Toward The Unified Theory of Decision Making under Uncertainty: Self-Conflicting Decision

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Abstract

A brand-new viewpoint on decision research is presented on the basis of the Bayesian decision theory. The Bayesian theory can be made more realistic and more usable whether for *prescriptive* or *descriptive* only by doing the following: (1) Eliminate the formal fundamental asymmetry between the quantification of the qualitative preference order on options and the one of the qualitative belief order on events, (2) Introduce the factors on human psychology other than the preference and the belief, and the factors other than the events and the consequences on situations in which decisions are made, and (3) Unify the prescriptive and the descriptive theories on the basis of the spirit of the "Relationism." This is now my realization. The core in this paper is that people's *safety preference disposition* and *risk preference disposition* play indispensable roles in actual decision making. The substantial content of the uncertainty is clarified and the following three basic concepts are introduced within the framework of the Bayesian decision theory: (1) *Uncertainty-Probability Equivalence Principle*, (2) *Safety Preference Disposition*, and (3) *Risk Preference disposition*. By using these concepts, a hypothesis on actual decision making under uncertainty is formulated, which is called *Self-Conflicting Decision* (SCD). It cannot be emphasized too strongly that SCD differs absolutely from the Bayesian decision maximizing the expected utility. It is shown in an insightful way that SCD can produce naturally the decisions to the Ellsberg 3-color-ball decision problem which are in conformity with *both* the experimental results and the Bayesian decisions. The implications of SCD are examined by using the Continuous Ellsberg problem. My perspective on the unified theory is presented, and the problem of the formal fundamental asymmetry in the Bayesian decision theory is posed. This paper is concluded with challenging remarks.

Keywords

Bayesian Decision, Continuous Ellsberg Problem, Formal Fundamental Asymmetry in Quantification, Risk Preference Disposition, Safety Preference Disposition, Self-Conflicting Decision, Uncertainty, Uncertainty-Probability Equivalence Principle.

I. Introduction

Charles Sanders Peirce wrote in 1877 "There must be a real and living doubt, and without this all discussion is idle," quoted from [1].

It is not a wise decision to start in tabula rasa toward the unification of prescriptive and descriptive decision theories under uncertainty[1]. We need to choose a firm starting basis.

[1]This paper is concerned with individual decisions. The viewpoint in this paper can also be useful for the inquiries into group decisions.

I take my departure toward the unification on the basis of the Bayesian decision theory [2] that is prescriptive because it has the firm and clear axiomatic system and has proved useful in many fields; among others, Bayesian statistics [3], pattern recognition [4], [5], image processing [6], medicine [7], flexible information processing [8], vision [9], cognitive science [10][2]. This paper is based on the axiomatic system of the Bayesian decision theory in [2] that is an excellent description of the Bayesian axiomatic system in my judgment. Its compact summary with additional explanations and corrections written in English is furnished on my Web site (Memos/) so that anybody can enjoy making oneself familiar with it.

However, the Bayesian decision theory has the following basic problems to be solved.

(1) There is formal fundamental asymmetry between the quantification of the qualitative preference order on options and the one of the qualitative belief order on events.

(2) There are also fundamental differences not to be negligible between actual decision making and the Bayesian decision theory. The Bayesian theory assumes that the essential elements of decision making are your preference order on options, your belief order on events, and the consequences to be yours. This is too simplified abstraction of actual decision making.

The Bayesian theory can be made more realistic and more usable whether for *prescriptive* or *descriptive* by eliminating the asymmetry and by introducing the factors on human psychology other than the preference and the belief, and the factors other than the events and the consequences on situations in which decisions are made. The factors should be multi-dimensional, quantitative and specific concepts. For the purpose, we need to do the unified research of the prescriptive and the descriptive theories. Our target, the unified theory on decision making under uncertainty that includes the prescriptive and the descriptive theories, will be attained in the inquiries based on the spirit of the "Relationism" in the philosophy of Hiromatsu [11], [12][3]. This is now my realization.

The view presented ten years ago by Pitz and Sachs [13] has become newly important from the viewpoint in this paper: "[A] broader formulation of prescriptive theory is needed in which the distinction between description and prescription is less important."

[2]This is on my Web site (Papers/)

[3]My memos, written in Japanese, on the philosophy of Hiromatsu are on my Web site (Memos/).

The recent attempt of Tversky and Fox [14] tries to explain people's decisions under uncertainty by using the notion of Weighting Function (a nonlinear transformation of the probability scale) and the principle of bounded subadditivity. Their attempt is technically interesting. However, it is impossible for me to expect that such technical devices can contribute toward the unification. We should shoot a look at the heart of the Bayesian theory.

On the other hand, Shigemasu and Yokoyama [15] try to make the Bayesian axiomatic system flexible to describe the psychological decision making process. Their approach, Flexible or Gentle Bayesian Approach, is very interesting because it looks straight at the heart of the Bayesian theory. However, their approach is still technical because it is confined within higher order probabilities and their fuzzy representations using the membership functions. Their research policy should be more radical to make for their goal.

Ellsberg [16] has proposed the objection, in the counterexamples which are now well known as the Ellsberg problems, to the sure-thing principle as an axiom for the prescriptive decision, which is necessary to make the degrees of belief the additive probability measure. This principle or axiom asserts that if two options have a common consequence under a particular event, then the preference order of the options should be independent of the value of that common consequence. The Ellsberg 3-color-ball decision problem under uncertainty is insightful so as to identify some missing psychological factors in decision making.

Schmeidler [17] asserts that his extension of the expected utility based on the non-additive probability measure can explain the people's preferences to the Ellsberg 3-color-ball problem that are inconsistent with the additive expected utility. His explanation, however, is not cognitive but mathematical. This paper will show in a cognitive way that the non-additive probability is actually unnecessary to explain the people's preferences. Shigemasu [15], [18] asserts that the higher order probability can explain the Ellsberg's Paradox[4] better than the non-additive measure explanation [17], [19], [20], and that he has

[4]Note that there is no paradox anywhere, and the fact is that people's decisions are out of accordance with the decisions based on the sure-thing principle. Schmeidler also uses this word in [17]. It should be called the Ellsberg's *phenomenon*.

demonstrated the reasons for the discordance in the experiments on the Ellsberg 2-color-ball-2-urn problem by using a mixed distribution. Note, however, that he has changed the original one-time decision problem to a repeated decision problem, although the later is interesting in its own right. This indicates that he has failed to explain the original discordance. Akaike [21] provides a statistical explanation of the people's decision to the Ellsberg 2-color-ball-2-urn problem on the basis of his own theory of statistics, not as a cognitive scientist but a statistician. All of these is not a genuine cognitive explanation but superficial, and is not promising. This will have been your own realization when you have read this paper.

With the hope that the viewpoint in this paper can prove to be useful for the inquiries of the researchers who are aiming and will aim at the same target, I will report the process of my thought in a clear and easy style and refer to facts that have led me to the viewpoint.

II. Problems to be Solved

A. Ellsberg Problem

It would be very helpful for you to consider this insightful Ellsberg 3-color-ball decision problem for yourself to your comprehension of the issues to be discussed in this paper. Read the problem carefully and write down your choices before you read the following analyses of the problem. Note that this is not a quiz; there is no single "right answer" to the problem.

There is an urn known to contain 90 balls. Thirty of the balls are red, the remaining 60 are black and yellow in unknown proportion. One ball is to be drawn at random from the urn.

Situation S_1: There are the following two options, A_1 and A_2. Which of these do you prefer to bet on?

(A_1) If the ball is red, you will receive \$100. Otherwise nothing.

(A_2) If the ball is black, you will receive \$100. Otherwise nothing.

YOUR CHOICE:

Situation S_2: There are the following two options, A_3 and A_4. Which of these do you

prefer to bet on?

(A_3) If the ball is red or yellow, you will receive \$100. Otherwise nothing.

(A_4) If the ball is black or yellow, you will receive \$100. Otherwise nothing.

YOUR CHOICE:

B. Ellsberg Problem and Bayesian Decision

Let us identify, making use of the Ellsberg 3-color-ball problem, the problems to be solved.

B.1 Issues Posed by Preference Combinations Based on the Sure-Thing Principle

First, let us represent the Ellsberg problem transparently.

Symbols:

R, B, Y:Events of Red, Black and Yellow, respectively

$P(\cdot)$: Probabilities

$c(> 0)$: Gains, say \$100

N_r, N_b, N_y: Numbers of red, black and yellow balls, respectively

Conditions: $N_r = 30, N_b + N_y = 60$

The situations S_1 and S_2 can be represented as follows:

S_1:

Option A_1

Event	R	B	Y
Gain	c	0	0

Option A_2

Event	R	B	Y
Gain	0	c	0

S_2:

Option A_3

Event	R	B	Y
Gain	c	0	c

Option A_4

Event	R	B	Y
Gain	0	c	c

Assume that A_1 is strictly preferred to A_2, $A_1 > A_2$, in S_1. This preference should have nothing to do with the occurrence of Y because under Y the gain coming from A_1 is the same as that of A_2. The same thing holds in S_2. So, the preferences in S_1 and S_2 should be based on the possibilities of R and B. Since the gain structure under R and B is the same between S_1 and S_2, A_3 should be strictly preferred to A_4 ($A_3 > A_4$) in S_2 because we have assumed that $A_1 > A_2$ in S_1. As with this, if $A_2 > A_1$, then $A_4 > A_3$.

These patterns or combinations of the preferences do not contain $\{A_1 > A_2, A_4 > A_3\}$ and $\{A_2 > A_1, A_3 > A_4\}$ that are the dominant ones in the experiments [22], [23], [19]. That is to say, people's decision behavior is out of accordance with the sure-thing principle. This implies that people's probabilities, psychological probabilities, are *not additive* (Bear it in mind till Section IV.*B*.) [19], [20]. This poses the following issues:

(1) Are people's decisions wrong?

(2) Should the validity of the sure-thing principle as an axiomatic basis for prescriptive decision be seriously reconsidered?

B.2 Bayesian Decision: Maximizing Expected Utility

The Bayesian prescription for quantitative and coherent decision making is: Choose the option with the greatest expected utility [2]. This means that the Bayesian decision theory assumes that you can determine your subjective probabilities without ambiguity. Under uncertainty, however, by definition, you do not have enough information to specify them without ambiguity. Therefore, this prescription does not work under uncertainty. Let us see it.

Assume the following:

$$\begin{aligned} P(R|N_b) &= 1/3 \\ P(B|N_b) &= N_b/90 \\ P(Y|N_b) &= (60 - N_b)/90. \end{aligned}$$

N_b distributes uniformly in $\{0,1,...,60\}$, i.e., $P(N_b) = 1/61$ for all N_b. These assumptions are so natural in this problem that nobody can make any persuadable objections to them. We can assume without loss of generality that $u(0) = 0$ and $u(c) = 1$. $u(\cdot)$ is your utility of the gains.

The expected utilities of A_1, A_2, A_3, and A_4 are as follows.

Conditional Expected Utilities:

$$\begin{aligned}
\overline{u}(A_1|N_b) &= u(c)P(R|N_b) + u(0)P(B|N_b) + u(0)P(Y|N_b) \\
&= 1/3 \\
\overline{u}(A_2|N_b) &= N_b/90 \\
\overline{u}(A_3|N_b) &= 1/3 + (60 - N_b)/90 \\
\overline{u}(A_4|N_b) &= 2/3
\end{aligned}$$

Unconditional Expected Utilities:

$$\begin{aligned}
\overline{u}(A_i) &= \sum_{N_b=0}^{60} \overline{u}(A_i|N_b)P(N_b), \ i = 1, \ldots, 4. \\
\overline{u}(A_1) &= \overline{u}(A_2) = 1/3 \\
\overline{u}(A_3) &= \overline{u}(A_4) = 2/3
\end{aligned}$$

Therefore, you *cannot* rely on the maximization of your expected utility to choose one between A_1 and A_2, and one between A_3 and A_4. But you would like to choose one of them in each of the situations because you cannot lose at all but have the possibilities to gain some money. This implies that the Bayesian decision theory fails to abstract some indispensable factors in decision making.

Thus, the problems to be solved have been identified.

Problem 1: What are prescriptive decisions under uncertainty?

Problem 2: How do people make decisions under uncertainty?

C. Issues to be Discussed

First, to attack Problem 2, the substantial content of the uncertainty will be clarified and the following three basic concepts will be introduced within the framework of the Bayesian theory in Section III: (1) *Uncertainty-Probability Equivalence Principle*, (2) *Safety Preference Disposition*, and (3) *Risk Preference Disposition*. By using these concepts, a hypothesis concerning actual decision making under uncertainty will be formulated, which will be called *Self-Conflicting Decision* (SCD). It will be clarified how SCD differs from the Bayesian decision. It will be shown in Section IV that SCD can produce naturally the

decisions to the Ellsberg 3-color-ball decision problem which are in accordance with *both* the experimental results and the Bayesian decisions. Then, the implications of SCD will be examined by using the Continuous Ellsberg problem. My perspective on the unified theory will be presented. In Section V, the problem of the formal fundamental asymmetry in the Bayesian decision theory will be posed. Finally, the concluding remarks will be made in Section VI.

III. Self-Conflicting Decision

Let us begin with the clarification of the substantial content of the uncertainty.

A. Substantial Content of Uncertainty

In this paper, the options with the probability of 1 will be called *sure options*, the options with certain probabilities in $(0, 1)$ *probabilistic options*, and the options with uncertain probabilities *uncertain options*.

Let $\overline{u}(A_p)$ denote the unconditional expected utility of a *probabilistic* option, A_p. Let $\overline{u}(A_u|x)$, given x, denote the conditional expected utility of an *uncertain* option, A_u. Here, x denotes a generic random quantity. When the following relation holds,

$$\min_x \overline{u}(A_u|x) < \overline{u}(A_p) < \max_x \overline{u}(A_u|x) \tag{1}$$

that is, the uncertain option is *risky*, the uncertainty is *significant* with respect to choice between the probabilistic option and the uncertain option. In this case, the probabilistic option is called a *safe* option and the uncertain option a *risky* option.

For the case of choice between *two uncertain* options, A_1 and A_2, let $\overline{u}(A_1|x)$ and $\overline{u}(A_2|x)$ denote the conditional expected utilities of A_1 and A_2, respectively. When the following relations hold,

$$\min_x \overline{u}(A_2|x) < \min_x \overline{u}(A_1|x) < \max_x \overline{u}(A_1|x) < \max_x \overline{u}(A_2|x) \tag{2}$$

A_2 is risky and the uncertainty is significant with respect to choice between the uncertain options. In this case, A_1 is called a safe option and A_2 a risky option.

B. Uncertainty-Probability Equivalence Principle

First, let us review the essence of the principle or axiom of precise measurement concerning options in the Bayesian theory [2]. This principle asserts that there exists a probabilistic option, $\{c_2|p, c_1|1-p\}$, being equivalent to a sure option, c. Its formal representation is as follows. There exists a probability p such that

$$c \sim \{c_2|p, c_1|1-p\} \quad for \quad c_1 \leq c \leq c_2,$$

where c, c_1 and c_2 are consequences, "$\sim$" denotes an equivalence relation in preference, and "$c_1 \leq c$" signifies that c_1 is not preferred to c. The p is a subjective probability of the occurrence of c_2. This principle can be called the Principle of Sureness-Probability Equivalence.

Let us extend the principle of sureness-probability equivalence between a *sure* option and a *probabilistic* option to the one between an *uncertain* option and a *probabilistic* option. This new principle, *Uncertainty-Probability Equivalence Principle (UPEP)*, asserts that there exists a probabilistic option, $\{c_2|p^*, c_1|1-p^*\}$, where c_2 occurs with a subjective probability p^*, being equivalent to an uncertain option, $\{c_2|[p_1, p_2], c_1|[1-p_2, 1-p_1]\}$, where c_2 occurs with a probability, say p, in $[p_1, p_2]$ and c_1 occurs with a probability, say $1-p$, in $[1-p_2, 1-p_1]$. It is defined formally as follows.

Definition 1: (**Uncertainty-Probability Equivalence Principle**) *There exists a subjective probability p^* such that*

$$\{c_2|[p_1, p_2], c_1|[1-p_2, 1-p_1]\} \sim \{c_2|p^*, c_1|1-p^*\}, \tag{3}$$

where $c_1 \leq c_2$, $p_1 \leq p_2$, $p_1 < p^* < p_2$, and if $p_1 = p_2$, then $p^* = p_1 = p_2$. The p^* is called the subjective equivalent probability of the uncertain option.

C. Safety Preference Disposition

People like safety. They buy insurance. Fund managers hedge for stock portfolio protection. Politicians often use ambiguous words so that they cannot commit themselves to any sort of pledge. Or, animals have the natural disposition to avoid danger. You can also enumerate a lot of facts showing that people and animals have the natural disposition to

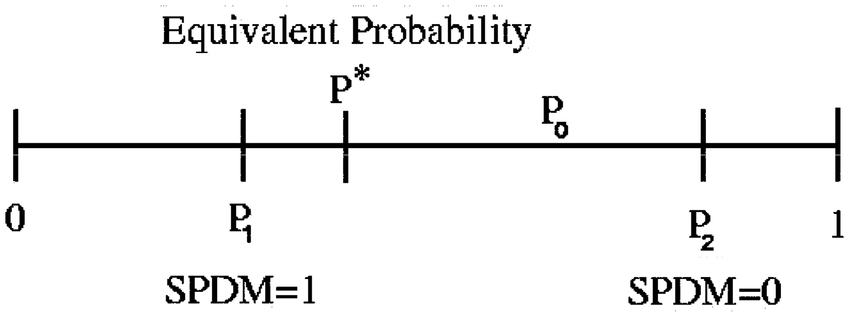

Fig. 1. Safety Preference Disposition Measure (SPDM)

like safety. I believe that this *safety preference disposition* has been created in the process of evolution.

The Bayesian decision theory does not realize the importance of this safety preference disposition in decision making. It is my natural feeling that their safety preference disposition plays an indispensable role in their decision making. This natural feeling is also grounded on my own actual serious stock investment decision experiences. To describe their decision behavior, it is essential to take the concept of the safety preference disposition into consideration in addition to the one of the preference between options.

Since the safety preference disposition is essentially personal and the decision under uncertainty is considered, the safety preference disposition measure is defined as follows.

Definition 2: (**Safety Preference Disposition Measure**) *The Safety Preference Disposition Measure (SPDM), α, is defined by using the subjective equivalent probability p^* as*

$$p^* \equiv \alpha p_1 + (1-\alpha)p_2, \tag{4}$$

where $0 < \alpha < 1$.

The safety preference disposition measure should not be considered a simple numeral but a function of the other psychological and situational factors to be discovered. Note that the more you want to play for safety, the larger your α is. Fig.1 illustrates the safety preference disposition measure. The probabilistic options $\{c_2|p_0, c_1|1-p_0\}$, where p_0's are greater than p^*, is preferred to the uncertain option $\{c_2|[p_1,p_2], c_1|[1-p_2, 1-p_1]\}$.

D. Risk Preference Disposition

People also like to take risks, although they have the safety preference disposition. They invest their own savings in stocks. Statesmen take risky political policies if necessary. See the late John Fitzgerald Kennedy, the 35th President of the United States, for the 13 Days of the Cuba Missile Crisis in October, 1962. Or, young animals challenge new food in risky situations. I believe that this *risk preference disposition* has also been created in the process of evolution.

Let us consider *preference* to find how to define the measure of the risk preference disposition. It is well known that preference is a mental disposition which plays an important role in decision making. When you prefer an option A_2 to an option A_1, there must be *strength* in your preference. In some cases, you prefer A_2 to A_1 *diffidently*, in other cases, *moderately*, *confidently*, etc. The strength of your preference is another mental factor that plays an indispensable role without all doubt in your decision making. The Bayesian decision theory fails to abstract this mental factor. Therefore, it is necessary to introduce a measure of the preference strength to describe actual decision making behavior. The preference strength is essentially personal. Hence, the measure of the preference strength is to be defined by using the subjective probability that A_2 is preferred to A_1. The definition of the measure of the preference strength is now straightforward. By using it, the measure of the *risk preference disposition* is defined as follows.

Definition 3: (**Risk Preference Disposition Measure**) *The measure of the Preference Strength on which A_2 is preferred to A_1, $I[A_2 > A_1]$, is defined by the subjective probability that $\overline{u}(A_2|x) > \overline{u}(A_1|x)$ as*

$$I[A_2 > A_1] \equiv Prob\{\overline{u}(A_2|x) > \overline{u}(A_1|x)\}, \tag{5}$$

where $\overline{u}(A_2|x)$ and $\overline{u}(A_1|x)$ are the conditional expected utilities, give x, of A_2 and A_1, respectively, and x denotes a generic random quantity. The Risk Preference Disposition Measure (RPDM) is defined as $I[A_2 > A_1]$ under the condition that A_2 is a risky option with respect to A_1. This is written $I_R[A_2 > A_1]$.

Note that if $\overline{u}(A_2|x)$ and $\overline{u}(A_1|x)$ are not random quantities, then $I[A_2 > A_1] = 1$ or 0, which mean that $A_2 > A_1$, $A_2 < A_1$, or $A_2 \sim A_1$ *for sure*.

E. Hypothesis

The safety preference disposition and the risk preference disposition are not one-dimensional concept but two-dimensional one. These are conflicting in your mind while you are struggling to make a decision. When a compromise between them has been made in your mind, you have a decision. Let us formalize it as a hypothesis concerning actual decision making.

Hypothesis: *People tend to choose one option between two options by making compromise between their Safety Preference Disposition and Risk Preference Disposition with themselves. If their Safety Preference Disposition is stronger (weaker) than their Risk Preference Disposition, then they tend to choose the safe option (risky option). This decision is named Self-Conflicting Decision (SCD).*

It cannot be emphasized too strongly that SCD differs absolutely from the Bayesian decision maximizing the expected utility.

Let us make this hypothesis operational with the following simple decision function:

$$\begin{aligned} If\ I_R[A_2 > A_1] &> \alpha,\ then\ A_2\ is\ chosen. \\ If\ I_R[A_2 > A_1] &= \alpha,\ then\ the\ choice\ is\ indecisive. \\ If\ I_R[A_2 > A_1] &< \alpha,\ then\ A_1\ is\ chosen. \end{aligned} \tag{6}$$

Here, A_2 is a risky option and A_1 is a safe one.

The decision function in the coming unified theory would not be complicated but complex. Some solutions of the decision function are to be prescriptive decisions, and other solutions are to be people's ones.

It should be emphasized here that the insightful view proposed more than ten years ago by Einhorn and Hogarth [24] has been actually realized in a promising way in the theory of SCD, although partially at present: "Conceptualizing decision ... as the clash between multiple selves is a potentially rich area of investigation and could provide useful conceptual links between phenomena of individual and group behavior. For example, individual irrationality might be seen as similar to the various voting paradoxes found in group decision making."

IV. Evaluation of Self-Conflicting Decision

The validity of the Self-Conflicting Decision Theory (SCDT) as a descriptive theory and its implications will be evaluated by using the following Continuous Ellsberg Problem (CEP). After that, my perspective on the unified theory will be presented.

A. Continuous Ellsberg Problem

Let us slightly extend the Ellsberg 3-color-ball problem to gain a deeper insight into decision under uncertainty. In CEP, the probability of R, r, is a known continuous non-random variable, and the probability of B, x, is a continuous random quantity. The others are the same as the Ellsberg 3-color-ball problem. Imagine a rotating "dartboard" of which face is painted red, black and yellow in proportion to the probability of each color, and that one dart is to be thrown at the rotating dartboard. This problem is represented as follows.

Event	R	B	Y
Probability	r	x	y
S_1 :			
Option			
A_1	c	0	0
A_2	0	c	0
S_2 :			
Option			
A_3	c	0	c
A_4	0	c	c

Conditions:

(1) r *is given,* $0 < r < 1$.

(2) $r + x + y = 1$

(3) There is no information to specify a particular value of x.

The conditional expected utilities of A_1, A_2, A_3 and A_4, given x, are

$$\overline{u}(A_1|x) = r$$

$$\begin{aligned}\overline{u}(A_2|x) &= x \\ \overline{u}(A_3|x) &= 1-x \\ \overline{u}(A_4|x) &= 1-r,\end{aligned}$$

where $0 \leq x \leq 1-r$, $u(0)=0$, and $u(c)=1$. We assume here that x distributes uniformly in $[0,1-r]$. The density function of x is $p(x)=1/(1-r)$ for x in $[0,1-r]$, $p(x)=0$ otherwise. I believe that nobody can make any decisive objections to this assumption in this problem. Note that the uncertainty in the situations S_1 and S_2 is *significant* for r in $(0,1/2)$.

(a) Bayesian Decision

The unconditional expected utilities are:

$$\begin{aligned}\overline{u}(A_1) &= r \\ \overline{u}(A_2) &= (1-r)/2 \\ \overline{u}(A_3) &= 1-(1-r)/2 \\ \overline{u}(A_4) &= 1-r.\end{aligned}$$

We obtain the following decisions.

$$\begin{aligned}&If \;\; r<1/3, \;\; then \;\; A_2 > A_1 \; and \; A_4 > A_3. \\ &If \;\; r=1/3, \;\; then \;\; A_1 \sim A_2 \; and \; A_3 \sim A_4. \\ &If \;\; r>1/3, \;\; then \;\; A_1 > A_2 \; and \; A_3 > A_4. \end{aligned} \tag{7}$$

Note that these combinations do not contain $\{A_2 > A_1, A_3 > A_4\}$ and $\{A_1 > A_2, A_4 > A_3\}$ which are obtained in the experiments, and the Bayesian theory is *indecisive* where $P(R)=r=1/3$. Let us call it *Bayesian Indecisiveness* (Keep it in mind till Section IV.*B*.).

(b) Self-Conflicting Decision

The risk preference disposition measures between these options are:

$$I_R[A_2 > A_1] = Prob\{x > r\}$$

$$= \begin{cases} 1 - r/(1-r) & for\ r < 1/2 \\ 0 & for\ r \geq 1/2 \end{cases}$$

$$I_R[A_3 > A_4] = Prob\{x < r\}$$
$$= \begin{cases} r/(1-r) & for\ r < 1/2 \\ 1 & for\ r \geq 1/2. \end{cases}$$

We obtain the following decisions.

S_1 : A_1 $vs.$ A_2 (A_2 is risky.)

For $r < 1/2$,

$$\begin{aligned} If\ \ 1 - r/(1-r) &< \alpha,\ then\ A_1 > A_2. \\ If\ \ 1 - r/(1-r) &= \alpha,\ then\ A_1 \sim A_2. \\ If\ \ 1 - r/(1-r) &> \alpha,\ then\ A_2 > A_1. \end{aligned}$$

For $r \geq 1/2$, $A_1 > A_2$.

S_2 : A_3 $vs.$ A_4 (A_3 is risky.)

For $r < 1/2$,

$$\begin{aligned} If\ \ r/(1-r) &< \alpha,\ then\ A_4 > A_3. \\ If\ \ r/(1-r) &= \alpha,\ then\ A_3 \sim A_4. \\ If\ \ r/(1-r) &> \alpha,\ then\ A_3 > A_4. \end{aligned}$$

For $r \geq 1/2$, $A_3 > A_4$.

Fig.2 summarizes the results.

A.1 Validation

Let us test the validity of SCDT as a descriptive theory in the special case where $P(R) = r = 1/3$, which is the Ellsberg 3-color-ball problem. The decisions with SCDT are as follows.

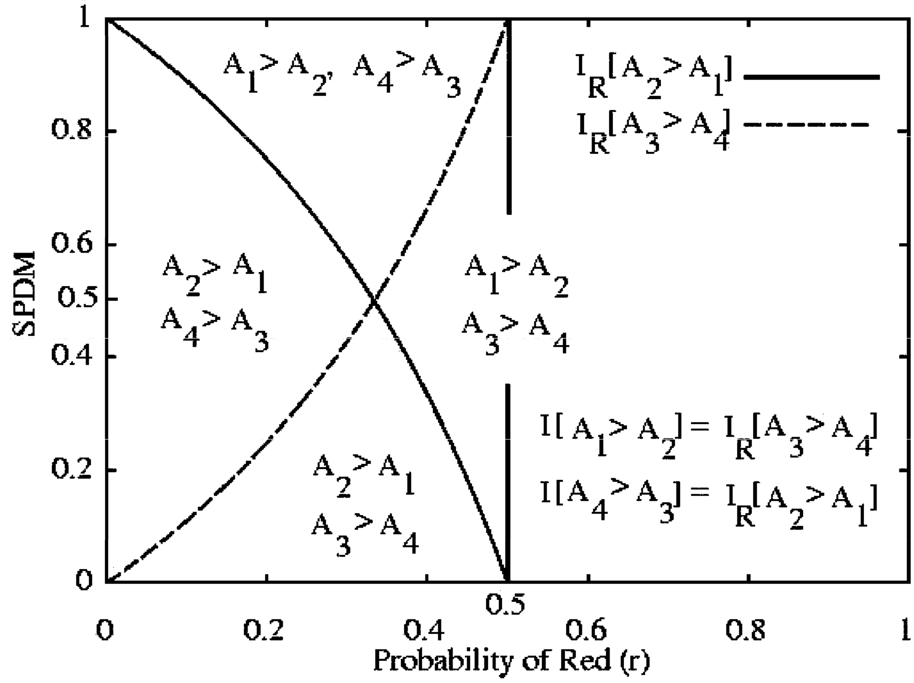

Fig. 2. Decision Space of Self-Conflicting Decision for CEP

S_1 : A_1 $vs.$ A_2 (A_2 is risky.)

$$\begin{aligned} If\ 1/2 &< \alpha,\ then\ A_1 > A_2. \\ If\ 1/2 &= \alpha,\ then\ A_1 \sim A_2. \\ If\ 1/2 &> \alpha,\ then\ A_2 > A_1. \end{aligned}$$

S_2 : A_3 $vs.$ A_4 (A_3 is risky.)

$$\begin{aligned} If\ 1/2 &< \alpha,\ then\ A_4 > A_3. \\ If\ 1/2 &= \alpha,\ then\ A_3 \sim A_4. \\ If\ 1/2 &> \alpha,\ then\ A_3 > A_4. \end{aligned}$$

See Fig.2. If $\alpha \neq 1/2$ and is kept constant in effect between the two situations S_1 and S_2, which are natural assumptions, then the possible combinations are $\{A_1 > A_2, A_4 > A_3\}$ and $\{A_2 > A_1, A_3 > A_4\}$. These are in accordance with the experimental results.

A.2 Implications

(a) Choice Based on the Equivalent Probability p^*

It is interesting to see the decisions based on the equivalent probability. Are they the same as those with SCDT? The answer is "Yes" in the case of CEP as you will see below. The equivalent probabilities in S_1 and S_2 concerning the events B and Y are:

$$p^* = \alpha p_1 + (1-\alpha)p_2,$$

where $p_1 = 0$ and $p_2 = 1 - r$. The unconditional expected utilities of A_1, A_2, A_4 and A_3 are:

S_1 : A_1 $vs.$ A_2 (A_2 is risky.)

$$\begin{aligned}\overline{u}(A_1) &= r \\ \overline{u}(A_2) &= p^* = (1-\alpha)(1-r)\end{aligned}$$

S_2 : A_3 $vs.$ A_4 (A_3 is risky.)

$$\begin{aligned}\overline{u}(A_3) &= r + p^* \\ &= r + (1-\alpha)(1-r) \\ \overline{u}(A_4) &= 1 - r.\end{aligned}$$

The decisions are as follows.

S_1 : A_1 $vs.$ A_2 (A_2 is risky.)
For $r < 1/2$,

$$\begin{aligned}\overline{u}(A_1) &> \overline{u}(A_2) \Longleftrightarrow 1 - r/(1-r) < \alpha \\ \overline{u}(A_1) &= \overline{u}(A_2) \Longleftrightarrow 1 - r/(1-r) = \alpha \\ \overline{u}(A_2) &> \overline{u}(A_1) \Longleftrightarrow 1 - r/(1-r) > \alpha\end{aligned}$$

For $r \geq 1/2$, $\overline{u}(A_1) > \overline{u}(A_2)$

S_2 : A_3 $vs.$ A_4 (A_3 is risky.)
For $r < 1/2$,

$$\begin{aligned}\overline{u}(A_4) &> \overline{u}(A_3) \Longleftrightarrow r/(1-r) < \alpha \\ \overline{u}(A_3) &= \overline{u}(A_4) \Longleftrightarrow r/(1-r) = \alpha \\ \overline{u}(A_3) &> \overline{u}(A_4) \Longleftrightarrow r/(1-r) > \alpha\end{aligned}$$

For $r \geq 1/2$, $\overline{u}(A_3) > \overline{u}(A_4)$.

We have obtained the same decisions as those with SCDT. It is worth while to examine the implications of this equivalence in detail to gain a deeper insight into the viewpoint in this paper. This is attractive future research.

(b) Case of $\alpha = I[A_1 > A_2]$ and $I[A_4 > A_3]$

A_1 and A_4 are safe options with respect to A_2 and A_3, respectively. $I[A_1 > A_2]$ and $I[A_4 > A_3]$ are in $(0,1)$ for the significant uncertainty. Therefore, $I[A_1 > A_2]$ and $I[A_4 > A_3]$ can be interpreted as a representation of the safety preference disposition. Then, whose safety preference disposition are these? Interestingly, these are Bayesian decision maker's, who is the rational, or consistent decision maker in the Bayesian theory. It's easy to see it.

$S_1 : A_1\ vs.\ A_2$ (A_2 is risky.)

$$\begin{aligned} \alpha &= I[A_1 > A_2] = I_R[A_3 > A_4] \\ RPDM &= I_R[A_2 > A_1] \end{aligned}$$

$S_2 : A_3\ vs.\ A_4$ (A_3 is risky.)

$$\begin{aligned} \alpha &= I[A_4 > A_3] = I_R[A_2 > A_1] \\ RPDM &= I_R[A_3 > A_4] \end{aligned}$$

The Bayesian decisions (7) can be derived easily from these relations by using the decision function (6).

Fig.2 summarizes all the results we have obtained.

B. Discussion

First, let us consider the issue "Are people's decisions wrong?" in Section II.B.1. Recognize here that SCDT has *two faces*, that is, the descriptive and the prescriptive. The difference between people's decision and the Bayesian one to the Ellsberg 3-color-ball decision problem is, in SCDT, due to the difference in the safety preference disposition between people and the Bayesian decision maker. This poses the following problems:

(1) What is the *prescriptive* safety preference disposition?

(2) What makes people' safety preference disposition different from the prescriptive one, provided that the latter exists?

It is quite possible that no prescriptive safety preference disposition will be found and the prescription *itself* will be personal, i.e., come to depend on your safety preference disposition.

The Bayesian indecisiveness (See Section IV.A.(a).) is caused by SPDM being exactly equal to RPDM. This exact equality seems a consequence of the mathematical idealization of actual decisions that the Bayesian axiomatic system involves because the Ellsberg 3-color-ball decision problem must not be prescriptively indecisive, and people and SCDT as a descriptive theory can decide it actually.

On the other hand, when we view the phenomenon of the Bayesian indecisiveness in CEP, $r = 1/3$ looks like a singular point in terms of mathematics. This view suggests that the Bayesian axiomatic system should be a map of a "higher dimensional" system to a lower one. What is the higher dimensional system? What axioms exist there? Is there the sure-thing principle in the higher system? Or, what is the form of the sure-thing principle there? Murofushi asserts from the viewpoint of the non-additive measure theory that the sure-thing principle is too strong to accept it as an axiom whether prescriptive or descriptive[5]. The non-additive measure theory attracts me to examine in detail if it can contribute to the settlement of this difficulty, the Bayesian indecisiveness, at the fundamental level (See Section II.B.1.) [17], [19], [20].

To solve all of these problems, we need to do the unified research of the prescriptive and the descriptive decision theories. Note that, in the unified research, it is essentially necessary for you to discriminate exactly between your prescriptive position and your descriptive one. If not so, confusion is inevitable.

Cohen and Jaffray [25] try to explain the preference combinations in the experiments on the Ellsberg 3-color-ball problem by introducing the concept of *pessimism* and *optimism*. Their definition of a pessimist (optimist) is: A subject is a pessimist (optimist) if his/her certainty-equivalent, sureness-equivalent in terms of this paper, to a probabilistic option is

[5]This is a private communication. Dr. Toshiaki Murofushi is an expert of the theory of fuzzy measure and is with the University of Electro-Communications in Japan. Email: cgstdm@etl.go.jp

greater (less) in French franc than the one to an uncertain option [23]. Their explanation is very interesting because it uses a psychological factor other than the preference and the belief. Compare, however, their definition with the definitions of the safety preference disposition measure and of the risk preference disposition measure in this paper. The difference is clear. Such one-dimensional concept as pessimism and optimism are too general and too simple to abstract the substantial contents of people's actual decision behavior.

Multi-dimensional, quantitative and specific concepts concerning human psychology and situations in which decisions are made are now needed to improve the Bayesian decision theory toward the more realistic and more usable unified theory. Let us call the measures of the psychological concepts *psychological quantities*, the ones of situational concepts *situational quantities*, and the ones of the both *cognitive quantities*. My current image of the unified theory is as follows. A whole of the dynamic relations among the cognitive quantities and the input information to the unified theory determines the characteristics of the cognitive quantities and of the input information. The whole of the dynamic relations produces *both* the prescriptive and people's decisions, as SCDT has done actually, that are the outputs from the unified theory. The situational measures *appear as* boundary conditions that are similar to the ones in physics. The psychological measures *appear as* personal constraint conditions such that there would not exist the corresponding things in physics. If you ask me to characterize the descriptive theory in the unified theory, I will answer that *a flying stone is simply flying without solving the differential equation.* Our target will be attained in the inquiries on the basis of the spirit of the "Relationism[6]" in the philosophy of Hiromatsu. The late Wataru Hiromatsu, Great Japanese Philosopher, learned much from the scientific works of Albert Einstein [26]. We have much to learn from their works. Science and Philosophy are the head and the tail, the tail and the head of a coin of the name of Inquiry.

V. Formal Fundamental Asymmetry in Quantification

According to the Bayesian decision theory, you must be willing to express your *sure* preference order between options *in Yes/No way.* Then, when you prefer an option to an-

[6]Consult Webster. It says that this is a doctrine holding that relations exist as real entities.

other ..., *diffidently*, *moderately*, *confidently*, ..., how should you do? March also points out that "people are often unsure about their preferences," quoted from [24]. Unfortunately, the Bayesian decision theory cannot satisfy you in such usual and important cases. This indicates that the qualitative preference order should be quantified so that it can represent the strength of your preference.

The following relations hold in CEP: $I[A_1 > A_2] = I[A_3 > A_4]$ and $I[A_2 > A_1] = I[A_4 > A_3]$. These relations suggest that the quantification could be made by the replacement of the qualitative preference order with the preference strength measure keeping the sure-thing principle by extending it over the preference strength measure. However, the Bayesian indecisiveness shown by the Ellsberg 3-color-ball problem must be eliminated at the fundamental level. See the discussion in Section IV.*B*.

Let us shoot a look at the heart of the Bayesian theory. In this paper, the preference strength measure has been defined by using the utility. In the Bayesian theory, however, first of all, the qualitative preference relation ($\leq$) is introduced as one of the primitive elements, and then the utility is introduced to represent it quantitatively for mathematical convenience, i.e., the compatibility of the expected utility with the qualitative preference relation. The canonical utility of a consequence c is defined as the probability $\mu(S)$ of any standard event S such that $c \sim \{c^*|S, c_*|S^c\}$, where S^c is the complement of S, c_* and c^* are the extreme consequences, and $c_* < c^*$ [2]. Note that this quantification has no intention of representing the preference strength quantitatively. Therefore, it might be necessary to reconstruct the axiomatic system radically in order to quantify the preference strength.

On the other hand, the degrees of belief, i.e., the subjective probabilities are derived from the qualitative belief relation between events making use of the principle on events that is similar to the precise measurement principle on options [2]. The concept of UPEP introduced in this paper seems to play a similar role in the quantification of the preference strength to that of the above principle in the quantification of the qualitative belief. This suggests that the reconstruction should be easier than in appearance.

There does not exist the concept of updating of the preference on options in the Bayesian theory, although there exists the one of the belief on events in the form of the subjective

probability. Your preference, including the strength, can change depending on your changing situations in which you are struggling for a decision. Imagine your stock investment decisions. Your preferable amount of investment capital, say \$1000 or \$10000, can be updated depending on your changing assets and liabilities even if your degrees of belief on the events do not change which affect the prices of the stock names you have chosen.

It should now be clear what is the formal fundamental asymmetry, suggested in Section I, between the quantification of the qualitative preference order on options and the one of the qualitative belief order on events. Let us call it the problem of Formal Fundamental Asymmetry in Quantification (FFAQ).

Pitz and Sachs [13] evaluate that "[t]he theoretical distinction between beliefs and preferences has been one of the most significant decision theory's contributions to the study of behavior." This suggests an origin of FFAQ. This theoretical distinction should not be a useful guiding principle in the inquiries toward the unified theory, rather should check the development of the unified theory.

I emphasize that the concepts of the safety preference disposition and of the risk preference disposition originated in this paper should be included in the coming axiomatic system and the problem of FFAQ should be solved in the inquiries toward the unification of the prescriptive and the descriptive decision theories.

Finally, let me pose a problem: Base the Self-Conflicting Decision theory on the coming axiomatic system.

VI. Concluding Remarks

This paper is my theoretical attack to make the Bayesian decision theory more realistic and more usable toward the coming unified decision theory under uncertainty that includes the prescriptive and the descriptive theories. The coming unified theory should be what is worth being called *Dynamic, Bayesian, Cognitive, and Relational Decision Theory*, where "Bayesian" means to be prescriptive, "Cognitive" means to use the cognitive quantities and to be descriptive, "relational" means to produce both the prescriptive and people's decisions by means of a whole of the relations that determines the characteristics of the cognitive quantities and of the input information, and "Dynamic" means that the relations are changeable or can be updated.

The viewpoint described in this paper by using the four concepts, (1) *Uncertainty-Probability Equivalence Principle*, (2) *Safety Preference Disposition* and (3) *Risk Preference Disposition*, and (4) *Formal Fundamental Asymmetry in Quantification*, is worth examining its possibilities in detail toward our target.

The empirical investigation of the Self-Conflicting Decision theory as a descriptive theory, in particular, to discover the other psychological and the situational factors that affect or provide the safety preference disposition measure, is also attractive and important future research work.

There is also another possibility that the viewpoint in this paper is applicable to animals because the safety preference disposition and the risk preference disposition must have been created in the process of evolution[7].

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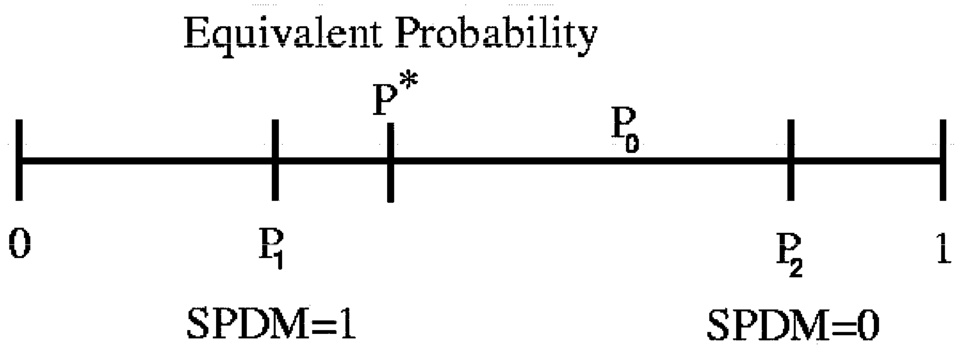

Fig. 3. Safety Preference Disposition Measure (SPDM)

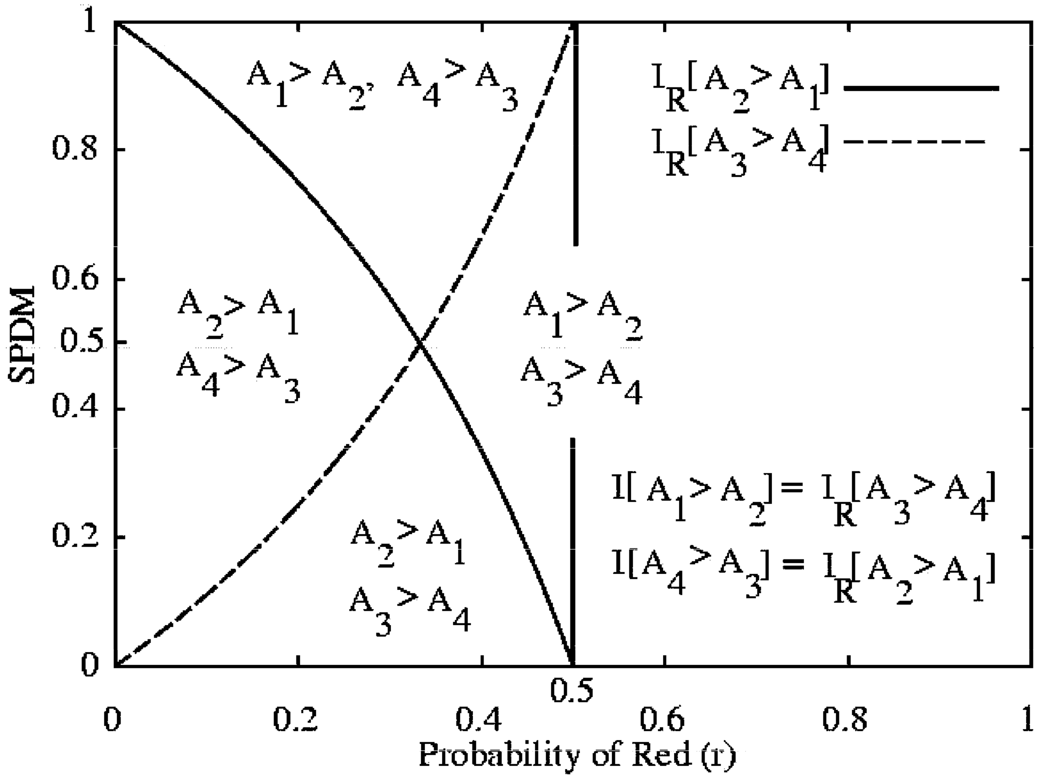

Fig. 4. Decision Space of Self-Conflicting Decision for CEP