CHAPTER 4 LAGRANGIAN MECHANICS ON MANIFOLDS

In this chapter we introduce the concepts of a differentiable manifold and its tangent bundle. A lagrangian function, given on the tangent bundle, defines a lagrangian "holonomic system" on a manifold. Systems of point masses with holonomic constraints (e.g., a pendulum or a rigid body) are special cases.

17 **Holonomic constraints**

In this paragraph we define the notion of a system of point masses with holonomic constraints.

A ***Example***

Let γ g be a smooth curve in the plane. If there is a very strong force field in a neighborhood of γ, directed towards the curve, then a moving point will always be close to γ. In the limit case of an infinite force field, the point must remain on the curve γ. In this case we say that a constraint is put on the system (Figure 54).

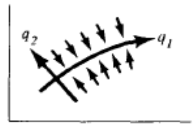

Figure 54 Constraint as an infinitely strong field

To formulate this precisely, we introduce curvilinear coordinates *q1* and *q2* on a neighborhood of γ *; q1* is in the direction of γ and *q2* is distance from the curve.

We consider the system with potential energy

$$U_N = Nq_2^2 + U_0(q_1, q_2),$$

depending on the parameter N (which we will let tend to infinity) (Figure 55).

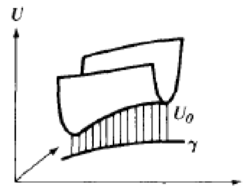

Figure 55 Potential energy U_N

We consider the initial conditions on γ *:*

$$q_1(0) = q_1^0,\ \dot{q}_1(0) = \dot{q}_1^0,\ q_2(0) = 0,\ \dot{q}_2(0) = 0.$$

Denote by $q_1 = \varphi(t, N)$ the evolution of the coordinate *q1* under a motion with these initial conditions in the field U_N.

Theorem. *The following limit exists, as* $N \to \infty$:

$$\lim_{N\to\infty} \varphi(t, N) = \psi(t).$$

The limit $q_1 = \psi(t)$ *satisfies Lagrange's equation*

$$\frac{d}{dt}\left(\frac{\partial L_*}{\partial \dot{q}_1}\right) = \frac{\partial L_*}{\partial q_1},$$

where $L_*(q_1, \dot{q}_1) = T\big|_{q_2=\dot{q}_2=0} - U_0\big|_{q_2=0}$ *(T is the kinetic energy of motion along* γ).

Thus, as $N \to \infty$, Lagrange's equations for *q1* and *q2* induce Lagrange's equation for *q* $q_1 = \psi(t)$.

We obtain exactly the same result if we replace the plane by the *3n*-dimensional configuration space of *n* points, consisting of a mechanical

system with metric $ds^2 = \sum_{i=1}^{n} m_i d\boldsymbol{r}_i^2$ (the m_i are masses), replace the curve γ by a submanifold of the 3n-dimensional space, replace *q1* by some coordinates ***q1*** on γ, and replace *q2* by some coordinates ***q2*** in the directions perpendicular to γ. If the potential energy has the form

$$U = U_0(\boldsymbol{q_1}, \boldsymbol{q_2}) + N\boldsymbol{q}_2^2$$

then as $N \to \infty$, a motion on γ is defined by Lagrange's equations with the lagrangian function

$$L_* = T\big|_{q_2=\dot{q}_2=0} - U_0\big|_{q_2=0}.$$

B *Definition of a system with constraints*

We will not prove the theorem above/5 but neither will we use it. We need it only to justify the following.

Definition. Let γ be an m-dimensional surface in the 3n-dimensional configuration space of the points $\boldsymbol{r_1}, \ldots, \boldsymbol{r_n}$ with masses $m_1, \ldots, m_n$. Let $\boldsymbol{q} = (q_1, \ldots, q_m)$ be some coordinates on $\gamma : \boldsymbol{r_i} = \boldsymbol{r_i}(\boldsymbol{q})$. The system described by the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) = \frac{\partial L}{\partial \boldsymbol{q}}, \quad L = \frac{1}{2}\sum m_i \dot{\boldsymbol{r}}_i^2 + U(\boldsymbol{q})$$

is called a system of *n* points with *3n - m* ideal *holonomic constraints.* The surface γ is called the *configuration space of the system with constraints.*

If the surface γ is given by *k = 3n-m* functionally independent equations $f_1(\boldsymbol{r}) = 0, \ldots, f_k(\boldsymbol{r}) = 0$, then we say that the system is constrained by the relations $f_1 = 0, \ldots, f_k = 0$.

Holonomic constraints also could have been defined as the limiting case of a system with a large potential energy. The meaning of these constraints in mechanics lies in the experimentally determined fact that many mechanical systems belong to this class more or less exactly.

From now on, for convenience, we will call ideal holonomic constraints simply constraints. Other constraints will not be considered in this book.

18 **Differentiable manifolds**

The configuration space of a system with constraints is a differentiable manifold. In this paragraph we give the elementary facts about differentiable manifolds.

A *Definition of a differentiable manifold*

A set *M* is given the structure of a differentiable manifold if *M* is provided with a finite or

countable collection of *charts,* so that every point is represented in at least one chart.

A chart is an open set U in the euclidean coordinate space $\boldsymbol{q} = (q_{1,} \dots, q_m)$, together with a one-to-one mapping φ of U onto some subset of M $\varphi : U \to \varphi U \subset M$.

We assume that if points **p** and **p'** in two charts U and U' have the same image in M, then **p** and **p'** have neighborhoods $V \subset U$ and $V' \subset U'$ with the same image in M (Figure 56). In this way we get a mapping $\varphi'^{-1}\varphi : V \to V'$.

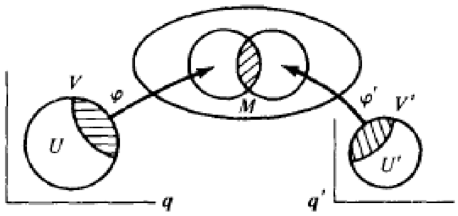

Figure 56 Compatible charts

Thts is a mapping of the region V of the euclidean space **q** onto the region V' of the euclidean space **q',** and it is given by n functions of n variables, $\boldsymbol{q}' = \boldsymbol{q}'(\boldsymbol{q})$, ($\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}')$ .The charts U and U' are called *compatible* if these functions are differentiable.

An *atlas* is a union of compatible charts. Two atlases are *equivalent* if their union is also an atlas.

A differentiable manifold is a class of equivalent atlases. We will consider only *connected* manifolds. Then the number n will be the same for all charts; it is called the *dimension* of the manifold.

A *neighborhood* of a point on a manifold is the image under a mapping $\varphi : U \to M$ of a neighborhood of the representation of this point in a chart U. We will assume that every two different points have non-intersecting neighborhoods.

B *Examples*

EXAMPLE 1. Euclidean space R^n is a manifold, with an atlas consisting of one chart.

EXAMPLE 2. The sphere

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

has the structure of a manifold. with atlas, for example, consisting of two charts ($U_i, \varphi_i, i = 1{,}2$) in stereographic projection (Figure 57). An analogous construction applies to the n-sphere.

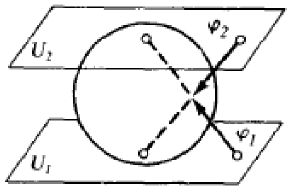

Figure 57 Atlas of a sphere

EXAMPLE 3. Consider a planar pendulum. Its configuration space - the circle S^1 - is a manifold. The usual atlas is furnished by the angular coordinates $\varphi : R^1 \to S^1$, $U_1 = (-\pi, \pi)$, $U_2 = (0, 2\pi)$ (Figure 58).

EXAMPLE 4. The configuration space of the "spherical" mathematical pendulum is the two-dimensional sphere S^2 (Figure 58).

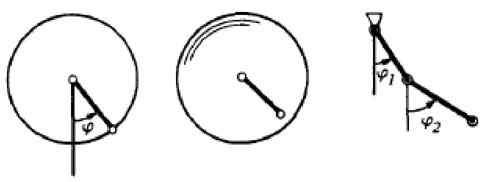

Figure 58 Planar, spherical and double planar pendulums

EXAMPLE 5. The configuration space of a "planar double pendulum" is the direct product of two circles, i.e., the two-torus $T^2 = S^1 \times S^1$ (Figure 58).

EXAMPLE 6. The configuration space of a spherical double pendulum is the direct product of two spheres, $S^2 \times S^2$.

EXAMPLE 7. A rigid line segment in the (q1, q2)-plane has for its configuration space the manifold $R^2 \times S^1$, with coordinates *q1, q2 , q3* (Figure 59). It is covered by two charts.

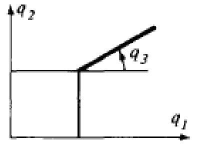

Figure 59 Configuration space of a segment in the plane

EXAMPLE 8. A rigid right triangle *OAB* moves around the vertex O. The position of the triangle is given by three numbers: the direction $OA \in S^2$ is given by two numbers, and if *OA* is given, one can rotate $OB \in S^1$ around the axis *OA* (Figure 60).

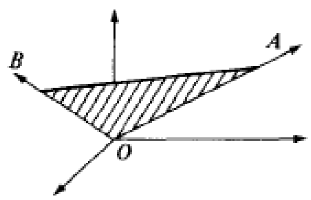

Figure 60 Configuration space of a triangle

Connected with the position of the triangle *OAB* is an orthogonal right-handed frame, $\boldsymbol{e_1} = OA/|OA|$, $\boldsymbol{e_2} = OB/|OB|$, $\boldsymbol{e_3} = [\boldsymbol{e_1}, \boldsymbol{e_2}]$. The correspondence is one-to-one: therefore the position of the triangle is given by an orthogonal three-by-three matrix with determinant 1.

The set of all three-by-three matrices is the nine-dimensional space R^9. Six orthogonality conditions select out two three-dimensional connected manifolds of matrices with determinant +1 and -1. The rotations of three-space (determinant +1) form a group, which we call *SO(3).* Therefore, the *configuration space of the triangle OAB is* SO(3).

PROBLEM. Show that SO(3) is homeomorphic to three-dimensional real projective space.

Definition. The dimension of the configuration space is called the *number of degrees of freedom.*

EXAMPLE 9. Consider a system of *k* rods in a closed chain with hinged joints.

PROBLEM. How many degrees of freedom does this system have?

EXAMPLE 10. *Embedded manifolds.* We say that *M* is an embedded k-dimensional sub-manifold of euclidean space R^n (Figure 61) if in a neighborhood *U* of every point $x \in M$ there are *n-k* functions $f_1 : U \to R$, $f_2 : U \to R$,…, $f_{n-k} : U \to R$ such that the intersection of *U* with *M* is given by the equations $f_1 = 0$, …, $f_{n-k} = 0$, and the vectors $grad f_1$, …, $grad f_{n-k}$ at **x** are linearly independent.

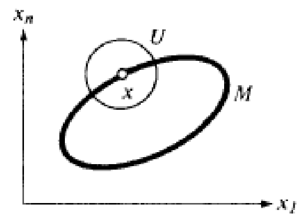

Figure 61 Embedded submanifold

It is easy to give M the structure of a manifold, i.e., coordinates in a neighborhood of x (how?).

It can be shown that every manifold can be embedded in some euclidean space. In Example 8, SO(3) is a subset of R^9.

PROBLEM. Show that SO(3) is embedded in R^9, and at the same time, that SO(3) is a manifold.

C *Tangent space*

If M is a k-dimensional manifold embedded in E^n, then at every point **x** we have a k-dimensional tangent space *TM* x. Namely, *TM*x is the orthogonal complement to $\{gradf_1,\ldots,gradf_{n-k}\}$ (Figure 62). The vectors of the tangent space TMx based at **x** are called tangent vectors to M at **x.** We can also define these vectors directly as velocity vectors of curves in M:

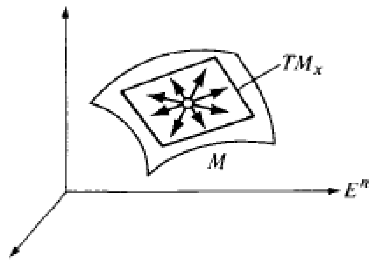

Figure 62 Tangent space

$\dot{\boldsymbol{x}} = \lim_{t\to 0}\frac{\varphi(t)-\varphi(0)}{t}$, where $\varphi(0)=\boldsymbol{x}$, $\varphi(t)\in M$.

The definition of tangent vectors can also be given in intrinsic terms, independent of the embedding of M into E^n.

We will call two curves $\boldsymbol{x}=\varphi(t)$ and $\boldsymbol{x}=\boldsymbol{\psi}(t)$ *equivalent* if $\varphi(0)=\boldsymbol{\psi}(0)=\boldsymbol{x}$ and $\lim_{t\to 0}(\varphi(t)-\psi(t))/t=0$ in some chart. Then this tangent relationship is true in any chart (prove this!).

Definition. A *tangent vector* to a manifold M at the point x is an equivalence class of curves $\varphi(t)$, with $\varphi(0)=\boldsymbol{x}$.

It is easy to define the operations of multiplication of a tangent vector by a number and addition of tangent vectors. The set of tangent vectors to M at x forms a *vector space TM*x . This space is also called the *tangent space* to M at x.

For embedded manifolds the definition above agrees with the previous definition. Its advantage lies in the fact that it also holds for abstract manifolds, not embedded anywhere.

Definition. Let U be a chart of an atlas for M with coordinates $q_1,\ldots,q_n$. Then the *components* of the tangent vector to the curve $\boldsymbol{q}=\varphi(t)$ are the $\xi_1,\ldots,\xi_n$, where $\xi_i=(d\varphi_i/dt)\big|_{t=0}$.

D *The tangent bundle*

The union of the tangent spaces to M at the various points, $\cup_{x\in M}TM_x$, has a natural differentiable manifold structure, the dimension of which is twice the dimension of M.

This manifold is called the *tangent bundle* of *M* and is denoted by *TM.* A point of *TM* is a vector ξ, tangent to *M* at some point **x**. Local coordinates on *TM* are constructed as follows. Let $q_1,\ldots,q_n$ be local coordinates on *M,* and $\xi_1,\ldots,\xi_n$ components of a tangent vector in this coordinate system. Then the *2n* numbers $(q_1,\ldots,q_n,\xi_1,\ldots,\xi_n)$ give a local coordinate system on *TM.* One sometimes writes dq_i for ξ_i.

The mapping $p: TM \to M$ which takes a tangent vector ξ to the point $\boldsymbol{x} \in M$ at which the vector is tangent to *M* ($\xi \in TM_x$), is called the *natural projection.* The inverse image of a point $\boldsymbol{x} \in M$ under the natural projection, $p^{-1}(\boldsymbol{x})$, is the tangent space *TM*x. This space is called th*e fiber of the tangentbundle over the point* **x.**

E ***Riemannian manifolds***

If *M* is a manifold embedded in euclidean space, then the metric on euclidean space allows us to measure the lengths of curves, angles between vectors, volumes, etc. All of these quantities are expressed by means of the lengths of tangent vectors, that is, by the positive definite quadratic form given on every tangent space *TM*x (Figure 63):

$$TM_x \to R, \quad \xi \to \langle \xi, \xi \rangle .$$

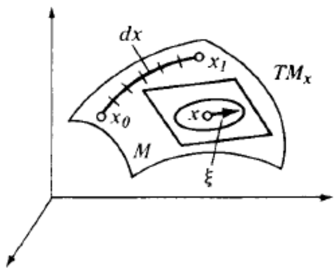

Figure 63 Riemannian metric

For example, the length of a curve on a manifold is expressed using this form as $l(\gamma) = \int_{xL0}^{x_1} \sqrt{\langle dx, dx \rangle}$, or, if the curve is given parametrically, $\gamma : [t_0, t_1] \to M$, $t \to \boldsymbol{x}(t) \in M$, then $l(\gamma) = \int_{t_0}^{t_1} \sqrt{\langle \dot{x}, \dot{x} \rangle} dt$.

Definition. A differentiable manifold with a fixed positive definite quadratic form $\langle \xi, \xi \rangle$ on every tangent space *TMx* is called a *Riemannian manifold.* The quadratic form is called the *Riemannian metric.*

Remark. Let *U* be a chart of an atlas for *M* with coordinates $q_1,\ldots,q_n$. Then a Riemannian metric is given by the formula

$$ds^2 = \sum_{i,j=1}^{n} a_{ij}(q) dq_i dq_j , \quad a_{ij} = a_{ji},$$

where dq_i are the coordinates of a tangent vector.

The functions $a_{ij}(q)$ are assumed to be differentiable as many times as necessary.

F ***The derivative map***

Let $f: M \to N$ be a mapping of a manifold *M* to a manifold *N. f* is called *differentiable* if in local coordinates on *M* and *N* it is given by differentiable functions.

Definition. The *derivative* of a differentiable mapping $f: M \to N$ at a point $x \in M$ is the linear map of the tangent spaces

$$f_{*x} : TM_x \to TN_{f(x)} ,$$

which is given in the following way (Figure 64):

Let $\boldsymbol{v} \in TM_x$. Consider a curve $\varphi : R \to M$ with $\varphi(0) = \boldsymbol{x}$ and velocity

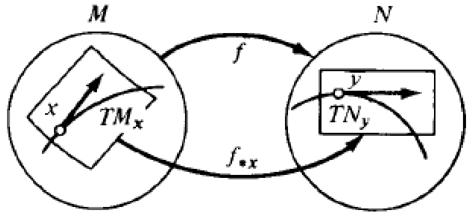

Figure 64 Derivative of a mapping

vector $(d\varphi / dt)\big|_{t=0} = \boldsymbol{v}$. Then $f_{*x}\boldsymbol{v}$ is the velocity vector of the curve $f \circ \varphi : R \to N$,

$$f_{*x}\boldsymbol{v} = \left.\frac{d}{dt}\right|_{t=0} f(\varphi(t)) .$$

PROBLEM. Show that the vector $f_{*x}\boldsymbol{v}$ does not depend on the curve φ, but only on the vector **v**.

PROBLEM. Show that the map $f_{*x} : TM_x \to TN_{f(x)}$ is linear.

PROBLEM. Let $x = (x_{1,} \dots, x_m)$ be coordinates in a neighborhood of $x \in M$, and $y = (y_{1,} \dots, y_n)$ be coordinates in a neighborhood of $y \in N$. Let ξ be the set of components of the vector **v**, and $\boldsymbol{\eta}$ the set of components of the vector $f_{*x}\boldsymbol{v}$. Show that

$$\boldsymbol{\eta} = \frac{\partial y}{\partial \boldsymbol{x}} \xi \text{ , i.e., } \eta_i = \sum_j \frac{\partial y_i}{\partial x_j} \xi_j$$

Taking the union of the mappings f_{*x} for all x, we get a mapping of the whole tangent bundle

$$f_* : TM \to TN , \quad f_*\boldsymbol{v} = f_{*x}\boldsymbol{v} \quad \text{for} \quad \boldsymbol{v} \in TM_x .$$

PROBLEM. Show that f_* is a differentiable map.

PROBLEM. Let $f : M \to N$, $g : N \to K$, and $h = g \circ f : M \to K$. Show that $h_* = g_* f_*$.

19 **Lagrangian dynamical systems**

In this paragraph we define lagrangian dynamical systems on manifolds. Systems with Holonomic constraints are a particular case.

A ***Definition of a lagrangian system***

Let M be a differentiable manifold, TM its tangent bundle, and $L : TM \to R$ a differentiable function. A map $\gamma : R \to M$ is called a *motion in the lagrangian system with configuration manifold M and lagrangian function L* if γ is an extremal of the functional

$$\boldsymbol{\Phi}(\gamma) = \int_{t_0}^{t_1} L(\dot{\gamma}) dt ,$$

where $\dot{\boldsymbol{\gamma}}$ is the velocity vector $\dot{\boldsymbol{\gamma}}(t) \in TM_{\gamma(t)}$.

EXAMPLE. Let M be a region in a coordinate space with coordinates $\boldsymbol{q} = (q_1, \ldots, q_n)$. The lagrangian function $L: TM \to R$ may be written in the form of a function $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of the *2n* coordinates. As we showed in Section 12, the evolution of coordinates of a point moving with time satisfies Lagrange's equations.

Theorem. *The evolution of the local coordinates* $\boldsymbol{q} = (q_1, \ldots, q_n)$ *of a point* $\boldsymbol{\gamma}(t)$ *under motion in a lagrangian system on a manifold satisfies the Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \boldsymbol{q}},$$

where $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ *is the expression for the function* $L: TM \to R$ *in the coordinates* $\boldsymbol{q}$ *and* $\dot{\boldsymbol{q}}$ *on TM.*

We often encounter the following special case.

B *Natural systems*

Let M be a Riemannian manifold. The quadratic form on each tangent space,

$$T = \frac{1}{2}\langle \boldsymbol{v}, \boldsymbol{v} \rangle, \quad \boldsymbol{v} \in TM_x,$$

is called the *kinetic energy.* A differentiable function $U: M \to R$ is called a *potential energy.*

Definition. A lagrangian system on a Riemannian manifold is called *natural* if the lagrangian function is equal to the difference between kinetic and potential energies: $L = T - V$.

EXAMPLE. Consider two mass points *m1* and *m2* joined by a line segment of length l in the *(x,* y)-plane. Then a configuration space of three dimensions

$$M = R^2 \times S^1 \subset R^2 \times R^2$$

is defined in the four-dimensional configuration space $R^2 \times R^2$ of two free points (x_1, y_1) and (x_2, y_2) by the condition $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = l$ (Figure 65).

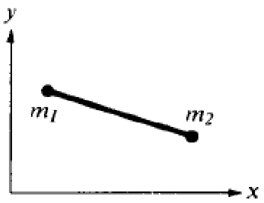

Figure 65 **Segment in the plane**

There is a quadratic form on the tangent space to the four-dimensional space (x_1, x_2, y_1, y_2):

$$m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2).$$

Our three-dimensional manifold, as it is embedded in the four-dimensional one, is provided with a Riemannian metric. The holonomic system thus obtained is called in mechanics a line segment of fixed length in the *(x,* y)-plane. The kinetic energy is given by the formula

$$T = m_1 \frac{\dot{x}_1^2 + \dot{y}_1^2}{2} + m_2 \frac{\dot{x}_2^2 + \dot{y}_2^2}{2}.$$

C *Systems with holonomic constraints*

In Section 17 we defined the notion of a system of point masses with Holonomic constraints. We will now show that such a system is natural.

Consider the configuration manifold *M* of a system with constraints as embedded in the *3n*-dimensional configuration space of a system of free points. The metric on the 3n-dimensional space is given by the quadratic form $\sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i^2$. The embedded Riemannian manifold *M* with potential energy *U* coincides with the system defined in Section 17 or with the limiting case of the system with potential $U + N\boldsymbol{q}_2^2$, $N \to \infty$, which grows rapidly outside of *M*.

D *Procedure for solving problems with constraints*

1. Determine the configuration manifold and introduce coordinates $q_1, \ldots, q_n$ (in a neighborhood of each of its points).

2. Express the kinetic energy $T = \sum \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2$ as a quadratic form in the generalized velocities

$$T = \frac{1}{2} \sum a_{ij}(\boldsymbol{q}) \dot{\boldsymbol{q}}_i \dot{\boldsymbol{q}}_j .$$

3. Construct the lagrangian function $L = T - U(\boldsymbol{q})$ and solve Lagrange's equations.

EXAMPLE. We consider the motion of a point mass of mass 1 on a surface of revolution in three-dimensional space. It can be shown that the orbits are geodesics on the surface. In cylindrical coordinates r, φ, z the surface is given (locally) in the form $r = r(z)$ or $z = z(r)$. The kinetic energy has the form (Figure 66)

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}\left[(1 + r_z'^2)\dot{z}^2 + r^2(z)\dot{\varphi}^2\right]$$

in coordinates φ and z, and

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}\left[(1 + z_r'^2)\dot{r}^2 + r^2\dot{\varphi}^2\right]$$

in coordinates r and φ. (We have used the identity $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\varphi}^2$.)

The lagrangian function *L* is equal to *T*. In both coordinate systems φ is a cyclic coordinate. The corresponding momentum is preserved; $p_\varphi = r^2\dot{\varphi}$ is nothing other than the z-component of angular momentum. Since the system has two degrees of freedom, knowing the cyclic coordinate φ is sufficient for integrating the problem completely (cf.

Corollary 3, Section 15).

We can obtain easily a clear picture of the orbits by reasoning slightly differently. Denote be α the angle of the orbit with a meridian. We have $r\dot{\varphi} = |v|\sin\alpha$, where $|v|$ is the magnitude of the velocity vector (Figure 66).

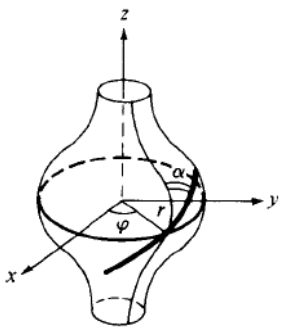

Figure 66 Surface of revolution

By the law of conservation of energy, H=L=T is preserved. Therefore, $|v| = const$, so the conservation law for p_φ takes the form $r\sin\alpha = const$ ("Clairaut's theorem").

This relationship shows that the motion takes place in the region $|\sin\alpha| \le 1$, i.e., $r \ge r_0 \sin\alpha_0$. Furthermore, the inclination of the orbit from the meridian increases as the radius r decreases. When the radius reaches the smallest possible value, $r = r_0 \sin\alpha_0$, the orbit is reflected and returns to the region with larger r (Figure 67).

PROBLEM. Show that the geodesics on a convex surface of revolution are divided into three classes: meridians, closed curves, and geodesics dense in a ring $r \ge c$.

PROBLEM. Study the behavior of geodesics on the surface of a torus ($(r-R)^2 + z^2 = \rho^2$).

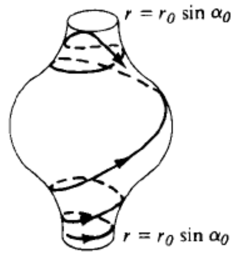

Figure 67 Geodesics on a surface of revolution

E *Non-autonomous systems*

A *lagrangian non-autonomous system* differs from the autonomous systems, which we have been studying until now, by the additional dependence of the lagrangian function on time:

$$L : TM \times R \to R, \quad L = L(q, \dot{q}, t).$$

In particular, both the kinetic and potential energies can depend on time in a non-autonomous natural system:

$$T : TM \times R \to R, \quad U : M \times R \to R, \quad T = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t), \quad U = U(\boldsymbol{q}, t).$$

A system of *n* mass points, constrained by holonomic constraints dependent on time, is defined with the help of a time-dependent submanifold of the configuration space of a free system. Such a manifold is given by a mapping

$$i : M \times R \to E^{3n}, \quad i(\boldsymbol{q}, t) = \boldsymbol{x},$$

which, for any fixed $t \in R$, defines an embedding $M \to E^{3n}$. The formula of

section D remains true for non-autonomous systems.

EXAMPLE. Consider the motion of a bead along a vertical circle of radius r (Figure 68) which rotates with angular velocity w around the vertical axis passing through the center 0 of the circle. The manifold M is the circle. Let q be the angular coordinate on the circle, measured from the highest point.

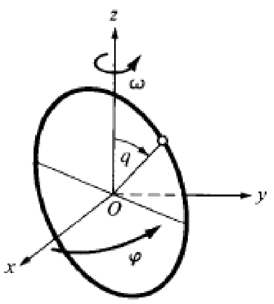

Figure 68 Bead on a rotating circle

Let x, y, and z be Cartesian coordinates in E^3 with origin 0 and vertical axis z. Let φ be the angle of the plane of the circle with the plane xOz. By hypothesis, $\varphi = \omega t$. The mapping $i : M \times R \to E^3$ is given by the formula

$$i(q,t) = (r \sin q \cos \omega t, r \sin q \sin \omega t, r \cos q) .$$

From this formula (or, more simply, from an "infinitesimal right triangle") we find that

$$T = \frac{m}{2}(\omega^2 r^2 \sin^2 q + r^2 \dot{q}^2) , \quad U = mgr \cos q .$$

In this case the lagrangian function $L = T\text{-}U$ turns out to be independent of r, although the constraint does depend on time. Furthermore, the lagrangian function turns out to be the same as in the one-dimensional system with kinetic energy

$$T_0 = \frac{M}{2}\dot{q}^2 , \quad M = mr^2 ,$$

and with potential energy

$$V = A \cos q - B \sin^2 q , \quad A = mgr , \ B = \frac{m}{2}\omega^2 r^2 .$$

The form of the phase portrait depends on the ratio between A and B. For 2B<A (i.e., for a rotation of the circle slow enough that $\omega^2 r < g$), the lowest position of the bead ($q = \pi$) is stable and the characteristics of the motion are generally the same as in the case of mathematical pendulum ($\omega = 0$).

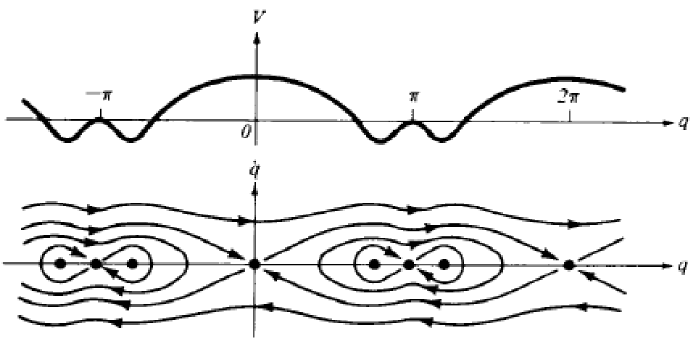

Figure 69 Effective potential energy and phase plane of the bead

For $2B > A$, i.e., for sufficiently fast

rotation of the circle, the lowest position of the bead becomes unstable; on the other hand, two stable positions of the bead appear on the circle, where $q=-A/2B=-g/\omega^2 r$. The behavior of the bead under all possible initial conditions is clear from the shape of the phase curves in the $(q,\dot{q})$-plane (Figure 69).

20 E. Noether's theorem

Various laws of conservation (of momentum, angular momentum, etc.) are particular cases of one general theorem: to every one-parameter group of diffeomorphisms of the configuration manifold of a lagrangian system which preserves the lagrangian function, there corresponds a first integral of the equations of motion.

A *Formulation of the theorem*

Let *M* be a smooth manifold, $L: TM \to R$ a smooth function on its tangent bundle *TM*. Let $h: M \to M$ be a smooth map.

Definition. A lagrangian system (*M, L) admits the mapping h* if for any tangent vector $\boldsymbol{v} \in TM$,

$$L(h_* \boldsymbol{v}) = L(\boldsymbol{v}).$$

EXAMPLE. Let $M=\{x_1,x_2,x_3)\}$, $L=(m/2)(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2)-U(x_2,x_3)$. The system admits the translation $h:(x_1,x_2,x_3)\to(x_1+s,x_2,x_3)$ along the x_1 axis and does not admit, generally speaking, translations along the x_2 axis.

Noether's theorem. If the system (*M, L*) admits the one-parameter group of diffeomorphisms $h^s: M\to M$, $s\in R$, then the lagrangian system of equations corresponding to L has a first integral $I: TM\to R$. In local coordinates q on M the integral I is written in the form

$$I(\boldsymbol{q},\dot{\boldsymbol{q}})=\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\frac{dh^s(\boldsymbol{q})}{ds}\bigg|_{s=0}.$$

B *Proof*

First, let $M=R^n$ be coordinate space. Let $\varphi: R\to M$, $\boldsymbol{q}=\varphi(t)$ be a solution to Lagrange's equations. Since h_*^s preserves *L,* the translation of a solution, $h^s\circ\varphi: R\to M$ also satisfies Lagrange's equations for any s.

We consider the mapping $\boldsymbol{\Phi}: R\times R\to R^n$, given by $\boldsymbol{q}=\boldsymbol{\Phi}(s,t)=h^s(\varphi(t))$ (Figure 70). We will denote derivatives with respect to *t* by

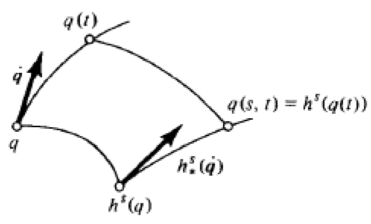

Figure 70 Noether's theorem

dots and with respect to s by primes. By hypothesis

(1) $$0=\frac{\partial L(\boldsymbol{\Phi},\dot{\boldsymbol{\Phi}})}{\partial s}=\frac{\partial L}{\partial \boldsymbol{q}}\boldsymbol{\Phi}'+\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{\Phi}}'$$

where the partial derivatives of L are taken at the point $\boldsymbol{q}=\boldsymbol{\Phi}(s,t)$, $\dot{q}=\dot{\Phi}(s,t)$.

As we stated above, the mapping $\Phi|_{s=const}:R\to R^n$ for any fixed s satisfies Lagrange's equation

$$\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{\Phi}(s,t),\dot{\boldsymbol{\Phi}}(s,t))\right]=\frac{\partial L}{\partial \boldsymbol{q}}(\boldsymbol{\Phi}(s,t),\dot{\boldsymbol{\Phi}}(s,t)).$$

We introduce the notation $\boldsymbol{F}(s,t)=(\partial L/\partial\dot{\boldsymbol{q}})(\boldsymbol{\Phi}(s,t),\dot{\boldsymbol{\Phi}}(s,t))$ and substitute $\partial \boldsymbol{F}/\partial t$ for $\partial L/\partial \boldsymbol{q}$ in (1).

Writing $\dot{\boldsymbol{q}}'$ as $d\boldsymbol{q}'/dt$, we get

$$0=\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\boldsymbol{q}'+\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\left(\frac{d}{dt}\boldsymbol{q}'\right)=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\boldsymbol{q}'\right)=\frac{dI}{dt}.$$

Remark. The first integral $I=(\partial L/\partial\dot{\boldsymbol{q}})\boldsymbol{q}'$ defined above using local coordinates **q.** It turns out that the *value of I(v) does not depend on the choice of coordinate system* **q.**

In fact, I is the rate of change of *L(v)* when the vector $v\in TM_x$ varies inside *TMx* with velocity $(d/ds)|_{s=0}h^s\boldsymbol{x}$. Therefore, *I(v)* is well defined as a function of the tangent vector $v\in TM_x$. Noether's theorem is proved in the same way when M is a manifold.

C ***Examples***

EXAMPLE 1. Consider a system of point masses with masses m_i:

$$L=\sum m_i\frac{\dot{\boldsymbol{x}}_i^2}{2}-U(\boldsymbol{x}),\quad x=x_{i1}\boldsymbol{e_1}+x_{i2}\boldsymbol{e}_2+x_{i3}\boldsymbol{e}_3,$$

constrained by the conditions $f_j(\boldsymbol{x})=0$. We assume that the system admits translations along the $\mathbf{e}_1$ axis:

$$h^s:\boldsymbol{x}_i\to\boldsymbol{x}_i+s\boldsymbol{e_1}\quad \text{for all } i.$$

In other words, the constraints admit motions of the system as a whole along the $\mathbf{e}_1$ axis, and the potential energy does not change under these.

By Noether's theorem we conclude: If a system admits translations along the $\mathbf{e}_1$ axis, then the projection of its center of mass on the $\mathbf{e}_1$ axis moves linearly and uniformly.

In fact, $(d/ds)|_{s=0}h^s\overset{1}{\boldsymbol{x}}_i=\boldsymbol{e}_i$. According to the remark at the end of B, the quantity

$$I = \sum \frac{\partial L}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{i}}} \boldsymbol{e}_{\boldsymbol{i}} = \sum m_i \dot{x}_{i1}$$

is preserved, i.e., the first component P_1 of the momentum vector is preserved. We showed this earlier for a system without constraints.

EXAMPLE 2. If a system admits rotations around the $\mathbf{e}_1$ axis, then the angular momentum with respect to this axis,

$$M1 = \sum_i ([\boldsymbol{x}_i, m_i \dot{\boldsymbol{x}}_i], \boldsymbol{e}_1)$$

is conserved.

It is easy to verify that if h^s is rotation around the $\mathbf{e}_1$ axis by the angle *s,* then $(d/ds)|_{s=0} h^s \boldsymbol{x}_{\boldsymbol{i}} = [\boldsymbol{e}_{\boldsymbol{1}}, \boldsymbol{x}_{\boldsymbol{i}}]$, from which it follows that

$$I = \sum_i \frac{\partial L}{\partial \dot{\boldsymbol{x}}_{\boldsymbol{i}}} [\boldsymbol{e}_{\boldsymbol{1}}, \boldsymbol{x}_i] = \sum_i (\boldsymbol{m}_i \dot{\boldsymbol{x}}_i, [\boldsymbol{e}_{\boldsymbol{1}}, \boldsymbol{x}_{\boldsymbol{i}}]) = \sum_i ([\boldsymbol{x}_{\boldsymbol{i}}, m_i \dot{\boldsymbol{x}}_{\boldsymbol{i}}], \boldsymbol{e}_{\boldsymbol{1}}) .$$

PROBLEM 1. Suppose that a particle moves in the field of the uniform helical line $x = \cos\varphi$, $y = \sin\varphi$, $z = c\varphi$. Find the law of conservation corresponding to this helical symmetry.

ANSWER. In any system which admits helical motions leaving our helical line fixed, the quantity $I = cP_3 + M_3$ is conserved.

PROBLEM 2. Suppose that a rigid body is moving under its own inertia. Show that its center of mass moves linearly and uniformly. If the .center of mass is at rest, then the angular momentum with respect to it is conserved.

PROBLEM 3. What quantity is conserved under the motion of a heavy rigid body if it is fixed at some point 0? What if, in addition, the body is symmetric with respect to an axis passing through 0?

PROBLEM 4. Extend Noether's theorem to non-autonomous lagrangian systems.

Hint. Let $M1 = M \times R$ be the extended configuration space (the direct product of the configuration manifold M with the time axis R).Define a function $L_1 : TM \to R$ by $L\dfrac{dt}{d\tau}$; i.e., in local coordinates q, *t* on M_1 we define it by the formula

$$L_1\left(\boldsymbol{q}, t, \frac{dq}{d\tau}, \frac{dt}{d\tau}\right) = L\left(\boldsymbol{q}, \frac{d\boldsymbol{q}/d\tau}{dt/d\tau}, t\right)\frac{dt}{d\tau} .$$

We apply Noether's theorem to the lagrangian system (M_1, L_1).

If L1 admits the transformations $h^s : M_1 \to M_1$, we obtain a first integral $I_1 : TM_1 \to R$. Since $\int L dt = \int L_1 d\tau$, this reduces to a first integral $I : TM \times R \to R$ of the original system. If in local coordinates **(q,** *t)* on *M1* we have

$I_1 = I_1(\boldsymbol{q}, t, d\boldsymbol{q}/d\tau, dt/d\tau)$, then $I(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = I_1(\boldsymbol{q}, t, \dot{\boldsymbol{q}}, I)$. In particular, if L does not depend on time, L1 admits translations along time, $h^s(\boldsymbol{q}, t) = (\boldsymbol{q}, t+s)$. The corresponding first integral I is the energy integral.

21 D'Alembert's principle

We give here a new definition of a system of point masses with holonomic constraints and prove its equivalence to the definition given in Section 17.

A *Example*

Consider the holonomic system (M, *L),* where *M* is a surface in three-dimensional space $\{\mathbf{x}\}$:

$$L = \frac{1}{2} m \dot{\boldsymbol{x}}^2 - U(\boldsymbol{x})$$

In mechanical terms, "the mass point **x** of mass *m* must remain on the smooth surface *M."* Consider a motion of the point, **x**(t). If Newton's equations $m\ddot{x} + (\partial U / \partial \boldsymbol{x}) = 0$ were satisfied, then in the absence of external forces *(U* = 0) the trajectory would be a straight line and could not lie on the surface *M.* From the point of view of Newton, this indicates the presence of a new force "forcing the point to stay on the surface."

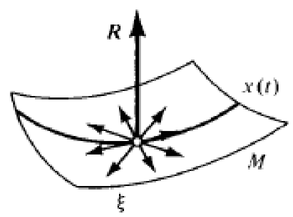

Figure 71 Constraint force

Definition. The quantity

$$\boldsymbol{R} = m\ddot{\boldsymbol{x}} + \frac{\partial U}{\partial \boldsymbol{x}}$$

is called the *constraint force* (Figure 71).

If we take the constraint force $\boldsymbol{R}(t)$ into account, Newton's equations are obviously satisfied:

$$m\ddot{\boldsymbol{x}} = -\frac{\partial U}{\partial \boldsymbol{x}} + \boldsymbol{R}.$$

The physical meaning of the constraint force becomes clear if we consider our system with constraints as the limit of systems with energy $U + NU_1$ as $N \to \infty$, where $U_1(\boldsymbol{x}) = \rho^2(\boldsymbol{x}, M)$. For large N the constraint potential NU_1 produces a rapidly changing force $\boldsymbol{F} = -N\partial U_1 / \partial \boldsymbol{x}$; when we pass to the limit ($N \to \infty$) the average value of the force $\boldsymbol{F}$ under oscillations of **x** near M is **R**. The force is perpendicular to M. Therefore, the constraint force **R** is perpendicular to $M : (\boldsymbol{R}, \xi) = 0$ for every tangent vector ξ.

B *Formulation of the D'Aiembert-Lagrange principle*

In mechanics, tangent vectors to the configuration manifold are called *virtual variations.* The D'Alembert- Lagrange principle states:

$$\left(m\ddot{\boldsymbol{x}}+\frac{\partial U}{\partial \boldsymbol{x}},\boldsymbol{\xi}\right)=0$$

for any virtual variation $\boldsymbol{\xi}$, or stated differently, the *work of the constraint force on any virtual variation is zero.*

For a system of points **X**; with masses m_i the constraint forces $\boldsymbol{R_i}$ are defined by $\boldsymbol{R_i}=m_i\ddot{\boldsymbol{x}}_i+(\partial U/\partial \boldsymbol{x}_i)$, and D'Aiembert's principle has the form $\sum(\boldsymbol{R_i},\xi_i)=0$, *or* $\sum((m_i\ddot{\boldsymbol{x}}_i+(\partial U/\partial \boldsymbol{x}_i),\boldsymbol{\xi_i})=0$, i.e., the sum of the works of the constraint forces on any virtual variation $\{\xi_i\}\in TM_x$ is zero.

Constraints with the property described above are called *ideal.*

If we define a system with holonomic constraints as a limit as $N\to\infty$, then the D'Aiembert-Lagrange principle becomes a theorem: its proof is sketched above for the simplest case.

It is possible, however, to *define* an ideal holonomic constraint using the D'Aiembert-Lagrange principle. In this way we have three definitions of holonomic systems with constraints:

1. The limit of systems with potential energies $U+NU_1$ as $N\to\infty$.
2. A holonomic system *(M, L),* where *M* is a smooth submanifold of the configuration space of a system without constraints and *L* is the lagrangian.
3. A system which complies with the D'Alembert-Lagrange principle.

All three definitions are mathematically equivalent.

The proof of the implications $(1)\Rightarrow(2)$ and $(1)\Rightarrow(3)$ is sketched above and will not be given in further detail. We will now show that $(2)\Rightarrow(3)$.

C The equivalence of the D' Alembert-Lagrange principle and the variational principle

Let M be a submanifold of euclidean space, $M\subset R^N$, and $\boldsymbol{x}:R\to M$ a curve, with **x** (t0)= **x** 0, **x** (t1)= **x**1.

Definition. The curve x is called a *conditional extrernal* of the action functional

$$\boldsymbol{\Phi}=\int_{t_0}^{t_1}\left\{\frac{\dot{\boldsymbol{x}}^2}{2}-U(\boldsymbol{x})\right\}dt,$$

if the differential $\delta\boldsymbol{\Phi}$ is equal to zero *under the condition* that the variation consists of nearby curves joining $\mathbf{x}_0$ to $\mathbf{x}_1$ in *M.*

We will write

(1) $$\delta_M\boldsymbol{\Phi}=0.$$

Clearly, Equation (1) is equivalent to the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial L}{\partial \boldsymbol{q}}, \quad L = \frac{\dot{\boldsymbol{x}}^2}{2} - U(\boldsymbol{x}), \ \ \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{q}),$$

in some local coordinate system **q** on *M*.

Theorem. *A curve* $\boldsymbol{x} : R \to M \subset R^N$ *is a conditional extremal of the action (i.e., satisfies Equation* (1)) *if and only if it satisfies D'Alembert's equation*

(2) $$\left(\ddot{\boldsymbol{x}} + \frac{\partial U}{\partial \boldsymbol{x}}, \boldsymbol{\xi} \right) = 0, \quad \forall \boldsymbol{\xi} \in TM_x .$$

Lemma. *Let* $f : \{t : t_0 \le t \le t_1\} \to R^N$ *be a continuous vector field. If, for every continuous tangent vector field* ξ *, tangent to M along* **x** *(i.e.,* $\xi(t) \in TM_{x(t)}$ *with* $\xi(t) = 0$ *for* $t = t_0, t_1$*), we have*

$$\int_{t_0}^{t_1} f(t)\xi(t)dt = 0,$$

then the field f(t) *is perpendicular to M at every point* x(t) *(i.e.,* (f(t), **h)** = 0 *for every vector* $h \in TM_{x(t)}$ *(Figure* 72).

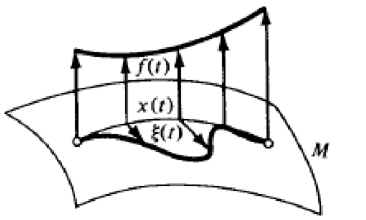

Figure 72 Lemma about the normal field

The proof of the lemma repeats the argument which we used to derive the Euler-Lagrange equations in Section 12.

PROOF OF THE THEOREM. We compare the value of Φ on the two curves x(t) and x(t)+$\xi(t)$, where $\xi(t_0) = \xi(t_1) = 0$. Integrating by parts, we obtain

$$\delta\boldsymbol{\Phi} = \int_{t_0}^{t_1} \left(\dot{\boldsymbol{x}}\dot{\xi} - \frac{\partial U}{\partial \boldsymbol{x}} \xi \right) dt = -\int_{t_0}^{t_1} \left(\ddot{\boldsymbol{x}} + \frac{\partial U}{\partial \boldsymbol{x}} \right) \xi dt .$$

It is obvious from this formula that Equation (1), $\delta_M \boldsymbol{\Phi} = 0$, is equivalent to the collection of equations

(3) $$\int_{t_0}^{t_1} \left(\ddot{\boldsymbol{x}} + \frac{\partial U}{\partial \boldsymbol{x}} \right) \xi dt = 0 .$$

for all tangent vector fields $\xi(t) \in TM_{x(t)}$ with $\xi(t_0) = \xi(t_1) = 0$. By the lemma (where we must set $\boldsymbol{f} = \ddot{\boldsymbol{x}} + (\partial U / \partial \boldsymbol{x})$) the collection of equations (3) is equivalent to the D'Alembert-Lagrange equation (2).

D *Remarks*

Remark 1. We derive the *D'Alembert-Lagrange principle for a system of n points* $x_i \in R^3$, $i = 1, \ldots, n$ with masses m_i, with holonomic constraints, from the above theorem.

In the coordinates $\bar{x} = \{\bar{x}_i = \sqrt{m_i}\, x_i\}$, the kinetic energy takes the form

$$T = \frac{1}{2}\sum m_i \dot{x}_i^2 = \frac{1}{2}\dot{\bar{x}}^2 .$$

By the theorem, the extremals of the principle of least action satisfy the condition

$$\left(\ddot{\bar{\boldsymbol{x}}} + \frac{\partial U}{\partial \bar{\boldsymbol{x}}}, \xi\right) = 0$$

(the D'Alembert-Lagrange principle for points in R^{3n}: 3n-dimensional reaction force is orthogonal to the manifold M in the metric T). Returning to the coordinates Xi, we get

$$0 = \left(\sqrt{m_i}\,\hat{x}_i + \frac{\partial U}{\partial \sqrt{m_i}\, x_i}, \sqrt{m_i}\,\xi_i\right) = \sum\left(m_i \ddot{x}_i + \frac{\partial U}{\partial x_i}, \xi_i\right),$$

i.e., the D'Alembert-Lagrange principle in the form indicated earlier: the sum of the work of the reaction forces on virtual variations is zero.

Remark 2. The D'Alembert-Lagrange principle can be given in a slightly different form if we turn to statics. An *equilibrium position* is a point x_0 which is the orbit of a motion: $x(t) = x_0$.

Suppose that a point mass moves along a smooth surface M under the influence of the force $\boldsymbol{f} = -\partial U / \partial \boldsymbol{x}$.

Theorem. The point x_0 in M is an equilibrium position if and only if the force is orthogonal to the surface at $\boldsymbol{x_0} : (\boldsymbol{f}(\boldsymbol{x}_0), \xi) = 0$ for all $\xi \in TM_{x0}$.

This follows from the D'Alembert-Lagrange equations in view of the fact that $\ddot{\boldsymbol{x}} = 0$.

Definition. $-m\ddot{\boldsymbol{x}}$ is called the *force of inertia.*

Now the D'Alembert-Lagrange principle takes the form:

Theorem. If the forces of inertia are added to the acting forces, **x** becomes an equilibrium position.

PROOF. D'Alembert's equation

$$(-m\ddot{\boldsymbol{x}} + \boldsymbol{f}, \xi) = 0$$

expresses the fact, as in the preceding theorem, that **x** is an equilibrium position of a system with forces $-m\ddot{\boldsymbol{x}} + \boldsymbol{f}$.

Entirely analogous statements are true for systems of points: If **x** = {**xi**} are equilibrium positions, then the sum of the work of the forces acting on the virtual

variations is equal to zero. If the forces of inertia $m_i\ddot{\boldsymbol{x}}_i(t)$ are added to the acting forces, then the position **x**(t) becomes an equilibrium position.

Now a problem about motions can be reduced to a problem about equilibrium under actions of other forces.

Remark 3. Up to now we have not considered cases when the constraints depend on time. All that was said above carries over to such constraints without any changes.

EXAMPLE. Consider a bead sliding along a rod which is tilted at an angle α to the vertical axis and is rotating uniformly with angular velocity ω around this axis (its weight is negligible). For our coordinate q we take the distance from the point 0 (Figure 73). The kinetic energy and lagrangian are

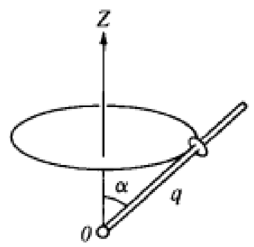

Figure 73 Bead on a rotating rod

$$L = T = \frac{1}{2}mv^2 = \frac{1}{2}\dot{q}^2 + \frac{1}{2}m\omega^2 r^2\,,\quad r = q\sin\alpha\,.$$

Then lagrange's equation is

$$m\ddot{q} = m\omega^2 q\sin^2\alpha\,.$$

The constraint force at each moment is orthogonal to virtual variations (i.e., to the direction of the rod), but is not at all orthogonal to the actual trajectory.

Remark 4. It is easy to derive conservation laws from the D'Alembert-Lagrange equations. For example, if translation along the x 1 axis $\xi_i = \boldsymbol{e_1}$ is among the virtual variations, then the sum of the work of the constraint forces on this variation is equal to zero:

$$\sum(\boldsymbol{R_i}, \boldsymbol{e}_1) = (\sum \boldsymbol{R_i}, \boldsymbol{e_1}) = 0\,.$$

Ifwe now consider constraint forces as external forces, then we notice that the sum of the first components of the external forces is equal to zero. This means that the first component, P_1, of the momentum vector is preserved.

We obtained this same result earlier from Noether's theorem.

Remark 5. We emphasize once again that the holonomic character of some particular physical constraint or another (to a given degree of exactness) is a question of experiment. From the mathematical point of view, the holonomic character of a constraint is a postulate of physical origin; it can be introduced in various equivalent forms, for example, in the form of the principle of least action (1) or the

D'Alembert-Lagrange principle (2), but, when defining the constraints, the term always refers to experimental facts which go beyond Newton's equations.

Remark 6. Our terminology differs somewhat from that used in mechanics textbooks, where the D'Alembert-Lagrange principle is extended to a wider class of systems (" non-holonomic systems with ideal constraints"). In this book we will not consider non-holonomic systems. We remark only that one example of a non-holonomic system is a sphere rolling on a plane without slipping. In the tangent space at each point of the configuration manifold of a non-holonomic system there is a fixed subspace to which the velocity vector must belong.

Remark 7. If a system consists of mass points connected by rods, hinges, etc., then the need may arise to talk about the constraint force of some particular constraint.

We defined the total "constraint force of all constraints" $\boldsymbol{R}_i$ for every mass point m_i. The concept of a constraint force for an individual constraint is *impossible* to define, as may be already seen from the simple example of a beam resting on three columns. If we try to define constraint forces of the columns, $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$ by passing to a limit (considering the columns as very rigid springs), then we may become convinced that the result depends on the distribution of rigidity.

Problems for students are selected so that this difficulty does not arise.

PROBLEM. A rod of weight *P,* tilted at an angle of 60° to the plane of a table, begins to fall with initial velocity zero (Figure 74). Find the constraint force of the table at the initial moment, considering the table as (a) absolutely smooth and (b) absolutely rough. (In the first case, the holonomic constraint holds the end of the rod on the plane of the table, and in the second case, at a given point.)

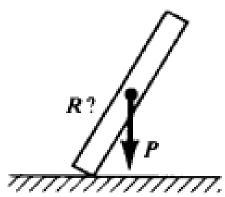

Figure 74 Constraint force on a rod

END OF CHAPTER 4.