〔RESEARCH〕

How the Eight Decision Models Can or Cannot Explain Boiney's Emotional Skewness Effects in Decision Making under Ambiguity

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Boiney has recently reported the interesting emotional effects of skewed second-order probability on decision making under ambiguity. This paper examines, using a generalized Boiney problem, how the following eight decision models can or cannot explain Boiney's emotional skewness effects: Keynes Conventional Coefficient Model (1921), Ellsberg Model (1961), Gärdenfors-Sahlin Model (1982), Einhorn-Hogarth Model (1985), Kahn-Sarin Model (1988), Boiney Model (1993), Flexible Bayesian Approach (1994) and Self-Conflicting Decision (1995). Among them, Boiney Model, Flexible Bayesian Approach and Self Conflicting decision (SCD) can only accommodate the emotional skewness effects. The seven models except for SCD are a maximization type model. SCD is not based on the maximization principle but principally a psychological hypothesis (Self-Conflicting Hypothesis). This paper provides an explanation of SCD because it is a new decision model.

§1 Introduction

John Maynard Keynes wrote in 1921 (Keynes, 1943, p.71): "As the relevant evidence at our disposal increases, the magnitude of the probability of the argument may either decrease or increase, according as the knowledge strengthens the unfavourable or favourable evidence; but *something* seems to have increased in either case, — we have a more substantial basis upon which to rest our conclusion. I express this by saying that an accession of new evidence increases the *weight* of an argument. New evidence will sometimes decrease the probability of an argument, but it will always increase its 'weight.' "

Ellsberg, forty years later since then, rediscovered the importance of the ambiguity about probabilities in decision making (Ellsberg, 1961). He has proposed the objection, in the coun- terexamples, to the sure-thing principle in decisions under ambiguity. Becker and Brownson (Becker & Brownson, 1964), Curley and Yates (Cruley & Yates, 1985) and Kahn and Sarin (Kahn & Sarin, 1988) examined the ambiguity effects practical importance of decision making under uncertainty and ambiguity is nowadays widely recognized (Camerer & Weber, 1992).

The above researchers deal with symmetric second-order probabilities and do not examine emotional effects on decision making. Boiney, however, has recently reported an interesting experiment concerning the emotional effects of skewed second-order probabilities on attitude toward ambiguity preferences (Boiney, 1993).

The objective in this paper is to examine, using a generalized Boiney problem, how the following eight decision models can or cannot explain Boiney' s emotional skewness effects and to show the importance of introducing the capability to treat emotions into decision theory: Keynes's Conventional Coefficient Model (Keynes, 1943), Ellsberg Model (Ellsberg, 1961), Gärdenfors-Sahlin Model (Gärdenfors & Sahlin, 1982), Einhorn-Hogarth Model (Einhorn & Hogarth, 1985), Kahn-Sarin Model (Kahn & Sarin, 1988),

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Boiney Model (Boiney, 1993), Flexible Bayesian Approach (Shigemasu & Yokoyama, 1994), (Shigemasu, 1995) and Self- Conflicting Decision (Ihara, 1995).

The first section of this paper sums up Boiney experiment. The second section explains Self-Conflicting Decision (SCD) and presents a preliminary analysis of Boiney's sample lotteries with SCD. The third section generalizes the Boiney problem. The fourth section analyzes the generalized Boiney problem with the eight decision models. The last section provides concluding remarks on the results and suggestions for future work.

§2 Boiney Experriment

Boiney experiment is summed up below (Boiney, 1993).

Sample Lotteries:

The following are three sample lotteries in his experiment. These lotteries have the same mean (first-order) probability. Lottery A is negatively skewed and Lottery B is positively skewed. The first-order probability and the second-order probability are denoted by FOP and SOP, respectively.

Lottery A

EVENT :	Red		Black	
FOP :	0.02	0.22	0.98	0.78
SOP :	0.1	0.9	0.9	0.1
GAIN :	$200		$0	

Lottery B

EVENT :	Red		Black	
FOP :	0.18	0.38	0.82	0.62
SOP :	0.9	0.1	0.1	0.9
GAIN :	$200		$0	

Lottery C

EVENT :	Red	Black
FOP :	0.20	0.80
SOP :	1	1
GAIN :	$200	$0

Lottery A and Lottery B were paired with Lottery C.

Results:

1. Approximately 40% exhibit no systematic preference pattern; they are not consistently ambiguous seeking, averse, or neutral, nor do their shifts in preference coincide regularly with one of the independent variables.
2. About 23% of the subjects exhibit a consistent attitude toward ambiguity: 19 subjects are ambiguity neutral, 5 are ambiguity averse, and 6 are ambiguity seeking.
3. The main result (Skew Sensitivity):The largest group to exhibit a clear pattern reveals an intriguing effect. 48 of 130 subjects (about 37%) are "skew sensitive"; they are ambiguity seeking under positive skewness, yet ambiguity averse under negative skewness. This preference pattern can only be explained by changes in the direction of skewness of SOP since the pattern occurs over both outcome conditions, for all levels of mean, and the range and variance of the negatively and positively skewed gambles are identical.

§3 Preliminary Analysis of The Sample Lotteries with Self-Conflicting Decision

Self-Conflicting Decision

Self-Conflicting Decision (SCD) will be summarized below. See Ihara (Ihara, 1995) for further details. Let us begin with the clarification of the substantial content of uncertainty.

Substantial Content of Uncertainty

The options with the probability of 1 will be called *sure options*, the options with certain probabilities in (0, 1) *probabilistic options*, and the options with uncertain probabilities *uncertain options.*

Let $\bar{u}\,(A_p)$ denote the unconditional expected utility of a *probabilistic* option, A_p . Let $\bar{u}\,(A_u|x)$, given x, denote the conditional expected utility of an *uncertain* option, A_u . Here, x denotes a generic random quantity. When the following relation holds,

$$\min_x \bar{u}(A_u|x) < \bar{u}(A_p) < \max_x \bar{u}(A_u|x) \tag{1}$$

that is, when the uncertain option is *risky*, the uncertainty is *significant* with respect to choice between the probabilistic option and the uncertain option. In this case, the probabilistic option is called a *safe* option and the uncertain option a *risky* option.

For the case of choice between two uncertain options, A_1 and A_2, let $\bar{u}(A_1|x)$ and $\bar{u}(A_2|x)$ denote the conditional expected utilities of A_1 and A_2, respectively. When the following relations hold,

$$\min_x \bar{u}(A_2|x) < \min_x \bar{u}(A_1|x) < \max_x \bar{u}(A_1|x) < \max_x \bar{u}(A_2|x) \tag{2}$$

A_2 is risky and the uncertainty is significant with respect to choice between the uncertain options. In this case, A_1 is called a safe option and A_2 a risky option.

Uncertainty-Probability Equivalence Principle

First, let us review the essence of the principle or axiom of precise measurement concerning options in Bayesian theory (Bernardo & Smith, 1994). This principle asserts that there exists a probabilistic option, $\{c_2|p, c_1|1-p\}$, being equivalent to a sure option, c. Its formal representation is as follows. There exists a probability p such that

$$c \sim \{c_2|p, c_1|1-p\} \quad \text{for } c_1 \le c \le c_2,$$

where c, c_1 and c_2 are consequences, " ~" denotes an equivalence relation in preference, and "$c_1 \le c$" signifies that c_1 is not preferred to c. The p is a subjective probability of the occurrence of c_2. This principle can be called the Principle of Sureness-Probability Equivalence.

Let us extend the principle of sureness-probability equivalence between a *sure* option and a *probabilistic* option to the one between an *uncertain* option and a *probabilistic* option. This new principle, *Uncertainty-Probability Equivalence Principle* (*UPEP*), asserts that there exists a probabilistic option, $\{c_2|p^*, c_1|1-p^*\}$, where c_2 occurs with a subjective probability p^*, being equivalent to an uncertain option $\{c_2|[p_1, p_2], c_1|[1-p_2, 1-p_1]\}$, where c_2 occurs with a probability, say p, in $[p_1, p_2]$ and c_1 occurs with a probability, $1-p$, in $[1-p_2, 1-p_1]$. It is defined formally as dollows.

Definition 1 (Uncertainty-Probability Equivalence Principle) *There exists a subjective probability p^* such that*

$$\{c_2|[p_1, p_2], c_1|[1-p_2, 1-p_1]\} \sim \{c_2|p^*, c_1|1-p^*\}, \tag{3}$$

where $c_1 \le c_2$, $p_1 \le p_2$, $p_1 < p^* < p_2$, and if $p_1 = p_2$, then $P^* = p_1 = p_2$. The p^* is called the subjective equivalent probability of the uncertain option.

Safety Preference Disposition

People like safety. They buy insurance. Fund managers hedge for stock portfolio protection. Politicians often use ambiguous words so that they cannot commit themselves to any sort of pledge. Or, animals have the natural disposition to avoid danger. You can also enumerate a lot of facts showing that people and animals have the natural disposition to like safety. I believe that this *safety preference disposition* has been created in the process of evolution.

Bayesian decision theory does not realize the importance of this safety preference disposition in decision making. It is my natural feeling that their safety preference disposition plays an indispensable role in their decision making. To describe their decision behavior, it is essential to take the concept of the safety preference disposition into consideration in addition to the one of the preference between options.

Since the safety preference disposition is essentially personal and the decision under uncertainty is considered, the safety preference disposition measure is defined as follows.

Definition 2 (Safety Preference Disposition Measure) *The Safety Preference Disposition Measure* (*SPDM*), α, *is defined by using the subjective equivalent probability p^* as*

$$p^* \equiv \alpha p_1 + (1-\alpha) p_2, \tag{4}$$

where $0 < \alpha < 1$.

The safety preference disposition measure should not be considered a simple numeral. It is a function of the other psychological and situational factors to be discovered. Note that the more you want to play for safety, the larger your α is. **Fig.1** illustrates the safety preference disposition measure. The probabilistic options $\{c_2|p_0, c_1|1-p_0\}$, where p_0 is greater than p^*, is preferred to the uncertain option $\{c_2|[p_1, p_2], c_1|1-p_2, 1-p_1]\}$.

Risk Preference Disposition

People also like to take risks, although they have the safety preference disposition. They invest their own savings in stocks. Statesmen take risky political policies if necessary. See the late John Fitzgerald Kennedy, the 35th President of the United States, for the 13 Days of the Cuba Missile Crisis in October, 1962. Or, young animals challenge new food in risky situations. I believe that this *risk preference disposition* has also been created in the process of evolution.

Let us consider *preference* to find how to define a

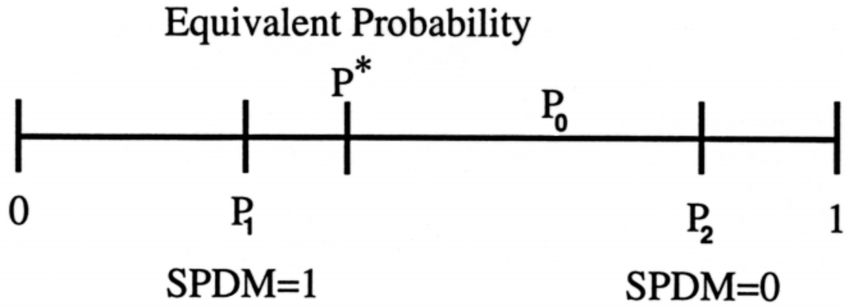

Fig.1 Safety Preference Disposition Measure (SPDM)

measure of the risk preference disposition. It is well known that preference is a mental disposition which plays an important role in decision making. When you prefer an option A_2 to an option A_1, there must be *strength* in your preference. In some cases, you prefer A_2 to A_1 *diffidently*, in other cases, *moderately*, *confidently*, etc. The strength of your preference is another mental factor that plays an indispensable role without all doubt in your decision making. Bayesian decision theory fails to abstract this mental factor. Therefore, it is necessary to introduce a measure of the preference strength to describe actual decision making behavior. The preference strength is essentially personal. Hence, the measure of the preference strength is to be defined by using the subjective probability that A_2 is preferred to A_1 . The definition of the measure of the preference strength is now straightforward. By using it, the measure of the *risk preference disposition* is defined as follows.

Definition 3 (Risk Preference Disposition Measure) *The measure of the Preference Strength on which A_2 is preferred to A_1 , $I[A_2 \succ A_1]$, is defined by the subjective probability that $\bar{u}(A_2|x) > \bar{u}(A_1|x)$ as*

$$I[A_2 \succ A_1] \equiv Prob\{\bar{u}(A_2|x) > \bar{u}(A_1|x)\}, \qquad (5)$$

where $\bar{u}(A_2|x)$ and $\bar{u}(A_1|x)$ are the conditional expected utilities, given x, of A_2 and A_1 , respectively, and x denotes a generic random quantity. The Risk Preference Disposition Measure (RPDM) is defined as $I[A_2 \succ A_1]$ under the condition that A_2 is a risky option with respect to A_1 . This is written $I_R[A_2 \succ A_1]$.

Note that if $\bar{u}(A_2|x)$ and $\bar{u}(A_1|x)$ are not random quantities, then $I[A_2 \succ A_1]$ = 1 or 0, which mean that $A_2 \succ A_1$, $A_2 \prec A_1$, or $A_2 \sim A_1$ *for sure.*

Hypothesis

The safety preference disposition and the risk preference disposition are not one-dimensional concept but two-dimensional one. These are conflicting in your mind while you are struggling to make a decision. When a compromise between them has been made in your mind, you have a decision. Let us formalize it as a hypothesis concerning actual decision making.

Self-Conflicting Hypothesis: *People tend to choose one option between two options by making compromise between their Safety Preference Disposition and Risk Preference Disposition with themselves. If their Safety Preference Disposition is stronger (weaker) than their Risk Preference Disposition, then they tend to choose the safe option (risky option). This decision is named Self-Conflicting Decision (SCD).*

It cannot be emphasized too strongly that SCD differs absolutely from the Bayesian decision maximizing the expected utility.

Let us make this hypothesis operational with the following simple decision function:

$$\begin{aligned} &\textit{If } I_R[A_2 \succ A_1] > \alpha, \textit{ then } A_2 \textit{ is chosen.} \\ &\textit{If } I_R[A_2 \succ A_1] = \alpha, \textit{ then the choice is indecisive.} \\ &\textit{If } I_R[A_2 \succ A_1] < \alpha, \textit{ then } A_1 \textit{ is chosen.} \end{aligned} \qquad (6)$$

Here, A_2 is a risky option and A_1 is a safe one.

It should be emphasized here that the insightful view proposed more than fifteen years ago by Einhorn and Hogarth (Einhorn & Hogarth, 1981) has been actually realized in a promising way in the theory of SCD, although partially at present: "Conceptualizing decision ... as the clash between multiple selves is a potentially rich area of investigation and could provide useful conceptual links between phenomena of individual and group behavior. For example, individual irrationality might be seen as similar to the various voting paradoxes found in group decision making."

Preliminary Analysis of the Sample Lotteries

Let P_1 , P_2 , r and P^* denote the larger FOP, smaller FOP, the second-order probability of P_1 and the equivalent probability, respectively. The mean probability is $P_m = r \cdot P_1 + (1–r) \cdot P_2 = 0.20$. Without loss of generality we can set $\bar{u}(0)$ = and $\bar{u}(200) = 1$. We have the following.

$\bar{u}(A|FOP) = \{0.02, 0.22\}$

$\bar{u}(B|FOP) = \{0.18, 0.38\}$

$\bar{u}(C|FOP) = 0.20$

$I_R[A \succ C] = Prob\{\{0.02, 0.22\} > 0.20\} = 0.9$

$I_R[B \succ C] = Prob\{\{0.18, 0.38\} > 0.20\} = 0.1$

$\alpha = (P_2 - P^*) / (P_2 - P_1)$

I propose a hypothesis to explain the skewness sffect (Ihara, 1997). The hypothesis consists of two parts. The first and second parts are called"Focus Hypothesis" and " Emotion Hypothesis", respectively.

Focus Hypothesis: The ambiguous lottery (Lottery A or Lottery B) will be evaluated against the unambiguous lottery (Lottery C). A decision maker will behave as though he/she places a disproportionate weight on the first-order probability *far from the probabilitry of the unambiguous lottery* (Lottery C), thereby focus on the far first-order probability (0.02 in Lottery A or 0.38 in Lottery B).

Emotion Hypothesis: The focused probability (0.02 in Lottery A or 0.38 in Lottery B) will be compared with the probability (0.2) of the unambiguous lottery (Lottery C). When the focused probability is much less (larger) than the probability of the unambiguous lottery, there is the potential for extreme "disappointment" ("elation") if the decision maker feels that the less (larger) chance of winnng may be the true state of nature. Ambiguity aversion (ambiguity seeking) will result in consequence.

The focused probability in Lottery A is 0.02 and the probability of Lottery C is 0.2. Hence Lottery C is preferred over Lottery A that is negatively skewed. The focused probability of Lottery B is 0.38 and the probability of Lottery C is 0.2. Hence Lottery B, which is positively skewed, is preferred to the Lottery C.

The emotions of " disappointment" and " elation " are close to Safety Preference Disposition in Self-Conflicting Decision (Ihara, 1995). People who dislike to be disappointed are in the mental state of the *high* safety preference disposition and those who like to be elated are in the mental state of the *low* safety preference disposition. Let us assume the following here.

Assumption 1: The focused probability is the first-order probability of an ambiguous lottery far from the probability of an unambiguous lottery.

Assumption 2: The equivalent probability in SCD is the focused probability.

Let us continue the analysis by using **Assumptions 1, 2**. For Lottery A $P^*(A) = P_1(A) = 0.02$, so $\alpha(A) = 1$. For Lottery B $P^*(B) = P_2(B) = 0.38$, so $\alpha(B) = 0$. We obtain $A \prec C$, $C \prec B$. Lottery A is not risky with respect to Lottery B, vice versa. SCD dose not define a decision rule in this case. Let us try the following decision rule: the preference in A and B is determined based on the equivalent probabilities, $P^*(A)$ and $P^*(B)$. $A \prec B$ because $P^*(A) = 0.02 < P^*(B) = 0.38$. We have obtained $A \prec C \prec B$ (skew sensitivity).

Let us define a decision, according to the above success, in the case where two options are not risky each other as follows. Let us call the decision **Non-Risky Decision**.

Non-Risky Decision: The option is chosen which has the largest expected utility with the equivalent probability in the case where two options are not risky each other.

§4 Generalized Boiney Problem

Let us generalize the Boiney problem to examine how the decision models can or cannot explain the skewness effects.

Conditions

The problem satisfies the following conditions.

1. $P_1(R|*) < P_2(R|*)$ (first-order probabilities of event Red, $*$ is Lottery A or Lottery B.)
2. $0 < r < 1$ (second-order probability of $P_1(R|*)$, r is r_a for Lottery A and r_b for Lottery B.)
3. $P_m(R|A) = P_m(R|B) = P(R|C)$ (mean first-order probabilities of event Red for Lotteries A, B and C, respectively)
4. $P_2(R|A) - P_1(R|A) = P_2(R|B) - P_1(R|B)$
5. $X \neq 0$ ($X > 0$ for the gain and $X < 0$ for the loss damain.)
6. Lottery A is negatively skewed on Red. Lottery B is positively skewed on Red.

Lotteries

The problem is represented by the following lotteries.

```
LOTTERY              A or B
EVENT:         Red                  Black
FOP  : P1(R|*)  P2(R|*)  1-P1(R|*)  1-P2(R|*)
SOP  :    r        1-r       r          1-r
GAIN :        $X                   $0
LOTTERY                 C
EVENT:         Red                  Black
FOP  :       P(R|C)              1-P(R|C)
SOP  :          1                    1
GAIN :         $X                   $0
```

Each of the three lotteries is paired with the others and one lottery is chosen from each of the three pairs.

r_a and r_b

The measure of skewness on event Red in Lottery $*$ (A or B), $S(R|*)$, is defined as

$$S(R|*) \equiv \frac{\boldsymbol{E}[\{P(R|*) - P_m(R|*)\}^3]}{\sigma^3(R|*)},$$

where $P(R|*)$ is the first-order probability of Red, $P_m(R|*)$ is its mean, $\sigma(R|*)$ is its standard deviation and the expectation is taken over the second-order probability. The measure can be written

$$S(R|*) = \frac{2r_* - 1}{\sqrt{r_*(1-r_*)}},$$

where r_* is r_a or r_b. In order that Lottery A is negatively skewed on Red, $r_a > 1/2$. In order that Lottery B is positively skewed on Red, $r_b < 1/2$. Hence Lottery A is positively skewed on Black, and Lottery B is negatively skewed on Black.

FOPs of Event Black

The first-order probabilities of event Black (denoted as BK) are as follows.

Lottery A:

$$P_1(BK|A) = 1 - P_1(R|A)$$
$$P_2(BK|A) = 1 - P_2(R|A)$$
$$P_m(BK|A) = 1 - P_m(R|A)$$
$$P_1(BK|A) > P_m(BK|A) > P_2(BK|A)$$

Lottery B:

$$P_1(BK|B) = 1 - P_1(R|B)$$
$$P_2(BK|B) = 1 - P_2(R|B)$$
$$P_m(BK|B) = 1 - P_m(R|B)$$
$$P_1(BK|B) > P_m(BK|B) > P_2(BK|B)$$

Lottery C:

$$P(BK|C) = 1 - P(R|C)$$
$$P_m(BK|A) = P_m(BK|B) = P(BK|C)$$

Relations between FOPs

The following relations between the FOPs hold.

Proposition 1:

$$P_1(R|A) < P_1(R|B) < P(R|C) < P_2(R|A) < P_2(R|B)$$
$$P_1(BK|A) > P_1(BK|B) > P(BK|C) > P_2(BK|A) > P_2(BK|B)$$

Proof:

Using $P_m(R|A) = P_m(R|B)$ and $P_2(R|A) - P_1(R|A) = P_2(R|B) - P_1(R|B)$ we obtain

$$P_1(R|B) - P_1(R|A) = P_2(R|B) - P_2(R|A) = (r_b - r_a)\{P_2(R|A) - P_1(R|A)\} > 0$$
$$P_1(R|A) < P_1(R|B)$$
$$P_2(R|A) < P_2(R|B)$$

Using $P_m(R|A) = P_m(R|B)$ we obtain

$$P_1(R|B) < P_2(R|A).$$

Q.E.D.

§5 Analyses of Models

Keynes's Conventional Coefficient Model (1921)

Keynes's decision theory is a Weighted Monetary Value (WMV) approach (Brady, 1993).

WMV is defined as

$$WMV = \sum_{i=1}^{N} c_i A_i$$

$$c_i = P_i \cdot [1 / (2 - P_i)]^j \cdot [2w_i / (1 + w_i)]^k,$$

where $j, k \in \{-1, 0, 1\}$, P_i is the probability of A_i, A_i is a monetary outcome, c_i is called Keynes's conventional coefficient of risk and weight and $0 \leq w_i \leq 1$. Keynes's weight of evidence, w_i, represents the amount of unreliability or ambiguity or completeness of the information or the body of knowledge upon which the probability calculations are being based. $[1 / (2 - P_i)]^j$ is a weight which allows the decision maker to deal with the 'riskiness' of a probability bet (Brady, 1993), (Brady, 1994).

$j = -1, 0, 1$ represent risk seeking, resk neutrality and risk aversion, respectively. $k = -1, 0, 1$ represent ambiguity seeking, ambiguity neutrality and ambiguity aversion, respectively. $j = k = 0,1$ are Keynes's original

version and j = k = -1 is Bradv's extension (Brady, 1994).

Keynes's model can express all the cases of risk and ambiguity preferences, but it is not specified how to represent SOP in terms of Keynes's weight of evidence, w_i .

Ellsberg Model (1961)

Let ρ be decision maker's degree of confidence, in a given state of information or ambiguity, in the estimated distribution y^0 . Let min_x be the minimum expected pay-off to an act x as the probability distribution ranges the set Y^0 ; let est_x be the expected pay-off to the act corresponding to the estimated distribution y^0 .

Ellsberg Model is: *Associate with each x the index*

$$\rho \cdot est_x + (1 - \rho) \cdot min_x.$$

Choose that act with the highest index

Gain

$$Index(R|A) = \rho P_m(R|A) + (1-\rho)P_1(R|A)$$
$$Index(R|B) = \rho P_m(R|B) + (1-\rho)P_1(R|B)$$
$$Index(R|C) = P(R|C)$$

We obtain

$$A \prec B \prec C$$

for $0 \le \rho < 1$.

Loss

$$Index(BK|A) = \rho P_m(BK|A) + (1-\rho)P_1(BK|A)$$
$$Index(BK|B) = \rho P_m(BK|B) + (1-\rho)P_1(BK|B)$$
$$Index(BK|C) = P(BK|C)$$

We obtain

$$A \succ B \succ C$$

for $0 \le \rho < 1$.

Therefore, Ellsberg Model can produce ambiguity aversion in the gain domain and ambiguity seeking in the loss domain. However, it can accommodate the skew sensitivity in neither of the domains.

Gärdenfors-Sahlin Model:MMEU (1982)

The maximin criterion for expected utilities (MMEU): The alternative with the largest minimal expected utility ought to be chosen. A agent selects a desired level ρ_0 of epistemic reliability (ρ). Only those second-order probabilities which pass this ρ_0 -level are taken into consideration in making decision by the agent. Let $\bar{u}_{ik}$ be the expected utility of the alternative A_i by the k-th first-order probability distribution which passes the ρ_0 -level. MMEU can be represented as follows.

$$\max_i \min_k \{\bar{u}_{ik}\} = \bar{u}_{i'k'},$$

where the alternative $A_{i'}$ ought to be chosen.

Gain

(1) *Lottery A vs. Lottery C*

$$\bar{u}_a(R|r_a) = P_1(R|A)$$
$$\bar{u}_a(R|1-r_a) = P_2(R|A)$$
$$\bar{u}_c(R|r_c = 1) = P(R|C)$$

For $\rho_0 < r_a$,

$$Min\{\bar{u}_a(R|r_a),\ \bar{u}_a(R|1-r_a)\} = P_1(R|A)$$
$$Max\{P_1(R|A), P(R|C)\} = P(R|C)$$

Lottery C is chosen, which means ambiguity aversion.

For $r_a < \rho_0 < 1 - r_a$,

$$Min\{\bar{u}_a(R|1-r_a)\} = P_2(R|A)$$
$$Max\{P_2(R|A), P(R|C)\} = P_2(R|A)$$

Lottery A is chosen, which means ambiguity seeking.

(2) *Lottery B vs. Lottery C*

$$\bar{u}_b(R|r_b) = P_1(R|B)$$
$$\bar{u}_b(R|1-r_b) = P_2\, P(R|B)$$
$$\bar{u}_c(R|r_c = 1) = P(R|C)$$

For $\rho_0 < 1 - r_b$,

$$Min\{\bar{u}_b(R|r_b), \bar{u}_b(R|1-r_b)\} = P_1(R|B)$$
$$Max\{P_1(R|B), P(R|C)\} = P(R|C)$$

Lottery C is chosen, which means ambiguity aversion.

For $1 - r_b < \rho_0 < r_b$,

$$Min\{\bar{u}_b(R|r_b)\} = P_1(R|B)$$
$$Max\{P_1(R|B), P(R|C)\} = P(R|C)$$

Lottery C is chosen, which means ambiguity aversion. MMEU cannot produce the ambiguity seeking for positive skewness.

(3) *Lottery A vs. Lottery B*

For $\rho_0 < Min\{r_a, 1 - r_b\}$

$$Max\{P_1(R|A), P_1(R|B)\} = P_1(R|B)$$

Lottery B is chosen.

Loss

(1) *Lottery A vs. Lottery C*

$$\bar{u}_a(BK|r_a) = P_1(BK|A)$$
$$\bar{u}_a(BK|1-r_a) = P_2(BK|A)$$
$$\bar{u}_a(BK|r_c = 1) = P_m(BK|C)$$

For $\rho_0 < r_a$

$$Min\{\bar{u}_a(BK|r_a), \bar{u}_a(BK|1-r_a)\} = P_2(BK|A)$$
$$Max\{P_1(BK|A), P_m(BK|C)\} = P_m(BK|C)$$

Lottery C is chosen, which means ambiguity aversion.

For $r_a < \rho_0 < 1 - r_a$,

$$Min\{\bar{u}_a(BK|1-r_a)\} = P_2(BK|A)$$
$$Max\{P_2(BK|A), P_m(BK|C)\} = P_m(BK|C)$$

Lottery C is chosen, which means ambiguity aversion.

(2) *Lottery B vs. Lottery C*

$$\bar{u}_b(BK|r_b) = P_1(BK|B)$$
$$\bar{u}_b(BK|1-r_b) = P_2(BK|B)$$
$$\bar{u}_c(BK|r_c=1) = P_m(BK|C)$$

For $< \rho_0 < 1 - r_b$,

$$Min\{\bar{u}_b(BK|r_b), \bar{u}_b(BK|1-r_b)\} = P_2(BK|B)$$
$$Max\{P_2(BK|B), P_m(BK|C)\} = P_m(BK|C)$$

Lottery C is chosen, which means ambiguity aversion.

For $1 - r_b < \rho_0 < r_b$,

$$Min\{\bar{u}_b(BK|r_b)\} = P_1(BK|B)$$
$$Max\{P_1(BK|B),\ P_m(BK|C)\} = P_1(BK|B)$$

Lottery B is chosen, which means ambiguity seeking. MMEU cannot produce the ambiguity seeking for positive skewness.

(3) *Lottery A vs. Lottery B*

For $\rho_0 < Min\{r_a, 1 - r_b\}$

$$Max\{P_2(BK|A),\ P_2(BK|B)\} = P_2(BK|A)$$

Lottery A is chosen.

Therefore MMEU cannot accommodate the skew sensitivity to this problem. This suggests that Boiney's subjects were not sensitive to the values of SOP because the epistemic reliability plays a central role in MMEU.

Einhorn-Hogarth Model: Anchoring-and-Adjustment Model (1985)

Let P_A be an initial assessment (anchor) of probability under consideration. The judged probability $S(P_A)$ is defined as

$$S(P_A) = P_A + \theta\,(1 - P_A - P_A^{\beta}),$$

where θ ($0 \le \theta \le 1$) expresses the amount of ambiguity perceived in the situation, and β ($\beta \ge 0$) expresses the person's attitude toward ambiguity in the circumstances.

In this model

$$0 \le S(P_A) \le 1$$

is assumed. In the problem, the judged probability in the gain domain should be in $[P_1, P_2]$. Therefore, this model cannot be applied to the problem. It is, however, easy to extend this model to $P_1 \le S(P_A) \le P_2$. An extended judged probability $S'(P_A)$ is

$$S'(P_A) = P_A + \theta\{(P_2 - P_A) - (P_A - P_1)^{\beta}(P_2 - P_1)\}.$$

For $P_1 = 0$ and $P_2 = 1$, $S'(P_A)$ reduces $S(P_A)$. The extended judged probability $S'(P_A)$ will be used in an analysis with Self-Conflicting Decision below.

Kahn-Sarin Model (1988)

The value function for lottery L is

$$V(L) = w(E)u(X).$$

The decision weight is defined as

$$w(E) = P_m + \int_0^1 (P - P_m)\, e^{-\frac{\lambda(P-P_m)}{\sigma}} f(P)\, dP,$$

where $f(P)$ is the second-order probability density function of P, λ reflects an individual's attitude toward ambiguity in an given context. $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ mean ambiguity seeking (prone), ambiguity neutral, and ambiguity averse, respectively.

This model is a kind of the anchoring and adjustment model with the mean probability as the anchor.

<u>Gain</u>

$$w_a(R) = P_m(R|A) + \left(e^{-\lambda\sqrt{\frac{r_a}{1-r_a}}} - e^{\lambda\sqrt{\frac{1-r_a}{r_a}}}\right) r_a(1-r_a)$$
$$\{P_2(R|A) - P_1(R|A)\}$$

$$w_b(R) = P_m(R|B) + \left(e^{-\lambda\sqrt{\frac{r_b}{1-r_b}}} - e^{\lambda\sqrt{\frac{1-r_b}{r_b}}}\right) r_b(1-r_b)$$
$$\{P_2(R|B) - P_1(R|B)\}$$
$$w_c(R) = P(R|C)$$

<u>Loss</u>

$$w_a(BK) = P_m(BK|A) + \left(e^{-\lambda\sqrt{\frac{1-r_a}{r_a}}} - e^{\lambda\sqrt{\frac{r_a}{1-r_a}}}\right) r_a(1-r_a)$$
$$\{P_1(BK|A) - P_2(BK|A)\}$$

$$w_b(BK) = P_m(BK|B) + \left(e^{-\lambda\sqrt{\frac{1-r_b}{r_b}}} - e^{\lambda\sqrt{\frac{r_b}{1-r_b}}}\right) r_b(1-r_b)$$
$$\{P_1(BK|B) - P_2(BK|B)\}$$
$$w_c(BK) = P_m(BK|C)$$

If λ has the same sign in Lottery A and Lottery B, then this model implies that the ambiguity preferences for Lottery A are the same as the ones for Lottery B.

To enable this model to describe the skewness sensitivity to the problem, λ must be a function of skewness. For instance $\lambda = -(mean-mode)/\sigma$ can make it possible formally, although its psychological base is unknown.

Note that the range $|P_2-P_1|$ and $r(1-r)$ accentuate the attitude toward ambiguity and r_a=1/2 does not imply the weight of P_m .

Boiney Model (1993)

Define a lottery L as awarding $X if event E obtains, otherwise awarding nothing, with u($0) = 0. Let P be the probability of winning, which is a random variable with density function f(P) and mean P_m . The proposed model will replace the probability of an event E with a subjective decision weight, w(E), so that the value function for L is

$$V(L) = w(E)u(X)$$

A decision weight w(E) must capture two key phenomena observed in the data. The first phenomenon is that decision makers behave as though they place a disproportionate amount of weight on probabilities far from the mean. To reflect this behavior, one component of the decision weight will be the mean squared deviation, $(P-P_m)^2$. The second phenomenon is that individuals have different attitudes toward ambiguity, which will be expressed by introducing two individual-specific parameters, e and d. These are simply constants, $e \geq 0$ and $d \leq 0$, designed to measure the potential "elation" and potential "disappointment" experienced due to probabilities above and below the mean, respectively.

In oder to reflect both phenomena, a function A(P) is defined to represent the attitude toward a given probability. More specifically,

$$A(P) = \begin{cases} e(P-P_m)^2 & if\ P \geq P_m \\ d(P-P_m)^2 & if\ P < P_m \end{cases}$$

Taking the expectation over this function results in the "expected attitude" toward the second-order distribution f(P): $\int_0^1 A(P)\, f(P)\, dP$. In essence, the decision weight will adjust the mean probability P_m up and down according to this expected attitude toward the ambiguous distribution. The decision weight for event E is defined as

$$w(E) = P_m + \int_0^1 A(P)\, f(P)\, dP.$$

Boiney's model is also a kind of the anchoring and adjustment model with the mean probability as the anchor. The model does not use the interesting concept of the focused probability that Boiney has introduced in (Boiney, 1993). The reason seems that Boiney's basic idea behind the model development is constrained within the old idea of the maximization of some quantity like the expected utility.

(A) Gain

$$w_a(R) = P_m(R|A) + \{er_a + d(1-r_a)\}\,\sigma_a^2$$
$$w_b(R) = P_m(R|B) + \{er_b + d(1-r_b)\}\,\sigma_b^2$$
$$w_c(R) = P(R|C)$$

(B)Loss

$$w_a(BK) = P_m(BK|A) + \{dr_a + e(1-r_a)\}\,\sigma_a^2$$
$$w_b(BK) = P_m(BK|B) + \{dr_b + e(1-r_b)\}\,\sigma_b^2$$
$$w_c(BK) = P(BK|C)$$

Here, σ^2 is the variance of the FOP.

$\sigma_a^2 = r_a(1 - r_a)\{P_1(R|A) - P_2(R|A)\}^2 = r_a(1 - r_a)\{P_1(BK|A) - P_2(BK|A)\}^2$

$\sigma_b^2 = r_b(1 - r_b)\{P_1(R|B) - P_2(R|B)\}^2 = r_b(1 - r_b)\{P_1(BK|B) - P_2(BK|B)\}^2$

<u>Skew Sensitivity</u>

The model needs an assumption about e and d to accommodate the skew sensitivity. Boiney assumes that $-d = e \neq 0$. Using $r_a < 1/2$ and $r_b < 1/2$ and $-d = e$, we obtain $A \prec C \prec B$ in the gain domain and $A \succ C \succ B$ in the loss domain.

<u>Ambiguity Aversion</u>

By using Boine's assumption of $e = 0$ and $d \neq 0$, we can obtain $A,\ B \prec C$ in both domain. The preference order between A and B depends upon the difference between $d(1-r_a)\}\,\sigma_a^2$ and $d(1-r_b)\}\,\sigma_b^2$ in the gain domain and the difference between $dr_a\,\sigma_a^2$ and $dr_b\,\sigma_b^2$ in the loss domain. If $\sigma_a = \sigma_b$, which means $r_a + r_b = 1$, $A \prec B$ in the gain domain and $A \succ B$ in the loss domain.

<u>Ambiguity Seeking</u>

By using Boiney's assumption of $d = 0$ and $e \neq 0$, we can obtain $A, B \succ C$ in both domains. The preference order between A and B depends upon the difference between $er_a\,\sigma_a^2$ and $er_b\,\sigma_b^2$ in the gain domain and the difference between $e(1-r_a)\,\sigma_a^2$ and $e(1-r_b)\,\sigma_b^2$ in the loss domain. If $\sigma_a = \sigma_b$, which means $r_a + r_b = 1$, $A \prec B$ in the gain domain and $A \succ B$ in the loss domain.

Ambiguity Neutrality

By using Boiney's assumption of $e = d = 0$, we can obtain $A \sim B \sim C$ in both domains. This assumption is exactly equivalent to the expected utility maximization. This indicates that the expected utility maximization is emotionless and the expected utility maximization has a fundamental drawback in describing human decision-making behaviors. Note that if $e = -d$, $r_a = r_b = 1/2$, then we obtain $A \sim B \sim C$.

Flexible Bayesian Approach (1994)

Shigemasu proposes the flexible Bayesian approach to describe the psychological decision making process (Shigemasu & Yokoyama, 1994), (Shigemasu, 1995). Its essence is as follows.

1. *The subjective probability of an uncertain event,* E_j (j = 1; 2, …, n), *depends on the situation under consideration. The subjective probability* $P(E_j)$ *is not determined uniquely but its higher oder probabilities only are obtainable.*
2. *The utility,* $u(a_i, E_j)$, *of the consequence,* $c(a_i, E_j)$, *that an action,* a_i(i = 1, 2,…, m), *and an event,* E_j, *bring to the decision maker depends upon* $\pi = \{P(E_1), P(E_2), \ldots, P(E_n)\}$. *that is,*
$$u(a_i, E_j) = u(c(a_i, E_j) \mid \pi).$$
3. *The optimal action is the one that maximizes the following expected utility.*
$$\bar{u}(a_i) = \sum_{j=1}^{n} \int u(c(a_i, E_j) \mid \pi)\, f(P(E_j))\, dP(E_j),$$
where $f(P(E_j))$ *is the second-order probability density function of* $P(E_j)$.

Let us apply the flexible Bayesian approach to the problem to see whether it can accommodate the skewness effects. We assume that the utility is also a function of the second-order probability.

$$u(*, R) = u(c(*, R) \mid \pi'_*)$$
$$\pi'_* = \{P(R \mid *), P(BK \mid *), r_*\}$$

Here, $*$ = A, B, C; $P(R \mid *) = P_1(R \mid A)$, $P_2(R \mid B)$; $P(BK \mid *) = P_1(BK \mid A)$, $P_2(BK \mid B)$.

In the gain domain, we assume the following utility.

$$u(c(*, R) \mid \pi'_*) = \begin{cases} |X|\,[P_m(R \mid *) + e \cdot \{P(R \mid *) - P_m(R \mid *)\}^2] \\ \quad if\ P(R \mid *) \geq P_m(R \mid *) \\ |X|\,[P_m(R \mid *) + d \cdot \{P(R \mid *) - P_m(R \mid *)\}^2] \\ \quad if\ P(R \mid *) < P_m(R \mid *) \end{cases}$$

$$P_m(R \mid *) = r_* P_1(R \mid *) + (1 - r_*) P_2(R \mid *)$$
$$u(c(*, BK) \mid \pi'_*) = 0$$

In the loss domain, we assume the following utility.

$$u(c(*, R) \mid \pi'_*) = \begin{cases} |X|\,[-P_m(R \mid *) + d \cdot \{P(R \mid *) - P_m(R \mid *)\}^2] \\ \quad if\ P(R \mid *) \geq P_m(R \mid *) \\ |X|\,[-P_m(R \mid *) + e \cdot \{P(R \mid *) - P_m(R \mid *)\}^2] \\ \quad if\ P(R \mid *) < P_m(R \mid *) \end{cases}$$

$$u(c(*, BK) \mid \pi'_*) = 0$$

The second-order probabilities are as follows.

$$f(P(R \mid *)) = \begin{cases} r_* & for\ P_1(R \mid *) \\ 1 - r_* & for\ P_2(R \mid *) \end{cases}$$

Using the above assumptions we obtain in the gain domain

$$\bar{u}(A) = |X| \left[P_m(R \mid A) + \{e r_a + d(1 - r_a)\}\sigma_a^2\right]$$

$$\bar{u}(B) = |X| \left[P_m(R \mid B) + \{e r_b + d(1 - r_b)\}\sigma_b^2\right]$$

$$\bar{u}(C) = |X|\, P(R \mid C) = |X|\, P_m(R \mid A) = |X|\, P_m(R \mid B),$$

in the loss domain

$$\bar{u}(A) = |X| \left[-P_m(R \mid A) + \{d r_a + e(1 - r_a)\}\sigma_a^2\right]$$

$$\bar{u}(B) = |X| \left[-P_m(R \mid B) + \{d r_b + e(1 - r_b)\}\sigma_b^2\right]$$

$$\bar{u}(C) = -|X|\, P(R \mid C) = -|X|\, P_m(R \mid A) = -|X|\, P_m(R \mid B).$$

We have obtained the exactly same results as those of Boiney model. It is interesting to note that Boiney model is a weighting function model and the flexible Bayesian approach is a utility model. It seems easier to understand the the disappointment and elation parameters, d and e, in the utility model than in the weighting function model.

Self-Conflicting Decision (1995)

Without loss of generality, in the gain domain ($X > 0$), we can set $\bar{u}(0) = 0$ and $\bar{u}(X) = 1$. Hence we can ignore event Black in the gain domain. Lottery A and Lottery B are negatively and positively skewed on event Red in the gain domain, respectively.

In the loss domain ($X < 0$), without loss of generality, we can set $\bar{u}(X) = 0$ and $\bar{u}(0) = 1$. Hence we can ignore event Red in the loss domain. Lottery A and Lottery B are positively and negatively skewed on event Black in the loss domain, respectively.

Gain

Let us begin with the analysis of the skew sensitivity in the gain domain.

(1) Skew Sensitivity

$$\bar{u}_a(R|FOP) = FOP = \{P_1(R|A), P_2(R|A)\}$$
$$\bar{u}_b(R|FOP) = FOP = \{P_1(R|B), P_2(R|B)\}$$
$$\bar{u}_c(R|FOP) = FOP = P(R|C)$$

Since $P_1(R|A) < P(R|C) < P_2(R)$, A and B are risky with respect to C.

$I_R[A \succ C|R] = Prob[\{P_1(R|A), P_2(R|A)\} > P(R|C)] = 1 - r_a$

$I_R[B \succ C|R] = Prob[\{P_1(R|B), P_2(R|B)\} > P(R|C)] = 1 - r_b$,

where $1/2 < 1 - r_a < 1$ and $0 < 1 - r_b < 1/2$.

$r_a < 1/2$ implies that

$$P_m(R|A) - P_1(R|A) > P_2(R|A) - P_m(R|A).$$

This means that the probability far from the mean for Lottery A is $P_1(R|A)$. $r_b > 1/2$ implies that

$$P_m(R|A) - P_1(R|B) < P_2(R|B) - P_m(R|A).$$

This means that the probability far from the mean for Lottery B is $P_2(R|B)$.

By Assumption 1 the focused probability in Lottery A is $P_1(R|A)$ and the one in Lottery B is $P_2(R|B)$. By Assumption 2 the equivalent probability for Lottery A, $P^*(R|A)$, is $P_1(R|A)$. Hence $\alpha(R|A) = 1$. By Assumption 2 the equivalent probability for Lottery B, $P^*(R|B)$, is $P_2(R|B)$. Hence $\alpha(R|B) = 0$. We obtain $A \prec C, C \prec B$.

Proposition 1 means that A is not risky with respect to B, vice versa. So, by using **Non-Risky Decision** for the choice that is not risky, we can obtain $A \prec C \prec B$ (skew sensitivity).

It is easy to see that Self-Conflicting Decision can produce the skew sensitivity for the following equivalent probabilities:

$P_1(R|A) \le P^*(R|A) < P_1(R|A) + r_a\{P_2(R|A) - P_1(R|A)\}$

$P_2(R|B) - (1-r_b)\{P_2(R|B) - P_1(R|B)\} < P^*(R|B) \le P_2(R|B)$.

Note that $P^*(R|A) < P^*(R|B)$. Therefore Assumption 2 can be relaxed as follows.

Assumption 3: The equivalent probability is the focused probability or its adjusted one with the focused probability as the anchor.

(2) Other Possible Focused Probability Combinations and Preference Patterns

In this problem the FOP for gain is $P_1(R)$ or $P_2(R)$. Hence the focused probabilities, if any, would be $P_1(R)$ or $P_2(R)$. The values between $P_1(R)$ and $P_2(R)$ could not be the focused probabilities.

Assumption 4-1: The focused probability is $P_1(R)$ or $P_2(R)$ in the gain domain.

There are the following 4 possible combinations in the focused probabilities.

1. $P_1(R|A)$ and $P_1(R|B)$
2. $P_2(R|A)$ and $P_2(R|B)$
3. $P_1(R|A)$ and $P_2(R|B)$
4. $P_2(R|A)$ and $P_1(R|B)$

Combination 1 implies that the safety preference disposition is high. This corresponds to Boiney's assumption of $e = 0$ and $d \neq 0$. Combination 2 implies that the safety preference disposition is low. This corresponds to Boiney's assumption of $e \neq 0$ and $d = 0$. Combination 3 consists of the focused probabilities that are the FOPs far from the mean. Combination 4 consists of the focused probabilities that are the FOPs with the highest SOPs.Under these combinations the possible preference patterns are as follows:

1. $A \prec B \prec C$ (ambiguity aversion)
2. $C \prec A \prec B$ (ambiguity seeking)
3. $A \prec C \prec B$ (skew sensitivity)
4. $B \prec C \prec A$.

Preference Patterns 1-3 are reported but Preference Pattern 4 is not reported in (Boiney, 1993). This suggests that the focused probabilities are not determined based on the values of SOP but based on individual safety preference disposition and the skewness of SOP. This is a reason that Gärdenfors-Sahlin model cannot accommodate the skew sensitivity.

(3) Ambiguity Neutrality

Boiney's nineteen subjects were ambiguity neutral. If the equivalent probabilities for gain are:

for Lottery A

$P^*(R|A) = (1 - r_a)P_1(R|A) - r_aP_2(R|A) = P_1(R|A) + r_a\{P_2(R|A) - P_1(R|A)\}$,

for Lottery B

$P^*(R|B) = (1 - r_b)P_1(R|B) - r_bP_2(R|B) = P_2(R|B) + (1 - r_b)\{P_1(R|B) - P_2(R|B)\}$,

then Self-Conflicting Decision can produce ambiguity neutrality.

This equivalent probabilities can also be obtained by setting $P_A = P_1(R|A)$ or $P_2(R|B)$, $\theta = r_a$ or $1 - r_b$

and $\beta = 0$ for Lottery B in the extended judged probability $S'(P_A)$. The anchoring and adjustment processes can be interpreted as follows. A focused probability is chosen as the anchor, and the anchor is adjusted based on the SOP of the anchor and the range of the FOP. It is natural to assume according to the Boiney's results that the focused probability for Lottery A is $P_1(R|A)$ and the one for Lottery B is $P_2(R|B)$. This is a possible explanation of the ambiguity neutrality.

On the other hand, Boiney (Boiney, 1993) writes that his subjects are second-semester MBA students and had been exposed to a few weeks of decision analysis in a required Quantitative Analysis for Management course and many clearly indicated that they were neither ignorant of expected utility nor making some kind of a mistake. Therefore, it would be reasonable to interpret that the 19 subjects used the expected utility maximization.

<u>Loss</u>

Let us begin with the analysis of the skew sensitivity in the loss domain.

(1) Skew Sensitivity

$$\bar{u}_a(BK|FOP) = FOP = \{P_1(BK|A), P_2(BK|A)\}$$
$$\bar{u}_b(BK|FOP) = FOP = \{P_1(BK|B), P_2(BK|B)\}$$
$$\bar{u}_c(BK|FOP) = FOP = P_m(BK|C)$$

Since $P_1(BK) > P(BK|C) > P_2(BK)$, A and B are risky with respect to C.

$I_{BK}[A \succ C | BK] = Prob[\{P_1(BK|A), P_2(BK|A) > P_m(BK|C)] = r_a$

$I_{BK}[B \succ C | BK] = Prob[\{P_1(BK|B), P_2(BK|B) > P_m(BK|C)] = r_b$,

where $0 < r_a < 1/2$ and $1/2 < r_b < 1$.

$r_a < 1/2$ implies that

$$P_m(BK|A) - P_2(BK|A) > P_1(BK|A) - P_m(BK|A).$$

This means that the probability far from the mean for Lottery A is $P_1(BK|A)$. $r_b > 1/2$ implies that

$$P_m(BK|A) - P_2(BK|B) > P_1(BK|B) - P_m(BK|A).$$

This means that the probability far from the mean for Lottery B is $P_2(BK|B)$

By Assumption 1 the focused probability in Lottery A is $P_1(BK|A)$ and the one in Lottery B is $P_2(BK|B)$. By Assumption 2 the equivalent probability for Lottery A, $P^*(BK|A)$, is $P_1(BK|A)$. Hence $\alpha(BK|A) = 0$. By Assumption 2 the equivalent probability for Lottery B, $P^*(BK|B)$, is $P_2(BK|B)$. Hence $\alpha(BK|B) = 1$. We obtain $A \succ C, C \succ B$.

Proposition 1 means that A is not risky with respect to B, vice versa. So, by using Non-Risky Decision for the choice that is not risky, we can obtain $A \succ C \succ B$ (skew sensitivity).

It is easy to see that Self-Conflicting Decision can produce the skew sensitivity for the following equivalent probabilities:

$P_1(BK|A) - r_a\{P_1(BK|A) - P_2(BK|A)\} < P^*(BK|A) \le P_1(BK|A)$

$P_2(BK|B) \le P^*(BK|B) < P_2(BK|B) + (1-r_b)\{P_1(BK|B) - P_2(BK|B)\}$

Note that $P^*(BK|A) > P^*(BK|B)$. Therefore Assumption 3 holds in the loss domain.

(2) Other Possible Focused Probability Combinations and Preference Patterns

In this problem the FOP for loss is $P_1(BK)$ or $P_2(BK)$. Hence the focused probabilities, if any, would be $P_1(BK)$ or $P_2(BK)$. The values between $P_1(BK)$ and $P_2(BK)$ could not be the focused probabilities.

Assumption 4-2: The focused probability is $P_1(BK)$ or $P_2(BK)$ in the loss domain.

There are the following 4 possible combinations in the focused probabilities.

1. $P_1(BK|A)$ and $P_1(BK|B)$
2. $P_2(BK|A)$ and $P_2(BK|B)$
3. $P_1(BK|A)$ and $P_2(BK|B)$
4. $P_2(BK|A)$ and $P_1(BK|B)$

Combination 1 implies that the safety preference disposition is low. This corresponds to Boiney's assumption of $d = 0$ and $e \neq 0$. Combination 2 implies that the safety preference disposition is high. This corresponds to Boiney's assumption of $d \neq 0$ and $e = 0$. Combination 3 consists of the focused probabilities that are the FOPs far from the mean. Combination 4 consists of the focused probabilities that are the FOPs with the highest SOPs. Under these combinations the possible preference patterns are as follows:

1. $A \succ B \succ C$ (ambiguity seeking)
2. $C \succ A \succ B$ (ambiguity aversion)
3. $A \succ C \succ B$ (skew sensitivity)
4. $B \succ C \succ A$.

Preference Patterns 1-3 are reported but Preference Pattern 4 is not reported in (Boiney, 1993). This suggests that the focused probabilities are not determined based on the values of SOP but based on individual safety preference disposition and the

skewness of SOP. This is a reason that Gärdenfors-Sahlin model cannot accommodate the skew sensitivity.

(3) Ambiguity Neutrality

Boiney's nineteen subjects were ambiguity neutral. If the equivalent probabilities for loss are:

for Lottery A

$P^{*}(BK|A) = P_1(BK|A) - r_a\{P_1(BK|A) - P_2(BK|A)\}$,

for Lottery B

$P^{*}(BK|B) = P_2(BK|B) + (1 - r_b)\{P_1(BK|B) - P_2(BK|B)\}$,

then Self-Conflicting Decision can produce ambiguity neutrality.

This equivalent probabilities can also be obtained by setting $P_A = P_1(BK|A)$ or $P_2(BK|B), \theta = r_a$ or $1 - r_b$ and $\beta = 0$ for Lottery A in the extended judged probability $S'(P_A)$. The anchoring and adjustment processes can be interpreted as follows. A focused probability is chosen as the anchor, and the anchor is adjusted based on the SOP of the anchor and the range of the FOP. It is natural to assume according to the Boiney's results that the focused probability for Lottery A is $P_1(BK|A)$ and the one for Lottery B is $P_2(BK|B)$. This is a possible explanation of the ambiguity neutrality.

However, it would be reasonable, based on the same argument as the one in the gain domain, to interpret that the 19 subjects used the expected utility maximization.

Summary of Self-Conflicting Decision Analysis

First, to identify the equivalent probability theoretically, the focused probability introduced by Boiney has been examined. The focused probability has been assumed to be the first-order probability far from the mean probability, which is the probability of a probabilistic option, in **Assumption 1**. In **Assumption 2** the equivalent probability has been assumed to be the focused probability. In **Assumption 3** this assumption has been extended by using the extended judged probability, i.e., the equivalent probability is the focused probability or its adjusted one with the focused probability as the anchor.

Secondly, a decision between two non-risky options has been defined (**Non-Risky Decision**): the option is chosen which has the largest expected utility with the equivalent probability in the case where two options are not risky each other.

It has been shown that SCD can accommodate the skew sensitivity for both the gain and loss domains on the basis of the assumptions and Non-Risky Decision.

Finally, based on **Assumption 4-1** for gain and **Assumption 4-2** for loss asserting that the focused probability is one of the 2 first-order probabilities but not between them, it has been shown that SCD can accommodate the ambiguity seeking, the ambiguity aversion and the ambiguity neutrality for both the gain and loss domains. One combination of the focused probabilities has been found which Boiney does not report.

§6 Concluding Remarks

We have examined, using the generalized Boiney problem, how the eight decision models can or cannot explain the Boiney emotional skewness effects. It has been shown that Keynes Conventional Coefficient Model (1921), Ellsberg Model (1961), Gärdenfors-Sahlin Model (1982), Einhorn-Hogarth Model(1985) and Kahn-Sarin Model(1988) cannot accommodate the skewness effects but Boiney Model (1993), Flexible Bayesian Approach (1994) and Self- Conflicting Decision (1995) can.

It has been shown that the concepts of disappointment and *elation* can be used successfully in both the weighting function model (Boiney model) and the utility model (Flexible Bayesian Approach). It is an interesting research problem to make clear in detail the relations between the two models.

The seven models except for SCD are a maximization type model. SCD is not based on the maximization principle but principally based on the psychological hypothesis, i.e., **Self- Conflicting Hypothesis**. The validity of the hypothesis has been suggested by showing that SCD can accommodate the skewness effects based on the reasonable assumption that the equivalent probability in SCD is the focused probability introduced by Boiney and by processing *elation* and *disappointment* mathematically with Safety Preference Disposition Measure (α). Moreover it has been shown that the equivalent probability in SCD is the *extended judged probability* with the *focused*

probability as the anchor. It has been found that there are the close relations between Boiney model and SCD. The relations indicate that the concept of the focused probability is important in decision-making research under ambiguity.

The above 3 successful models have the tools - *disappointment parameter* (d), *elation parameter* (e), *focused probabilities* or *Safety Preference Disposition Measure* (α) - to treat the emotions but the other 5 failed ones do not. This clearly shows that emotions are important factors in human decision making and can be included mathematically in decision theory. The construction of decision theory that can treat emotions is one of the most urgent agenda in decision-making research because decision makers are humans and man is a creature of emotion.

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The target of the research is under standing decision making in the framework of "Desision Making and Emotion."