# International Economics B

# 2. Basics in noncooperative game theory

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# What is "game theory"?

- **Consider situations in which more than one person ("agents") interact with each other.**
  - **One agent's decision affects other agents' well-being ("payoff").**
- **In the presence of such strategic interactions, study**
  - **How does each agent make a decision?**
  - **What outcome is produced as a consequence of the agents' behavior?**

# Basic concepts in noncooperative game theory

- **Noncooperative game theory: Players (=agents who participate in a game) make decisions independently.**
  - **cf. Cooperative game theory: Assuming the possibility of external enforcement of cooperative behavior (e.g., through contract law)**
- **Representations of games**
  - **Normal-form (or Strategic-form) game: Useful to describe games in which players simultaneously make decisions.**
  - **Extensive-form game: Useful to describe games in which sequentially make decisions.**

**Assumptions:**

- **All players have the accurate knowledge about the structure, rules, and payoffs of the game.**
- **Perfect information: Each player has all the information concerning the actions taken by other players earlier in the game that affect the player's decision about which action to choose at a particular time.**
    - **cf. Imperfect information**

# Formal definition of a normal-form game

## Definition

**A normal form game is described by:**

1. **A set of players: $I \equiv \{1, 2, \ldots, N\}$.**
2. **An action set of each player: $a^i \in A^i$, $A^i = \{a_1^i, a_2^i, \ldots, a_{k_i}^i\}$, which is the set of all actions available to player $i$;**
   **An outcome of the game: $a \equiv (a^1, a^2, \ldots, a^i, \ldots, a^N)$, which is a list of the actions chosen by each player.**
3. **A payoff function of each player: $\pi^i(a) = \pi^i(a^1, \ldots, a^N)$.**

# Examples of normal-form games

## Example 1: Peace-War game

| | | Country 2 | |
|---|---|---|---|
| | | WAR | PEACE |
| Country 1 | WAR | 1    1 | 3    0 |
| | PEACE | 0    3 | 2    2 |

**In this game,**

- **Set of players: {Country 1, Country 2}**
- **Each player's action set:** $A^1 = A^2 = \{\text{WAR}, \text{PEACE}\}$
- **Possible outcomes of the game: (W, W), (W, P), (P, W), (P, P)**
- **Payoff: when** $a = (\text{W}, \text{P})$, $\pi^1(a) = 3$ **and** $\pi^2(a) = 0$

## Solving the game

- **An equilibrium of a game: How will the game end up from the all possible outcomes?**
- **The most commonly used solution concept is the Nash equilibrium.**
  - **Nash, J. (1951), "Non-Cooperative Games,"** *Annals of Mathematics* **54(2), 286-295.**
- **A set of strategies is a Nash equilibrium if no player can do better by unilaterally changing his/her strategy.**

**Denote the list of actions by players except player $i$ by $a^{-i}$, i.e.,**

$$a^{-i} \equiv (a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^N).$$

**$\Rightarrow$ An outcome $a$ can be expressed as the list of player $i$'s action and actions of players other than player $i$: $a = (a^i, a^{-i})$.**

## Formal definition of a Nash equilibrium

### Definition

**An outcome $\hat{a} = (\hat{a}^1, \hat{a}^2, \ldots, \hat{a}^N) \in A^1 \times A^2 \times \cdots \times A^N$ is a Nash equilibrium if no player has an incentive to deviate from $\hat{a}^i$ provided that all other players do not deviate from $\hat{a}^{-i}$.**

**Formally, for every player $i = 1, 2, \ldots, N$,**

$$\pi^i(\hat{a}^i, \hat{a}^{-i}) \geq \pi^i(a^i, \hat{a}^{-i}) \quad \forall a^i \in A^i.$$

# Best-response function and Nash equilibrium

## Definition

**The best-response function of player $i$ is the function $R^i(a^{-i})$ that assigns, for given actions $a^{-i}$ of other players, an action $a^i = R^i(a^{-i})$ that maximizes player $i$'s payoff $\pi^i(a^i, a^{-i})$.**

## Theorem

**If $\hat{a} = (\hat{a}^1, \ldots, \hat{a}^N)$ is a Nash equilibrium outcome, then $\hat{a}^i = R^i(\hat{a}^{-i})$ holds for every player $i$.**

**Example 1 (Peace-War game):**

| | | Country 2 WAR | Country 2 PEACE |
|---|---|---|---|
| Country 1 | WAR | 1, 1 | 3, 0 |
| | PEACE | 0, 3 | 2, 2 |

- **Country 1's best-response function:**

$$R^1(a^2) = \begin{cases} \text{WAR} & \text{if } a^2 = \text{WAR} \\ \text{WAR} & \text{if } a^2 = \text{PEACE} \end{cases}$$

- **Country 2's best-response function:**

$$R^2(a^1) = \begin{cases} \text{WAR} & \text{if } a^1 = \text{WAR} \\ \text{WAR} & \text{if } a^1 = \text{PEACE} \end{cases}$$

**(WAR, WAR) is a (unique) Nash equilibrium.**

**The procedure for finding a Nash equilibrium:**

1. **Calculate the best-response function of each player.**
2. **Find outcomes that lie on the best-response functions of all players.**

**Not all games have a unique Nash equilibrium.**

- **Multiple Nash equilibria**
- **Nonexistence of a Nash equilibrium**

# Multiple Nash equilibria

## Example 2: Battle of the sexes

| | | Rachel | |
| --- | --- | --- | --- |
| | | **OPERA ($\omega$)** | **FOOTBALL ($\phi$)** |
| **Jacob** | **OPERA ($\omega$)** | **2 1** | **0 0** |
| | **FOOTBALL ($\phi$)** | **0 0** | **1 2** |

- **Both of them gain a higher utility if they go together to one of these events: "coordination game"**

**There are two Nash equilibria: $(\mathbf{OPERA}, \mathbf{OPERA})$ and $(\mathbf{FOOTBALL}, \mathbf{FOOTBALL})$.**

# Nonexistence of a Nash equilibrium

## Example 3: Battle of the sexes after 30 years of marriage

| | | Rachel | | | |
|---|---|---|---|---|---|
| | | OPERA ($\omega$) | | FOOTBALL ($\phi$) | |
| Jacob | OPERA ($\omega$) | 2 | 0 | 0 | 2 |
| | FOOTBALL ($\phi$) | 0 | 1 | 1 | 0 |

- J wants to be with R, but R wants to be alone.

**There is no (pure strategy) Nash equilibrium.**

- A Nash equilibrium in mixed strategy exists.
  - Mixed strategy: an assignment of a probability to each pure strategy (i.e., players randomly chooses each pure strategy).

## Extensive-form games

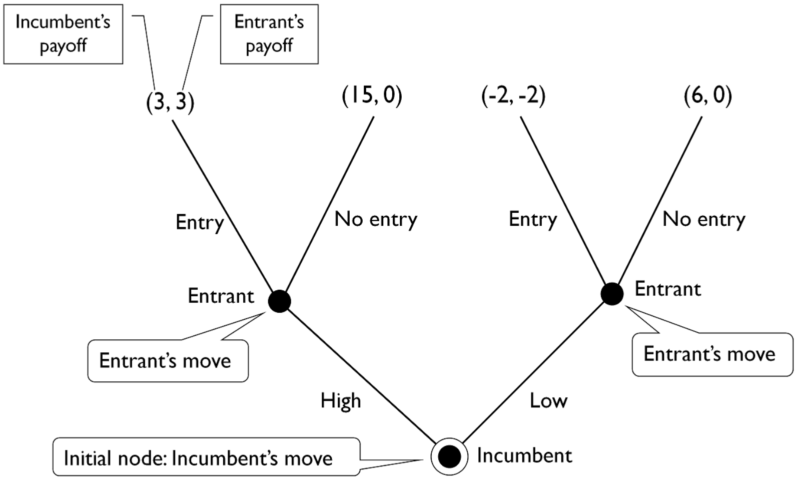
- **Games with dynamic interactions: represented by extensive forms (game trees).**
- **The extensive form of a game is a complete description of:**
  1. **The set of players;**
  2. **Who moves when and what their choices are;**
  3. **What players know when they move;**
  4. **The players' payoffs as a function of the choices that are made.**

## Definition

- An extensive form game consists of:
  1. A game tree containing
     - a starting node, other decision nodes, terminal nodes, and
     - branches linking each decision node to successor nodes.
  2. A list of players $i = 1, 2, \ldots, N$.
  3. For each decision node, the name of the player entitled to choose an action.
  4. Each player i's action set at each decision node.
  5. Each player i's payoff at each terminal node.

## Example 4: Entry deterrence

- **Players: an incumbent firm & a new entrant**
- **The order of play:**
  1. **Incumbent determines the price of its product: "High" or "Low".**
  2. **Entrant decides whether "Entry" or "No entry".**
- **Each player's payoff:**
  - **"High" & "Entry" $\rightarrow$ Both firms earn 3 million\$.**
  - **"High" & "No entry" $\rightarrow$ Incumbent = 15 million\$, Entrant = 0.**
  - **"Low" & "Entry" $\rightarrow$ Both firms lose money (-2 million\$).**
  - **"Low" & "No entry" $\rightarrow$ Incumbent = 6 million\$, Entrant = 0.**


Incumbent's payoff
Entrant's payoff
(3, 3)
(15, 0)
(-2, -2)
(6, 0)
Entry
No entry
Entry
No entry
Entrant
Entrant
Entrant's move
Entrant's move
High
Low
Initial node: Incumbent's move
Incumbent

# Strategies and outcomes in extensive form games

**Definition**

A **strategy** for player $i$, $s^i$, is a complete plan (list) of actions, one action for each decision node that the player is entitled to choose an action.

- Not what the player does at a single specific node but is a list of what the player does at every node where the player is entitled to choose an action.

**In Example 4 (Entry deterrence),**

- **Incumbent: One decision node (= initial node)**
  **$\Rightarrow$ Strategy: H or L**
- **Entrant: Two decision nodes (left and right)**
  **$\Rightarrow$ Specification of the precise action taking at each node.**
  - **E at both nodes $\rightarrow$ Strategy: (E, E)**
  - **E at left & NE at right $\rightarrow$ (E, NE)**
  - **NE at left & E at right $\rightarrow$ (NE, E)**
  - **NE at both nodes $\rightarrow$ (NE, NE)**
- **Possible outcomes:**
  **(H, (E, E)), (H, (E, NE)), (H, (NE, E)), (H, (NE, NE)), (L, (E, E)), (L, (E, NE)), (L, (NE, E)), (L, (NE, NE))**

# Solving the game

**Solution concept: Subgame perfect (Nash) equilibrium**

- **A refinement of the Nash equilibrium concept proposed by Selten (1965).**
  - **Selten, R. (1965), "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetragheit,"** *Zeitschrift für die gesamte Staatrwissenschaft* **121, 301–24 and 667–89.**
- **An equilibrium such that players' strategies constitute a Nash equilibrium in every subgame of the original game.**
  - **Subgame: A decision node from the original game along with the decision nodes and terminal nodes directly following this node.**
- **Subgame perfect equilibria eliminate noncredible threats.**

- **Rewrite the game in a normal form:**

| | | Entrant | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | (E,E) | | (E,NE) | | (NE,E) | | (NE,NE) | |
| Incumbent | H | 3 | 3 | 3 | 3 | 15 | 0 | 15 | 0 |
| | L | -2 | -2 | 6 | 0 | -2 | -2 | 6 | 0 |

- **Two Nash equilibria: (H,(E,E)) and (L,(E,NE))**
- **However, (H,(E,E)) includes noncredible threats.**
  - **(E,E): The entrant chooses "E" regardless of the incumbent's strategy.**
  - **Once the incumbent chooses "L", the entrant will prefer "NE" to "E".**

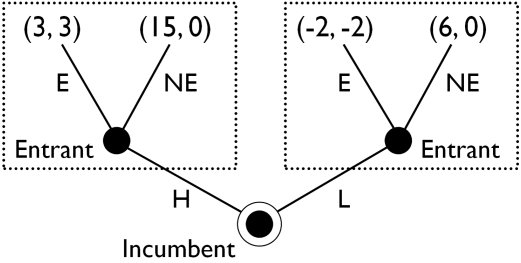
# Subgame

## Definition

- **A subgame is a decision node from the original game along with the decision nodes directly following this node.**
- **A subgame is called a proper subgame if it differs from the original game.**

**Example 4 (Entry deterrence game) has three subgames:**

- **A subgame after the incumbent chooses H**
- **A subgame after the incumbent chooses L**
- **The original game**



# Formal definition of the subgame perfect equilibrium

**Definition**

**An outcome is a subgame perfect equilibrium if it induces a Nash equilibrium in every subgame of the original game.**

- **A subgame perfect equilibrium outcome is also a Nash equilibrium of the original game.**

## Backward induction and the SPE

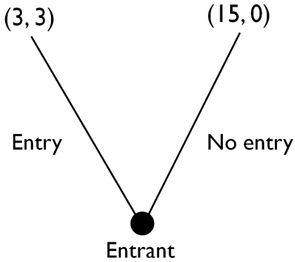
**Method for finding the SPE outcome: Backward induction**

1. **Find the NE of in the subgames leading to the terminal nodes (i.e., optimal strategy of the player who makes the last move of the game).**
2. **Find the NE for the subgames leading to the subgames leading to the terminal nodes (i.e., optimal choice of the next-to-last moving player), taking as given the NE actions played in the last subgames.**
3. **Continuing to solve in this way backwards in time until all players' actions have been determined.**

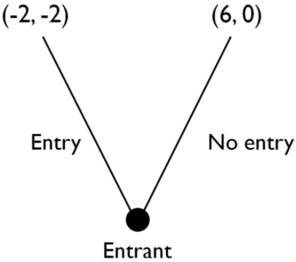
**Find the subgame perfect equilibrium in Example 4.**

1. **2nd stage (entrant's move): Two subgames**
    - **A subgame after the incumbent chooses H**
    - **A subgame after the incumbent chooses L**

   **$\Rightarrow$ In each subgame, the entrant chooses its optimal action: E or NE.**
2. **1st stage (incumbent's move): Taking the entrant's optimal choice in the 2nd stage, the incumbent chooses H or L.**

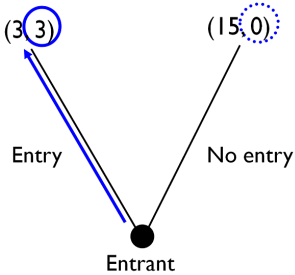



A subgame after "High"

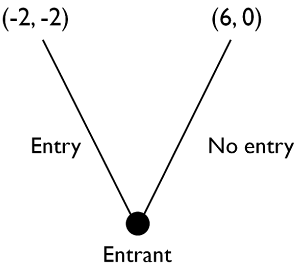



A subgame after "Low"

$\Rightarrow$ **Entrant's optimal strategy:**

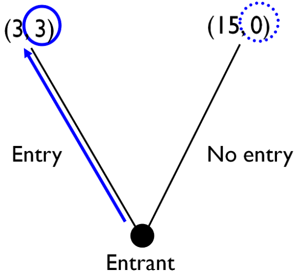



A subgame after "High"

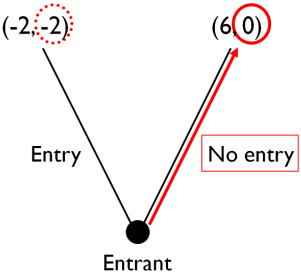



A subgame after "Low"

$\Rightarrow$ **Entrant's optimal strategy:**

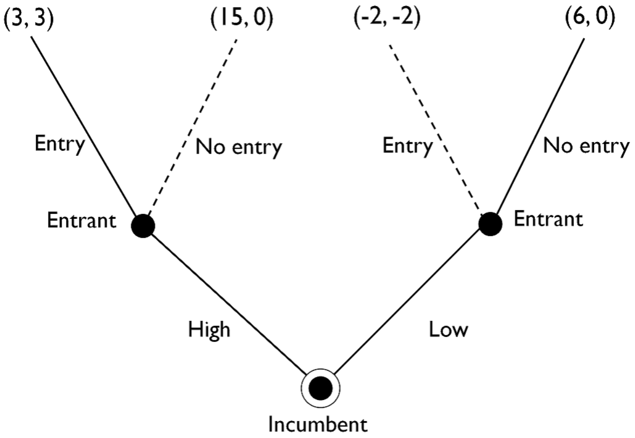



A subgame after "High"

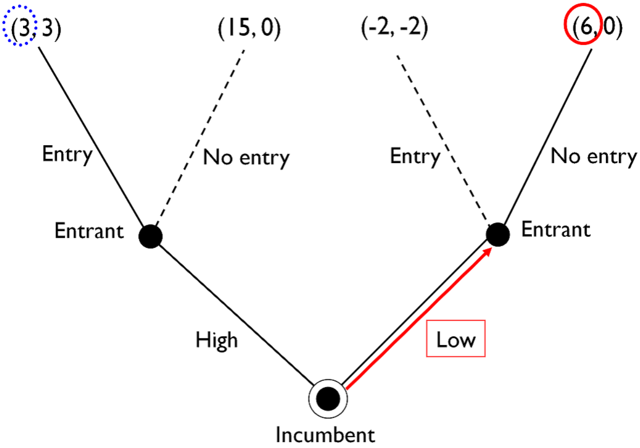



A subgame after "Low"

$\Rightarrow$ **Entrant's optimal strategy: (E, NE)**




$\Rightarrow$ **Incumbent's optimal strategy:**




⇒ **Incumbent's optimal strategy: L**

**In Example 4 (Entry-deterrence game),**

- **Subgame perfect equilibrium: $(\mathbf{L}, (\mathbf{E}, \mathbf{NE}))$**
- **Outcome of the game: The incumbent chooses "Low", and then the entrant chooses "No entry".**

# References


- **Mas-Colell, A., M.D. Whinston, and J.R. Green (1995),** *Microeconomic Theory*, **Oxford University Press.**
- **Shy, O. (1996),** *Industrial Organization: Theory and Applications*, **The MIT Press.**
- **Gibbons, R. (1992),** *Game Theory for Applied Economists*, **Princeton University Press.**