International Economics B
1. Mathematical preliminaries

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Definition

Given two sets X and Y, a **function** from X to Y is a rule that assigns each element of X to one and only one element of Y, and denoted by $f : X \mapsto Y$.

- The function is also written as $y = f(x)$, $x \in X$, where y is referred to as the **value** of the function f at x.
- X: **domain** of f; Y: **codomain** of f
- **Range**: the set of all the actual values a function has
- We usually consider cases where $X \subseteq \mathbb{R}^n$, where $\mathbb{R}^n$: real coordinate space.

- **Consider a function of one variable, $y = f(x)$, where $x \in \mathbb{R}$.**

Definition

A function f is concave if

$$f(\bar{x}) \geq \lambda f(x') + (1-\lambda) f(x''),$$

where $\bar{x} = \lambda x' + (1-\lambda)x''$ and $\lambda \in [0,1]$. It is strictly concave if the strict inequality holds when $\lambda \in (0,1)$.

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- **The definition of concavity/convexity can be applied to functions of n variables, $f(\mathbf{x})$, where $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$**

Homogeneous functions

- **Consider a function of n variables, $f(x_1, \cdots, x_n)$, defined for all nonnegative values $(x_1, \cdots, x_n) \geq 0$.**

Definition

A function $f(x_1, \cdots, x_n)$ is homogeneous of degree k if, for every $\lambda > 0$, we have

$$f(\lambda x_1, \cdots, \lambda x_n) = \lambda^k f(x_1, \cdots, x_n).$$

- **Examples:**
  - $f(x_1, x_2) = x_1/x_2$: **homogeneous of degree zero**
  - $f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$: **homogeneous of degree one**

- **Consider a function of one variable, $y = f(x)$, where $x \in \mathbb{R}$.**
- **Purpose of the derivative: to express how a change in x determines a change in y**

Definition

The derivative of a function $y = f(x)$ at a point $P = (x_1, f(x_1))$ is the slope of the tangent line at that point:

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}.$$

Definition

If $f'(x^0)$ is the derivative of a function $y = f(x)$ at the point x^0, then the total differential at a point x^0 is

$$dy = f'(x^0)dx.$$

Rules of differentiation

1. $f(x) = ax + b$ **(a, b: const.)** $\Rightarrow f'(x) = a$
2. $f(x) = x^n$ **(n: const.)** $\Rightarrow f'(x) = nx^{n-1}$
3. $h(x) = af(x) + bg(x)$ **(a, b: const.)** $\Rightarrow h'(x) = af'(x) + bg'(x)$
4. $h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x)$
5. $h(x) = \dfrac{f(x)}{g(x)}$ **($g(x) \neq 0$)** $\Rightarrow h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
6. $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x)$ **(Chain rule)**
7. $g(y) = f^{-1}(y)$ **($f'(x) \neq 0$)** $\Rightarrow g'(y) = \dfrac{1}{f'(x)}$**, where** $y = f(x)$
8. $f(x) = e^x \Rightarrow f'(x) = e^x$
9. $f(x) = \ln x \Rightarrow f'(x) = 1/x$

- **Derivative of a function is also a function $\Rightarrow$ we can write $dy/dx = f'(x)$, where f' is called the first derivative function of f.**
- **Derivative of f',**

$$\frac{d(dy/dx)}{dx} \quad \text{or} \quad \frac{d[f'(x)]}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

  is called the second derivative function.
- **If the first two derivatives of a function exist, we say that the function is twice differentiable.**

Convexity/concavity of differentiable functions:

Theorem

A twice-differentiable function f of a single variable defined on the interval $[a, b] \subset \mathbb{R}$ is

- **concave if and only if $f''(x) \leq 0$ for all $x \in (a, b)$;**
- **convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.**

Theorem

A twice-differentiable function f of a single variable defined on the interval $[a, b] \subset \mathbb{R}$ is

- **strictly concave if $f''(x) < 0$ for all $x \in (a, b)$;**
- **strictly convex if $f''(x) > 0$ for all $x \in (a, b)$.**

Differentiation: Functions of n variables

- **Consider a function of n variables, $f(\mathbf{x})$, where $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$.**

Definition

The partial differentiation of a function $y = f(x_1, \cdots, x_n)$ with respect to the variable x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, \cdots, x_i + \Delta x_i, \cdots, x_n) - f(x_1, \cdots, x_i, \cdots, x_n)}{\Delta x_i}.$$

- **Instead of $\partial f / \partial x_i$, the notations $\partial y / \partial x_i$, $f_i(\mathbf{x})$, or f_i are used interchangeably.**

- **Second-order partial derivatives:**

$$f_{ij} \equiv \frac{\partial f_i(\mathbf{x})}{\partial x_j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$$

- **If $n = 2$, there are $2^2 = 4$ second-order partial derivatives:**

$$f_{11} \equiv \frac{\partial f_1(x_1, x_2)}{\partial x_1}, \quad f_{12} \equiv \frac{\partial f_1(x_1, x_2)}{\partial x_2},$$
$$f_{21} \equiv \frac{\partial f_2(x_1, x_2)}{\partial x_1}, \quad f_{22} \equiv \frac{\partial f_2(x_1, x_2)}{\partial x_2}.$$

- $f_{ij} = f_{ji}$ **holds. (Young's theorem)**

Definition

The **first-order total differential** **for the function** $y = f(x_1, \cdots, x_n)$ **is**

$$dy = f_1 dx_1 + \cdots + f_n dx_n = \sum_{i=1}^{n} f_i dx_i.$$

Implicit function theorem

Let $F(x_1, \cdots, x_n, y) = 0$ **be an implicit function, with continuous first derivatives, which is satisfied at some point** $(x_1^0, \cdots, x_n^0, y^0)$ **and is defined in some neighborhood of this point. If** $F_y \neq 0$ **at this point, then there is a function** $y = f(x_1, \cdots, x_n)$ **defined in some neighborhood of** $\mathbf{x}^0 = (x_1^0, \cdots, x_n^0)$ **such that**

1. $y^0 = f(\mathbf{x}^0)$**, and**
2. $f_i(\mathbf{x}^0) = -F_{x_i}/F_y$**.**

Concavity/convexity of differentiable functions: case of $n = 2$ variables

Definition

The second-order total differentiation for the function $y = f(x_1, x_2)$ is

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2.$$

Theorem

If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable, then it is convex (concave) if and only if the second-order total differentiation is nonnegative (nonpositive).

Theorem

If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable, and the second-order total differentiation is positive (negative) whenever at least one of dx_1 or dx_2 is nonzero, then $y = f(x_1, x_2)$ is a strictly convex (strictly concave) function.

Differentiation of homogeneous functions:

Theorem

If a function $y = f(\mathbf{x})$ is homogeneous of degree k, then its first-order partial derivatives, $\partial f(\mathbf{x})/\partial x_i$, $i = 1, \cdots, n$, are homogeneous of degree $k-1$.

Euler's theorem

If a function $y = f(\mathbf{x})$ is homogeneous of degree k, then the following condition holds:

$$\sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i = k f(\mathbf{x}).$$

Definition

A matrix is a rectangular array of numbers enclosed in parentheses.

- **A matrix A of order $m \times n$ (m rows and n columns) can be explicitly written as**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

- **A matrix having only one row such as $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is called a row matrix or row vector.**
- **A matrix having only one column such as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is called a column matrix or column vector.**

Definition

An array that consists of only one row or one column is known as a vector.

Definition

An matrix that has the same number of rows and columns is called a square matrix.

Definition

The sum of two matrices is a matrix, the elements of which are the sums of the corresponding elements of the matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix}$$

$$\Rightarrow \quad A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & a_{m3}+b_{m3} & \cdots & a_{mn}+b_{mn} \end{bmatrix}.$$

- **In matrix algebra, real numbers are called scalars.**

Definition

Scalar multiplication is carried out by multiplying each element of the matrix by the scalar:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$\Rightarrow \quad \lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \lambda a_{m3} & \cdots & \lambda a_{mn} \end{bmatrix},$$

where λ is a scalar.

Matrix multiplication:

Definition

Two matrices A and B of dimensions $m \times n$ and $n \times q$ respectively are conformable to form the product matrix $AB = C$, since the number of columns of A is equal to the number of rows of B. The product matrix AB is of dimension $m \times q$, and its ij-th element, c_{ij}, is obtained by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

- **Example:**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$\Downarrow$$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix},$$

$$BA = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{12}b_{11} + a_{22}b_{12} & a_{13}b_{11} + a_{23}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} & a_{13}b_{21} + a_{23}b_{22} \\ a_{11}b_{31} + a_{21}b_{32} & a_{12}b_{31} + a_{22}b_{32} & a_{13}b_{31} + a_{23}b_{32} \end{bmatrix}$$

Theorem

Both of the product matrices AB and BA are well defined only if A and B are square matrices of the same order of for A of dimension $m \times n$ with B of dimension $n \times m$.

Determinants and the inverse matrix

Definition

The inverse matrix A^{-1} of a square matrix A of order n is the matrix that satisfies the condition that

$$AA^{-1} = A^{-1}A = I_n,$$

where I_n is the identity matrix (i.e., the square matrix with ones on the main diagonal and zeros elsewhere) of order n.

Definition

The matrix A for which A^{-1} exists is known as a nonsingular matrix. Any matrix A for which A^{-1} does not exist is known as a singular matrix.

The case of 2×2 matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{a_{22}a_{11} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- **If $a_{22}a_{11} - a_{21}a_{12} \neq 0$, A is nonsingular.**

Definition

The quantity $a_{22}a_{11} - a_{21}a_{12}$ is called the determinant of the 2×2 matrix A and is denoted by $|A|$ or $\det A$.

General $n \times n$ matrices:

Definition

The minor associated with each element a_{ij} of an $n \times n$ matrix A is denoted by M_{ij} and is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the i-th row and j-th column of the matrix A.

Definition

The cofactor associated with each element a_{ij} of an $n \times n$ matrix A is the minor of that element multiplied by $(-1)^{i+j}$:

$$C_{ij} = (-1)^{i+j} M_{ij}, \quad i, j = 1, \cdots, n.$$

Theorem

The determinant of an $n \times n$ matrix can be found by adding the product of each element a_{ij} and its associated cofactor C_{ij} along any row or column:

$$\begin{aligned} |A| &= \sum_{i=1}^{n} a_{ij} C_{ij} \quad \textbf{for any single } j = 1, \cdots, n \\ &= \sum_{j=1}^{n} a_{ij} C_{ij} \quad \textbf{for any single } i = 1, \cdots, n. \end{aligned}$$

- **Applying the theorem to 3×3 matrices**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  - **Focusing on the first row:**

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

  - **Focusing on the first column:**

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Systems of linear equations

- **Consider a system of n linear equations with n unknowns $(x_1, \cdots, x_n)$:**

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}
$$

  where a_{ij} and b_i, $i, j = 1, \cdots, n$, are parameters.

- **Using the matrix form, the linear system can be expressed as**

$$A\mathbf{x} = \mathbf{b},$$

  where x and b are column matrices.

Theorem

The solution for x of a matrix equation $A\mathrm{x} = \mathrm{b}$ is given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

if the matrix A is nonsingular.

Cramer's rule

Consider a matrix equation $A\mathrm{x} = \mathrm{b}$ and assume that $|A| \neq 0$. Then, individual values for the unknowns x_i, $i = 1, \cdots, n$, are given by

$$x_i = \frac{|A_i|}{|A|},$$

where A_i is the matrix formed by replacing the i-th column of A by the column vector b.

- **For example, x_1 is solved as**

$$x_1 = \frac{1}{|A|} \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Definition

At a global maximum x^*,

$$f(x^*) \geq f(x) \quad \textbf{for all} \;\; x$$

whereas at a local maximum $\hat{x}$,

$$f(\hat{x}) \geq f(x), \quad \textbf{for all} \;\; x \in [\hat{x} - \epsilon, \hat{x} + \epsilon]$$

and (possibly small) $\epsilon > 0$.

Theorem

Consider a differentiable function $y = f(x)$. If the function takes an extreme value (maximum or minimum) at a point x^*, then $f'(x^*) = 0$.

- **The condition $f'(x^*) = 0$ is referred to as the first-order condition.**
- **Second-order conditions:**

Theorem

If $f'(x^*) = 0$ and $f''(x^*) < 0$, then f has a local maximum at x^*.

Theorem

If $f'(x^*) = 0$ and $f''(x^*) > 0$, then f has a local minimum at x^*.

Application: Profit maximization of a monopolist

- **A monopolist having a cost function $C(x)$, where x is the output of a good, produces a good for a market, which has a demand function $D(p)$.**
  - $C(x)$**: twice differentiable and convex**
  - $D(p)$**: twice differentiable, with $D'(p) < 0$ for $p \geq 0$**
- **Market clearing condition:** $x = D(p)$
- **Monopolist's problem is to maximize its profit:**

$$\max_p \; pD(p) - C(D(p))$$

- **FOC:**

$$D(p) + pD'(p) - C'(D(p))D'(p) = 0$$

$\Rightarrow$ **Monopoly price** p^m

- **SOC:**

$$2D'(p) - C''(D(p))[D'(p)]^2 + [p - C'(D(p))]D''(p) < 0,$$

which is always satisfied if $D(p)$ is not too convex.

- **Rewriting the FOC:**

$$p\left[1 - \frac{1}{\epsilon(p)}\right] = C'(D(p)),$$

where

$$\epsilon(p) \equiv -\frac{dD(p)}{dp} \cdot \frac{p}{D(p)} > 0$$

is the price elasticity of demand.

  - **Assume $\epsilon(p) > 1$ for all $p \geq 0$.**

Theorem

The monopoly price p^m is higher than the marginal cost (= competitive equilibrium price).

Alternative representation of the monopoly equilibrium:

- $x = D(p) \Rightarrow$ **Inverse demand function:** $p = P(x)$
- **Monopolist's profit maximization problem:**

$$\max_x \; P(x)x - C(x)$$

- **FOC:**

$$P(x) + P'(x)x - C'(x) = 0$$

  - **SOC:** $2P'(x) + P''(x)x - C''(x) < 0$
  - $P(x) + P'(x)x$**: marginal revenue**
- **Example:** $P(x) = A - x$, $C(x) = cx$, $A > c \geq 0$

$$\Rightarrow \quad p^m = \frac{A+c}{2}, \;\; x^m = \frac{A-c}{2}$$

- **Focus on functions that are twice continuously differentiable, i.e., $f \in C^2$.**

Theorem

If at point $(x_1^*, \cdots, x_n^*)$ we have a local maximum or local minimum of the function f, then the conditions

$$\partial f(x_1^*, \cdots, x_n^*)/\partial x_1 = 0$$

$$\vdots$$

$$\partial f(x_1^*, \cdots, x_n^*)/\partial x_n = 0$$

hold simultaneously.

- $f_i(\mathbf{x}^*) = 0$, $i = 1, \cdots, n$: **first-order conditions**
- **Second-order conditions:**

Theorem

Suppose that $y = f(\mathbf{x})$ is a strictly concave (strictly convex) function defined on $\mathbf{x} \in \mathbb{R}^n$. If $f_i(\mathbf{x}^*) = 0$, $i = 1, \cdots, n$, at $\mathbf{x} = \mathbf{x}^*$, then $\mathbf{x}^*$ yields a unique global maximum (minimum).

Theorem

It is sufficient for $\mathbf{x}$ to yield a local maximum (local minimum) of the C^2 function $y = f(\mathbf{x})$ that $f_i(\mathbf{x}^*) = 0$, $i = 1, \cdots, n$, and the second-order total differential, $d^2y(\mathbf{x}^*) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(\mathbf{x}^*) dx_i dx_j$, is negative (positive).

Constrained optimization

Case of two variables and one constraint:

$$\max_{x_1,x_2} f(x_1,x_2) \quad \textbf{subject to} \;\; g(x_1,x_2)=0$$

or

$$\min_{x_1,x_2} f(x_1,x_2) \quad \textbf{subject to} \;\; g(x_1,x_2)=0$$

- **Define the Lagrange function (or Lagrangean):**

$$\mathcal{L}(x_1,x_2,\lambda)=f(x_1,x_2)+\lambda g(x_1,x_2),$$

where λ: Lagrange multiplier.

- **First-order conditions:**

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= f_1(x_1^*, x_2^*) + \lambda^* g_1(x_1^*, x_2^*) = 0\\ \frac{\partial \mathcal{L}}{\partial x_2} &= f_2(x_1^*, x_2^*) + \lambda^* g_2(x_1^*, x_2^*) = 0\\ \frac{\partial \mathcal{L}}{\partial \lambda} &= g(x_1^*, x_2^*) = 0\end{aligned}$$

- **Interpretation of λ:**

Theorem

The value of the Lagrange multiplier λ at the optimal solution always tells us the effect on the optimized value of the function f of a small relaxation of the constraint.

- **Second-order conditions:**

Theorem

If (x_1^*, x_2^*, λ) satisfies the FOCs, then it yields a maximum (minimum) if the determinant of the bordered Hessian $|H^*|$ is positive (negative), where

$$|H^*| = \begin{vmatrix} f_{11} + \lambda^* g_{11} & f_{12} + \lambda^* g_{12} & g_1 \\ f_{21} + \lambda^* g_{21} & f_{22} + \lambda^* g_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix}.$$

Case of n varibles and m constraints ($n > m$):

$$\max_{\mathbf{x}} \; f(\mathbf{x}) \;\; \textbf{or} \;\; \min_{\mathbf{x}} \; f(\mathbf{x}) \quad \textbf{s.t.} \; \begin{cases} g^1(\mathbf{x}) = 0 \\ \vdots \\ g^m(\mathbf{x}) = 0 \end{cases}$$

- **Lagrange function:**

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g^j(\mathbf{x})$$

- **First-order conditions:**

$$\frac{\partial \mathcal{L}}{\partial x_i} = f_i(\mathbf{x}^*) + \sum_{j=1}^{m} \lambda_j^* g_i^j(\mathbf{x}^*) = 0, \quad i = 1, \cdots, n$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = g^j(\mathbf{x}^*) = 0, \quad j = 1, \cdots, m$$

- **A consumer derives his/her utility by consuming two goods x_1 and x_2, represented by the utility function $U(x_1, x_2)$, and is facing a given prices p_1 and p_2 and a constant income I.**
  - $U(x_1, x_2)$: **twice continuously differentiable, with $U_i > 0$ for $x_i \geq 0$, $i = 1, 2$, and concave**
- **The consumer's utility maximization problem:**

$$\max_{x_1, x_2} U(x_1, x_2) \quad \textbf{s.t. } p_1 x_1 + p_2 x_2 = I$$

  - $p_1 x_1 + p_2 x_2 = I$: **budget constraint**

- **Lagrangean:**

$$\mathcal{L}(x_1, x_2, \lambda) = U(x_1, x_2) + \lambda(I - p_1x_1 - p_2x_2)$$

- **FOCs:**

$$\frac{\partial \mathcal{L}}{\partial x_1} = U_1(x_1^*, x_2^*) - \lambda^* p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = U_2(x_1^*, x_2^*) - \lambda^* p_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_1x_1^* - p_2x_2^* = 0$$

- **Determiant of the Hessian matrix of the Lagrangean:**

$$H^* = \begin{vmatrix} U_{11} & U_{12} & -p_1 \\ U_{21} & U_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} = -p_2^2U_{11} + 2p_1p_2U_{12} - p_1^2U_{22} > 0$$

$\Rightarrow$ **SOC is also satisfied.**

- **From the first two FOCs,**

$$\frac{U_1(x_1^*, x_2^*)}{U_2(x_1^*, x_2^*)} = \frac{p_1}{p_2},$$

where the LFS is equal to the marginal rate of substitution.

  - $u = U(x_1, x_2) \Rightarrow du = U_1 dx_1 + U_2 dx_2 = 0 \Rightarrow \dfrac{U_1}{U_2} = -\dfrac{dx_2}{dx_1}\Big|_{du=0}$

Theorem

At the optimal consumption levels $x_1^* > 0$ and $x_2^* > 0$, the consumer's marginal rate of substitution between two goods should be equal to the relative price of the two goods.

- (x_1^*, x_2^*) **is dependent on** (p_1, p_2, I)**:**

$$\begin{cases} \dfrac{U_1(x_1^*, x_2^*)}{U_2(x_1^*, x_2^*)} = \dfrac{p_1}{p_2} \\ p_1 x_1^* + p_2 x_2^* = I \end{cases}$$

$\Rightarrow$ **Demand functions:** $x_i^* = D^i(p_1, p_2, I)$**,** $i = 1, 2$

- **Example:** $U(x_1, x_2) = x_1^\alpha x_2^\beta$**,** $\alpha, \beta \in (0, 1)$

$$\Rightarrow \quad D^1(p_1, p_2, I) = \frac{\alpha I}{(\alpha + \beta) p_1}, \quad D^2(p_1, p_2, I) = \frac{\beta I}{(\alpha + \beta) p_2}$$

  - **The same demand functions can be derived from** $U(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$**.**

- **The derivative of a function $f(x)$ is $f'(x) \Rightarrow$ Conversely, the original function that has the derivative $f'(x)$ is $f(x)$.**
- **However, any function $g(x) = f(x) + C$, where C is an arbitrary constant, also has derivative function $f'(x)$.**
- **The process of finding the original function from the derivative function is called antidifferentiation and the antiderivative of a function is referred to as the indefinite integral.**

Definition

Let $F(x)$ be a function and $f(x)$ be its derivative (i.e., $f(x) = F'(x)$). Then, the indefinite integral of $f(x)$ is given by

$$\int f(x)dx = F(x) + C,$$

where C is called the constant of integration.

Rules of integration

1. **Power rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
2. **Integral of a sum:** $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$
3. **Integral of a constant multiple:** $\int k f(x) dx = k \int f(x) dx$
4. **Exponential rule:** $\int e^x dx = e^x + C$
5. **Logarithmic rule:** $\int \frac{1}{x} dx = \ln x + C, \quad x > 0$
6. **Integration by parts:** $\int g(x) f'(x) dx = f(x) g(x) - \int f(x) g'(x) dx$

- The **definite** or **Riemann** integral of a function $f(x)$ defined on a closed interval $[a, b]$ is the area underneath the curve over that interval.
- Let $x_0 = a$ and $x_n = b$. Then, a **partition** for the interval $[a, b]$ is a set of subintervals $[x_0, x_1]$, $[x_1, x_2]$,..., $[x_{n-1}, x_n]$ that satisfy the condition that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n < b$.

Definition

Let $\omega_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$ be an arbitrary set of points from the set of subintervals. Then,

$$S = \sum_{i=1}^{n} f(\omega_i)(x_i - x_{i-1})$$

is called a **Riemann sum** for the function $f(x)$ over the subinterval $[a, b]$.

Definition

If for every $\epsilon > 0$, there is some value $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(\omega_i)(x_i - x_{i-1}) - L \right| < \epsilon$$

for any partition on the closed interval $[a, b]$ such that $\max(x_i - x_{i-1}) < \delta$ and for any selection of points $\omega_i \in [x_{i-1}, x_i]$, $f(x)$ is said to be integrable on $[a, b]$. The value L is called the definite integral of $f(x)$ over the interval [a,b] and written as:

$$\int_a^b f(x)dx = L.$$

Theorem (fundamental theorem of integral calculus)

If the function $f(x)$ is continuous on the closed interval $[a.b]$ and if $F(x)$ is any indefinite integral of $f(x)$. Then, $f(x)$ is integrable on $[a, b]$ and it holds that

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(b)$ ($F(a)$) is the indefinite integral of $f(x)$ evaluated at $x = b$ ($x = a$).

Properties of integrals

1. **If a, b, and c are points in $\mathbb{R}$ such that $a < b < c$, then**

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

2. $\int_a^a f(x) \equiv \lim_{c \to a} \int_a^c f(x)dx = 0$

3. **Reversing the direction of integration changes the sign of the integral;**

$$\int_c^a f(x)dx = -\int_a^c f(x)dx$$

4. **If a function $f(x)$ is negatively valued on the interval $[a, c]$, $a < c$, then $\int_a^c f(x)dx < 0$ where $\left|\int_a^c f(x)dx\right|$ is the area of the region between the curve $f(x)$ and the x–axis between the points a and c.**

References

- **Hoy, M., J. Livernois, C. McKenna, R. Rees, and T. Stengos (2001),** *Mathematics for Economics*, **2nd edition, MIT Press.**
- **Dixit, A.K. and V. Norman (1980),** *Theory of International Trade*, **Cambridge University Press.**
- **Mas-Colell, A., M.D. Whinston, and J.R. Green (1995),** *Microeconomic Theory*, **Oxford University Press.**