CHAPTER 3 VARIATIONAL PRINCIPLES

In this chapter we show that the motions of a newtonian potential system are extremals of a variational principle, "Hamilton's principle of least action."

This fact has many important consequences, including a quick method for writing equations of motion in curvilinear coordinate systems, and a series of qualitative deductions-for example, a theorem on returning to a neighborhood of the initial point.

In this chapter we will use an n-dimensional coordinate space. A vector in such a space is a set of numbers $x=(x_1,\dots,x_n)$. Similarly, $\partial \boldsymbol{f}/\partial \boldsymbol{x}$ means $(\partial f/\partial x_1,\dots,\partial f/\partial x_n)$, and $(\boldsymbol{a},\boldsymbol{b})=a_1b_1+\cdots+a_nb_n$.

12 Calculus of variations

For what follows, we will need some facts from the calculus of variations. A more detailed exposition can be found in "A Course in the Calculus of Variations" by M. A. Lavrentiev and L. A. Lusternik, M. L., 1938, or G. E. Shilov, "Elementary Functional Analysis," MIT Press, 1974.

The calculus of variations is concerned with the extremals of functions whose domain is an infinite-dimensional space: the space of curves. Such functions are called functionals.

An example of a functional is the length of a curve in the euclidean plane: if $\gamma=\{(t,x):x(t)=x,t_0\le t\le t_1\}$, then $\Phi(\gamma)=\int_{t_0}^{t_1}\sqrt{1+\dot{x}^2}\,dt$.

In general, a functional is any mapping from the space of curves to the real numbers.

We consider an "approximation" γ' to γ , $\gamma'=\{(t,x):x=x(t)+h(t)\}$. We will call it $\gamma'=\gamma+h$. Consider the increment of Φ , $\Phi(\gamma+h)-\Phi(\gamma)$ (Figure 41).

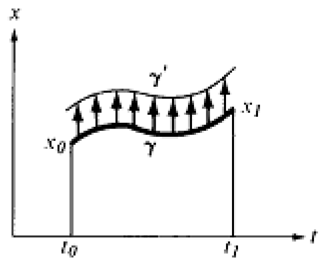

Figure 41 Variation of a curve

A *Variations*

Definition. A functional Φ is called differentiable if $\Phi(\gamma+h)-\Phi(\gamma)=F+R$, where F depends linearly on h (i.e., for a fixed γ , $F(h_1+h_2)=F(h_1)+F(h_2)$ and $F(ch)=cF(h)$, and $R(h,\gamma)=O(h)$, in the sense that, for $|h|<\varepsilon$ and $|dh/dt|<\varepsilon$, we have $|R|<C\varepsilon^2$. The linear part of the increment, F(h), is called the *differential*.

It can be shown that if Φ is differentiable, its differential is *uniquely* defined. The differential of a functional is also called its *variation*, and h is called a *variation of the curve*.

EXAMPLE. Let $\gamma=\{(t,x):x=x(t),t_0\le t\le t_1\}$ be a curve in the (t, x)-plane; $\dot{x}=dx/dt$; $L=L(a,b,c)$ a differentiable function of three variables. We define a functional Φ by

$$\Phi(\gamma)=\int_{t_0}^{t_1} L(x(t),\dot{x}(t),t)dt\ .$$

In case $L=\sqrt{1+b^2}$, we get the length of γ .

Theorem. The functional $\Phi(\gamma)=\int_{t_0}^{t_1} L(x(t),\dot{x}(t),t)dt$ is differentiable, and its derivative is given by the formula

$$F(h)=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}h-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]hdt+\left(\frac{\partial L}{\partial \dot{x}}h\right)\Bigg|_{t_0}^{t_1}\ .$$

PROOF.

$$\begin{aligned}\Phi(\gamma+h)-\Phi(\gamma)&=\int_{t_0}^{t_1}\left[L(x+h,\dot{x}+\dot{h},t)-L(x,\dot{x},t)\right]dt\\&=\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right]dt+O(h^2)\\&=F(h)+R\end{aligned}$$

where

$$F(h)=\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}h+\frac{\partial L}{\partial \dot{x}}\dot{h}\right)dt\quad\text{and}\quad R=O(h^2)$$

Integrating by parts, we find that

$$\int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{x}}\dot{h}dt=-\int_{t_0}^{t_1}h\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)dt+\left(h\frac{\partial L}{\partial \dot{x}}\right)\Bigg|_{t_0}^{t_1}\ .$$

B *Extremals*

Definition. An *extremal* of a differentiable functional $\Phi(\gamma)$ is a curve γ such that *F(h)* = 0 for all *h*. (In exactly the same way that γ is a stationary point of a function if the differential is equal to zero at that point.)

Theorem. *The curve* $\gamma : x = x(t)$ *is an extremal of the functional* $\Phi(\gamma)=\int_{t_0}^{t_1} L(x,\dot{x},t)dt$ *on the space of curves passing through the points* $x(t_0)=x_0$ *and* $x(t_1)=x_1$*, if and only if*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0\quad \textit{along the curve x(t).}$$

Lemma. */f a continuous function f(t),* $t_0\le t\le t_1$ *satisfies* $\int_{t_0}^{t_1} f(t)h(t)dt=0$ *for any continuous function* $h(t)$ *with* $h(t_0)=h(t_1)=0$*, then* $f(t)=0$.

PROOF OF THE LEMMA. Let $f(t^*)>0$ for some *t*, t0 < t* < t 1*• Since *f* is continuous, *f(t) >*

c in some neighborhood Δ of the point $t^*: t_0 < t^* - d < t < t^* + d < t_1$. Let *h(t)* be such that *h(t)* = 0 outside Δ, *h(t)* > 0 in Δ, and *h(t)* = 1 in Δ/2 (i.e., for *t* s.t. *t** - 1/2*d* < *t* < *t*+1/2d)*. Then, clearly, $\int_{t_0}^{t_1} f(t)h(t) \geq dc > 0$ (Figure 42). This contradiction shows that *f(t*)* = 0 for all *t*, t0 < t* < t 1*•

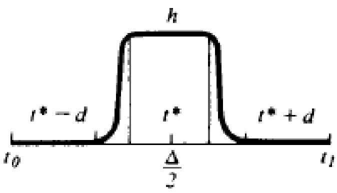

Figure 42 Construction of the function *h*

PROOF OF THE THEOREM. By the preceding theorem,

$$F(h) = -\int \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \right] h dt + \left(\frac{\partial L}{\partial \dot{x}} h\right)\Bigg|_{t_0}^{t_1}.$$

The term after the integral is equal to zero since *h(t0) = h(t0)* = 0. If γ is an extremal, then *F(h)* = 0 for all *h* with *h(t0) = h(t 1)* = 0. Therefore,

$$\int_{t_0}^{t_1} f(t)h(t)dt = 0$$

where

$$f(t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x},$$

for all such *h*. By the lemma, *f(t)*=0. Conversely, if *f(t)*=0, then clearly *F(h)*=0.

EXAMPLE. We verify that the extremals of length are straight lines. We have:

$$L = \sqrt{1+\dot{x}^2}\ ,\ \ \frac{\partial L}{\partial x} = 0\ ,\ \ \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\ ,\ \ \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = 0\ ,$$

$$\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = c\ ,\ \ \dot{x} = c_1\ ,\ \ x = c_1 t + c_2\ .$$

C *The Euler-Lagrange equation*

Definition. The equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

is called the *Euler-Lagrange equation* for the functional

$$\Phi = \int_{t_0}^{t_1} L(x, \dot{x}, t)dt\ .$$

Now let **x** be a vector in the n-dimensional coordinate space R^n,
$\gamma = \{(t, \boldsymbol{x}) : \boldsymbol{x} = \boldsymbol{x}(t), t_0 \leq t \leq t_1\}$ a curve in the (n+1)-dimensional space $R \times R^n$, and
$L : R^n \times R^n \times R \to R$ a function of *2n* + 1 variables. As before, we show:

Theorem. *The curve* γ *is an extremal of the functional* $\Phi(\gamma) = \int_{t_0}^{t_1} L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)dt$ *on the space of*

curves joining (t0 , x0) *and (t* 1, x1), *if and only if the Euler-Lagrange equation is satisfied along* γ .

This is a system of *n second-order* equations, and the solution depends on *2n* arbitrary constants. The *2n* conditions x(t0) = x0 , x(t 1) = x 1 are used for finding them.

PROBLEM. Cite examples where there are many extremals connecting two given points, and others where there are none at all.

D *An important remark*

The condition for a curve *y* to be an extremal of a functional does not depend on the choice of coordinate system.

For example, the same functional - length of a curve - is given in cartesian and polar coordinates by the different formulas

$$\Phi_{cart} = \int_{t_0}^{t_1} \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\, dt\ , \quad \Phi_{pol} = \int_{t_0}^{t_1} \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}\, dt\ .$$

The extremals are the same - straight lines in the plane. The equations of lines in cartesian and polar coordinates are given by different functions: $x_1 = x_1(t), x_2 = x_2(t)$, and $r = f(t), \varphi = \varphi(t)$.

However, both these vector functions satisfy the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

only in the first case, when $x_{cart} = x_1, x_2$ and $L_{art} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$, and in the second case when $x_{pol} = r, \varphi$ and $L_{pol} = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}$.

In this way we can easily describe in any coordinates a differential equation for the family of all straight lines.

PROBLEM. Find the differential equation for the family of all straight lines in the plane in polar coordinates.

13 Lagrange's equations

Here we indicate the variational principle whose extremals are solutions of Newton's equations of motion in a potential system.

We compare Newton's equations of dynamics

(1) $$\frac{d}{dt}(m_i\dot{r}_i) + \frac{\partial U}{\partial r} = 0$$

with the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0\ .$$

A ***Hamilton's principle of least action***

Theorem. *Motions of the mechanical system* (1) *coincide with extremals of the functional*

$$\Phi(\gamma) = \int_{t_0}^{t_1} L dt \,,$$

where $L = T - U$ *is the difference between the kinetic and potential energy.*

PROOF. Since $U = U(\boldsymbol{r})$ and $T = \Sigma m_i \dot{r}_i / 2$, we have $\partial L / \partial \dot{r}_i = \partial T / \partial \dot{r}_i = m_i \dot{r}_i$ and $\partial L / \partial r_i = -\partial U / \partial r_i$.

Corollary. *Let* $(q_1, \ldots, q_{3n})$ *be any coordinates in the configuration space of a system of n mass points. Then the evolution of* **q** *with time is subject to the Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = 0 \,,$$

where $L = T - U$.

PROOF. By the theorem above, a motion is an extremal of the functional $\int L dt$. Therefore, in any system of coordinates the Euler-Lagrange equation written in that coordinate system is satisfied.

Definition. In mechanics we use the following terminology: $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T - U$ is the *Lagrange function* or *lagrangian,* qLi are the *generalized coordinates,* $\dot{q}_i$ are *generalized velocities,* $\partial L / \partial \dot{q}_i = p_i$ are *generalized momenta,* $\partial L / \partial q_i$ are *generalized forces,* $\int_{t_0}^{t_1} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) dt$ is the *action,* $(d(\partial L / \partial \dot{q}_i) / dt - (\partial L / dq_i) = 0$ are *Lagrange's equations.*

The last theorem is called "Hamilton's form of the principle of least motion" because in many cases the action q(t) is not only an extremal but is also a *minimum* value of the action functional $\int_{t_0}^{t_1} L dt$.

B ***The simplest examples***

EXAMPLE 1. For a free mass point in E^3,

$$L = T = \frac{m \dot{r}^2}{2} \,;$$

in cartesian coordinates $q_i = r_i$ we find

$$L = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \,.$$

Here the generalized velocities are the components of the velocity vector, che generalized momenta $p_i m \dot{q}_i$ are the components of the momentum vector, and Lagrange's equations coincide with Newton's equations $d\boldsymbol{p} / dt = 0$. The extremals are straight lines. It follows from Hamilton's principle that straight lines are not only shortest (i.e., extremals of the length

$\int_{t_0}^{t_1}\sqrt{\dot q_1^2+\dot q_2^2+\dot q_3^2}\,dt$ but also extremals of the action $\int_{t_0}^{t_1}(\dot q_1^2+\dot q_2^2+\dot q_3^2)dt$.

PROBLEM. Show that this extremum is a minimum.

EXAMPLE 2. We consider planar motion in a central field in polar coordinates $q_1 = r$, $q_2 = \varphi$. From the relation $\dot{\boldsymbol r} = \dot r \boldsymbol{e_r} + \dot\varphi r \boldsymbol{e}_\varphi$ we find the kinetic energy $T = 1/2\cdot m\dot{\boldsymbol r}^2 = 1/2\cdot m(\dot r^2 + r^2\dot\varphi^2)$ and the lagrangian $L(\boldsymbol{q},\dot{\boldsymbol q}) = T(\boldsymbol q,\dot{\boldsymbol q}) - U(\boldsymbol q)$, where $U = U(q_1)$.

The generalized momenta will be $\boldsymbol p = \partial L/\partial\dot{\boldsymbol q}$, i.e.,

$$p_1 = m\dot r,\quad p_2 = mr^2\dot\varphi.$$

The first Lagrange equation $\dot p_1 = \partial L/\partial q_1$ takes the form

$$m\ddot r = mr\dot\varphi^2 - \frac{\partial U}{\partial r}.$$

We already obtained this equation in Section 8.

Since $q_2 = \varphi$ does not enter into *L,* we have $\partial L/\partial q_2 = 0$. Therefore, the second Lagrange equation will be $\dot p_2 = 0$, $p_2 = const$. This is the law of conservation of angular momentum.

In general, when the field is not central ($U = U(r,\varphi)$), we find $\dot p_2 = -\partial U/\partial\varphi$.

This equation can be rewritten in the form $d(\boldsymbol M, e_z)/dt = N$, where $N = ([\boldsymbol r,\boldsymbol F], \boldsymbol e_z)$ and $\boldsymbol F = -\partial U/\partial\boldsymbol r$. (The rate of change in angular momentum relative to the *z* axis is equal to the moment of the force **F** relative to the *z* axis.)

In fact, we have

$$dU = (\partial U/\partial\boldsymbol r)d\boldsymbol r + (\partial U/\partial\varphi)d\varphi = -(\boldsymbol F, d\boldsymbol r) = -(\boldsymbol F,\boldsymbol e_r)d\boldsymbol r - r(\boldsymbol F,\boldsymbol e_\varphi)d\varphi;$$

therefore,

$$-\partial U/\partial\varphi = r(F,e_\varphi) = r([e_r,F],e_z) = ([r,F],e_z).$$

This example suggests the following generalization of the law of conservation of angular momentum.

Definition. A coordinate q_i is called *cyclic* if it does not enter into the lagrangian: $\partial L/\partial q_i = 0$.

Theorem. *The generalized momentum corresponding to a cyclic coordinate is conserved:* $p_i = const$.

PROOF. By Lagrange's equation $dp_i/dt = \partial L/\partial q_i = 0$.

14 **Legendre transformations**

The Legendre transformation is a very useful mathematical tool: it transforms functions on a vector space to functions on the dual space. Legendre transformations are related to projective duality and tangential coordinates in algebraic geometry and the construction of dual Banach spaces in analysis. They are often encountered in physics (for example, in the definition of thermodynamic quantities).

A ***Definition***

Let *y = f(x)* be a convex function, *f''(x)* > 0. The *Legendre transformation* of the function *f* is a new

function g of a new variable p, which is constructed in the following way (Figure 43). We draw the graph of f in the x, y plane. Let p be a given number. Consider the straight line $y = px$. We take the point $x = x\{p)$ at which the curve is farthest from the straight line in the vertical direction: for each p the function $px - f(x) = F\{p, x)$ has a maximum with respect to x at the point $x\{p)$. Now we define $g(p) = F\{p, x\{p))$. The point $x(p)$ is defined by the extremal condition $\partial F / \partial x = 0$, i.e., $f'(x) = p$. Since f is convex, the point $x(p)$ is unique.

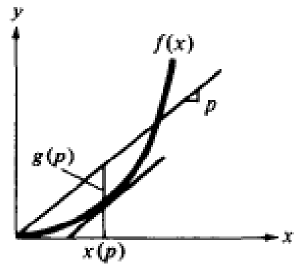

Figure 43 Legendre transformation

PROBLEM. Show that the domain of g can be a point, a closed interval, or a ray *if* f is defined on the whole x axis. Prove that if f is defined on a closed interval, then g is defined on the whole p axis.

B *Examples*

EXAMPLE 1. Let $f(x) = x^2$. Then $F(p,x) = px - x^2$, $x(p) = (1/2)p$, $g(p) = (1/4)p^2$.

EXAMPLE 2. Let $f(x) = mx^2/2$. Then $g(p) = p^2/2m$.

EXAMPLE 3. Let $f(x) = x^\alpha/\alpha$. Then $g(p) = p^\beta/\beta$, where $(1/\alpha) + (1/\beta) = 1$, $(\alpha > 1, \beta > 1)$.

EXAMPLE 4. *Let f(x)* be a convex polygon. Then *g{p)* is also a convex polygon, in which the vertices of *f(x)* correspond to the edges of *g{p)*, and the edges of *f(x)* to the vertices of *g(p)*. For example, the corner depicted in Figure 44 is transformed to a segment under the Legendre transformation.

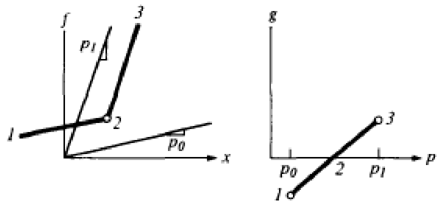

Figure 44 Legendre transformation taking an angle to a line segment

C *Involutivity*

Let us consider a function f which is differentiable as many times as necessary, with $f''(x) > 0$. It is easy to verify that a Legendre transformation takes convex functions to convex functions. Therefore, we can apply it twice.

Theorem. *The Legendre transformation* is *involutive, i.e., its square is the identity: if under the Legendre transformation f* is *taken to g, then the Legend re transform of g will again be f.*

PROOF. In order to apply the Legendre transform to g, with variable p, we must by definition look at a new independent variable (which we will call x), construct the function

$$G(x, p) = xp - g(p),$$

and find the point p(x) at which G attains its maximum: $\partial G / \partial p = 0$, i.e., $g'(p) = x$. Then the Legendre transform of $g(p)$ will be the function of x equal to G(x, p(x)). We will show that G(x, p(x)) = $f(x)$. To this end we notice that $G(x, p) = xp - g(p)$ has a simple geometric interpretation: it is the ordinate of the point with abscissa x on the line tangent to the graph of $f(x)$ with slope p (Figure 45). For fixed p, the function $G(x, p)$ is a linear function of x, with $\partial G / \partial x = p$, and for x

= x(p) we have G(x, *p) = xp - g(p) = f(x)* by the definition of *g(p).*Let us now fix *x = x0* and vary *p.* Then the values of G(x, *p)* will be the ordinates of the points of intersection of the line *x = x0* with the line tangent to the graph of *f(x)* with various slopes *p.* By the convexity of the graph it follows that all these tangents lie below the curve, and therefore the maximum of G(x, *p)* for a fixed x(p0) is equal to *f(x)* (and is achieved for *p* = p(x0) *=f'(xo)).*

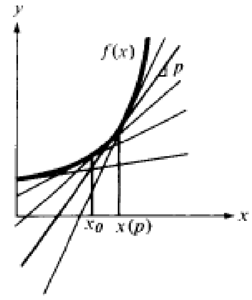

Figure 45 Involutivity of the Legendre transformation

Corollary. *Consider a given family of straight lines y = px - g(p). Then its envelope has the equation y = f(x), where f* is *the Legendre transform of g.*

D *Young's inequality*

Definition. Two functions, *f* and *g,* which are the Legendre transforms of one another are called *dual in the sense of Young.*

By definition of the Legendre transform, *F(x, p) = px - f(x)* is less than or equal to *g(p)* for any *x* and *p.* From this we have *Young's inequality:*

$$px \le f(x) + g(p).$$

EXAMPLE 1. If $f(x) = (1/2)x^2$, then $g(p) = (1/2)p^2$ and we obtain the well-known inequality $px \le (1/2)x^2 + (1/2)p^2$ for all *x* and *p.*

EXAMPLE 2. If $f(x) = x^\alpha / \alpha$, then $g(p) = p^\beta / \beta$, where $(1/\alpha) + (1/\beta) = 1$, and we obtain *Young's inequality* $px \le (x^\alpha / \alpha) + (p^\beta / \beta)$ *for all x>0, p>0,* $\alpha > 1$, $\beta > 1$, *and* $(1/\alpha) + (1/\beta) = 1$.

E *The case of many variables*

Now let f(x) be a convex function of the vector variable $\boldsymbol{x} = (x_{1,} \ldots, x_n)$ (i.e., the quadratic form $((\partial^2 f / \partial \boldsymbol{x}^2) d\boldsymbol{x}, d\boldsymbol{x})$ is positive definite). Then the Legendre transform is the function g(**p**) of the vector variable $\boldsymbol{p} = (p_1, \ldots, p_n)$, defined as above by the equalities $g(\boldsymbol{p}) = F(\boldsymbol{p}, \boldsymbol{x(p)}) = \max_{\boldsymbol{x}} F(\boldsymbol{p}, \boldsymbol{x})$, where F(**p**, x) = **(p**, x) - f(**x**) and $\boldsymbol{p} = \partial f / \partial \boldsymbol{x}$.

All of the above arguments, including Young's inequality, can be carried over without change to this case.

PROBLEM. Let $f : R^n \to R^n$ be a convex function. Let $R^n *$ denote the dual vector space. Show that the formulas above completely define the mapping $g : R^n * \to R$ (under the condition that the linear form $df\big|_x$ ranges over all of $R^n *$ when x ranges over R^n).

PROBLEM. Let f be the quadratic form $f(\boldsymbol{x}) = \Sigma f_{ij} x_i x_j$. Show that its Legendre transform is again a quadratic form $g(\boldsymbol{p}) = \Sigma g_{ij} p_i p_j$, and that the

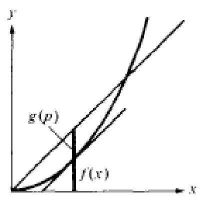

Figure 46 Legendre transformation of a quadratic form

values of both forms at corresponding points coincide (Figure 46):

$$f(\boldsymbol{x}(\boldsymbol{p})) = g(\boldsymbol{p}) \quad \text{and} \quad g(\boldsymbol{p}(\boldsymbol{x})) = f(\boldsymbol{x}) .$$

15 **Hamilton's equations**

By means of a Legendre transformation, a lagrangian system of second-order differential equations is converted into a remarkably symmetrical system of *2n* first-order equations called a hamiltonian system of equations (or canonical equations).

A ***Equivalence of Lagrange's and Hamilton's equations***

We consider the system of Lagrange's equations $\dot{\boldsymbol{p}} = \partial L / \partial \boldsymbol{q}$, where $\boldsymbol{p} = \partial L / \partial \dot{\boldsymbol{q}}$, with a given lagrangian function $L : R^n \times R^n \times R \to R$, which we will assume to be convex with respect to the second argument $\dot{\boldsymbol{q}}$.

Theorem. *The system of Lagrange's equations is equivalent to the system of 2n first-order equations (Hamilton's equations):*

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}},$$

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}},$$

where $H(\boldsymbol{p},\boldsymbol{q},t) = \boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ *is the Legendre transform of the lagrangian function viewed as a function of* $\dot{\boldsymbol{q}}$.

PROOF. By definition, the Legendre transform of $L(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ with respect to $\dot{\boldsymbol{q}}$ is the function $H(\boldsymbol{p}) = \boldsymbol{p}\dot{\boldsymbol{q}} - L(\dot{\boldsymbol{q}})$, in which $\dot{\boldsymbol{q}}$ is expressed in terms of $\boldsymbol{p}$ by the formula $\boldsymbol{p} = \partial L / \partial \dot{\boldsymbol{q}}$, and which depends on the parameters $\boldsymbol{q}$ and *t.* This function *H* is called the *hamiltonian.*

The total differential of the hamiltonian

$$dH = \frac{\partial H}{\partial p} d\boldsymbol{p} + \frac{\partial H}{\partial \boldsymbol{q}} d\boldsymbol{q} + \frac{\partial H}{\partial t} dt$$

is equal to the total differential of $\boldsymbol{p}\dot{\boldsymbol{q}} - L$ for $\boldsymbol{p} = \partial L / \partial \dot{\boldsymbol{q}}$:

$$dH = \dot{\boldsymbol{q}} d\boldsymbol{p} - \frac{\partial L}{\partial \boldsymbol{q}} d\boldsymbol{q} - \frac{\partial L}{\partial t} dt .$$

Both expressions for dH must be the same. Therefore,

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \frac{\partial H}{\partial \boldsymbol{q}} = -\frac{\partial L}{\partial \boldsymbol{q}}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} .$$

Applying Lagrange's equations $\dot{\boldsymbol{p}} = \partial L / \partial \boldsymbol{q}$, we obtain Hamilton's equations.

We have seen that, if **q**(t) satisfies Lagrange's equations, then (**p**(t), **q**(t)) satisfies Hamilton's equations. The converse is proved in an analogous manner. Therefore, the systems of Lagrange and Hamilton are equivalent.

Remark. The theorem just proved applies to all variational problems, not just to the lagrangian equations of mechanics.

B ***Hamilton's function and energy***

EXAMPLE. Suppose now that the equations are mechanical, so that the lagrangian has the usual form *L = T - U,* where the kinetic energy *T* is a quadratic form with respect to $\dot{\boldsymbol{q}}$:

$$T = \frac{1}{2}\sum a_{ij}\dot{q}_i\dot{q}_j \text{ , where } a_{ij} = a_{ij}(\boldsymbol{q},t) \text{ and } U = U(\boldsymbol{q}) .$$

Theorem. *Under the given assumptions, the hamiltonian H is the total energy H = T + U.*

The proof is based on the following lemma on the Legendre transform of a quadratic form.

Lemma. *The values of a quadratic form* f(x) *and of its Legendre transform* g(p) *coincide at corresponding points:* f(**x**) = g(**p**).

EXAMPLE. For the form $f(x) = x^2$ this is a well-known property of a tangent to a parabola. For the form $f(x) = (1/2)mx^2$ we have *p = mx* and

$$g(p) = p^2/2m = mx^2/2 = f(x) .$$

PROOF OF THE LEMMA By Euler's theorem on homogeneous functions $(\partial f/\partial \boldsymbol{x})\boldsymbol{x} = 2f$. Therefore,

$$g(\boldsymbol{p(x)}) = \boldsymbol{px} - f(\boldsymbol{x}) = (\partial f/\partial \boldsymbol{x})\boldsymbol{x} - f = 2f(\boldsymbol{x}) - f(\boldsymbol{x}) = f(\boldsymbol{x}) .$$

PROOF OF THE THEOREM. Reasoning as in the lemma, we find that

$$H = p\dot{q} - L = 2T - (T-U) = T + U .$$

EXAMPLE. For one-dimensional motion

$$\ddot{q} = -\frac{\partial U}{\partial q} .$$

In this case $T = (1/2)\dot{q}^2$, $U = U(q)$, $p = \dot{q}$, $H = (1/2)p^2 + U(q)$ and Hamilton's equations take the form

$$\dot{q} = p ,$$

$$\dot{p} = -\frac{\partial U}{\partial q} .$$

This example makes it easy to remember which of Hamilton's equations has a minus sign.

Several important corollaries follow from the theorem on the equivalence of the equations of motion to a hamiltonian system. For example, the law of conservation of energy takes the simple form:

Corollary 1. $d\boldsymbol{H}/dt = \partial \boldsymbol{H}/\partial t$ *. In particular, for a system whose hamiltonian function does not depend explicitly on time (* $\partial \boldsymbol{H}/\partial t = 0$ *), the law of conservation of the hamiltonian function holds:* $H(\boldsymbol{p}(t), \boldsymbol{q}(t)) = const$.

PROOF. We consider the variation in *H* along the trajectory *H(**p**(t),* **q**(t), *t).*Then, by Hamilton's equations,

$$\frac{dH}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}\left(-\frac{\partial H}{\partial \boldsymbol{q}}\right) + \frac{\partial H}{\partial \boldsymbol{q}}\frac{\partial H}{\partial \boldsymbol{p}} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} .$$

C ***Cyclic coordinates***

When considering central fields, we noticed that a problem could be reduced to a one-dimensional problem by the introduction of polar coordinates. It turns out that, given any symmetry of a problem allowing us to choose a system of coordinates **q** in such a way that the hamiltonian function is independent of some of the coordinates, we can find some first integrals and thereby reduce to a problem in a smaller number of coordinates.

Definition. If a coordinate q_1 does not enter into the hamiltonian function $H(p_1,\ldots,p_n;q_1,\ldots,q_n;t)$, i.e., $\partial H/\partial q_1 = 0$, then it is called *cyclic* (the term comes from the particular case of the angular coordinate in a central field).

Clearly, the coordinate q_1 is cyclic if and only if it does not enter into the lagrangian function ($\partial L/\partial q_1 = 0$). It follows from the hamiltonian form of the equations of motion that:

Corollary 2. *Let q_1 be a cyclic coordinate. Then p_1 is a first integral. In this case the variation of the remaining coordinates with time is the same as in a system with the n-1 independent coordinates $q_2,\ldots,q_n$ and with hamiltonian function*

$$H(p_2,\ldots,p_n,q_2,\ldots,q_n,,c),$$

depending on the parameter $c = p_1$.

PROOF. We set $\boldsymbol{p}' = (p_2,\ldots,p_n)$ and $q' = (q_2,\ldots,q_n)$. Then Hamilton's equation take the form

$$\frac{d}{dt}\boldsymbol{q}' = \frac{\partial H}{\partial \boldsymbol{p}'}, \qquad \frac{d}{dt}q_1 = \frac{\partial H}{\partial p_1},$$

$$\frac{d}{dt}\boldsymbol{p}' = -\frac{\partial H}{\partial \boldsymbol{q}'}, \qquad \frac{d}{dt}p_1 = 0.$$

The last equation shows that p1 = const. Therefore, in the system of equations for *p'* and *q',* the value of p1 enters only as a parameter in the hamiltonian function. After this system of *2n* - 2 equations is solved, the equation for q1 takes the form $\frac{d}{dt}q_1 = f(t)$, where

$$f(t) = \frac{\partial}{\partial p_1}H(p_1,\boldsymbol{p}'(t),\boldsymbol{q}'(t),t)$$

and is easily integrated.

Almost all the solved problems in mechanics have been solved by means of Corollary 2.

Corollary 3. *Every closed system with two degrees of freedom (n* = 2) *which has a cyclic coordinate is integrable.*

PROOF. In this case the system for *p'* and *q'* is one-dimensional and is immediately integrated by means of the integral *H(p',* q') = *c.*

16 **Liouville's theorem**

The phase flow of Hamilton's equations preserves phase volume. It follows, for example, that a hamiltonian system cannot be asymptotically stable.

For simplicity we look at the case in which the hamiltonian function does not depend explicitly on the time: $H = H(\mathbf{p}, \mathbf{q})$.

A *The phase flow*

Definition. The 2n-dimensional space with coordinates $p_1, \ldots, p_n; q_1, \ldots, q_n$ is called *phase space.*

EXAMPLE. In the case n = 1 this is the phase plane of the system $\ddot{x} = -\partial U / \partial x$, which we considered in Section 4.

Just as in this simplest example, the right-hand sides of Hamilton's equations give a vector field: at each point (**p, q**) of phase space there is a 2n-dimensional vector $(-\partial H / \partial \boldsymbol{q}, \partial H / \partial \boldsymbol{p})$. We assume that every solution of Hamilton's equations can be extended to the whole time axis.

Definition. The *phase flow* is the one-parameter group of transformations of phase space

$$g^t : (\boldsymbol{p}(0), \boldsymbol{q}(0)) \mapsto (\boldsymbol{p}(t), \boldsymbol{q}(t)),$$

where p(t) and q(t) are solutions of Hamilton's system of equations (Figure 47).

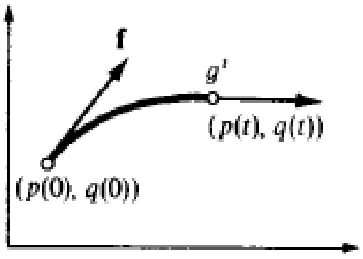

Figure 47 Phase flow

PROBLEM. Show that {g'} is a group.

B *Liouville' s theorem*

Theorem 1. *The phase flow preserves volume: for any region D we have (Figure 48)*

volume of $g^t D$ = *volume of D.*

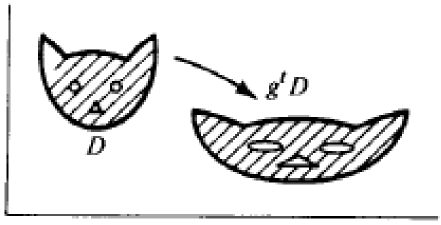

Figure 48 Conservation of volume

We will prove the following slightly more general proposition also due to Liouville.

Suppose we are given a system of ordinary differential equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, whose solution may be extended to the whole time axis. Let $\{g^t\}$ be the corresponding group of transformations:

(1) $$g^t(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x})t + O(t^2), \quad (t \to 0).$$

Let *D(0)* be a region in x-space and *v(0)* its volume;

$$v(t) = \text{volume of } D(t), \qquad D(t) = g^t D(0).$$

Theorem 2. *If div* f =0, *then* g^t *preserves volume: v(t) = v(0).*

C *Proof*

Lemma 1. $(dv/dt)\big|_{t=0} = \int_{D(0)} div\boldsymbol{f} dx$, ($dx = dx_1 \cdots dx_n$).

PROOF. For any *t,* the formula for changing variables in a multiple integral gives

$$v(t) = \int_{D(0)} \det \frac{\partial g^t \boldsymbol{x}}{\partial \boldsymbol{x}} dx$$

Calculating $\partial g^t \boldsymbol{x} / \partial \boldsymbol{x}$ by formula (1), we find

$$\frac{\partial g^t \boldsymbol{x}}{\partial \boldsymbol{x}} = E + \frac{\partial f}{\partial \boldsymbol{x}} t + O(t^2), \quad \text{as} \quad t \to 0 .$$

We will now use a well-known algebraic fact:

Lemma 2. *For any matrix A= (a;j),*

$$\det(E + At) = 1 + t \cdot trA + O(t^2), \quad t \to 0 ,$$

where $trA = \sum_{i=1}^{n} a_{ij}$ *is the trace of A (the sum of the diagonal elements).*

(The proof of Lemma 2 is obtained by a direct expansion of the determinant: we get 1 and *n* terms in *t;* the remaining terms involve t^2, t^3, etc.)

Using this, we have

$$\det \frac{\partial g^t \boldsymbol{x}}{\partial \boldsymbol{x}} = 1 + t \cdot tr \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} t + O(t^2) .$$

But $tr\partial \boldsymbol{f} / \partial x = \sum_{i=1}^{n} \partial f_i / \partial x_i = div\boldsymbol{f}$. Therefore,

$$v(t) = \int_{D(0)} \left[1 + tdiv\boldsymbol{f} + O(t^2)\right] dx$$

which proves Lemma **1.**

PROOF OF THEOREM 2. Since *t = t0* is no worse than *t* = 0, Lemma **1** can be written in the form

$$\left.\frac{dv(t)}{dt}\right|_{t=t0} = \int_{D(t0)} div\boldsymbol{f} dx ,$$

and if div **f**=0, *dv/d*=0.

In particular, for Hamilton's equations we have

$$div\boldsymbol{f} = \frac{\partial}{\partial \boldsymbol{p}}\left(-\frac{\partial H}{\partial \boldsymbol{q}}\right) + \frac{\partial}{\partial \boldsymbol{q}}\left(\frac{\partial H}{\partial \boldsymbol{p}}\right) = 0 .$$

This proves Liouville's theorem (Theorem 1).

PROBLEM. Prove Liouville's formula $W = W_0 e^{\int trAdt}$ for the Wronskian determinant of the linear system $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$.

Liouville's theorem has many applications.

PROBLEM. Show that in a hamiltonian system it is impossible to have asymptotically stable equilibrium positions and asymptotically stable limit cycles in the phase space.

Liouville's theorem has particularly important applications in statistical mechanics. Liouville's theorem allows one to apply methods of *ergodic theory* to the study of mechanics. We consider only the simplest example:

D ***Poincaré's recurrence theorem***

Let *g* be a volume-preserving continuous one-to-one mapping which maps a bounded region ***D*** of

euclidean space onto itself: $gD = \boldsymbol{D}$.

Then in any neighborhood U of any point of D there is a point $x \in U$ which returns to U, i.e., $g^n x \in U$ for some $n > 0$.

This theorem applies, for example, to the phase flow g^t of a two-dimensional system whose potential $U(x_1, x_2)$ goes to infinity as $(x_1, x_2) \to \infty$; in this case the invariant bounded region in phase space is given by the condition (Figure 49)

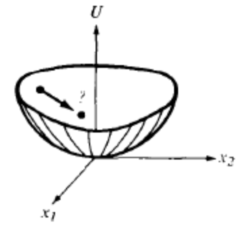

Figure 49 The way a ball will move in an asymmetrical cup is unknown; however Poincaré's theorem predicts that it will return to a neighborhood of the original position.

$$D = \{\boldsymbol{p}, \boldsymbol{q} : T + U \le E\}.$$

Poincaré's theorem can be strengthened, showing that almost every moving point returns repeatedly to the vicinity of its initial position. This is one of the few general conclusions which can be drawn about the character of motion. The details of motion are not known at all, even in the case

$$\ddot{\boldsymbol{x}} = -\frac{\partial U}{\partial \boldsymbol{x}}, \quad \text{where} \quad x = (x_1, x_2).$$

The following prediction is a paradoxical conclusion from the theorems of Poincaré and Liouville: if you open a partition separating a chamber containing gas and a chamber with a vacuum, then after a while the gas molecules will again collect in the first chamber (Figure 50).

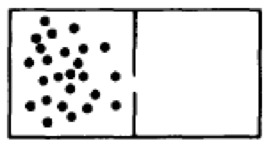

Figure 50 Molecules return to the first chamber.

The resolution of the paradox lies in the fact that "a while" may be longer than the duration of the solar system's existence.

PROOF OF POINCARE'S THEOREM. We consider the images of the neighborhood U (Figure 51):

$$U, gU, g^2U, \ldots, g^nU, \ldots$$

All of these have the same volume. If they never intersected, D would have infinite volume. Therefore, for some $k \ge 0$ and $l \ge 0$, with $k > I$,

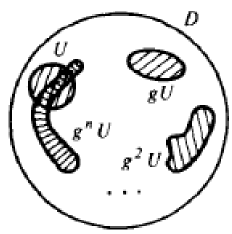

Figure 51 Theorem on returning

$$g^k U \cap g^l \ne \varnothing.$$

Therefore, $g^{k-1} U \cap U \ne \varnothing$. If y is in this intersection, then $y = g^n x$, with $x \in U$ $n = k$- 1). Then $x \in U$ and $g^n x \in U$ $(n = k$- /).

E *Applications of Poincaré's theorem*

EXAMPLE 1. Let D be a circle and g rotation through an angle α. If $\alpha = 2\pi(m/n)$, then g^n

is the identity, and the theorem is obvious. If α is not commensurable with 2π, then Poincare's theorem gives

$$\forall \delta > 0, \exists n : \left| g^n x - x \right| < \delta \quad \text{(Figure 52).}$$

It easily follows that

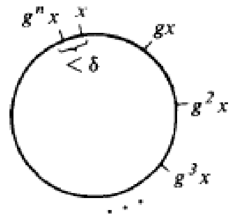

Figure 52 Dense set on the circle

Theorem. *If* $\alpha \neq 2\pi(m/n)$*, then the set of points* $g^k x$ *is dense on the circle (k* = 1, 2,…).

PROBLEM. Show that every orbit of motion in a central field with $U = r^4$ is either closed or densely fills the ring between two circles.

EXAMPLE 2. Let *D* be the two-dimensional torus and φ_1 and φ_2 angular coordinates on it (longitude and latitude) (Figure 53).

Consider the system of ordinary differential equations on the torus

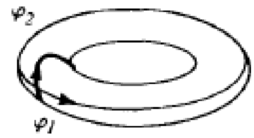

Figure 53 Torus

$$\dot{\varphi}_1 = \alpha_1, \quad \dot{\varphi}_2 = \alpha_2.$$

Clearly, div **f** = 0 and the corresponding motion

$$g^t : (\varphi_1, \varphi_2) \to (\varphi_1 + \alpha_1 t, \varphi_2 + \alpha_2 t)$$

preserves the volume $d\varphi_1 d\varphi_2$.

From Poincaré's theorem it is easy to deduce

Theorem. *If* α_1 / α_2 *is irrational, then the "winding line" on the torus,* $g^t(\varphi_1, \varphi_2)$*, is dense in the torus.*

PROBLEM. Show that if *w* is irrational, then the Lissajous figure (x = cos *t,y* =cos *wt)* is dense in the square $|x| \leq 1$, $|y| \leq 1$.

EXAMPLE 3. Let *D* be the n-dimensional torus T^n, i.e., the direct product of *n* circles:

$$D = S^1 \times S^1 \times \cdots \times S^1 = T^n.$$

A point on the n-dimensional torus is given by *n* angular coordinates $\varphi = (\varphi_1, \ldots, \varphi_n)$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, and let g^t be the volume-preserving transformation

$$g^t : T^n \to T^n, \quad \varphi \to \varphi + \boldsymbol{\alpha} t.$$

PROBLEM. Under which conditions α are the following sets dense:

(a) the trajectory $\{g^t \varphi\}$;

(b) the trajectory $\{g^k \varphi\}$

(*t* belongs to the group of real numbers R, *k* to the group of integers Z).

The transformations in Examples 1 to 3 are closely connected to mechanics. But since Poincaré's theorem is abstract, it also has applications unconnected with mechanics.

EXAMPLE 4. Consider the first digits of the numbers 2^n: 1,2,4,8,1,3,6,1,2,5,1,2,4,…

PROBLEM. Does the digit 7 appear in this sequence? Which digit appears more often, 7 or 8? How many times more often?

END OF CHAPTER 3.