thesis [1]. However, we solved black hole evaporation problem at the semi-classical level. So there remain the black hole quantization problem.

There have been many works concerning black hole evaporation, either in string theories[3][4][5], or semi-classical theory typically using apparent horizon [6]. Hiscok studied spherical model of the black hole evaporation using the Vaidya metric, which we also use in present work, to solve the black hole evaporation problem. However, he simply set a model not taking account of the field equation. Hajicek's work[8] treated the black hole mass more generally than our present case. However, he did not use the field equation either. One of the more recent studies is Sorkin and Piran's work [9] on charged black holes. And neutral case has been done by Hamade and Stewart[10]. Their conclusion is that black hole mass decreases or increases depending on initial condition. They used a model of the double null coordinates, and obtained a numerical result. But they did not consider the Hawking effect directly but they used massless scalar field as a matter. Brevik and Halnes calculated primordial black

hole evaporation[11]. Very recently Hayward studied black hole evaporation and formation using the Vaidya metric [12]. It seems no analytical equation has been proposed for the black hole mass with the Hawking effect taken into account.

The dynamics for the black hole mass with the Hawking effect is a long standing problem. Page[13] derived the equation of mass intuitively, that is $\dot{M} \propto -M^{-2}$. But it does not come from the first principle. We will comment on his intuitive result in the final section. To derive the equation of mass from the first principle we should treat the Einstein equation with the back reaction term by the Hawking radiation. However, the Einstein equation cannot be analytically solved, because the equation contains fourth derivative terms as back reaction. Recently Ashtekar and Krishnan derived an equation which describes how the horizon changes in time [14, 15, 16, 17, 18, 19, 20, 21, 22]. It needs only information of the horizon surface.

The black hole evaporation problem as back-reaction problem usually is not solved for the difficulty of the field equation. However, introducing three ideas we can solve it. First we use the dynamical horizon equation and enlarge it to timelike case using negative energy. And we use a fact that near the horizon the energy become negative. And we calculate the dynamical horizon equation in the Vaidya spacetime which meet the Einstein equation. With these ideas we can solve black hole evaporation problem.

In chapter II we introduce quantum field theory in the curved spacetime [23, 24, 25, 26]. And in this section we calculate the Green function and derive field equation which is solved for the back-reaction problem. And in this chapter we introduce negative energy or violation of energy condition [27, 28, 29, 30, 31, 32, 33, 34, 35, 36] which is used in this paper. In this chapter we derived the Green function which is used to derive negative energy near the black hole horizon. And negative energy is used to enlarge the definition of the dynamical horizon equation. In chapter III we introduce the dynamical horizon and equation of the dynamical horizon. The dynamical horizon equation is enlarged by negative energy to timelike case. And we use the timelike dynamical horizon equation to solve the black hole evaporation problem as back-reaction problem. In chapter IV we solve black hole evaporation problem using the dynamical horizon equation and negative energy in the Vaidya spacetime. In chapter V we discuss and comment on the obtained result.

Chapter 2

Quantum field theory (QFT) in curved spacetime

In this chapter we review quantum field theory (QFT) in curved spacetime. We start with basic formalism of this theory in section 2.1. The Hawking effect is explained in section 2.2. We derive the Green function in section 2.3. And finally we show what is negative energy in section 2.4. This chapter is mainly the derivation of the Green function. Once the Green function is derived we can calculate the energy in the limit of black hole horizon. This result is used in the chapter 3 and 4. Because of the negative energy we enlarge the definition of the dynamical horizon and we enlarge the dynamical horizon equation in the timelike case. And the energy near the event horizon is used in the integration of the dynamical horizon.

2.1 Basic Formalism and Particle Creation

In this section we treat basic formalism and particle creation by the background gravitational field. Particles are created by moving mirror or time dependent background spacetime. The particle creation by the background spacetime is one of main issues in this thesis, because the Hawking effect is the one of its phenomena. To derive the Hawking effect in the next section, we setup the background knowledge in this section.

In the subsection 2-1-1 we start with basic formalism of quantum field theory in curved spacetime and by derive second quantizing a scalar field. In the subsection 2-1-2 we calculate comformal transformation and derive conformal coupling factor in general dimensions. In the subsection 2-1-3 we treat particle creation from the background metric in a two dimensional model, while in the subsection 2-1-4 we treat particle creation by the moving mirror.

2.1.1 Second Quantization in Curved Space

We introduce a scalar field in the curved spacetime, and add an action for the scalar field term to the Einstein-Hilbert action. We start with following Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_a\varphi\partial^a\varphi - m^2\varphi^2 - \xi R\varphi^2), \tag{2.1}$$

where, signature is $(+---)$. Varying φ, we obtain the equation of motion as,

$$\Box\varphi + m^2\varphi + \xi R\varphi = 0. \tag{2.2}$$

Here m is the mass of the particle and ξ is a coupling constant to the Ricci scalar R. This coupling is only an assumption; which we ignore for simplicity. There are also other couplings such that $R_{ab}R^{ab}\varphi$ or $R_{abcd}R^{abcd}\varphi$. If ξ is zero we call that the scalar field is minimally coupled. If ξ is $1/6$ we call that the scalar field is conformally coupled. We define the inner product to the two solutions f_1 and f_2 of (1.2) as

$$(f_1, f_2) = i\int(f_2^* \overleftrightarrow{\partial}_\mu f_1)d\Sigma^\mu. \tag{2.3}$$

Here,$d\Sigma^\mu = d\Sigma\dot{n}^\mu$,$d\Sigma$ means the volume element of spacelike hypersurface, and $n^\mu$is the timelike vector orthogonal to this hypersurface with norm 1. Here the important point is the inner product does not depend on the choice of the hypersurface, that is,

$$(f_1, f_2)_{\Sigma_1} = (f_1, f_2)_{\Sigma_2}. \tag{2.4}$$

This fact is proved in a direct way. At first we assume that the V is the four dimensional submanifold between Σ_1, Σ_2, Then

$$(f_1, f_2)_{\Sigma_1} - (f_1, f_2)_{\Sigma_2} = i\oint_{\partial V}(f_2^* \overleftrightarrow{\partial}_\mu f_1)d\Sigma^\mu = i\int_V \nabla_\mu(f_2^* \overleftrightarrow{\partial}_\mu f_1)dV. \tag{2.5}$$

Here the last step follows from the four dimensional version of Gauss' law, and dV is the four dimensional volume element. We can prove that the way of integration (1.3) does not depend on the choice of the hypersurface Σ, because the following equation holds,

$$\begin{aligned}\nabla_\mu(f_2^* \overleftrightarrow{\partial}_\mu f_1) = \nabla_\mu(f_2^*\partial_\mu f_1 - f_1\partial_\mu f_2^*) &= f_2^*\Box f_1 - f_1\Box f_2^* \\ &-f_2^*(m^2+\xi R)f_1 + f_1(m^2+\xi R)f_2^* = 0,\end{aligned} \tag{2.6}$$

The quantization of the scalar field can be carried by the canonical quantization. The conjugate momentum is defined by

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}, \tag{2.7}$$

and the canonical commutation relation is given by

$$[\varphi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta(\mathbf{x}, \mathbf{x}'). \tag{2.8}$$

Here, the delta function appearing in the above equation is defined by

$$\int F(x')\delta(\mathbf{x},\mathbf{x}^{'})d\Sigma' = F(x), \tag{2.9}$$

for an arbitrary function $F(x)$. Then second quantization can be written by using annihilation and creation operators as,

$$\varphi = \sum_j (a_j f_j + a_j^\dagger f_j^*). \tag{2.10}$$

Here (f_j, f_j^*) form the complete set of solution of the equation of motion with f_j and f_j^* being the positive and negative frequency parts, respectively. And $[a_j, a_{j'}^\dagger] = \delta_{j,j'}$. For the second quantization we refer [37] to the reader.

In curved spacetime, the situation is quite different from the Minkowski vacuum. There is no unique choice of the positive frequency part $\{f_j\}$, and hence no unique notion of the vacuum state. This means that we cannot identify what constitutes a state without particle content, and the notion of particle becomes ambiguous. One possible resolution to this difficulty is to choose some quantities other than particle content to label quantum states. Possible choices might include local expectation values, such as $\langle\varphi\rangle, \langle\varphi^2\rangle$,etc. In the particular case of an asymptotically flat spacetime, we might use the particle content in an asymptotic region. Even this characterization is not unique. However, this non-uniqueness is an essential feature of the theory with physical consequences, namely the phenomenon of particle creation.

2.1.2 Conformal Factor

In this subsection we show that the coupling constant appearing in the previous Lagrangian ξ can determined if we demand the invariance under the conformal transformation. The conformal transformation is defined by the following transformation of the metrics as

$$\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}, \tag{2.11}$$

or,

$$\tilde{ds}^2 = \omega^2(x) ds^2. \tag{2.12}$$

By this transformation Christoffel symbols becomes,

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + C^\rho_{\mu\nu}. \tag{2.13}$$

Here, $C^\rho_{\mu\nu}$ is

$$C^\rho_{\mu\nu} = \omega^{-1}(\delta^\rho_\mu \nabla_\nu\omega + \delta^\rho_\nu \nabla_\mu\omega - g_{\mu\nu} g^{\rho\lambda}\nabla_\lambda\omega). \tag{2.14}$$

If we write the Riemann tensor with this $C^\rho_{\mu\nu}$,

$$\begin{aligned}\tilde{R}^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu} + \nabla_\mu C^\rho_{\nu\sigma} - \nabla_\nu C^\rho_{\mu\sigma} + C^\rho_{\mu\lambda} C^\lambda_{\nu\sigma} - C^\rho_{\nu\lambda} C^\lambda_{\mu\sigma} = \\ R^\rho_{\sigma\mu\nu} - 2(\delta^\rho_{[\mu}\delta^\alpha_{\nu]}\delta^\beta_\sigma - g_{\sigma[\mu}\delta^\alpha_{\nu]} g^{\rho\beta})\omega^{-1}(\nabla_\alpha\nabla_\beta\omega) \\ +2(2\delta^\rho_{[\mu}\delta^\alpha_{\nu]}\delta^\beta_\sigma - 2g_{\sigma[\mu}\delta^\alpha_{\nu]} g^{\rho\beta} + g_{\sigma[\mu}\delta^\rho_{\nu]} g^{\alpha\beta})\omega^{-2}(\nabla_\alpha\omega)(\nabla_\beta\omega).\end{aligned} \tag{2.15}$$

If we contract the above Riemann tensor with ρ and μ we can obtain the Ricci tensor as,

$$\tilde{R}_{\sigma\nu} = R_{\sigma\nu} - [(n-2)\delta^{\alpha}_{\sigma}\delta^{\beta}_{\nu} + g_{\sigma\nu}g^{\alpha\beta}]\omega^{-1}(\nabla_\alpha\nabla_\beta\omega)$$
$$+[2(n-2)\delta^{\alpha}_{\sigma}\delta^{\beta}_{\nu} - (n-3)g_{\sigma\nu}g^{\alpha\beta}]\omega^{-2}(\nabla_\alpha\omega)(\nabla_\beta\omega). \tag{2.16}$$

Here the n is the dimension of the universe $n = 4$ for the physical spacetime. The Ricci scalar is

$$\tilde{R} = \omega^{-2}R - 2(n-1)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\nabla_\beta\omega$$
$$-(n-1)(n-4)g^{\alpha\beta}\omega^{-4}(\nabla_\alpha\omega)(\nabla_\beta\omega)). \tag{2.17}$$

Similarly we can calculate covariant derivative as,

$$\tilde{\nabla}_\mu\tilde{\nabla}_\nu = \nabla_\mu\nabla_\nu\phi - (\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - g_{\mu\nu}g^{\alpha\beta})\omega^{-1}(\nabla_\alpha\omega)(\nabla_\beta\omega). \tag{2.18}$$

From the trace of above formula we obtain

$$\tilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi). \tag{2.19}$$

For the term of ω^{-3} to vanish the ξ in the action should be

$$\xi = \frac{(n-2)}{4(n-1)}. \tag{2.20}$$

If ξ is the above constant, we can say that the action is conformally coupled.

2.1.3 Particle Creation by Gravitational Field

In this section we treat particle creation effect by gravitation [38, 40] for scalar particles i.e. particles corresponding a scalar field The energy and momentum conservation law may not hold if we do not take the back reaction into account. The broken of the conservation low comes from fixing of background. We consider a spacetime which is asymptotically flat in the past and in the future, but non-flat in the intermediate region. The particle creation appeared from background spacetime. Now we write past positive frequency solution as $\{f_j\}$ and write future positive frequency as $\{F_j\}$. Let the orthogonal basis be $(f_j, f_j\ *)$ and (F_j, F_j^*) such that,

$$(f_j, f_{j'}) = (F_j, F_{j'}) = \delta_{jj'}$$
$$(f_j^*, f_{j'}^*) = (F_j^*, F_{j'}^*) = -\delta_{jj'}$$
$$(f_j, f_{j'}^*) = (F_j, F_{j'}^*) = 0. \tag{2.21}$$

These two orthogonal basis satisfy the following Fourier transformation as,

$$f_j = \sum_k(\alpha_{jk}F_k + \beta_{jk}F_k^*). \tag{2.22}$$

Here,

$$\alpha_{jk} = (f_j, F_k) \tag{2.23}$$
$$\beta_{jk} = (f_j, F_k^*) \tag{2.24}$$

Inserting the previous equation we obtain

$$\sum_k (\alpha_{jk}\alpha^*_{j'k} - \beta_{jk}\beta^*_{j'k}) = \delta_{jj'}, \tag{2.25}$$

and

$$\sum_k (\alpha_{jk}\alpha_{j'k} - \beta_{jk}\beta_{j'k}) = 0. \tag{2.26}$$

The inverse transformation to (2.2) is

$$F_k = \sum_j (\alpha^*_{jk} f_j - \beta_{jk} f_j^*). \tag{2.27}$$

The field operator φ is written in terms of $\{f_j\}$ or also $\{F_j\}$ as

$$\varphi = \sum_j (a_j f_j + a_j^\dagger f_j^*) = \sum_j (b_j F_j + b_j^\dagger F_j^\dagger). \tag{2.28}$$

Here a_j, and $a_j^\dagger$ are the creation and annihilation operators in the in-region that is the past infinity region, and $b_j, b_j^\dagger$ are the creation and annihilation operators in the out-region that is the future infinity region. In other words,$a_j|0\rangle_{in} = 0$ and $b_j|0\rangle_{out} = 0$. From the fact that $a_j = (\varphi, f_j)$ and $b_j = (\varphi, F_j)$, we see

$$a_j = \sum_k (\alpha^*_{jk} b - \beta^*_{jk} b_k^\dagger) \tag{2.29}$$
$$b_k = \sum_j (\alpha_{jk} a_j + \beta^*_{jk} a_j^\dagger). \tag{2.30}$$

We call this transformation as the Bogolubov transformation and the α_{jk} and β_{jk} as the Bogolubov coefficients. Now we are ready to describe the physical phenomenon of particle creation by a time-dependent gravitational field. Let us assume that no particles were present before the gravitational field is turned on. If the Heisenberg picture is adopted to describe the quantum dynamics, then $|0\rangle_{in}$ is the state of the system for all time. However, the occupation number operator which counts particles of the mode k in the out region is $N_k = b_k^\dagger b_k$. Thus the mean number of particles created in the mode k is

$$\langle N_k \rangle = {}_{in}\langle 0 | b_k^\dagger b_k | 0 \rangle_{in} = \sum_j |\beta_{jk}|^2. \tag{2.31}$$

We apply the above probability density in the Robertson-Walker universe [26][27]. The metric of this universe is,

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 = a^2(\eta)(d\eta^2 - d\mathbf{x}^2). \tag{2.32}$$

Here, η is the conformal time, as

$$d\eta = dt/a \tag{2.33}$$

$$t = \int^t dt = \int^\eta a(\eta^{'})d\eta^{'}. \tag{2.34}$$

The solution of the scalar field equation in the Friedmann universe is

$$f_{\mathbf{k}}(\mathbf{x}, \eta) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{a(\eta)\sqrt{(2\pi)^3}}\chi_k(\eta). \tag{2.35}$$

Here χ_k satisfies the following equation,

$$\frac{d^2\chi_k}{d\eta^2} + [k^2 - V(\eta)]\chi_k = 0, \tag{2.36}$$

where

$$V(\eta) := -a^2(\eta)[m^2 + (\xi - \frac{1}{6})R(\eta)]. \tag{2.37}$$

At this point, because the norm of $f_{\mathbf{k}}$ is one, we can derive the Wronskian condition as

$$\chi_k\frac{d\chi_k^*}{d\eta} - \chi_k^*\frac{d\chi_k}{d\eta} = i. \tag{2.38}$$

For simplicity we set $m = 0$, and search the solution in the past infinity, then

$$\chi_k(\eta) \sim \chi_k^{(in)}(\eta) = \frac{e^{-i\omega\eta}}{\sqrt{2\omega}}. \tag{2.39}$$

In the future infinity, the solution is

$$\chi_k(\eta) \sim \chi_k^{(out)}(\eta) = \frac{1}{\sqrt{2\omega}}(\alpha_k e^{-i\omega\eta} + \beta_k e^{i\omega\eta}). \tag{2.40}$$

Here the α_k and β_k are calculated by $\alpha_{\mathbf{k}\mathbf{k}'} = \alpha_k\delta_{\mathbf{k}\mathbf{k}'}$ and $\beta_{\mathbf{k}\mathbf{k}'} = \beta_k\delta_{\mathbf{k},-\mathbf{k}'}$. From the above result, the number density created in the unit volume is

$$N = \frac{1}{(2\pi a)^3}\int d^3k|\beta_k|^2. \tag{2.41}$$

And energy density is

$$\rho = \frac{1}{(2\pi a)^3 a}\int d^3k\omega|\beta_k|^2. \tag{2.42}$$

Because to calculate χ_k in analytical way is not easy, we calculate it in the perturbation way as

$$\chi_k(\eta) = \chi_k^{(in)}(\eta) + \omega^{-1}\int_{-\infty}^{\eta} V(\eta')\sin\omega(\eta - \eta')\chi_k(\eta')d\eta'. \tag{2.43}$$

For simplicity we treat the first order approximation and we replace $\chi_k(\eta')$ in the integration by $\chi_k^{(in)}(\eta')$. If we compare with (2.38),

$$\alpha_k \sim 1 + \frac{i}{2\omega}\int_{-\infty}^{\infty} V(\eta) d\eta \tag{2.44}$$

$$\beta_k \sim -\frac{i}{2\omega}\int_{-\infty}^{\infty} e^{-2i\omega\eta} V(\eta) d\eta. \tag{2.45}$$

If we insert V, mean number density and energy density are given as

$$N = \frac{(\xi - \frac{1}{6})^2}{16\pi a^3}\int_{-\infty}^{\infty} a^4(\eta) R(\eta) d\eta \tag{2.46}$$

$$\rho = \frac{(\xi - \frac{1}{6})^2}{32\pi^2 a^4}\int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \{\ln(\eta_1 - \eta_2)\mu \frac{d}{d\eta_1}[a^2(\eta_1)R(\eta_1)] \times \frac{d}{d\eta_2}[a^2(\eta_2)R(\eta_2)]\} \tag{2.47}$$

2.1.4 Particle Creation by Moving Mirrors

In this subsection we treat two dimensional spacetime. If a mirror is accelerating in the two dimensional spacetime and coupled to the universe [40, 41]. A simple example of quantum particle creation was given by Fulling and Davies [28][29]. This consists of a moving mirror in two-dimensional spacetime coupled to a massless scalar field, φ. The field is assumed to satisfy a boundary condition on the world line of the mirror, such as $\varphi = 0$. For a given mirror trajectory, it is possible to construct exact solutions of the wave equation which satisfy this boundary condition. Let $v = t + x$ and $u = t - x$ be null coordinates which are constant upon null rays moving to the left and to the right, respectively. A null ray of fixed v which reflects a moving mirror in two-dimensional spacetime accelerates for a finite period of time. The quantum radiation emitted to the right of the mirror propagates in the spacetime. First we define the orbit of the moving mirror as,

$$x = z(t), |\dot{z}(t)| < 1. \tag{2.48}$$

This is illustrate as following. The massless scalar field in the background satisfies the following equation in two dimensional spacetime as

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0. \tag{2.49}$$

Next we restrict the boundary condition as,

$$\phi(t, z(t)) = 0. \tag{2.50}$$

By the conformal scaling, the metric becomes

$$g_{\mu\nu} \to C(t, x) g_{\mu\nu}. \tag{2.51}$$

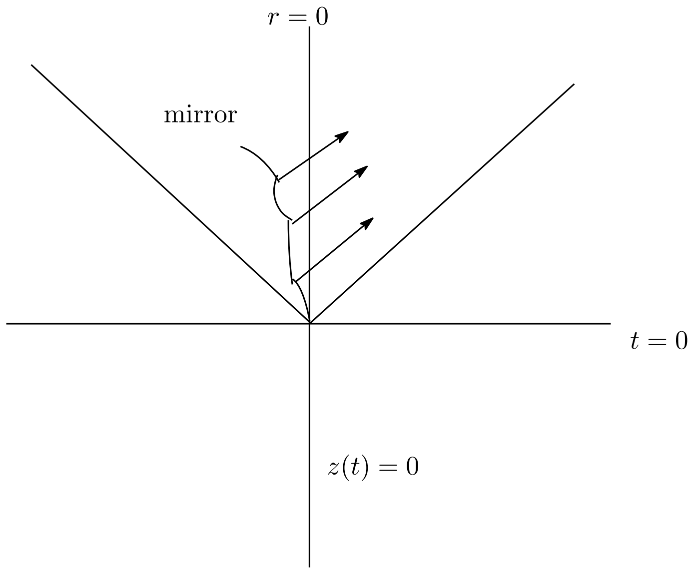

Figure 2.1: particle creation by moving mirrors. First mirror stays at $r = 0$ if $t < 0$. In the time $t > 0$, the mirror accelerates and emit particles.

We transform the null basis to $w \pm s$ by the following two functions as,

$$t - x = f(w - s),\ t + x = g(w + s). \tag{2.52}$$

Then the metric conformally transform as,

$$dt^2 - dx^2 = f'(w - s)g'(w + s)(dw^2 - ds^2). \tag{2.53}$$

Then the wave equation becomes,

$$\frac{\partial^2 \phi}{\partial w^2} - \frac{\partial^2 \phi}{\partial s^2} = 0. \tag{2.54}$$

Now we assume that the moving mirror is at $s = 0$ and that boundary condition there is,

$$\phi(w, 0) = 0. \tag{2.55}$$

Then the following equation holds,

$$\frac{1}{2}[g(w) - f(w)] = z(\frac{1}{2}[g(w) + f(w)]). \tag{2.56}$$

We assume the initial condition as

$$z(t) = 0\ for\ t < 0. \tag{2.57}$$

Then above condition is the same as,

$$f(w) = g(w) = w \ for \ w < 0 \tag{2.58}$$
$$g(w) = w \ for \ all \ w \tag{2.59}$$
$$\frac{1}{2}[w - f(w)] = z(\frac{1}{2}[w + f(w)]). \tag{2.60}$$

The final equation determines the $f(w)$. The complete solution of (2.54) and (2.55) is

$$\phi_\omega(w, s) = (\pi\omega)^{-\frac{1}{2}} \sin \omega s e^{-i\omega w} \ (\omega > 0), \tag{2.61}$$

or in other words

$$\phi_\omega(t, x) = i(4\pi\omega)^{-\frac{1}{2}}[e^{-i\omega g^{-1}(t+x)} - e^{-i\omega f^{-1}(t-x)}]. \tag{2.62}$$

This equation can be rewritten by DeWitt[25] as

$$\phi_\omega(t, x) = i(4\pi\omega)^{-\frac{1}{2}}[e^{-i\omega v} - e^{-\omega(2\tau_u - u)}]. \tag{2.63}$$

Here,

$$u \equiv t - x, \ \ v \equiv t + x \tag{2.64}$$

are the advanced and retarded coordinate respectively. And τ_u is defined by the following equation as,

$$\tau_u - z(\tau_u) = u. \tag{2.65}$$

Moreover u is obtained via the inverse function of f as

$$2\tau_u - u = f^{-1}(u) \equiv p(u). \tag{2.66}$$

We write $T_{\mu\nu}$ as,

$$\frac{1}{2}\begin{bmatrix} (\frac{\partial\phi}{dt})^2 + (\frac{\partial\phi}{\partial x})^2 & \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t} & (\frac{\partial\phi}{dt})^2 + (\frac{\partial\phi}{\partial x})^2 \end{bmatrix}. \tag{2.67}$$

If we write ϕ,

$$\phi(t, x) = \int_0^\infty d\omega[a_\omega^{in}\phi_\omega + a_\omega^{in\dagger}\phi_\omega^*], \tag{2.68}$$

the corresponding operator $T_{\mu\nu}$ is

$$T_{\mu\nu} =: T_{\mu\nu} : + \langle T_{\mu\nu} \rangle. \tag{2.69}$$

Here $: T_{\mu\nu} :$ means normal ordering and $\langle\rangle$ means the expectation value by the vacuum $|0\rangle$ with $a_\omega^{in}|0\rangle = 0$. From these results we obtain

$$\langle T_{\mu\nu} \rangle = \int_0^\infty d\omega T_{\mu\nu}(\phi_\omega, \phi_\omega^*). \tag{2.70}$$

If we use the point-splitting method to evaluate (2.70) by noting

$$\partial\phi_\omega/\partial t = (\frac{\omega}{4\pi})^{\frac{1}{2}}[e^{-i\omega v} - p'(u)e^{-i\omega p(u)}], \tag{2.71}$$

$$\partial\phi_\omega/\partial x = (\frac{\omega}{4\pi})^{\frac{1}{2}}[e^{-i\omega v} + p'(u)e^{-i\omega p(u)}], \tag{2.72}$$

$$\partial\phi^*_\omega/\partial t = (\frac{\omega}{4\pi})^{\frac{1}{2}}[e^{-i\omega(v+\varepsilon)} - p'(u+\varepsilon)e^{-i\omega p(u+\varepsilon)}], \tag{2.73}$$

$$\partial\phi^*_\omega/\partial x = (\frac{\omega}{4\pi})^{\frac{1}{2}}[e^{-i\omega(v+\varepsilon)} + p'(u+\varepsilon)e^{-i\omega p(u+\varepsilon)}]. \tag{2.74}$$

We obtain

$$\langle T_{00}\rangle = \langle T_{11}\rangle = \frac{1}{4\pi}\int_0^\infty \omega d\omega\{e^{i\omega\varepsilon} + p'(u)p'(u+\varepsilon)e^{i\omega[p(u+\varepsilon)-p(u)]}\}, \tag{2.75}$$

$$\langle T_{10}\rangle = \langle T_{01}\rangle = \frac{1}{4\pi}\int_0^\infty \omega d\omega\{e^{i\omega\varepsilon} - p'(u)p'(u+\varepsilon)e^{i\omega[p(u+\varepsilon)-p(u)]}\}. \tag{2.76}$$

If we use Taylor expansion, we obtain finally

$$\langle T_{00}\rangle = -(2\pi\varepsilon^2)^{-1} - \langle T_{01}\rangle \tag{2.77}$$

$$\langle T_{01}\rangle = (24\pi)^{-1}[\frac{p'''}{p'} - \frac{3}{2}(\frac{p''}{p'})^2] + O(\varepsilon), \tag{2.78}$$

$$= -(12\pi)^{-1}(p')^{\frac{1}{2}}[(p')^{-\frac{1}{2}}]'' + O(\varepsilon). \tag{2.79}$$

Unfortunately, the simple solution for the moving mirror radiation of a massless field in two-dimensional spacetime depends upon the special conformal properties in this case and does not generalize to massive field or to four dimensional spacetime. In the four dimensional case, there are exact solutions available for special trajectories [34][35], and approximate solutions for general trajectories [36], but no general, exact solutions. However, the technique of mapping between ingoing and outgoing ray is crucial in the derivation of particle creation by black holes.

2.2 The Hawking Effect

In this section, we will apply the notions of particle creation by gravitational fields to black hole spacetime. This leads to the Hawking effect [2, 42], the process by which black holes emit a thermal spectrum of particles. For the sake of definiteness、we will concentrate on the case of massless, scalar field in the Schwarzschild spacetime, but the basic ideas may by applied to any quantum field in a general black hole spacetime. For the most part, we will follow the original derivation given by Hawking [2]. We imagine that the black hole was formed at some time in the past by gravitational collapse. We assume gravitational collapse of the massive star and in the null past spacetime is close to the Minkowski spacetime.

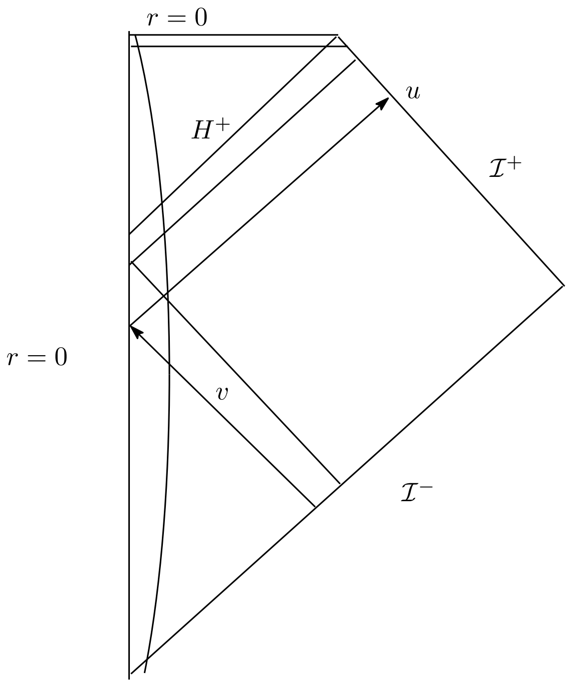

Figure 2.2: The Penrose diagram for the spacetime of a black hole formed by gravitational collapse. The $r = 0$ line on the left is the world line of the center of this body, the $r = 0$ line at the top of the diagram is the curvature singularity, and H^+ is the future event horizon. An ingoing light ray from $\mathcal{I}^-$ passes through the body and escapes to $\mathcal{I}^+$ as a $u =$ constant light ray.

Now we introduce the following two null vectors as,

$$v = t + r^* \tag{2.80}$$

$$u = t - r^* \tag{2.81}$$

$$r^* = r + 2M \ln \frac{|r - 2M|}{2M}. \tag{2.82}$$

Here, M is the black hole mass. Then the spacetime metric in the spacetime region outside of the black hole is Schwarzschild metric

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2 d\Omega^2 = (1 - \frac{2M}{r})dudv - r^2 d\Omega^2. \tag{2.83}$$

If we write in-mode wave function as $f_{\omega lm}$ and out-mode wave function $F_{\omega lm}$, then in the null infinity the wave function becomes as,

$$f_{\omega lm} \sim \frac{Y_{lm}(\theta, \phi)}{\sqrt{4\pi\omega r}} e^{-i\omega v} \ on \ \mathcal{I}^-, \tag{2.84}$$

$$f_{\omega lm} \sim \frac{Y_{lm}(\theta, \phi)}{\sqrt{4\pi\omega r}} e^{-i\omega G(u)} \ on \ \mathcal{I}^+, \tag{2.85}$$

$$F_{\omega lm} \sim \frac{Y_{lm}(\theta, \phi)}{\sqrt{4\pi\omega r}} e^{-i\omega u} \ on \ \mathcal{I}^+, \tag{2.86}$$

$$F_{\omega lm} \sim \frac{Y_{lm}(\theta, \phi)}{\sqrt{4\pi\omega r}} e^{-i\omega g(v)} \ on \ \mathcal{I}^-. \tag{2.87}$$

By the geometrical optics approximation and the boundary condition we see $u = g(v), v = g^{-1}(u) \equiv G(u)$ at $r = 0$. Although the u and v are originally independent, by the reflection at the center of black hole these two vectors depend.

Hawking gives a general rey-tracing argument which leads to the result that

$$u = g(v) = -4M \ln (\frac{v_0 - v}{C}) \tag{2.88}$$

or

$$v = G(u) = v_0 - Ce^{-u/4M}, \tag{2.89}$$

in the geometrical optics approximation. Here C is a constant and v_0 is the limiting value of v for rays which pass through the body before the horizon forms. To derive this equation we assume that the inner metric becomes as

$$ds^2 = dT^2 - dr^2 - r^2 d\Omega^2. \tag{2.90}$$

Because of Eq. (2.83) and Eq. (2.90), at the event horizon following equation holds as

$$1 - (\frac{dR}{dT})^2 = (\frac{R - 2M}{R})(\frac{dt}{dT})^2 - (\frac{R - 2M}{R})^{-1}(\frac{dR}{dT})^2. \tag{2.91}$$

Now we use approximation that R is close to $2M$, and set the time T_0 at $R = 2M$. Then approximately we can obtain the equation as

$$R(T) \approx 2M + A(T_0 - T). \tag{2.92}$$

Inserting this equation to Eq.2.91, we obtain the following approximate equation as

$$(\frac{R - 2M}{2M})^2 \approx (\frac{R - 2M}{2M})^{-2}(\frac{dR}{dT}) \approx \frac{(2M)^2}{(T - T_0)^2}. \tag{2.93}$$

From the above formula, we obtain

$$t \sim -2M \ln(\frac{T_0 - T}{B}), \; T \; \rightarrow \; T_0. \tag{2.94}$$

Similar calculation can be carried out for r^* and we obtain

$$r^* \sim 2M \ln(\frac{r - 2M}{2M}) \sim 2M \ln \frac{A(T_0 - T)}{2M}. \tag{2.95}$$

In this limit the approximation

$$U = T - r = T - R(T) \sim (1 + A)T - 2M - AT_0 \tag{2.96}$$

holds. Now we write two null coordinates in the shell as V, U. Here $V = T + r$ and $U = T - r$. There are three conditions to be determined: the relation between the values of the null coordinates v and V for the ingoing ray, the relation between V and U at the center of the shell, and finally the relation between U and u for the outgoing ray; see figure 3. Let us suppose that our null ray enters the shell at a radius of R_1, which is finitely larger than $2M$. At this point, both $R/(R-2M)$ and dR/dt are finite and approximately constant. Thus dt/dT is approximately constant, so $t \propto T$. Similarly r^* is a linear function near $r = R_1$. Then,

$$V(v) = av + b, \quad a, b = const. \tag{2.97}$$

The V coincides with the U is at $r = 0$,

$$U = V. \tag{2.98}$$

Inserting Eq.(2.94) and Eq. (2.95) to the Eq. (2.97) and the Eq. (2.98), we arrive at Eq.(2.88) and Eq. (2.89). Although we have performed our explicit calculation for the special case of a thin shell, the result is more general, as is revealed by Hawking's derivation. We can understand why this is this case; the crucial logarithmic dependence which governs the asymptotic form of $u(v)$ comes from the last step in the above sequence of matchings. This step through the collapsing body, which is essentially independent of the interior geometry. We could imagine dividing a general spherically symmetric star into a sequence of collapsing shells. As the null ray enters and exits each shell, each null coordinate is a linear function of the proceeding one, until we come to the exit from the last shell. At this point, the retarded time u in the exterior spacetime is a

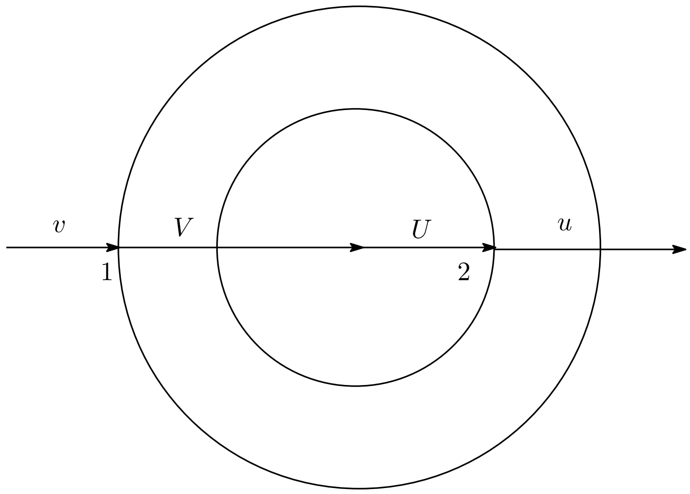

Figure 2.3: An ingoing ray enters the collapsing shell at point 1, passes through the origin, and exits as an out going ray at point 2, when the shell has shrunk to a smaller radius. Note that the rays in question are actually imploding or exploding spherical shells of light.

logarithmic function of the previous coordinate, and hence also a logarithmic function of v as given by Eq.(2.88)

From (2.87) the out-mode function is obtained as

$$F_{\omega lm} \sim e^{4Mi\omega \ln [(v_0 - v)/C]}, \ v < v_0 \tag{2.99}$$

$$F_{\omega lm} \sim 0 \ v > v_0 \tag{2.100}$$

By the Bogolubov transformation of the above out-mode function, we obtain

$$F_{\omega lm} = \int_0^\infty d\omega^{'} (\alpha^*_{\omega^{'} \omega lm} f_{\omega^{'} lm} - \beta_{\omega^{'} \omega lm} f^*_{\omega^{'} lm}). \tag{2.101}$$

Here $\alpha_{\omega^{'} \omega lm} = \alpha_{\omega^{'} lm, \omega lm}$ and $\beta_{\omega^{'} \omega lm} = \beta_{\omega^{'} lm, \omega lm}$. From this equation we obtain as

$$\alpha^*_{\omega^{'} \omega lm} = \frac{1}{2\pi} \sqrt{\frac{\omega^{'}}{\omega}} \int_{-\infty}^{v_0} dv e^{i\omega^{'} v} e^{4Mi\omega \ln [(v_0 - v)/C]}, \tag{2.102}$$

$$\beta_{\omega^{'} \omega lm} = -\frac{1}{2\pi} \sqrt{\frac{\omega^{'}}{\omega}} \int_{-\infty}^{v_0} dv e^{-i\omega^{'} v} e^{4Mi\omega \ln [(v_0 - v)/C]}, \tag{2.103}$$

or equivalently,

$$\alpha^*_{\omega^{'} \omega lm} = \frac{1}{2\pi} \sqrt{\frac{\omega^{'}}{\omega}} \int_0^\infty dv e^{-i\omega^{'} v^{'}} e^{4Mi\omega \ln (v^{'}/C)}, \tag{2.104}$$

$$\beta_{\omega^{'} \omega lm} = \frac{1}{2\pi} \sqrt{\frac{\omega^{'}}{\omega}} \int_0^\infty dv e^{i\omega^{'} v^{'}} e^{4Mi\omega \ln (v^{'}/C)}. \tag{2.105}$$

Noting that the integration along the closed loop C becomes

$$\oint_C dv' e^{-i\omega' v'} e^{4Mi\omega \ln{(v'/C)}} = 0, \tag{2.106}$$

we obtain

$$\begin{aligned}\int_0^\infty dv' e^{-i\omega' v'} e^{4Mi\omega \ln{(v'/C)}} &= -\int_0^\infty dv' e^{i\omega' v'} e^{4Mi\omega \ln{(-v'/C - i\varepsilon)}} \\ &= -e^{4\pi M\omega} \int_0^\infty dv' e^{i\omega' v'} e^{4Mi\omega \ln{(v'/C)}}. \end{aligned} \tag{2.107}$$

In the first line we have used Eq.(2.106) and make variable transformation $v' \to -v'$. In the second line we have used the relation of $\ln{(-v'/C - i\varepsilon)} = -\pi + i\ln{(v'/C)}$.
From these result we obtain the result as

$$|\alpha_{\omega'\omega lm}| = e^{4\pi M\omega} |\beta_{\omega'\omega lm}|. \tag{2.108}$$

From the relation of the Bogolubov coefficients, we obtain

$$\sum_{\omega'} (|\alpha_{\omega'\omega lm}|^2 - |\beta_{\omega'\omega lm}|^2) = \sum_{\omega'} (e^{8\pi M\omega} - 1)|\beta_{\omega'\omega lm}|^2 = 1. \tag{2.109}$$

Finally the Number density in the future null infinity becomes

$$N_{\omega lm} = \sum_{\omega'} |\beta_{\omega'\omega lm}|^2 = \frac{1}{e^{8\pi M\omega} - 1}. \tag{2.110}$$

If we identify the above result with the Bose-Einstein statistical thermal radiation, the temperature is

$$T_H = \frac{1}{8\pi M}. \tag{2.111}$$

We call this temperature as the Hawking temperature.

2.3 Green Function and stress-tensor Renormalization

In this section we derive Green function [51, 52, 53, 54, 55, 56, 57] of the scalar field created from the coupling to the background metric. The Green function is important issue because we use Green function in the derivation of negative energy near black hole. In the subsection 2.3.1, we derive the Green function with complicated calculation and we re-normalize its trace. In the subsection 2.3.2 we treat infrared behaviour.

2.3.1 Ultraviolet Behavior

In this subsection we treat the stress-energy tensor. At the first time, we define the Green function in a abstract way. The Green function is defined by

$$G_1(x, x') \equiv \langle 0|\phi(x)\phi(x')|0\rangle. \tag{2.112}$$

The Hadamard expansion of the Green function can be written as

$$G_1(x, x') = \frac{U(x, x')}{\rho} + V(x, x')\ln\rho + W(x, x'). \tag{2.113}$$

Here $\rho = \frac{1}{2}y_a y^a$ and y^a is the vector whose curve is a geodesics curve connecting x and x'. If we look at a small region in which x and x' are contained. The relation $\rho = \frac{1}{2}(x - x')^2$ holds. We now explain the way to derive an expression for the stress-energy tensor in terms of the Green function which is given in (2.111). The action for the scalar field coupled to background is

$$S[\phi] = \frac{1}{2}\int d^4x g^{1/2}\phi(\Box - \xi R - m^2)\phi, \tag{2.114}$$

and the field equation is

$$g^{-1/2}\frac{\delta S}{\delta\phi} = (\Box - \xi R - m^2)\phi = 0. \tag{2.115}$$

The stress-energy tensor from the action can be written as

$$\begin{aligned} T^{ab} &\equiv 2g^{-1/2}\frac{\delta S}{\delta g_{ab}} \\ &= (1-2\xi)\phi^{;a}\phi^{;b} + (2\xi - \frac{1}{2})g^{ab}\phi_{;c}\phi^{;c} - 2\xi\phi\phi^{;ab} + 2\xi g^{ab}\phi\Box\phi \\ &\quad +\xi(R^{ab} - \frac{1}{2}Rg^{ab})\phi^2 - \frac{1}{2}m^2 g^{ab}\phi^2. \end{aligned} \tag{2.116}$$

If we use the point-splitting method, the stress-energy tensor becomes

$$T^{ab} = [\vec{\tau}^{ab}(\phi(x)\phi(x'))] \equiv \lim_{x\to x'}\vec{\tau}^{ab}(\phi(x)\phi(x')). \tag{2.117}$$

Here $\vec{\tau}^{ab}$ is defined as the following,

$$\begin{aligned} \vec{\tau}^{ab} &= (1-2\xi)g^b_{b'}\nabla^a\nabla^{b'} + (2\xi - \frac{1}{2})g^{ab}g^c_{c'}\nabla^c\nabla^{c'} - 2\xi\nabla^a\nabla^b \\ &\quad +2\xi g^{ab}\nabla_c\nabla^c + \xi(R^{ab} - \frac{1}{2}Rg^{ab}) - \frac{1}{2}m^2 g^{ab}. \end{aligned} \tag{2.118}$$

For the case of massless minimally coupled scalar field,

$$T_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,a}\phi^{,a}. \tag{2.119}$$

Using the definition of the Green function (2.111) we obtain as

$$\langle T_{ab}\rangle = \frac{1}{2}\lim_{x'\to x}\{[\partial_a\partial_{b'} - \frac{1}{2}\partial_a\partial^{a'}]G^{(1)}(x,x')\}. \tag{2.120}$$

From now on we study the Green function. At first we decompose the metric in the Riemann normal coordinate around x' as,

$$\begin{aligned} g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}y^\alpha y^\beta - \frac{1}{6}R_{\mu\alpha\nu\beta;\gamma}y^\alpha y^\beta y^\gamma \\ +(-\frac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \frac{2}{45}R_{\alpha\mu\beta\lambda}R^\lambda{}_{\gamma\mu\delta})y^\alpha y^\beta y^\gamma y^\delta. \end{aligned} \tag{2.121}$$

In the same way we can calculate its determinant as,

$$\begin{aligned} g = 1 - \frac{1}{3}R_{\alpha\beta}y^\alpha y^\beta - \frac{1}{6}R_{\alpha\beta;\gamma}y^\alpha y^\beta y^\gamma \\ +(\frac{1}{18}R_{\alpha\beta}R_{\gamma\delta} - \frac{1}{90}R_{\lambda\alpha\beta}{}^\kappa R_{\lambda\gamma\delta\kappa} - \frac{1}{20}R_{\alpha\beta;\gamma\delta})y^\alpha y^\beta y^\gamma y^\delta. \end{aligned} \tag{2.122}$$

In the next step we define $\bar{G}(x,x')$ following as,

$$\begin{aligned} G(x,x') &= g^{-1/4}(x)\bar{G}(x,x')g^{-1/4}(x') \\ &= g^{-1/4}\bar{G}(x,x'). \end{aligned} \tag{2.123}$$

Inserting these results into the equation of the Green function,

$$(\Box - m^2 + \xi R)G(x,x') = g^{-1/2}\delta(x-x'), \tag{2.124}$$

then after the complicated calculations we obtain

$$\begin{aligned} \eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G} - [m^2 + (\xi - \frac{1}{6})R]\bar{G} - \frac{1}{3}R_\alpha{}^\nu y^\alpha\partial_\nu\bar{G} \\ +\frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta y^\alpha y^\beta\partial_\mu\partial_\nu\bar{G} - (\xi - \frac{1}{6})R_{;\alpha}y^\alpha\bar{G} \\ +(\frac{1}{3}R_\alpha{}^\nu{}_{;\beta} + \frac{1}{6}R_{\alpha\beta}{}^{;\nu})y^\alpha y^\beta\partial_\nu\bar{G} + \frac{1}{6}R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma}y^\alpha y^\beta y^\gamma\partial_\mu\partial_\nu\bar{G} \\ -\frac{1}{2}(\xi - \frac{1}{6})R_{;\alpha\beta}y^\alpha y^\beta\bar{G} + (-\frac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} + \frac{1}{60}R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda} \\ +\frac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta} - \frac{1}{120}R_{;\alpha\beta} + \frac{1}{40}\Box R_{\alpha\beta})y^\alpha y^\beta\bar{G} \\ +(-\frac{3}{20}R^\nu{}_{\alpha;\beta\gamma} + \frac{1}{10}R_{\alpha\beta}{}^{;\nu}{}_\gamma - \frac{1}{60}R^\kappa{}_\alpha{}^\nu{}_\beta R_{\kappa\gamma} \\ +\frac{1}{15}R^\kappa{}_{\alpha\lambda\beta}R_\kappa{}^\nu{}_\gamma{}^\lambda)y^\alpha y^\beta y^\gamma\partial_\nu\bar{G} + (\frac{1}{20}R^\mu{}_\alpha{}^\mu{}_{\beta;\gamma\delta} \\ +\frac{1}{15}R^\mu{}_{\alpha\lambda\beta}R^\lambda{}_\gamma{}^\nu{}_\delta)y^\alpha y^\beta y^\gamma y^\delta\partial_\mu\partial_\nu\bar{G} = -\delta(y). \end{aligned} \tag{2.125}$$

Extending above equation to an arbitrary dimensions, and performing the Fourier transformation, we obtain,

$$\bar{G}(x,x') = \int\frac{d^nk}{(2\pi)^n}e^{iky}\bar{G}(k). \tag{2.126}$$

Here $ky \equiv k_\alpha y^\alpha = \eta^{\alpha\beta} k_\alpha y_\beta$. Expanding the $\bar{G}(k)$, we can write

$$\bar{G}(k) = \bar{G}_0(k) + \bar{G}_1(k) + \bar{G}_2(k) + \cdots . \tag{2.127}$$

The relation to the $\bar{G}_i(x, x')$ is given by

$$\bar{G}_i(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{iky} \bar{G}_i(k), \tag{2.128}$$

where

$$\bar{G}_0(k) = (k^2 + m^2)^{-1} \tag{2.129}$$

and the first order is

$$\bar{G}_1(k) = 0. \tag{2.130}$$

And the second order term is

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \bar{G}_2 - m^2 \bar{G}_2 + (\xi - \frac{1}{6}) R \bar{G}_0 - \frac{1}{3} R_\alpha{}^\nu y^\alpha \partial_\nu \bar{G}_0 + \frac{1}{3} R^\mu{}_\alpha{}^\nu{}_\beta y^\alpha y^\beta \partial_\mu \partial_\nu \bar{G}_0 = 0. \tag{2.131}$$

If we take into account of the Lorentz invariance for y^a in the above equation,

$$-\frac{1}{3} R_\alpha{}^\nu y^\alpha \partial_\nu \bar{G}_0 + \frac{1}{3} R^\mu{}_\alpha{}^\nu{}_\beta y^\alpha y^\beta \partial_\mu \partial_\nu \bar{G}_0 \equiv 0 \tag{2.132}$$

Then the Eq. (2.131) becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \bar{G}_2 - m^2 \bar{G}_2 + (\xi - \frac{1}{6}) R \bar{G}_0 = 0, \tag{2.133}$$

and the second order term is

$$\bar{G}_2(k) = (\frac{1}{6} - \xi) R / (k^2 + m^2)^2. \tag{2.134}$$

In the similar calculations, the condition of the Lorentz invariance for the y^a is

$$(\frac{1}{3} R_\alpha{}^\nu{}_{;\beta} + \frac{1}{6} R_{\alpha\beta}{}^{;\nu}) y^\alpha y^\beta \partial_\nu \bar{G}_0 + \frac{1}{6} R^\mu{}_\alpha{}^\nu{}_{\beta;\gamma} y^\alpha y^\beta y^\gamma \partial_\mu \partial_\nu \bar{G}_0 \equiv 0, \tag{2.135}$$

and

$$(-\frac{3}{20} R^\nu{}_{\alpha;\beta\gamma} + \frac{1}{10} R_{\alpha\beta}{}^{;\nu}{}_\gamma - \frac{1}{60} R^\kappa{}_\alpha{}^\nu{}_\beta R_{\kappa\gamma} + \frac{1}{15} R^\kappa{}_{\alpha\lambda\beta} R_\kappa{}^\nu{}_\gamma{}^\lambda) y^\alpha y^\beta y^\gamma \partial_\nu \bar{G}_0 + (\frac{1}{20} R^\mu{}_\alpha{}^\mu{}_{\beta;\gamma\delta} + \frac{1}{15} R^\mu{}_{\alpha\lambda\beta} R^\lambda{}_\gamma{}^\nu{}_\delta) y^\alpha y^\beta y^\gamma y^\delta \partial_\mu \partial_\nu \bar{G}_0 \equiv 0. \tag{2.136}$$

Though Eq.(2.83) becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G} - [m^2 + (\xi - \frac{1}{6})R]\bar{G} - (\xi - \frac{1}{6})R_{;\alpha}y^\alpha\bar{G}$$
$$-\frac{1}{2}(\xi - \frac{1}{6})R_{;\alpha\beta}y^\alpha y^\beta\bar{G} + (-\frac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} + \frac{1}{60}R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda}$$
$$+\frac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta} - \frac{1}{120}R_{;\alpha\beta} + \frac{1}{40}\Box R_{\alpha\beta})y^\alpha y^\beta\bar{G} = -\delta(y). \tag{2.137}$$

If we write it in the momentum space with k being the momentum, we obtain

$$[k^2 + m^2 + (\xi - \frac{1}{6})R]\bar{G} + i(\xi - \frac{1}{6})R_{;\alpha}\partial^\alpha\bar{G}$$
$$+[-\frac{1}{2}(\xi - \frac{1}{6})R_{;\alpha\beta} - \frac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} + \frac{1}{60}R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda}$$
$$+\frac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta} - \frac{1}{120}R_{;\alpha\beta} + \frac{1}{40}\Box R_{\alpha\beta}]\partial^\alpha\partial^\beta\bar{G}(k) = 1. \tag{2.138}$$

If we solve this equation for the fourth order, we obtain

$$\bar{G}(k) = (k^2 + m^2)^{-1} + (\frac{1}{6} - \xi)R(k^2 + m^2)^{-2}$$
$$+i(\frac{1}{6} - \xi)R_{;\alpha}(k^2 + m^2)^{-1}\partial^\alpha(k^2 + m^2)^{-1}$$
$$+(\frac{1}{6} - \xi)^2R^2(k^2 + m^2)^{-3}$$
$$+a_{\alpha\beta}(k^2 + m^2)^{-1}\partial^\alpha\partial^\beta(k^2 + m^2)^{-1}. \tag{2.139}$$

Here

$$a_{\alpha\beta} = \frac{1}{2}(\xi - \frac{1}{6})R_{;\alpha\beta} + \frac{1}{30}R_\alpha{}^\lambda R_{\lambda\beta} - \frac{1}{60}R^\kappa{}_\alpha{}^\lambda{}_\beta R_{\kappa\lambda}$$
$$-\frac{1}{60}R^{\lambda\mu\kappa}{}_\alpha R_{\lambda\mu\kappa\beta} + \frac{1}{120}R_{;\alpha\beta} - \frac{1}{40}\Box R_{\alpha\beta}. \tag{2.140}$$

In the next step we define the differential relation as,

$$(k^2 + m^2)^{-1}\partial^\alpha(k^2 + m^2)^{-1} \equiv \frac{1}{2}\partial^\alpha(k^2 + m^2)^{-2} \tag{2.141}$$

$$(k^2 + m^2)^{-1}\partial^\alpha\partial^\beta(k^2 + m^2)^{-1} \equiv \frac{1}{3}\partial^\alpha\partial^\beta(k^2 + m^2)^{-2}$$
$$-\frac{2}{3}\eta^{\alpha\beta}(k^2 + m^2)^{-3}. \tag{2.142}$$

The Eq.(2.139) becomes

$$\bar{G}(k) = (k^2 + m^2)^{-1} + (\frac{1}{6} - \xi)R(k^2 + m^2)^{-2}$$
$$+\frac{1}{2}i(\frac{1}{6} - \xi)R_{;\alpha}\partial^\alpha(k^2 + m^2)^{-2}$$
$$+\frac{1}{3}a_{\alpha\beta}\partial^\alpha\partial^\beta(k^2 + m^2)^{-2}$$
$$+[(\frac{1}{6} - \xi)^2R^2 - \frac{2}{3}a_\lambda{}^\lambda](k^2 + m^2)^{-3}. \tag{2.143}$$

If we derive $\bar{G}(x,x')$, we carry out the inverse Fourier transformation of the Green function, we obtain

$$\bar{G}(x,x') = \int \frac{d^n k}{(2\pi)^n} e^{iky}[1 + f_1(x,x')(-\frac{\partial}{\partial m^2}) \\ +f_2(x,x')(\frac{\partial}{\partial m^2})^2]\frac{1}{k^2+m^2}. \tag{2.144}$$

Here,

$$f_1(x,x') = (\frac{1}{6}-\xi)R + \frac{1}{2}(\frac{1}{6}-\xi)R_{;\alpha}y^\alpha \\ -\frac{1}{3}a_{\alpha\beta}y^\alpha y^\beta \tag{2.145}$$

$$f_2(x,x') = \frac{1}{2}(\frac{1}{6}-\xi)^2R^2 - \frac{1}{3}a^\lambda{}_\lambda. \tag{2.146}$$

We have used the following integration relation

$$(k^2+m^2)^{-1} = \int_0^\infty ids \exp[-is(k^2+m^2)], \tag{2.147}$$

where m^2 should be replaced by $m^2 - i\varepsilon$ precisely speaking. And we use following definition as

$$F(x,x';is) = 1 + f_1(x,x')is + f_2(x,x')(is)^2. \tag{2.148}$$

Moreover we use the identity,

$$\int \frac{d^n k}{(2\pi)^n} \exp[-is(k^2+m^2)+iky] \\ = \frac{i}{(4\pi)^{n/2}}(is)^{-n/2}\exp(-im^2 s - \frac{\sigma}{2is}), \tag{2.149}$$

where $\sigma = \frac{1}{2}y_\alpha y^\alpha$. Then we obtain

$$\bar{G}(x,x') = \frac{i}{(4\pi)^{n/2}} \\ \times \int_0^\infty \frac{ids}{(is)^{n/2}} \exp(-im^2 s - \frac{\sigma}{2is})F(x,x';is). \tag{2.150}$$

Using van Vleck determinant,

$$\Delta(x,x') = -g^{-1/2}(x)\det[-\partial_\mu\partial_{\mu'}\sigma(x,x')]g^{-1/2}(x'). \tag{2.151}$$

we obtain the Green function as

$$G(x,x') = \frac{i\Delta^{1/2}(x,x')}{(4\pi)^{n/2}} \\ \times \int_0^\infty \frac{ids}{(is)^{n/2}} \exp(-im^2 s - \frac{\sigma}{2is})F(x,x';is). \tag{2.152}$$

In what follows we show

$$\frac{2}{(-g)^{\frac{1}{2}}}\frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu}\rangle, \tag{2.153}$$

where we write the *effective action* as W, The generating function as Z is given by

$$Z[J] = \int \mathcal{D}[\phi] \exp\{iS_m[\phi] + i\int J(x)\phi(x)d^n x\} \tag{2.154}$$

By definition we have

$$Z[0] \equiv \langle \text{out,0}|0, \text{in}\rangle = \langle 0|0\rangle = 1 \tag{2.155}$$

Since the stress-energy tensor $T_{\mu\nu}$ is given by

$$\frac{2}{(g)^{\frac{1}{2}}}\frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu}, \tag{2.156}$$

and the derivative of the generating function is

$$\begin{aligned}\delta Z[0] &= i\int \mathcal{D}[\phi]\delta S_m e^{iS_m[\phi]} \\ &= i\langle \text{out,0}|0, \text{in}\rangle.\end{aligned} \tag{2.157}$$

We obtain

$$\frac{2}{(-g)^{\frac{1}{2}}}\frac{\delta Z[0]}{\delta g^{\mu\nu}} = i\langle \text{out,0}|T_{\mu\nu}|0, \text{in}\rangle \tag{2.158}$$

In terms of

$$W = -i \ln \langle \text{out,0}|0, \text{in}\rangle. \tag{2.159}$$

We obtain

$$\frac{2}{(-g)^{\frac{1}{2}}}\frac{\delta W}{\delta g^{\mu\nu}} = \frac{\langle \text{out,0}|T_{\mu\nu}|0, \text{in}\rangle}{\langle \text{out,0}|0, \text{in}\rangle}. \tag{2.160}$$

By using the replacement of the delta function

$$\int d^n x[-g(x)]^{\frac{1}{2}}\delta^n(x-y)[-g(y)]^{-\frac{1}{2}} = 1, \tag{2.161}$$

and we define K_{xy} by

$$K_{xy} = (\Box + m^2 - i\varepsilon + \zeta R)\delta^n(x-y)[-g(y)]^{-\frac{1}{2}}, \tag{2.162}$$

then the following equation is satisfied as

$$\int d^n y[-g(y)]^{\frac{1}{2}} K_{xy}K^{-1}_{yz} = \delta(x-z)[-g(z)]^{-\frac{1}{2}}. \tag{2.163}$$

The above equation means

$$K^{-1}_{xz} = -G_F(x,z) \tag{2.164}$$

Combining the results (2.164) and (2.165) we obtain

$$Z[0] \propto [\det(-G_F)]^{\frac{1}{2}}, \tag{2.165}$$

and

$$W = -i \ln Z[0] = -\frac{1}{2} i \mathrm{tr}[\ln(-G_F)]. \tag{2.166}$$

Furthermore from the normalization condition, we see

$$\langle x|x'\rangle = \delta^n(x - x')[-g(x)]^{\frac{1}{2}}. \tag{2.167}$$

So

$$G_F(x, x') = \langle x|G_F|x'\rangle. \tag{2.168}$$

By the definition the following equation holds

$$G_F = -K^{-1} = i \int_0^\infty e^{-iKs} ds, \tag{2.169}$$

and

$$\langle x|e^{-iKs}|x'\rangle = i(4\pi)^{-n/2} \Delta^{\frac{1}{2}}(x, x') e^{im^2 s + \sigma/2is} F(x, x'; is)(is)^{-n/2}. \tag{2.170}$$

If the K has small negative imaginary part, the (2.170) can be calculate

$$\int_\Lambda^\infty e^{-iKs}(is)^{-1} i ds = \mathrm{Ei}(-i\Lambda K) \tag{2.171}$$

Here, Ei is the exponential integral function. This function can be expand as

$$\mathrm{Ei}(x) = \gamma + \ln(-x) + O(x) \tag{2.172}$$

Here γ is the Euler constant.

Inserting this equation to Eq. (2.169) and set $\Lambda = 0$, we obtain

$$\ln(-G_F) = -\ln K = \int_0^\infty e^{iKs}(is)^{-1} i ds \tag{2.173}$$

From the above result we obtain

$$\langle x| \ln(-G_F^{DS})|x'\rangle = -\int_{m^2}^\infty G_F^{DS}(x, x') dm^2. \tag{2.174}$$

If we express W by F, we obtain

$$W = \frac{1}{2} i \int d^n x [-g(x)]^{\frac{1}{2}} \lim_{x' \to x} \int_{m^2}^\infty dm^2 G_F^{DS}(x, x'). \tag{2.175}$$

Furthermore if we take a limit $x' \to x$, we obtain

$$W = \frac{1}{2} i \int_{m^2}^\infty dm^2 \int d^n x [-g(x)]^{\frac{1}{2}} G_F^{DS}(x, x). \tag{2.176}$$

We define *effective Lagrangian density* $\mathcal{L}_{eff}$ as follows

$$W = \int \mathcal{L}_{eff}(x) d^n x \equiv \int [-g(x)]^{\frac{1}{2}} L_{eff}(x) d^n x, \tag{2.177}$$

where

$$L_{eff}(x) = [-g(x)]^{-\frac{1}{2}} \mathcal{L}_{eff}(x) = \frac{1}{2} i \lim_{x' \to x} \int_{m^2}^{\infty} dm^2 G_F^{DS}(x, x'). \tag{2.178}$$

Inserting G_F^{DS},

$$L_{eff} \approx \lim_{x' \to x} \frac{\Delta^{\frac{1}{2}}(x, x')}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x, x') \int_0^{\infty} (is)^{j-1-n/2} e^{-i(m^2 s - \sigma/2s)} i ds. \tag{2.179}$$

Taking the limit $x' \to x$, we obtain

$$L_{eff} \approx \frac{1}{2}(4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x) \int_0^{\infty} (is)^{j-1-n/2} e^{-im^2 s} i ds \tag{2.180}$$

$$= \frac{1}{2}(4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x) (m^2)^{n/2-j} \Gamma(j - n/2). \tag{2.181}$$

Here $a_j(x) = a_j(x, x')$. We can rewrite this equation by using the arbitrary mass scale μ as,

$$L_{eff} \approx \frac{1}{2}(4\pi)^{-n/2} (m/\mu)^{n-4} \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma(j - n/2). \tag{2.182}$$

The divergence appear in the case of $j = 2$. So we can write the divergence term of W as

$$W_{div} = \frac{1}{2}(4\pi)^{-n/2} (m/\mu)^{n-4} \Gamma(2 - n/2) \int d^n x [-g(x)]^{\frac{1}{2}} a_2(x) \tag{2.183}$$

$$= \frac{1}{2}(4\pi)^{-n/2} (m/\mu)^{n-4} \Gamma(2 - n/2) \int d^n x [-g(x)]^{\frac{1}{2}} [\alpha F(x) + \beta G(x)] + O(n-4). \tag{2.184}$$

Here,

$$F = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{3} R^2 \tag{2.185}$$

$$G = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2. \tag{2.186}$$

And here, the constant numbers α, β are

$$\alpha = \frac{1}{120}, \quad \beta = -\frac{1}{360} \tag{2.187}$$

Using the next relations

$$\frac{2}{(-g)^{\frac{1}{2}}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int (-g)^{\frac{1}{2}} F d^n x = -(n-4)(F - \frac{2}{3}\Box R) \tag{2.188}$$

$$\frac{2}{(-g)^{\frac{1}{2}}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int (-g)^{\frac{1}{2}} G d^n x = -(n-4)G, \tag{2.189}$$

then we obtain

$$\begin{aligned} \langle T_\mu{}^\mu \rangle = \frac{2}{(-g)^{\frac{1}{2}}} g^{\mu\nu} \frac{\delta W_{div}}{\delta g^{\mu\nu}} &= \frac{1}{2}(4\pi)^{-n/2}(m/\mu)^{n-4}(4-n)\Gamma(2-n/2) \\ &\times[\alpha(F - \frac{2}{3}\Box R) + \beta G] + O(n-4). \end{aligned} \tag{2.190}$$

Then the divergence term of stress-energy tensor i.e. T_{div} is

$$\langle T_\mu{}^\mu \rangle_{div} = (1/16\pi^2)[\alpha(F - \frac{2}{3}\Box R) + \beta G]. \tag{2.191}$$

If we remove the divergence term, we obtain

$$\langle T_\mu{}^\mu \rangle_{ren} = -(1/16\pi^2)[\alpha(F - \frac{2}{3}\Box R) + \beta G] \tag{2.192}$$

$$= -a_2/16\pi^2 \tag{2.193}$$

$$= -(1/2880\pi^2)[R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta} - \Box R] \tag{2.194}$$

$$= -(1/2880\pi^2)[C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2 - \Box R]. \tag{2.195}$$

We call the above formula as the conformal anomaly because the renormalization of a quantum stress tensor breaks conformal invariance. A conformally invariant classical theory, such as electromagnetism or the conformally coupled massless scalar field, has the property lost in the renormalized quantum theory, and the expectation value of T^μ_μ acquires a nonzero trace. This anomalous trace is independent of the choice of quantum state and is a local geometrical quantity.

2.3.2 Infrared Behavior

We consider massless scalar field in flat four-dimensional spacetime. In our discussion of the Hadamard form, we noted that it is a common, although not universal property of quantum states. In a state in which the two-point function does not have the Hadamard form, the renormalization procedure outlined above will not remove all of the infinities from the stress tensor. In flat spacetime, a state which does not have the Hadamard form would have to be considered to be unphysical if the normal-ordered energy density were infinite. Fulling, Sweeny and Wald [58]have shown that a two point function which has the Hadamard form at one time will have it at all times. In particular, in any spacetime which is asymptotically flat in the past or in the future, the Hadamard form will hold if it holds in the flat region [59]. Thus, it seems reasonable to require that the

two point function having the Hadamard form be a criterion for a physically acceptable state.

Examples of state which do not have the Hadamard form may be constructed even in flat spacetime. Let us first consider a massless scalar field in flat four-dimensional spacetime, which has the mode expansion

$$\varphi = \sum_{\mathbf{k}} (a_{\mathbf{k}} f_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger} f_{\mathbf{k}}^{*}). \tag{2.196}$$

If we box normalize the mode function, as

$$f_{\mathbf{k}} = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2\omega V}} [\alpha(\omega) e^{-i\omega t} + \beta(\omega) e^{i\omega t}], \tag{2.197}$$

we must require

$$|\alpha(\omega)|^2 - |\beta(\omega)|^2 = 1. \tag{2.198}$$

We define the two-point function by

$$\langle\varphi|\phi(x)\phi(x')|\varphi\rangle = \frac{1}{2(2\pi)^3} \int d^3k\omega^{-1} \{[\alpha(\omega)e^{-i\omega t} + \beta(\omega)e^{i\omega t}] \\ \times [\alpha^*(\omega)e^{i\omega t'} + \beta^*(\omega)e^{-i\omega t'}] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{2.199}$$

Here, φ is defined by (2.197). If we assume ω is small,

$$\langle\varphi|\phi(x)\phi(x')|\varphi\rangle \sim \frac{1}{(2\pi)^2} \int d\omega\omega |\alpha(\omega) + \beta(\omega)|^2. \tag{2.200}$$

For example, let

$$\beta(\omega) = \omega^{-c} \quad \alpha(\omega) = (1 + \omega^{-2c})^{\frac{1}{2}}. \tag{2.201}$$

Then we obtain

$$|\alpha(\omega) + \beta(\omega)| \sim \omega^{-c}, \quad \omega \longrightarrow 0. \tag{2.202}$$

In this case it diverges for $c > 1$.

In the case of two dimensions, the two-point function is

$$\langle\varphi|\phi(x)\phi(x')|\varphi\rangle \sim \frac{1}{4\pi} \int d\omega\omega^{-1} |\alpha(\omega) + \beta(\omega)|^2. \tag{2.203}$$

If we assume similarly as

$$\beta(\omega) = -\omega^{-c}, \quad \alpha(\omega) = (1 + \omega^{-2c})^{\frac{1}{2}}. \tag{2.204}$$

Then

$$|\alpha(\omega) + \beta(\omega)| \sim \frac{1}{4}\omega^{2c}, \quad \omega \longrightarrow 0. \tag{2.205}$$

In this case the above equation is finite if $c > 0$. If we choose $\alpha = 1, \beta = 0$, the two-point function in two-dimensional case diverges. One may show that in any

state which is free of infrared divergences, $\langle\phi^2\rangle$ must be a growing function of time [47] In the case of $t \to \infty$

$$\langle\phi^2\rangle \sim t^{2c}. \tag{2.206}$$

Similar divergence appearing in the case of background is de-Sitter space-time. Then the metric is

$$ds^2 = \frac{1}{(H\eta)^2}(d\eta^2 - d\mathbf{x}^2) = dt^2 - e^{2Ht}d\mathbf{x}^2, \tag{2.207}$$

and

$$\Box\phi = 0. \tag{2.208}$$

The mode function satisfying the above equation is given by Hankel function as,

$$f_{\mathbf{k}} \propto e^{i\mathbf{k}\cdot\mathbf{x}}[c_2 H^{(2)}_{\frac{3}{2}}(k\eta) + c_1 H^{(1)}_{\frac{3}{2}}(k\eta)]. \tag{2.209}$$

If we set $c_2 = 1, c_1 = 0$, this state is infrared divergent, because the Hankel function behaves as,

$$H^{(2)}_{\frac{3}{2}}(k\eta) \sim k^{-\frac{3}{2}}, \quad k \to 0. \tag{2.210}$$

In this case in the future infinity,

$$\langle\phi^2\rangle \sim \frac{H^3 t}{4\pi^2}, \quad t \to \infty. \tag{2.211}$$

This result is applicable to the Goldstone model of $U(1)$ symmetry breaking. At first we define the Lagrangian density by

$$\mathcal{L} = \partial_\alpha\Phi^*\partial^\alpha\Phi - V(\Phi), \tag{2.212}$$

where the potential V is given by

$$V(\Phi) = -\frac{1}{2}m^2\Phi^*\Phi + \frac{1}{4}\lambda(\Phi^*\Phi)^2. \tag{2.213}$$

The potential is minimal at the point of

$$\Phi = \sigma e^{i\phi}, \quad \sigma = m\lambda^{-1/2}. \tag{2.214}$$

The equation of motion of Φ is $\Box\phi = 0$. If we divide positive and negative frequency part of this wave function, we obtain $\phi = \phi^+ + \phi^-$, and $\phi^+|0\rangle = 0, and\langle 0|\phi^- = 0$. If we use annihilation and creation operator, $\phi^+ = \sum_j a_j f_j, \phi^- = \sum_j a_j^\dagger f_j^*$. We use the Campbell-Baker-Hausdorff formula

$$e^{i\phi} = ei(\phi^+ + \phi^-) = e^{i\phi^-}e^{-\frac{1}{2}[\phi^+,\phi^-]}e^{i\phi^+}. \tag{2.215}$$

And if we use the identity.

$$[\phi^+, \phi^-] = \sum_j f_j f_j^* = \langle\phi^2\rangle, \tag{2.216}$$

we obtain

$$\langle \Phi \rangle = \sigma \langle e^{i\phi} \rangle = \sigma e^{-\frac{1}{2}\langle \phi^2 \rangle}. \tag{2.217}$$

The ultraviolet divergence in $\langle \phi^2 \rangle$ is absorbed in a rescaling of Φ. In spacetimes, such as four dimensional flat space, where one can have $\langle \phi^2 \rangle$ constant in a physically acceptable state, then there are stable broken symmetry states in which $\langle \Phi \rangle \neq 0$. However, in two dimensional flat spacetime or in four dimensional de-Sitter spacetime, the growth of $\langle \phi^2 \rangle$ forces $\langle \Phi \rangle \neq 0$ to decay in time. In this cases, the infrared behavior of the massless scalar field prevents the existence of a stable state of broken symmetry.

2.4 Negative Energy

2.4.1 Casimir effect

In this section we consider electrodynamic stress-energy tensor in vacuum. An electromagnetic field will be modified in the presence of conducting surfaces. Then the stress-energy tensor operator is given using

$$\begin{aligned} T^{\mu\nu}(x,\varepsilon) &= F^{\mu\lambda}(x+\frac{1}{2}\varepsilon)F^{\nu}_{\ \lambda}(x-\frac{1}{2}\varepsilon) \\ &\quad -\frac{1}{4}g^{\mu\nu}F^{\lambda\kappa}(x+\frac{1}{2}\varepsilon)F_{\lambda\kappa}(x-\frac{1}{2}\varepsilon), \end{aligned} \tag{2.218}$$

or

$$T^{\mu\nu} = \lim_{\varepsilon\to 0}(1+\frac{1}{4}\varepsilon^{\lambda}\frac{\partial}{\partial\varepsilon^{\lambda}})T^{\mu\nu}(x,\varepsilon) \tag{2.219}$$

Because of these definitions we can remove the divergence of the term $(\varepsilon^2)^{-2}$. From now on we consider the property of the stress-energy tensor. From the fact that the divergence is zero, we obtain

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.220}$$

From the fact that the trace of stress-energy tensor is zero we obtain

$$T^{\mu}_{\ \mu} = 0. \tag{2.221}$$

Now we consider the parallel and infinite complete condensation. The length of two condensation is a. We assume the direction is $\hat{z}^\mu = (0,0,0,1)$. Then the stress-energy tensor becomes as

$$\langle T^{\mu\nu} \rangle_{(0)} = (\frac{1}{4}g^{\mu\nu} - \hat{z}^\mu\hat{z}^\nu)f(z). \tag{2.222}$$

However, because its trace is zero, $f(z)$ should be a constant factor and it represents the energy in unit volume. Then the stress-energy tensor becomes

$$\langle T^{\mu\nu} \rangle_{(0)} = (\frac{1}{4}g^{\mu\nu} - \hat{z}^\mu\hat{z}^\nu)(\hbar c/a^4)\gamma. \tag{2.223}$$

Here,

$$\gamma = \frac{1}{2\pi^2} \sum_{l=1}^{\infty} l^{-4} = (1/2\pi^2)\zeta(4) = \pi^2/180. \tag{2.224}$$

The energy density is

$$\langle T^{00} \rangle_{(0)} = -(\pi^2/720)(\hbar c/a^4), \tag{2.225}$$

and the pressure is

$$\langle T^{33} \rangle_{(0)} = -(\pi^2/240)(\hbar c/a^4). \tag{2.226}$$

In this particular case the Casimir effect gives negative energy and pressure.

2.4.2 Negative Energy: a simple example

We can illustrate the basic phenomenon of negative energy arising from quantum coherence with a very simple example. Let the quantum state of the system be a superposition of the vacuum and a two particle state:

$$|\Phi\rangle = \frac{1}{\sqrt{1+\varepsilon^2}}(|0\rangle + \varepsilon|2\rangle). \tag{2.227}$$

Here we take the relative amplitude ε to be a real number. Let the energy density operator be normal ordered:

$$\rho =: T_{tt} : . \tag{2.228}$$

Then the averaged value of energy becomes as

$$\langle \rho \rangle = \frac{1}{1+\varepsilon^2}[2\varepsilon \mathrm{Re}(\langle 0|\rho|2\rangle) + \varepsilon^2 \langle 2|\rho|2\rangle]. \tag{2.229}$$

We may always choose ε to be sufficiently small that the first term on the right hand side dominates the second term. However, the former term may be either positive or negative. At any given point, we could choose the sign of ε so as to make $\langle \rho \rangle < 0$ at that point.

In the next step we show more general result. We use a fact that the energy of the squeezed state becomes negative value at some point. General squeezed state for a single mode can be written as

$$|z, \zeta\rangle = D(z)S(\zeta)|0\rangle. \tag{2.230}$$

Here, $D(z)$ is the displacement operator defined as

$$D(z) \equiv \exp(za^\dagger - z^* a) = e^{-|z|^2/2} e^{za^\dagger} e^{-z^* a}, \tag{2.231}$$

and $S(\zeta)$ is the squeeze operator defined as

$$S(\zeta) \equiv \exp[\frac{1}{2}\zeta^* a^2 - \frac{1}{2}\zeta(a^\dagger)^2]. \tag{2.232}$$

Here,

$$z = se^{i\gamma}, \quad \zeta = re^{i\delta}. \tag{2.233}$$

With z being an arbitrary complex number. The displacement operator and the squeeze operator satisfy the relation as

$$\begin{aligned} D^\dagger(z) a D(z) &= a + z, && (2.234) \\ D^\dagger(z) a^\dagger D(z) &= a^\dagger + z, && (2.235) \\ S^\dagger(\zeta) a S(\zeta) &= a \cosh r - a^\dagger e^{i\delta} \sinh r, && (2.236) \\ S^\dagger(\zeta) a^\dagger S(\zeta) &= a^\dagger \cosh r - a e^{-i\delta} \sinh r. && (2.237) \end{aligned}$$

If $\zeta = 0$, we obtain coherent state $|z\rangle = |z, 0\rangle$. In this case the averaged value of the quantum field is

$$\langle \phi \rangle = zf + z^* f^*. \tag{2.238}$$

Furthermore the fluctuation of this state is minimal and

$$\langle : \phi^2 : \rangle = \langle \phi \rangle^2. \tag{2.239}$$

If we consider the squeezed state in the region $|0, \zeta\rangle$, then $\langle \rho \rangle < 0$.

From now on we apply above discussion to the curved spacetime. We assume the Bogolubov transformation as

$$a = \alpha^* b - \beta^* b^\dagger, \tag{2.240}$$

with $|\alpha|^2 - |\beta|^2 = 1$. The annihilation operators satisfy, $a|0\rangle_{in} = 0$ and $b|0\rangle_{out} = 0$ Now we assume in-state and out-state are related as

$$|0\rangle_{in} = \Sigma |0\rangle_{out}, \tag{2.241}$$

where Σ is a certain operator with $\Sigma^\dagger a \Sigma = b = \alpha a + \beta^* a^\dagger$. If we multiply this equation by $\Sigma^\dagger a$, we obtain

$$\Sigma^\dagger a \Sigma |0\rangle = 0. \tag{2.242}$$

If we set $\alpha = \cosh r, \beta = -e^{-i\delta}$, $\Sigma = S$ and the $|0\rangle_{in}$ becomes the squeezed vacuum. Then the averaged value of the energy may become negative.

By the Hawking effect, the energy is extracted from black hole so that the area of the horizon decreases. We would like to consider this effect. Although we used dominant energy condition, the energy condition is broken near the horizon. For example we consider two dimensional black hole [50]. The metric is

$$ds^2 = (1 - 2M/r) dt^2 - (1 - 2M/r)^{-1} dr^2. \tag{2.243}$$

By the coordinate transformation we obtain

$$\begin{aligned} u &= t - r^* \\ v &= t + r^* && (2.244) \\ r^* &= r + 2M \ln(r/2M - 1) \\ ds^2 &= (1 - 2M/r) du dv. && (2.245) \end{aligned}$$

Then the stress-energy tensor is

$$T_{\mu\nu} = -(\frac{\varepsilon^{-2}}{4\pi t_a t^a} + \frac{R}{24\pi})(\frac{t_\mu t_\nu}{t_a t^a} - \frac{1}{2} g_{\mu\nu}) + \theta_{\mu\nu} + O(\varepsilon) \tag{2.246}$$

Here, $\theta_{\mu\nu}$ can be written using conformal factor as,

$$\begin{aligned} \theta_{\bar{u}\bar{u}} &= -(12\pi)^{-1} C^{1/2} (C^{-1/2})_{,\bar{u}\bar{u}} \\ \theta_{\bar{v}\bar{v}} &= -(12\pi)^{-1} C^{1/2} (C^{-1/2})_{,\bar{v}\bar{v}} \\ \theta_{\bar{u}\bar{v}} &= \theta_{\bar{v}\bar{u}} = 0. \end{aligned} \tag{2.247}$$

If we re normalize the stress-energy tensor and we assume $t^a t_a = \pm 1$, then we obtain

$$T_{\mu\nu} = \theta_{\mu\nu} + \frac{R}{48\pi} g_{\mu\nu} \tag{2.248}$$

Calculating this term we obtain as

$$\begin{aligned} T_{uu} = T_{vv} &= (24\pi)^{-1}(\frac{3M^2}{2r^4} - \frac{M}{r^3}) \\ T_{uv} = T_{vu} &= (24\pi)^{-1}(\frac{2M^2}{r^4} - \frac{M}{r^3}) \\ T_{tt} &= (24\pi)^{-1}(\frac{7M^2}{r^4} - \frac{4M}{r^3}) \\ T_{tr} &= T_{rt} = 0 \\ T_{rr} &= -(24\pi)^{-1}(1 - 2M/r)^{-2} \frac{M^2}{r^4}. \end{aligned} \tag{2.249}$$

This example shows the fact that the energy becomes negative near the horizon. There is also negative energy inequality [48].

Chapter 3

Dynamical Horizon

In this chapter we introduce the concept of the dynamical horizon. The dynamical horizon has the equation which describes how the black hole radius changes. The equation has been normally used only in the spacelike case. If the horizon is spacelike, black hole all ways increases by the apparent horizon analysis. However, by using the result derived in chapter 2, we can extend the definition of the dynamical horizon to timelike case. Then the timelike dynamical horizon equation can be applied to black hole evaporation problem in the next chapter.

3.1 Spacelike and timelike dynamical horizons

Ashtekar and Krishnan considered dynamical horizon [15][20], and derived a new equation that dictates how the dynamical horizon radius changes. Apparent horizon is a time slice of the dynamical horizon. The definition of dynamical horizon is,

Definition. A smooth, three-dimensional, spacelike submanifold H in a space-time is said to be a *dynamical horizon* if it is foliated by preferred family of 2-spheres such that, on each leaf S, the expansion $\Theta_{(l)}$ of a null normal l^a vanishes and the expansion $\Theta_{(n)}$ of the other null normal n^a is strictly negative.

The requirement that one of the null expansions is zero comes from the intuition that black hole does not emit even light. And the requirement that other null expansion is strictly negative comes from that null matter goes in black holes inward.

In this section we recapitulate the important formula which gives a change of the dynamical horizon radius by the matter flow, using 3+1 and then 2+1 decompositions and also the Gauss-Bonet theorem. Decomposing the Einstein-Hilbert action in 3+1 dimensions, we obtain the constraint equations, scalar

constraint and vector constraint as

$$H_S \equiv \mathcal{R} + K^2 - K^{ab}K_{ab} = 16\pi G\bar{T}_{ab}\hat{\tau}^a\hat{\tau}^b \tag{3.1}$$
$$H_V^a \equiv D_b(K^{ab} - Kq^{ab}) = 8\pi G\bar{T}^{bc}\hat{\tau}_c q_b^a. \tag{3.2}$$

where, K_{ab} is the extrinsic curvature defined by $K_{ab} := q_a^c q_b^d \nabla_c \hat{\tau}_d$, and K is its trace,K_a^a. Here $\hat{\tau}^a$ and $\hat{r}^a$ are unit vectors in the time and radial directions. We choose the vector $\hat{r}^a$ along the dynamics of the horizon, and $\hat{\tau}^a$ is defined by the orthogonality $\hat{r}^a\hat{\tau}_a = 0$, so that there are two choices of time vector, future or past. q_{ab} is three dimensional spatial metric, $\mathcal{R}$ is the three dimensional scalar curvature, and D_a is three dimensional covariant derivative. ΔH is the volume of the dynamical horizon between two trapped surfaces. We set

$$\bar{T}_{ab} = T_{ab} - \frac{1}{8\pi G}\Lambda g_{ab}, \tag{3.3}$$

with T_{ab} being the matter stress-energy tensor in the case that the cosmological constant Λ is present. We denote the flux of matter energy across ΔH by $\mathcal{F}_{matter}^R$

$$\mathcal{F}_{matter}^R := \int_{\Delta H} T_{ab}\hat{\tau}^a \xi_{(R)}^b d^3V. \tag{3.4}$$

By the Einstein equation, we can rewrite the right hand side in terms of the geometrical quantities as

$$\mathcal{F}_{matter}^{(R)} = \frac{1}{16\pi G}\int_{\Delta H} N_R(H_S + 2\hat{r}_a H_V^a)d^3V$$
$$= \frac{1}{16\pi G}\int_{\Delta H} N_R(\mathcal{R} + K^2 - K^{ab}K_{ab} + 2\hat{r}_a D_b P^{ab})d^3V. \tag{3.5}$$

Here, $\xi_{(R)}^a := N_R l^a$ $(N_R := |\partial R|)$ and R is the radius of the dynamical horizon, and

$$P^{ab} := K^{ab} - Kq^{ab}. \tag{3.6}$$

Now, we decompose $\mathcal{R}$ in 2+1 dimensions

$$\mathcal{R} = \tilde{\mathcal{R}} + \tilde{K}^2 - \tilde{K}_{ab}\tilde{K}^{ab} + 2D_a\alpha^a, \tag{3.7}$$

here $\tilde{K}_{ab} := \tilde{q}_a^c \tilde{q}_b^d D_c \hat{r}_d$, and $\alpha^a := \hat{r}^b D_b \hat{r}^a - \hat{r}^a D_b \hat{r}^b$.

Then we also rewrite P^{ab} as

$$\hat{r}_b D_a P^{ab} = D_a\beta^a - P^{ab}D_a\hat{r}_b, \tag{3.8}$$

with

$$\beta^a := K^{ab}\hat{r}_b - K\hat{r}^a. \tag{3.9}$$

Putting together the equations (6)-(9), we obtain

$$H_S + 2\hat{r}_a H_V^a = \tilde{\mathcal{R}} + \tilde{K}^2 - \tilde{K}_{ab}\tilde{K}^{ab}$$
$$+K^2 - K_{ab}K^{ab} - 2P^{ab}D_a\hat{r}_b + 2D_a\gamma^a, \tag{3.10}$$

with

$$\gamma^a := \alpha^a + \beta^a. \tag{3.11}$$

Now, we use the fact that the null expansion $\Theta_{(l)}$ can be written as

$$\Theta_{(l)} = K - K_{ab}\hat{r}^a\hat{r}^b + \tilde{K}, \tag{3.12}$$

we further decompose the extrinsic curvature K_{ab} into 2+1 dimensions as,

$$\tilde{K}_{ab} = \frac{1}{2}\tilde{K}\tilde{q}_{ab} + \tilde{S}_{ab} \tag{3.13}$$

$$K_{ab} = A\tilde{q}_{ab} + S_{ab} + 2\tilde{W}_{(a}\hat{r}_{b)} + B\hat{r}_a\hat{r}_b. \tag{3.14}$$

Here $\tilde{K}_{ab}$is the extrinsic curvature in 2+1 dimensions, $\tilde{K}$ is its trace ($\tilde{K} = \tilde{K}^a_a$) $\tilde{S}_{ab}$ is the traceless part of $\tilde{K}_{ab}$, S_{ab} is the projection of traceless part on S, and $\tilde{W}_a$ is the projection of $K_{ab}\hat{r}^b$ on S. And also we define $A := \frac{1}{2}K_{ab}\tilde{q}^{ab}$, $B = K_{ab}\hat{r}^a\hat{r}^b$, where $\tilde{q}_{ab}$ is two dimensional metric $\tilde{q}_{ab} := q_{ab} - \hat{r}_a\hat{r}_b$. Inserting these decompositions into the previous equation, we obtain

$$\begin{aligned} H_S + 2\hat{r}_a H^a_V = \tilde{\mathcal{R}} - \sigma_{ab}\sigma^{ab} - 2\tilde{W}_a\tilde{W}^a - 2\tilde{W}^a\hat{r}^b D_b\hat{r}_a \\ + \frac{1}{2}\Theta_{(l)}(\Theta_{(l)} + 4B) + 2D_a\gamma^a. \end{aligned} \tag{3.15}$$

Here $\sigma_{ab} := S_{ab} + \tilde{S}_{ab}$ is shear of l^a, that is, $\sigma_{ab} = \tilde{q}^m_a\tilde{q}^n_b\nabla_m l_n - \frac{1}{2}\tilde{q}_{ab}\tilde{q}^{ab}\nabla_m l_n$. Using

$$\begin{aligned} \gamma^a = \alpha^a + \beta^a = \hat{r}^a D_b\hat{r}^a - \hat{r}^a D_b\hat{r}^b + K^{ab}\hat{r}_b - K\hat{r}_a \\ = \hat{r}^b D_b\hat{r}^a + \tilde{W}^a - \Theta_{(l)}\hat{r}^a, \end{aligned} \tag{3.16}$$

we can rewrite the acceleration term, as

$$\hat{r}^b D_b\hat{r}_a = (N_R)^{-1}\tilde{D}_b N_R. \tag{3.17}$$

Finally we get

$$\begin{aligned} H_S + 2\hat{r}_a H^a_V = \tilde{\mathcal{R}} - \sigma_{ab}\sigma^{ab} - 2\zeta^a\zeta_a + 2\tilde{D}_a\zeta^a \\ + \frac{1}{2}\Theta_{(l)}(\Theta_{(l)} + 4B - 4\tilde{K}), \end{aligned} \tag{3.18}$$

where

$$\zeta^a := \tilde{W}^a + \tilde{D}^a \ln N_R = \tilde{q}^{ab}\hat{r}^c\nabla_c l_b, \tag{3.19}$$

and therefore

$$\mathcal{F}^{(R)}_{matter} = \frac{1}{16\pi G}\int_{\Delta H} N_R(\tilde{\mathcal{R}} - \sigma_{ab}\sigma^{ab} - 2\zeta^a\zeta_a)d^3V. \tag{3.20}$$

To evaluate the right hand side of Eq. (3.20) we note that equation (3.5) reduces to

$$\begin{aligned} \int_{\Delta H} N_R\tilde{\mathcal{R}}d^3V = 16\pi G\int_{\Delta H}\bar{T}_{ab}\hat{\tau}^a\xi^b_{(R)}d^3V \\ + \int_{\Delta H}(|\sigma|^2 + 2|\zeta|^2)d^3V. \end{aligned} \tag{3.21}$$

Here we put, $|\sigma|^2 = \sigma_{ab}\sigma^{ab}, |\zeta|^2 = \zeta^a\zeta_a$. We see that the second term of right hand side of this equation is the form of the Bondi energy, therefore positive. If we assume dominant energy condition, the right hand side would be positive, and therefore the horizon radius would increase. Using the Gauss-Bonet theorem, the left hand side becomes,

$$\int_{\Delta H} N_R \tilde{\mathcal{R}} d^3V = \int_{R_1}^{R_2} dr(\oint_S \tilde{\mathcal{R}} d^2V) = 8\pi(R_2 - R_1). \tag{3.22}$$

Substituting equation (22) back in equation (3.21) one obtains

$$\begin{aligned}(\frac{R_2}{2G} - \frac{R_1}{2G}) &= \int_{\Delta H} \bar{T}_{ab}\hat{\tau}^a \xi^b_{(R)} d^3V \\ &+ \frac{1}{16\pi G}\int_{\Delta H}(|\sigma|^2 + 2|\zeta|^2)d^3V. \end{aligned} \tag{3.23}$$

This is the dynamical horizon equation that tells how the horizon radius changes by the matter flow, shear and expansion. In the spherically symmetric case that we shall consider in what follows the second term of the right hand side vanishes. Although in the case of quantum field theory in curved space time, the dominant energy condition does not hold[48][24], we can use the dynamical horizon equation because it is valid even when the black hole radius decreases. And the dynamical horizon equation is a consequence of the Einstein equation. We use the dynamical horizon equation in place of the Einstein equation. Because of the negative energy, we can expand the definition of the dynamical horizon in the case of timelike. Then the definition of the dynamical horizon is now,

Definition (modified version). A smooth, three-dimensional, spacelike or **timelike** submanifold H in a space-time is said to be a *dynamical horizon* if it is foliated by preferred family of 2-spheres such that, on each leaf S, the expansion $\Theta_{(l)}$ of a null normal l^a vanishes and the expansion $\Theta_{(n)}$ of the other null normal n^a is strictly negative.

We can easily calculate the timelike dynamical horizon equation only replacing $\hat{r}^a$ and $\hat{\tau}^a$. And the way of 3+1 and 2+1 decomposition is replaced.

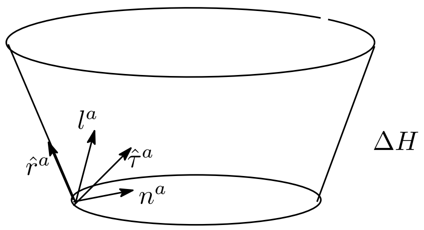

Figure 3.1: This shows spacelike dynamical horizon.$\hat{r}^a$ is normal vector parallel to the dynamics. $\hat{\tau}^a$ is chosen as orthogonal to $\hat{r}^a$.l^a is the null normal whose expansion is zero. n^a is the null normal whose expansion is strictly negative. ΔH is the region between two trapped surface.

Chapter 4

Black Hole Evaporation

In this chapter we study the consequence of negative energy introduced in chapter 2 to the dynamical horizon equation introduced in chapter3. On the basis of the timelike dynamical horizon equation and negative energy near black hole horizon in the Vaidya metric, we can solve the black hole evaporation problem as a back-reaction problem. We introduce the Vaidya metric because the Einstein equation and the dynamical horizon equation usually are not consistent.

4.1 Radius of the dynamical horizon

This chapter is mainly based on Sawayama [1]. The Vaidya metric is of the form

$$ds^2 = -Fdv^2 + 2Gdvdr + r^2 d\Omega^2, \tag{4.1}$$

where F and G are functions of v and r, and v^a is null vector and r is the area radius, and M is the mass defined by $M = \frac{r}{2}(1 - \frac{F}{G^2})$, a function of v and r. This metric is spherically symmetric. By substituting the Vaidya metric (2) into the Einstein equation so that we can identify the energy-momentum tensor T_{ab} as

$$\begin{aligned} 8\pi T_{vv} &:= \frac{2}{r^2}(FM_{,r} + GM_{,v}) && (4.2) \\ 8\pi T_{vr} &:= -\frac{2G}{r^2}M_{,r} && (4.3) \\ 8\pi T_{rr} &:= \frac{2G_{,r}}{rG}. && (4.4) \end{aligned}$$

We do not need to check that the solution of the dynamical horizon equation satisfies the Einstein equation. Because we would like to consider the Schwarzschild like metric, we set $v = t + r^*$, where r^* is tortoise coordinate with dynamics

$$r^* = r + 2M(v)\ln\left(\frac{r}{2M(v)} - 1\right). \tag{4.5}$$

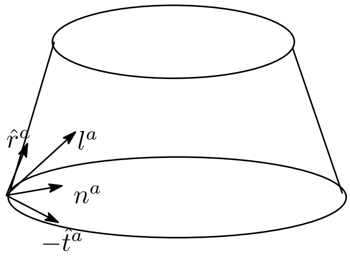

Figure 4.1: For the case that the dynamical horizon decreases, we should choose $l^a = -\hat{t}^a + \hat{r}^a$ so that l^a points into the dynamical horizon.

For later convenience, we write,

$$a = \left. \frac{\partial r}{\partial r^*} \right|_v . \tag{4.6}$$

There are two null vectors,

$$l^a = \begin{pmatrix} l^t \\ l^{r*} \\ l^\theta \\ l^\phi \end{pmatrix} = \begin{pmatrix} -a^{-1} \\ a^{-1} \\ 0 \\ 0 \end{pmatrix}, \tag{4.7}$$

corresponding to the null vector v^a, and the other is

$$n^a = \begin{pmatrix} n^t \\ n^{r*} \\ n^\theta \\ n^\phi \end{pmatrix} = \begin{pmatrix} -a^{-1} \\ -\frac{F}{F-2Ga} a^{-1} \\ 0 \\ 0 \end{pmatrix}. \tag{4.8}$$

Here we multiply a^{-1} so that $l^a = v^a$. This choice of the null vector l^a is explained in figure 4.1. From now on we put,

$$F = \left(1 - \frac{2M(v)}{r}\right) \tag{4.9}$$

$$G = 1, \tag{4.10}$$

in a similar form to the Schwarzschild metric, assuming that $M(v)$ is a function of v only. For a constant M, the metric coincides with the Schwarzschild metric. We calculate the expansions $\Theta_{(l)}$ and $\Theta_{(n)}$ of the two null vectors l^a, n^a, because the definition of the dynamical horizon requires one of the null expansions to be zero and the other to be minus. The result is,

$$\Theta_{(l)} = \frac{1}{r}(2F - a) \tag{4.11}$$

$$\Theta_{(n)} = \frac{1}{ar}\left(\frac{-2F^2 + aF - 2a^2}{-F + 2a}\right). \tag{4.12}$$

From $\Theta_{(l)} = 0$ we get,

$$2F - a = 0. \tag{4.13}$$

we can check that the other null expansion $\Theta_{(n)}$is strictly negative. Therefore in this case, we can apply the dynamical horizon equation. In the usual Schwarzschild metric with dynamics, both expansions become zero. This is one of the reasons why we choose the Vaidya metric. By inserting equation (4.6) to equation (4.13), we obtain

$$\begin{aligned} a &= F\Bigg(1 - 2M_{,v}\ln\left(\frac{r}{2M} - 1\right) \\ &+ \frac{r}{M(r/2M-1)}M_{,v}\Bigg). \end{aligned} \tag{4.14}$$

Note that a is proportional to F. Now we solve $\Theta_{(l)} = 0$, to determine the dynamical horizon radius as

$$\begin{aligned} 2F - a &= 2F \\ &- F\Bigg(1 - 2M_{,v}\ln\left(\frac{r}{2M} - 1\right) \\ &+ \frac{r}{M(r/2M-1)}M_{,v}\Bigg) \\ &= 0. \end{aligned} \tag{4.15}$$

From this equation we obtain,

$$\begin{aligned} &1 + \Bigg(- 2M_{,v}\ln\left(\frac{r_D}{2M} - 1\right) \\ &+ \frac{r_D}{M(r_D/2M-1)}M_{,v}\Bigg) = 0. \end{aligned} \tag{4.16}$$

Here $F = 0$ is also the solution of the dynamical horizon. The dynamical horizon radius r_D is given by solving (4.16) as

$$r_D = 2M + 2Me^{-v/2M}. \tag{4.17}$$

Note that this dynamical horizon radius is outside the $r = 2M$, that is other solution.

4.2 Dynamical horizon with only Vaidya matter

First, we should derive the energy-momentum tensor $T_{\hat{t}l}$ for the integration of the dynamical horizon equation. For this end we derive it from the given Vaidya matter. For $G = 1,\ F = 1 - \frac{2M(v)}{r}$, the non-vanishing components of

the energy-momentum tensor becomes

$$
\begin{aligned}
T_{vv} &= \frac{1}{4\pi r^2}(FM_{,r} + M_{,v}) && (4.18)\\
T_{lr^*} &= -\frac{1}{4\pi r^2}M_{,r}a && (4.19)\\
T_{r^*r^*} &= 0. && (4.20)
\end{aligned}
$$

Here we have made the coordinate transformation from r to r^*. Writing T_{tl} in terms of T_{vv} and T_{vr^*} given by (4.18)(4.19) with $l^a = v^a$, we see

$$
\begin{aligned}
T_{tl} &= -T_{vv} + T_{vr^*}\\
&= \frac{1}{4\pi r^2}(-FM_{,r} - M_{,v} - aM_{,r})\\
&= -\frac{1}{4\pi r^2}\frac{5}{2}M_{,v}. && (4.21)
\end{aligned}
$$

With $\hat{t}^a$ being the unit vector in the direction of t^a, we obtain

$$T_{\hat{t}l} = -\frac{1}{4\pi r^2}\frac{5}{2}M_{,v}F^{-1}. \tag{4.22}$$

For the dynamical horizon integration (3.23), we get

$$\int_{r_1}^{r_2} 4\pi r_D^2 T_{\hat{t}l} dr_D = \frac{5}{2}\int_{M_1}^{M_2}(1 + e^{-v/2M})dM, \tag{4.23}$$

where we have used

$$F = \frac{e^{-v/2M}}{1 + e^{-v/2M}}, \tag{4.24}$$

and the fact

$$\frac{dM}{dv} = -e^{-v/2M}\left(2(1 + e^{-v/2M}) + \frac{v}{M}e^{-v/2M}\right)^{-1}, \tag{4.25}$$

changing the integration variable from r_D to M. In the above calculation, we treat $M_{,v}$ and F^{-1} with r_D fixed, because these functions are used only in the integration. Inserting equation (4.23) to the dynamical horizon equation (3.23), we obtain

$$
\begin{aligned}
&\frac{1}{2}(2M + 2Me^{-v/2M})\Big|_{M_1}^{M_2}\\
&= \int_{M_1}^{M_2}\frac{5}{2}(1 + e^{-v/2M})dM. && (4.26)
\end{aligned}
$$

Taking the limit $M_2 \to M_1 = M$, we obtain

$$-\frac{3}{2}(1 + e^{-v/2M}) + \frac{v}{2M}e^{-v/2M} = 0. \tag{4.27}$$

This equation is the dynamical horizon equation in the case that only the Vaidya matter is present. There is no solution of this equation, except the trivial one ($F = 0$ or $r = 2M$), so

$$r_D = 2M(v). \tag{4.28}$$

Here $M(v)$ is the arbitrary function only of the v, which represent the Vaidya black hole spacetime.

4.3 Dynamical horizon with Hawking matter

Next, we take into account the Hawking radiation. To solve this problem, we use two ideas that is to use the dynamical horizon equation, and to use the Vaidya metric. The reason to use the dynamical horizon equation comes from the fact that we need only information of matter near horizon, without solving the full Einstein equation with back reaction being the fourth order differential equations, for a massless scalar field. For the matter on the dynamical horizon, we use the result of Candelas [60], which assumes that spacetime is almost static and is valid near the horizon, $r \sim 2M$.

$$\begin{aligned} T_{tl} &= -T_{tt} \\ &= \frac{1}{2\pi^2(1-2M/r)} \int_0^\infty \frac{d\omega\omega^3}{e^{8\pi M\omega}-1} \\ &= \frac{1}{2cM^4\pi^2(1-2M/r)}, \end{aligned} \tag{4.29}$$

where we have used a well known result,

$$\int_0^\infty \frac{d\omega\omega^3}{e^{a\omega}-1} = \frac{\pi^4}{15a^4}, \tag{4.30}$$

and where $c = 61440$. This matter energy is negative near the event horizon. In the dynamical horizon equation, if black hole absorbs negative energy, black hole radius decreases. This is one of the motivations to use the negative energy tensor. Next we replace length of t to unit length, because in the dynamical horizon equation $\hat{t}$ is used, so

$$\hat{t}^0 = F^{-1/2}, \quad l^0 = F^{-1/2}, \tag{4.31}$$

and therefore, the energy tensor becomes

$$T_{\hat{t}l} = \frac{1}{2M^4c\pi^2(1-2M/r)^2}. \tag{4.32}$$

Calculating the integration on the right hand side of (3.23) for this matter,

$$
\begin{aligned}
& b\int \frac{r_D^2}{M^4(1-2M/r_D)^2}dr_D \\
&= b\int \frac{4M^2(1+e^{-v/2M})^4}{M^4 e^{-v/M}}\frac{dr_D}{dM}dM \\
&= b\int_{R_1}^{R_2} \frac{4(1+e^{-v/2M})^4}{M^2}e^{-v/M}\left(2(1+e^{-v/2M})+\frac{v}{M}e^{-v/2M}\right)dM. \quad (4.33)
\end{aligned}
$$

Here we insert the expression for r_D (4.17)in the first line, and the expression for $dr_D/dM = 2(1+e^{-v/2M}) + \frac{v}{M}e^{-v/2M}$ is used. Here b is a constant calculated in [60]

$$
b = \frac{1}{30720\pi}. \quad (4.34)
$$

If we also take account of the contribution of the Vaidya matter, and inserting this into the integration to the dynamical horizon equation (3.23), we obtain

$$
\begin{aligned}
& \frac{1}{2}(2M + 2Me^{-v/2M})\Big|_{M_1}^{M_2} \\
&= b\int_{M_1}^{M_2} \frac{2^2(1+e^{-v/2M})^4}{M^2}e^{-v/M}\left(2(1+e^{-v/2M})+\frac{v}{M}e^{-v/2M}\right)dM \\
&\quad + \int_{M_1}^{M_2}\frac{5}{2}(1+e^{-v/2M})dM. \quad (4.35)
\end{aligned}
$$

Taking the limit $M_2 \to M_1 = M$, we finally get

$$
\begin{aligned}
& -\frac{3}{2}(1+e^{-v/2M}) + \frac{v}{2M}e^{-v/2M} \\
&= b\frac{2^2(1+e^{-v/2M})^4}{M^2}e^{v/M}\left(2(1+e^{-v/2M})+\frac{v}{M}e^{-v/2M}\right), \quad (4.36)
\end{aligned}
$$

or

$$
M^2 = \frac{8b(1+e^{-v/2M})^4 e^{v/M}}{-\frac{3}{2}(1+e^{-v/2M})+\frac{v}{2M}e^{-v/2M}}\left((1+e^{-v/2M})+\frac{v}{2M}e^{-v/2M}\right). \quad (4.37)
$$

This is the main result of the present work that describes how the mass of black hole decreases. This equation is the transcendental equation, so usually it cannot be solved analytically. However, with the right hand side depending only on $-v/2M$, we can easily treat Eq.(4.37) numerically. Figure 4.3 is a graph of M as a function of v

If the dynamical horizon were inside the event horizon, the dynamical horizon radius would be

$$
r_D = 2M - 2Me^{-v/2M}.
$$

In this case, the dynamical horizon equation would become

$$M^2 = \frac{-8b(1-e^{-v/2M})^4 e^{v/M}}{\frac{7}{2}(1-e^{-v/2M}) - \frac{v}{2M}e^{-v/2M}} \left((1-e^{-v/2M}) - \frac{v}{2M}e^{-v/2M} \right).$$

The singular behavior of this expression excludes its physical relevance.

Now we show an approximation of Eq.(4.37) in particular limiting case. Taking the limit $M \to 0$, and $-v/2M = \text{const}$, we can see that (4.37) becomes,

$$\frac{bC_1}{M^2} + \frac{bvC_2}{M^3} = 0. \tag{4.38}$$

Where C_1, C_2 are positive constants. or

$$M = -\frac{C_2 v}{C_1}. \tag{4.39}$$

So, in the vanishing process the mass is proportional to v. For $M \to$ large

$$\dot{M} = -\frac{C_3}{\log M}, \tag{4.40}$$

where C_3 is a positive constant. It comes from the limit $M \to \infty$ and $-v/2M \to \infty$. In this limit the equation (4.37) becomes $v = -2M \log M$. This is different from Page's result [13]. Because if M goes to large, the dynamical horizon radius increases as M^2, so absorbed energy also become large. From this reason derivative of M by v changes. If we do not consider next order, the derivative of M becomes $\dot{M} = -C_4$, so that $4\pi r_D^2 T^4 \approx 1$, contradicting with Page's intuition.

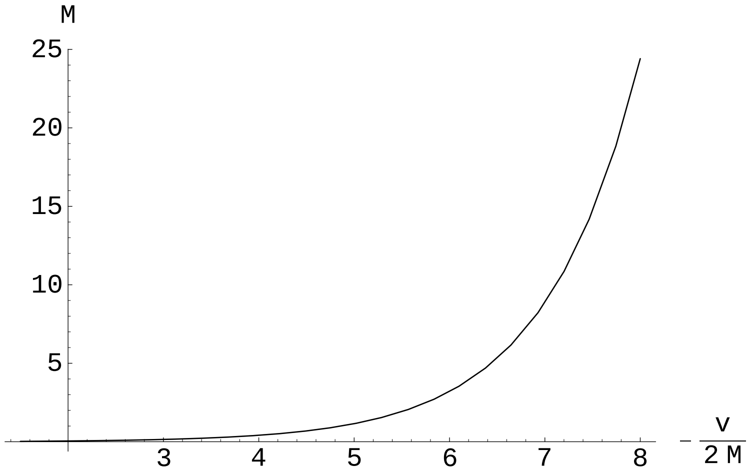

Figure 4.2: Numerical calculation of (4.37) The present approximation is justified for small M in the Planck unit.

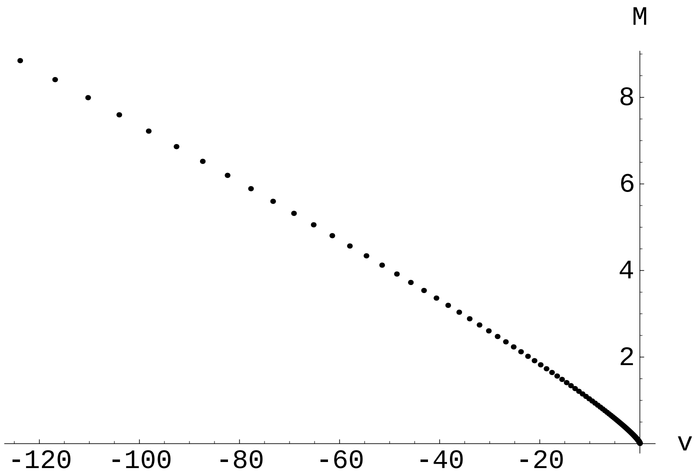

Figure 4.3: Numerical calculation of the black hole mass M as a function of v from the equation (4.37).The unit is the Planck unit.

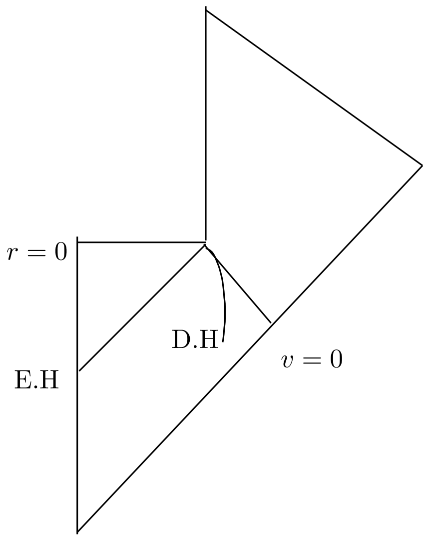

Figure 4.4: The Penrose diagram of the dynamical horizon. E.H means event horizon and D.H means the dynamical horizon.

Chapter 5

Conclusion

We have derived an equation which describes how the black hole mass changes taking into account of the Hawking radiation, in the special Vaidya spacetime which becomes the Schwarzschild spacetime in the static limit. From the analysis of the transcendental equation (4.37), we have shown that the black hole mass eventually vanishes and the spacetime becomes the Minkowski spacetime independent of the initial black hole mass size.

The dynamical horizon method in this paper can take into account of the back reaction of the Hawking radiation without solving the field equation which contains the fourth order differentials.

In the limit of the black hole mass going to zero, the derivative of the mass becomes small in proportion to the null coordinate ($v = t + r^*$). On the other hand as the black hole mass becomes large, the derivative behaves the minus of the inverse of the logarithm of the mass. Our result, which is different from Page's result, comes from the fact that in the large mass limit, the black hole radius behaves like quadratic of the black hole mass. This probably comes from when large mass limit that the approximation $r \to 2M$ is broken.

We would like to compare the present work to the preceding works. Sorkin and Piran or Hamade and Stewart used a massless scalar field instead of the Hawking radiation as the back reaction directly. The conclusion of their paper is that black hole starts with the small mass and it evaporates or increases. However, it is shown in the present work that even if the black hole starts with a large mass it always vanishes.

Although we have treated the black hole evaporation semi-classically, we hope this work will give an intuition to quantization of black holes.

Appendix A

Proof of the formula for the dynamical horizon

A.1 Dynamical horizon

We use that expansion of null vector can be written,

$$\Theta_{(l)} = K - K_{ab}\hat{r}^a\hat{r}^b + \tilde{K}. \tag{A.1}$$

Now we prove it.

$$\begin{aligned}
\tilde{K} + K &= \frac{1}{2}\tilde{q}^{ab}D_a(l_b - n_b) + \frac{1}{2}q^{ab}\nabla_a(l_b + n_b) \\
&= \frac{1}{2}\tilde{q}^{ab}D_a(l_b - n_b) + \frac{1}{2}(\tilde{q}^{ab} + \hat{r}^a\hat{r}^b)\nabla_a(l_b + n_b) \\
&= \tilde{q}^{ab}\nabla_a l_b + \hat{r}^a\hat{r}^b D_a\hat{\tau}_b && \text{(A.2)} \\
\hat{r}^a\hat{r}^b D_a\hat{\tau}_b &= \hat{r}^a\hat{r}^b K_{ab} && \text{(A.3)}
\end{aligned}$$

At the first line, we use $l^a = \hat{\tau}^a + \hat{r}^a$ and $n^a = \hat{\tau}^a - \hat{r}^a$. And second line, we use $q^{ab} = \tilde{q}^{ab} + \hat{r}^a\hat{r}^b$. My choice of l^a and n^a is not that case, but it is no problem. Because in the spherical symmetry, K and $\hat{r}^a\hat{r}^b K_{ab}$ cancels, and null expansion of l^a become simply proportional to $\tilde{K}$.

Proof of (3.15).

$$\begin{aligned}
\tilde{K}_{ab}\tilde{K}^{ab} &= (\frac{1}{2}\tilde{K}\tilde{q}_{ab} + \tilde{S}_{ab})(\frac{1}{2}\tilde{K}\tilde{q}^{ab} + \tilde{S}^{ab}) \\
&= \frac{1}{4}\tilde{K}^2\tilde{q}_{ab}\tilde{q}^{ab} + \tilde{S}_{ab}\tilde{S}^{ab} \\
&= \frac{1}{2}\tilde{K}^2 + \tilde{S}_{ab}\tilde{S}^{ab} && \text{(A.4)}
\end{aligned}$$

At the first line we simply insert (3.13), and second line, we use $\tilde{q}^{ab}\tilde{S}_{ab} = 0$ from definition.

$$\begin{aligned} K_{ab}K^{ab} &= (A\tilde{q}_{ab} + S_{ab} + 2\tilde{W}_{(a}\hat{r}_{b)} + B\hat{r}_a\hat{r}_b) \\ &\times (A\tilde{q}^{ab} + S^{ab} + 2\tilde{W}^{(a}\hat{r}^{b)} + B\hat{r}^a\hat{r}^b) \\ &= 2A^2 + S_{ab}S^{ab} + 4\tilde{W}_{(a}\hat{r}_{b)}\tilde{W}^{(a}\hat{r}^{b)} + B^2 \end{aligned} \tag{A.5}$$

This is almost all same as last equation, we use the definition of $\tilde{q}^{ab}$ and S^{ab}. And there are many cancellation terms.

Now we calculate the equation (A.5) term by term. At first A can be written by,

$$\begin{aligned} A &= \frac{1}{2}K_{ab}\tilde{q}^{ab} \\ &= \frac{1}{2}K_{ab}(q^{ab} - \hat{r}^a\hat{r}^b) \\ &= \frac{1}{2}(K - K_{ab}\hat{r}^a\hat{r}^b) \\ &= \frac{1}{2}(K - B). \end{aligned} \tag{A.6}$$

So first term of right hand side of equation (A.5) is,

$$\begin{aligned} 2A^2 &= \frac{1}{4}(K - B)^2 \\ &= \frac{1}{2}(K^2 - 2KB + B^2). \end{aligned} \tag{A.7}$$

Third term of equation (A.5) is written by,

$$\begin{aligned} \tilde{W}_{(a}\hat{r}_{b)}\tilde{W}^{(a}\hat{r}^{b)} &= \frac{1}{2}\tilde{W}_a\tilde{W}^a\hat{r}_b\hat{r}^b + \frac{1}{2}\tilde{W}_a\hat{r}^a\tilde{W}^b\hat{r}_b \\ &= \frac{1}{2}\tilde{W}_a\tilde{W}^a + 0. \end{aligned} \tag{A.8}$$

Here we use a definition of $\tilde{W}^a$, that is two dimension vector. Finally we changes term of $P^{ab}D_a\hat{r}_b$.

$$\begin{aligned} P^{ab}D_a\hat{r}_b &= (K^{ab} - Kq^{ab})D_a\hat{r}_b \\ &= (A\tilde{q}^{ab} + S^{ab} + 2\tilde{W}^{(a}\hat{r}^{b)} + B\hat{r}^a\hat{r}^b - Kq^{ab})D_a\hat{r}_b \\ &= A\tilde{q}^{ab}D_b\hat{r}_a + S^{ab}D_a\hat{r}_b + 2\tilde{W}^{(a}\hat{r}^{b)}D_a\hat{r}_b + B\hat{r}^a\hat{r}^bD_a\hat{r}_b - Kq^{ab}D_a\hat{r}_b \\ &= A\tilde{K} + S^{ab}D_a\hat{r}_b + \tilde{W}^a\hat{r}^bD_b\hat{r}_a - Kq^{ab}D_a\hat{r}_b \\ &= \frac{1}{2}(K - B)\tilde{K} + S^{ab}q^c_bD_a\hat{r}_c + \tilde{W}^a\hat{r}^bD_b\hat{r}_a - K(\tilde{q}^{ab} + \hat{r}^a\hat{r}^b)D_a\hat{r}_b \\ &= \frac{1}{2}K\tilde{K} - \frac{1}{2}B\tilde{K} + S^{ab}(\tilde{q}^c_b + \hat{r}^c\hat{r}_b)D_a\hat{r}_c + \tilde{W}^a\hat{r}^bD_b\hat{r}_a - K\tilde{K} \\ &= -\frac{1}{2}K\tilde{K} - \frac{1}{2}B\tilde{K} + S^{ab}\tilde{K}_{ab} + \tilde{W}^a\hat{r}^bD_b\hat{r}_a \\ &= -\frac{1}{2}K\tilde{K} - \frac{1}{2}B\tilde{K} + S^{ab}\tilde{S}_{ab} + \tilde{W}^a\hat{r}^bD_b\hat{r}_a \end{aligned} \tag{A.9}$$

At the second line we simply insert the decomposition of K^{ab} (3.14). And at the fourth line, we use definition of $\tilde{K}$ and the fact that $\hat{r}^a$ is normal vector. And sixth line we insert definition of A and definition of q^a_b.

Proof of (3.18). Now we simply insert the definition of γ^a and ζ^a.

$$\begin{aligned}\gamma^a = \alpha^a + \beta^a &= \hat{r}^a D_b \hat{r}^a - \hat{r}^a D_b \hat{r}^b + K^{ab}\hat{r}_b - K\hat{r}_a \\ &= \hat{r}^b D_b \hat{r}^a + \tilde{W}^a - \Theta_{(l)}\hat{r}^a \end{aligned} \tag{A.10}$$

$$\begin{aligned}\zeta_a\zeta^a &= (\tilde{W}_a + \hat{r}^b D_b \hat{r}_a)(\tilde{W}^a + \hat{r}^b D_b \hat{r}^a) \\ &= \tilde{W}_a\tilde{W}^a + 2\tilde{W}_a\hat{r}^b D_b\hat{r}^a + \hat{r}^b(D_b\hat{r}_a)\hat{r}^c(D_c\hat{r}^a). \end{aligned} \tag{A.11}$$

Next we show $\tilde{D}_a\zeta^a$ can be written by follows.

$$\begin{aligned}\tilde{D}_a\zeta^a &= \tilde{q}^c_a D_c\zeta^a \\ &= \tilde{q}^c_a D_c(\tilde{W}^a + \hat{r}^b D_b\hat{r}_a) \\ &= (q^c_a - \hat{r}_a\hat{r}^c)D_c(\tilde{W}^a + \hat{r}^b D_b\hat{r}^a) \\ &= D_a\tilde{W}^a - \hat{r}_a\hat{r}^c D_c\tilde{W}^a + (q^c_a - \hat{r}_a\hat{r}^c)D_c(\hat{r}^b D_b\hat{r}^a) \\ &= D_a\tilde{W}^a - \hat{r}_a\hat{r}^c D_c\tilde{W}^a + (q^c_a - \hat{r}_a\hat{r}^c)((D_c\hat{r}^b)(D_b\hat{r}^a) + \hat{r}^b D_c D_b\hat{r}^a \\ &= D_a\tilde{W}^a - \hat{r}_a\hat{r}^c D_c\tilde{W}^a \\ &+ (D_a\hat{r}^b)(D_b\hat{r}^a) + \hat{r}^b D_a D_b\hat{r}^a - \hat{r}_a\hat{r}^c(D_c\hat{r}^b)(D_b\hat{r}^a) - \hat{r}_a\hat{r}^c\hat{r}^b D_c D_b\hat{r}^a \\ &= D_a\tilde{W}^a + \tilde{W}_a\hat{r}^c D_c\hat{r}^a + (D_a\hat{r}^b)(D_b\hat{r}^a) + \hat{r}^b D_a D_b\hat{r}^a \\ &- \hat{r}_a\hat{r}^c(D_c\hat{r}^b)(D_b\hat{r}^a) + \hat{r}^c\hat{r}_b(D_c\hat{r}_a)(D_b\hat{r}^a) \end{aligned} \tag{A.12}$$

It is long calculation, but we only use the definition of $\tilde{W}^a$ and $\tilde{q}^{ab}$ and the fact $\hat{r}^a$ is normal vector.

Now I show $D_a\gamma^a$ is written by follows.

$$\begin{aligned}D_a\gamma^a &= D_a(\hat{r}^b D_b\hat{r}^a + \tilde{W}^a - \Theta_{(l)}\hat{r}^a) \\ &= (D_a\hat{r}^b)(D_b\hat{r}^a) + \hat{r}^b D_a D_b\hat{r}^a + D_a\tilde{W}^a - (D_a\Theta_{(l)})\hat{r}^a - \Theta_{(l)}D_a\hat{r}^a \\ &= (D_a\hat{r}^b)(D_b\hat{r}^a) + \hat{r}^b D_a D_b\hat{r}^a + D_a\tilde{W}^a - \hat{r}^a D_a\Theta_{(l)} - \Theta_{(l)}\tilde{K} \end{aligned} \tag{A.13}$$

Next I show ζ_a is written by follows.

$$\begin{aligned}\zeta_a &= \tilde{W}_a + \hat{r}^b D_b\hat{r}_a \\ &= K_{ab}\hat{r}^b - B\hat{r}_a + \hat{r}^b D_b\hat{r}_a \\ &= \hat{r}^b D_a\hat{\tau}_b - \hat{r}_a\hat{r}^b\hat{r}^c D_c\hat{\tau}_b + \hat{r}^b D_b\hat{r}_a \\ &= \tilde{q}^b_a\hat{r}^c\nabla_c l_b \end{aligned} \tag{A.14}$$

Above two equation, we use same method of derivation of equation (A.12).

Proof that σ_{ab} is shear.

$$\begin{aligned}
\tilde{S}_{ab} &= \tilde{K}_{ab} - \frac{1}{2}\tilde{K}\tilde{q}_{ab} \\
&= \tilde{q}_a^c \tilde{q}_b^d D_c \hat{r}_d - \frac{1}{2}\tilde{q}^{cd}\tilde{q}_{ab} D_c \hat{r}_a \qquad \text{(A.15)} \\
S_{ab} &= K_{ab} - \frac{1}{2}\tilde{q}_{ab} - W_{(a}r_{b)} - Br_a r_b \\
&= (\tilde{q}_b^d - r_b r^d) D_a \hat{\tau}_d - \frac{1}{2}(\tilde{q}^{cd} + r^c r^d)\tilde{q}_{ab} D_c \hat{\tau}_d \\
&+ \hat{r}^d \hat{r}_{(b} D_{a)} \hat{\tau}_d + \hat{r}^c \hat{r}^d \hat{r}_{(b} D_{a)} \hat{\tau}_d - \hat{r}^d \hat{r}^c \hat{r}_a \hat{r}_b D_c \hat{\tau}_d \\
&= \tilde{q}_b^d D_a \hat{\tau}_d - \frac{1}{2}\tilde{q}^{ac}\tilde{q}_{ab} D_a \hat{\tau}_d \qquad \text{(A.16)} \\
\tilde{S}_{ab} + S_{ab} &= \tilde{q}_a^c \tilde{q}_b^d D_c l_d - \frac{1}{2}\tilde{q}^{cd}\tilde{q}_{ab} D_c l_a \qquad \text{(A.17)}
\end{aligned}$$

In the calculation of equation (A.15) and (A.16), we insert the decomposition of $\tilde{S}^{ab}$ and S^{ab}.

A.2 Vaidya calculation

We write the inverse metric of the Vaidya, because we use it calculation of expansion of null vectors. Here we write only t ant r^* components.

$$\begin{pmatrix} -F & a-F \\ a-F & -F+2a \end{pmatrix}^{-1} = \frac{-1}{a^2}\begin{pmatrix} -F+2a & -a+F \\ -a+F & -F \end{pmatrix} \qquad \text{(A.18)}$$

Next I show follow of dynamical horizon radius. From equation $2F - a = 0$,

$$2 - 1 + \left(2M_{,v} \ln\left(\frac{r_D}{2M} - 1\right) + \frac{r_D}{M(1 - 2M/r_D)} M_{,v}\right) = 0. \qquad \text{(A.19)}$$

$$2M_{,v} \ln\left(\frac{r_D}{2M} - 1\right) + \frac{r_D}{M(1 - 2M/r_D)} M_{,v} = -1 \qquad \text{(A.20)}$$

It is rewritten by follows.

$$\partial_v (2M \ln(r_D/2M - 1)) = -1 \qquad \text{(A.21)}$$

To integrate this equation, we get,

$$2M \ln(r_D/2M - 1) = -v \qquad \text{(A.22)}$$

From this equation,

$$r_D = 2M + 2Me^{-v/2M}. \qquad \text{(A.23)}$$

Next I show how T_{tl} is written by T_{vv} and T_{vr}. I use the fact $l^a = -v^a$ (now I renormalize l^a).

$$
\begin{aligned}
T_{ll} &= T_{tt} - 2T_{tr^*} + T_{r^*r^*} \\
T_{tl} &= -T_{ll} - T_{tr^*} \\
T_{vv} &= T_{tt} - 2T_{tr^*} + T_{r^*r^*} \\
T_{ll} &= T_{vv} \\
T_{vr^*} &= -T_{tr^*} + T_{r^*r^*} \\
T_{tl} &= -T_{vv} + T_{vr^*}
\end{aligned} \tag{A.24}
$$

To insert the specific form of T_{vv} and T_{vt},

$$
\begin{aligned}
T_{tl} &= \frac{1}{4\pi r^2}(-3FM_{,r} - M_{,v}) \\
&= \frac{1}{4\pi r^2}\left(-\frac{3}{2}M_{,v} - M_{,v}\right) \\
&= -\frac{1}{4\pi r^2}\frac{5}{2}M_{,v}.
\end{aligned} \tag{A.25}
$$

I show integration of $T_{\hat{t}l}$.

$$
\begin{aligned}
\int_{r_1}^{r_2} 4\pi r_D^2 T_{\hat{t}\hat{l}} dr_D &= -\int_{r_1}^{r^2} \frac{5}{2} M_{,v} F^{-1} dr_D \\
&= -\int_{r_1}^{r_2} \frac{5}{2} F^{-1} \frac{dr_D}{dv}\frac{dM}{dr_D} dr_D \\
&= \int_{M_1}^{M_2} \frac{5}{2} \frac{1+e^{-v/2M}}{e^{-v/2M}} e^{-v/2M} dM \\
&= \frac{5}{2}\int_{M_1}^{M_2} (1+e^{-v/2M}) dM
\end{aligned} \tag{A.26}
$$

In the second line, I use the fact that $M_{,v} = dr_D/dv \times dM/dr_D$. In the third line, I use the relation $dr_D = dr_D/dM \times dM$.

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