On Boiney's Hypothesis and Model for Explaining the Skewness Effects

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Abstract

Boiney has recently proposed a hypothesis and model for explaining the skewness effects due to skewed second-order probability on decision making under ambiguity. This paper examines the hypothesis and model. It is shown that the hypothesis consists of two parts, Focus Hypothesis and Emotion Hypothesis. The Focus Hypothesis is psychologically unlikely and produces abnormal preference patterns in Boiney model in the case where the mean probability of an ambiguous gamble is not equal to the probability of an unambiguous gamble. An alternative for the Focus Hypothesis is proposed. It is shown that the alternative produces the same preference patterns as Boiney's for the same mean probability and normal preference patterns in the case where the mean probability of the ambiguous gamble is different from the probability of the unambiguous gamble.

Keywords

Decision Making, Skewness Effects, Ambiguity, Uncertainty, Focus Hypothesis, Emotion Hypothesis.

I. Introduction

Boiney has recently reported an interesting experiment concerning the effects of *skewed* second-order probabilities on attitudes toward ambiguity preferences and proposed a hypothesis and model for explaining the effects [1]. The objective in this paper is to examine the hypothesis and model and to propose an alternative hypothesis.

Boiney has confirmed the following two hypotheses for choice between an ambiguous and unambiguous gamble with the same mean probability and outcomes experimentally [1].

Hypothesis 1 The unambiguous gamble will be preferred if the ambiguous gamble is negatively skewed or symmetric.

Hypothesis 2 The ambiguous gamble will be preferred to the unambiguous gamble under positive skewness.

Experimental Design:

The subjects were 130 first-year MBA students in a production and operations management class. The experiment was a fully crossed $2 \times 3 \times 3$, OUTCOME (\$200, - \$200) $\times$ MEAN (0.20, 0.50, 0.8) $\times$ SKEW (Negative, Symmetric, Positive) design. OUTCOME

was a between-subject factor. MEAN and SKEW were within-subject factors. The range of the second-order probability rather than the variance, was held constant at 0.20 in all SKEW conditions.

Sample Lotteries:

The following are three sample lotteries in his experiments. These lotteries have the same mean (first-order) probability. Lottery A is negatively skewed and Lottery B is positively skewed. The first-order probability and the second-order probability are denoted by FOP and SOP, respectively.

Lottery A

EVENT:	Red		Black	
SOP :	0.1	0.9	0.9	0.1
FOP :	0.02	0.22	0.98	0.78
GAIN :	$200		$0	

Lottery B

EVENT:	Red		Black	
SOP :	0.9	0.1	0.1	0.9
FOP :	0.18	0.38	0.82	0.62
GAIN :	$200		$0	

Lottery C

EVENT:	Red	Black
SOP :	1	1
FOP :	0.20	0.80

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GAIN :     $200          $0
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Results:

1. Approximately 40% exhibit no systematic preference pattern; they are not consistently ambiguous seeking, averse, or neutral, nor do their shifts in preference coincide regularly with one of the independent variables.
2. About 23% of the subjects exhibit a consistent attitude toward ambiguity: 19 subjects are ambiguity neutral, 5 are ambiguity averse, and 6 are ambiguity seeking.
3. The main result (Skew Sensitivity): Negative Skewness $\prec$ Unambiguous $\prec$ Positive Skewness for both gain/loss.

The largest group to exhibit a clear pattern reveals an intriguing effect. 48 of 130 subjects (about 37%) are "skew sensitive"; they are ambiguity seeking under positive skewness, yet ambiguity averse under negative skewness. This preference pattern can only be explained by changes in the direction of skewness of SOP since the pattern occurs over both outcome conditions, for all levels of mean, and the range and variance of the negatively and positively skewed gambles are identical.

II. Boiney's Hypothesis for explaining the skewness effects

We can understand well Boiney's hypothesis by separating it into two parts. The first and second parts can be called "Focus Hypothesis" and "Emotion Hypothesis", respectively.

Focus Hypothesis: The ambiguous lottery (Lottery A or Lottery B) will be evaluated independently of the unambiguous lottery (Lottery C). Many will behave as though they place a disproportionate weight on the first-order probabilities *far from the mean of the ambiguous lottery* (Lottery A or Lottery B), thereby focus on the far first-order probability (0.02 in Lottery A and 0.38 in Lottery B).

Emotion Hypothesis: The focused probability (0.02 in Lottery A and 0.38 in Lottery B) will be compared with the probability of the unambiguous lottery (Lottery C). When the focused probability is much less (larger) than the probability of the unambiguous lottery, 0.2 in Lottery C, there is the potential for extreme "disappointment" ("elation") if the decision maker believes that the less (larger) chance of winning was the true state of nature, resulting in ambiguity aversion (ambiguity seeking).

The focused probability in Lottery A is 0.02 and the probability of Lottery C is 0.2. Hence Lottery C is preferred over Lottery A that is negatively skewed. The focused probability of Lottery B is 0.38 and the probability of Lottery C is 0.2. Hence Lottery B, which is positively skewed, is preferred to the Lottery C.

I agree to the Emotion Hypothesis. But I do not agree to the the Focus Hypothesis. It is unlikely that decision makers will evaluate the ambiguous lottery independently of the unambiguous lottery and calculate the mean probability of an ambiguous lottery by using the second-order probabilities. Even if the mean probability is calculated to evaluate the ambiguous lottery, it is unclear why the mean is compared with each of the first-order probabilities of the ambiguous lottery.

It is more likely that the ambiguous lottery (Lottery A or Lottery B) will be evaluated against the unambiguous lottery (Lottery C) and many will compare each of the first-order probabilities of the ambiguous lottery with the probability of the unambiguous lottery. The first-order probability of the ambiguous lottery that is close to the probability of the unambiguous lottery will be less weighted and the probability far from the probability of the unambiguous lottery will be more weighted and focused on.

In Lottery A that is negatively skewed (Lottery B that is positively skewed) the probability far from the probability of Lottery C is less (greater) than the probability of Lottery C. Hence Lottery C (Lottery B) is preferred over Lottery A (Lottery C) according to the Emotion Hypothesis.

I modify the Focus Hypothesis as follows.

Modified Focus Hypothesis: The ambiguous lottery (Lottery A or Lottery B) will be evaluated against the unambiguous lottery (Lottery C) and many will behave as though they place a disproportionate weight on the first-order probabilities *far from the probability of the unambiguous lottery* (Lottery C), thereby focus on the far first-order probability (0.02 in Lottery A and 0.38 in Lottery B).

Since the Boiney problems have the same mean probability, the Modified Focus Hypothesis and the Emotion Hypothesis give the same results in Boiney model as those of Boiney model using the Focus Hypothesis and the Emotion Hypothesis for the Boiney problems. However, when the probability of Lottery C is different from the mean proba-

bility of Lottery A or Lottery B, the Focus Hypothesis and the Modified Focus Hypothesis will produce different preference patterns.

III. Generalized Boiney Problems

Let us generalize the Boiney problems to examine the implications of the Focus Hypothesis and Modified Focus Hypothesis.

Conditions:

1. $P_1(R|*) < P_2(R|*)$ (first-order probabilities of event Red, $*$ is Lottery A or Lottery B.)
2. $0 < r < 1$ (second-order probability, r is r_a for Lottery A and r_b for Lottery B.)
3. $P_m(R|A) = P_m(R|B) = P(R|C)$ (mean first-order probabilities of event Red for Lotteries A, B and C, respectively)
4. $P_2(R|A) - P_1(R|A) = P_2(R|B) - P_1(R|B)$
5. $\$X \neq 0$
6. Lottery A is negatively skewed on Red. Lottery B is positively skewed on Red.

Lotteries:

LOTTERY		A or B		
EVENT:	Red		Black	
SOP :	r	1 - r	r	1 - r
FOP :	P1(R|*)	P2(R|*)	1 - P1(R|*)	1 - P2(R|*)
GAIN :	$X		$0	

LOTTERY	C	
EVENT:	Red	Black
SOP :	1	1
FOP :	P(R|C)	1 - P(R|C)
GAIN :	$X	$0

A. Skewness

The measure of skewness on event Red, $S(R|*)$, is defined as

$$S(R|*) \equiv \frac{\boldsymbol{E}[\{P(R|*) - P_m(R|*)\}^3]}{\sigma^3(R|*)},$$

where $P(R|*)$ is the first-order probability of Red, $P_m(R|*)$ is its mean, $\sigma(R|*)$ is its standard deviation and the expectation is taken over the second-order probability. The measure can be written

$$S(R|*) = \frac{2r-1}{\sqrt{r(1-r)}}.$$

In order that Lottery A is negatively skewed on Red, $r_a < 1/2$. Hence $P_m(R|A) - P_1(R|A) > P_2(R|A) - P_m(R|A)$ for Lottery A. In order that Lottery B is positively skewed on Red, $r_b > 1/2$. Hence $P_m(R|B) - P_1(R|B) < P_2(R|B) - P_m(R|B)$ for Lottery B.

IV. Boiney Model

Boiney has expressed his hypothesis mathematically in the form of a weighted model as follows [1].

Define a lottery L as awarding \$$X$ if event E obtains, otherwise awarding nothing, with $u(\$0) = 0$. Let P be the probability of winning, which is a random variable with density function $f(P)$ and mean P_m. The proposed model will replace the probability of an event E with a subjective decision weight, $w(E)$, so that the value function for L is

$$V(L) = w(E)u(X).$$

A decision weight $w(E)$ must capture two key phenomena observed in the data. The first phenomenon is that decision makers behave as though they place a disproportionate amount of weight on probabilities far from the mean. To reflect this behavior, one component of the decision weight will be the mean squared deviation, $(P - P_m)^2$. The second phenomenon is that individuals have different attitudes toward ambiguity, which will be expressed by introducing two individual-specific parameters, e and d. These are simply constants, $e, d \geq 0$, designed to measure the potential "elation" and potential "disappointment" experienced due to probabilities above and below the mean, respectively.

In oder to reflect both phenomena, a function $A(P)$ is defined to represent the attitude toward a given probability. More specifically,

$$A(P) = \begin{cases} e(P-P_m)^2 & if\ P \geq P_m \\ -d(P-P_m)^2 & if\ P < P_m. \end{cases} \tag{1}$$

Taking the expectation over this function results in the "expected attitude" toward the second-order distribution $f(P)$: $\int_0^1 A(P)f(P)dP$. In essence, the decision weight will adjust the mean probability P_m up and down according to this expected attitude toward the ambiguous distribution. The decision weight for event E is defined as

$$w(E) = P_m + \int_0^1 A(P)f(P)dP.$$

V. Implications

A. Same Mean Probability

It will be shown that the Focus Hypothesis and the Modified Focus Hypothesis applied to Boiney model produce the same preference patterns for the Boiney problems.

A.1 Using Focus Hypothesis

(1) Gain

The decision weights for event R (Red) in the gain domain, $w_a(R)$ for Lottery A, $w_b(R)$ for Lottery B and $w_c(R)$ for Lottery C, are given by

$$w_a(R) = P_m(R|A) + \{er_a - d(1-r_a)\}r_a(1-r_a)(P_2-P_1)^2$$

$$w_b(R) = P_m(R|B) + \{er_b - d(1-r_b)\}r_b(1-r_b)(P_2-P_1)^2$$

$$w_c(R) = P(R|C).$$

The following d and e imply ambiguity aversion.

$$d > \frac{r}{1-r}e, \quad for \;\; r = r_a \;\; or \;\; r_b$$

The following d and e imply ambiguity seeking.

$$d < \frac{r}{1-r}e, \quad for \;\; r = r_a \;\; or \;\; r_b$$

(2) Loss

The decision weights for event BK (Black) in the loss domain, $w_a(BK)$ for Lottery A, $w_b(BK)$ for Lottery B and $w_c(BK)$ for Lottery C, are given by

$$w_a(BK) = P_m(BK|A) + \{-dr_a + e(1-r_a)\}r_a(1-r_a)(P_2 - P_1)^2$$

$$w_b(BK) = P_m(BK|A) + \{-dr_b + e(1-r_b)\}r_b(1-r_b)(P_2 - P_1)^2$$

$$w_c(BK) = P(BK|C).$$

The following d and e imply ambiguity aversion.

$$d > \frac{1-r}{r}e, \quad for \;\; r = r_a \;\; or \;\; r_b$$

The following d and e imply ambiguity seeking.

$$d < \frac{1-r}{r}e, \quad for \;\; r = r_a \;\; or \;\; r_b$$

A.2 Using Modified Focus Hypothesis

Using the Modified Focus Hypothesis the function $A(P)$ (1) is written as

$$A(P) = \begin{cases} e(P-P_c)^2 & if \; P \geq P_c \\ -d(P-P_c)^2 & if \; P < P_c. \end{cases} \tag{2}$$

Here P_c is the probability of the unambiguous lottery. Since the mean probabilities of the Boiney problems are identical, $P_m(R|A) = P_m(R|B) = P(R|C)$, the preference patterns generated by the Modified Focus Hypothesis are the same as those by the Focus Hypothesis.

B. Different Mean Probabilities

Let us examine the implications of the Focus Hypothesis and the Modified Focus Hypothesis in Boiney model in the case where $P_m(R|A)$ and $P_m(R|B)$ are not equal to $P(R|C)$. For simplicity the following notation will be used:

$$P_1 = P_1(R|A) \;\; or \;\; P_1(R|B)$$

$$P_2 = P_2(R|A) \;\; or \;\; P_2(R|B)$$

$$P_m = P_m(R|A) \;\; or \;\; P_m(R|B)$$

$$P_c = P(R|C)$$

$$r = r_a \;\; or \;\; r_b$$

We are interested in P_c for $P_1 \leq P_c \leq P_2$.

B.1 Using Focus Hypothesis

The following relation holds in the gain domain for the Focus Hypothesis.

$$d < \frac{r}{1-r}e + \frac{P_m - P_c}{r(1-r)^2(P_2-P_1)^2} \Longrightarrow A, B \succ C \tag{3}$$

The elation term (the first term with e in the right-hand side of the antecedent in (3)) leaves unchanged while P_c is approaching P_1 or P_2. It is unlikely. The skewness-emotion effect, including the skewness and individual attitudes toward ambiguity, would diminish as P_c is approaching P_1 or P_2.

At $P_c = P_1$ the following relation holds.

$$d > \frac{r}{1-r}e - \frac{1}{r(1-r)(P_2-P_1)} \Longrightarrow A, B \prec C \tag{4}$$

At $P_c = P_2$ the following relation holds.

$$d < \frac{r}{1-r}e - \frac{1}{(1-r)^2(P_2-P_1)} \Longrightarrow A, B \succ C \tag{5}$$

These show that the skewness-emotion effect remains at $P_c = P_1$ and $P_c = P_2$. For example, a sufficiently small (large) elation parameter prefers Lottery C (Lottery A and Lottery B) over Lottery A and Lottery B (Lottery C) at $P_c = P_1$ (P_2). However, it is abnormal and all decision makers would definitely prefer Lottery A and Lottery B (Lottery C) to Lottery C (Lottery A and Lottery B) at $P_c = P_1$ (P_2) regardless of the skewness and their attitudes toward ambiguity. A similar argument can be made in the loss domain.

B.2 Using Modified Focus Hypothesis

The following relations hold in the gain domain for the Modified Focus Hypothesis.

$$e > \frac{1}{1-r}\frac{(P_1-P_c)^2}{(P_2-P_c)^2}d - \frac{P_m-P_c}{(1-r)(P_2-P_c)^2} \Longrightarrow A, B \succ C \tag{6}$$

$$d > \frac{1-r}{r}\frac{(P_2-P_c)^2}{(P_1-P_c)^2}e + \frac{P_m-P_c}{r(P_1-P_c)^2} \Longrightarrow A, B \prec C \tag{7}$$

The disappointment term (the first term with d in the right-hand side of the antecedent in (6)), *i.e.*, the skewness-emotion effect diminishes as P_c is approaching P_1. Similarly, the elation term (the first term with e in the right-hand side of the antecedent in (7)), *i.e.*, the skewness-emotion effect diminishes as P_c is approaching P_2.

At $P_c = P_1$ the following relation holds.

$$e > -\frac{1}{P_2-P_1} \Longrightarrow A, B \succ C \tag{8}$$

At $P_c = P_2$ the following relation holds.

$$d > -\frac{1}{P_2-P_1} \Longrightarrow A, B \prec C \tag{9}$$

These show that Lottery A and Lottery B (Lottery C) are preferred over Lottery C (Lottery A and Lottery B) at $P_c = P_1$ (P_2) regardless of the skewness and individual attitudes toward ambiguity. Similar results can be obtained in the loss domain.

VI. Conclusion

This paper has examined Boiney's hypothesis and model for explaining the skewness effects. It has been shown that the hypothesis consists of two parts, the Focus Hypothesis and the Emotion Hypothesis. The Focus Hypothesis is psychologically unlikely and produces abnormal preference patterns in Boiney model in the case where the mean probability of an ambiguous gamble is not equal to the probability of an unambiguous gamble. The Modified Focus Hypothesis has been proposed. It has been shown that the Modified Focus Hypothesis produces the same preference patterns as Boiney's for the same mean probability and normal preference patterns in the case where the mean probability of the ambiguous gamble is different from the probability of the unambiguous gamble.

References

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Jiro Ihara was born in Tokyo, Japan, on February 14, 1943. He received the B. Eng., M. Eng., and Dr. Eng. degrees in electrical engineering from Seikei University, Tokyo, Japan, in 1967, 1969, and 1983, respectively. He joined the Electrotechnical Laboratory (ETL) in 1969 and is now a senior researcher of ETL. From 1977 to 1978 he spent ten months with the Institute of Cybernetics, the Ukrainian Academy of Sciences, in Kiev, U.S.S.R. He is the author of the IEEE papers: A Fitting Characteristic Vector Based on the Mean Square Error with an Application, IEEE Trans., SMC**9**-8, pp. 405-410, 1979, A Structural Analysis of Criteria for Selecting Model Variables, IEEE Trans., SMC**10**-8, pp.460-466, 1980, and Extension of Conditional Probability and Measures of Belief and Disbelief in a Hypothesis Based on Uncertain Evidence, IEEE Trans., PAMI**9**-4, pp.561-568, 1987. His research interests include decision making, emotions, human doubt and belief, human information use and Bayesian statistics.