CHAPTER 6 RIGID BODIES

In this chapter we study in detail some very special mechanical problems. These problems are traditionally included in a course on classical mechanics, first because they were solved by Euler and Lagrange, and also because we live in three-dimensional euclidean space, so that most of the mechanical systems with a finite number of degrees of freedom which we are likely to encounter consist of rigid bodies.

26 Motion in a moving coordinate system

In this paragraph we define angular velocity.

A *Moving coordinate systems*

We look at a lagrangian system described in coordinates q,t by the lagrangian function $L(q,\dot{q},t)$. It will often be useful to shift to a moving coordinate system $\boldsymbol{Q}=\boldsymbol{Q}(q,t)$.

To write the equations of motion in a moving system, it is sufficient to express the lagrangian function in the new coordinates.

Theorem. *If the trajectory $\gamma: q=\varphi(t)$ of Lagrange's equations $d(\partial L/\partial\dot{q})/dt=\partial L/\partial q$ is written as $\gamma: Q=\Phi(t)$ in the local coordinates Q,t (where $Q=Q(q,t)$), then the function $\Phi(t)$ satisfies Lagrange's equations $d(\partial L'/\partial\dot{\boldsymbol{Q}})/dt=\partial L'/\partial\boldsymbol{Q}$, where $L'(Q,\dot{Q},t)=L(q,\dot{q},t)$.*

PROOF. The trajectory γ is an extremal: $\delta\int_\gamma L(\boldsymbol{q},\dot{\boldsymbol{q}},t)dt=0$. Therefore, $\delta\int_\gamma L(\boldsymbol{Q},\dot{\boldsymbol{Q}},t)dt=0$ and $\Phi(t)$ satisfies Lagrange's equations.

B *Motions, rotations, and translational motions*

We consider, in particular, the important case where **q** is the cartesian radius vector of a point relative to an inertial coordinate system k (which we will call *stationary)*, and **Q** is the cartesian radius vector of the same point relative to a *moving* coordinate system K.

Definition. Let k and K be oriented euclidean spaces. A *motion* of K relative to k is a mapping smoothly depending on t:

$$D_t: K\to k,$$

which preserves the metric and the orientation (Figure 103).

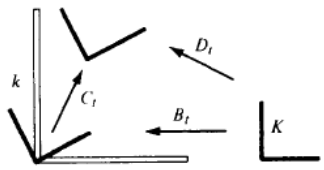

Figure 103 The motion D_t decomposed as the product of a rotation B_t and translation C_t

Definition. A motion D, is called a *rotation* if it takes the origin of K to the origin of k, i.e., if D_t is a linear operator.

Theorem. *Every motion D_t can be uniquely written as the composition of a rotation $B_t: K\to k$ and a translation $C_t: k\to k$:*

$$D_t=C_tB_t$$

where $C_t\boldsymbol{q}=\boldsymbol{q}+\boldsymbol{r}(t)$, $(q,r\in k)$.

PROOF. We set $\boldsymbol{r}(t)=D_t\boldsymbol{0}$, $B_t=C_t^{-1}D_t$. Then $B_t\boldsymbol{0}=\boldsymbol{0}$.

Definition. A motion D_t is called *translational* if the mapping $B_t: K\to k$ corresponding to it does not

depend on $t: B_t = B_0 = B$, $D_t\boldsymbol{Q} = B\boldsymbol{Q} + \boldsymbol{r}(t)$.

We will call k a stationary coordinate system, K a moving one, and $\boldsymbol{q}(t) \in k$ the radius-vector of a point moving relative to the stationary system; if

(1) $$\boldsymbol{q}(t) = D_t\boldsymbol{Q}(t) = B_t\boldsymbol{Q}(t) + \boldsymbol{r}(t)$$

(Figure 104), $\boldsymbol{Q}(t)$ is called the radius vector of the point relative to the moving system.

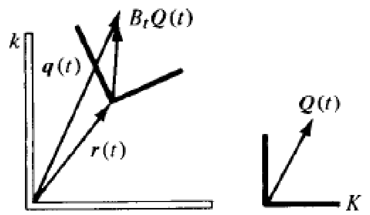

Figure 104 Radius vector of a point with respect to stationary (**q**) and moving (**Q**) coordinate systems

Warning. The vector $B_t\boldsymbol{Q}(t) \in k$ should not be confused with $Q(t) \in K$. T hey lie in different spaces!

C *Addition of velocities*

We will now express the "absolute velocity" $\dot{\boldsymbol{q}}$ in terms of the relative motion $\boldsymbol{Q}(t)$ and the motion of the coordinate system, D_t . By differentiating with respect to t in formula (1) we find a formula for the addition of velocities

(2) $$\dot{\boldsymbol{q}} = \dot{B}\boldsymbol{Q} + B\dot{\boldsymbol{Q}} + \dot{\boldsymbol{r}} .$$

In order to clarify the meaning of the three terms in (2), we consider the following special cases.

The case of translational motion ($\dot{B} = 0$)

In this case Equation (2) gives $\dot{\boldsymbol{q}} = B\dot{\boldsymbol{Q}} + \dot{\boldsymbol{r}}$. In other words, we have shown

Theorem. *If the moving system K has a translational motion relative to k, then the absolute velocity is equal to the sum of the relative velocity and the velocity of the motion of the system K:*

(3) $$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v_0} ,$$

where

$v = \dot{q} \in k$ *is the absolute velocity,*

$v' = B\dot{Q} \in k$ *is the relative velocity (distinct from* $\dot{\boldsymbol{Q}} \in K$!)

$\boldsymbol{v_0} = \dot{\boldsymbol{r}} \in k$ *is the velocity of motion of the moving coordinate system.*

D *Angular velocity*

In the case of a rotation of K the relationship between the relative and absolute velocities is not so simple. We first consider the case when our point is at rest in K (i.e., $\dot{\boldsymbol{Q}} = 0$) and the coordinate system K rotates (i.e., $\boldsymbol{r} = 0$). In this case the motion of the point $\boldsymbol{q}(t)$ is called a *transferred rotation.*

EXAMPLE. *Rotation with fixed angular velocity* $\omega \in k$. Let $U(t): k \to k$ be the rotation of the space k around the $\boldsymbol{\omega}$-axis through the angle $|\boldsymbol{\omega}|t$. Then $B(t) = U(t)B(0)$ is called a *uniform rotation of K with angular velocity* $\boldsymbol{\omega}$.

Clearly, the velocity of the transferred motion of the point **q** in this case is given by the formula (Figure 105):

$$\dot{\boldsymbol{q}} = [\boldsymbol{\omega}, \boldsymbol{q}].$$

We now turn to the general case of a rotation of K **(** $\boldsymbol{r} = 0, \dot{\boldsymbol{Q}} = 0$).

Theorem. *At every moment of time t, there is a vector* $\boldsymbol{\omega}(t) \in k$ *such that the transferred velocity is expressed by the formula*

(4) $$\dot{\boldsymbol{q}} = [\boldsymbol{\omega}, \boldsymbol{q}], \quad \forall \boldsymbol{q} \in k.$$

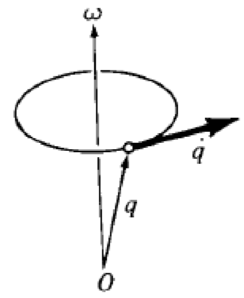

Figure 105 Angular velocity

The vector $\boldsymbol{\omega}$ is called the *instantaneous angular velocity;* clearly, it is defined uniquely by Equation (4).

Corollary. *Suppose that a rigid body K rotates around a stationary point* 0 *of the space k. Then at every moment of time there exists an instantaneous axis of rotation - the straight line in the body passing through* 0 *such that the velocity of its points at the given moment of time is equal to zero. The velocity of the remaining points is perpendicular to this straight line and is proportional to the distance from it.*

The instantaneous axis of rotation in k is given by its vector $\boldsymbol{\omega}$; in K the corresponding vector is denoted by $\boldsymbol{\Omega} = B^{-1}\boldsymbol{\omega} \in K$; $\boldsymbol{\Omega}$ is called *the vector of angular velocity in the body.*

EXAMPLE. The angular velocity of the earth is directed from the center to the North Pole; its length is equal to $2\pi / 3600 \cdot 24 \sec^{-1} \approx 7.3 \times 10^{-5} \sec^{-1}$.

PROOF OF THE THEOREM. By (2) we have

$$\dot{\boldsymbol{q}} = \dot{B}\boldsymbol{Q}.$$

Therefore, if we express $\boldsymbol{Q}$ in terms of $\boldsymbol{q}$, we get $\dot{\boldsymbol{q}} = \dot{B}B^{-1}\boldsymbol{q} = A\boldsymbol{q}$, where $A = \dot{B}B^{-1}: k \to k$ is a linear operator on *k.*

Lemma 1. *The operator A is skew-symmetric:* $A^t + A = 0$.

PROOF. Since $B: K \to k$ is an orthogonal operator from one euclidean space to another, its transpose is its inverse: $B^t = B^{-1}: k \to K$. By differentiating the relationship $BB^t = E$ with respect to *t*, we get

$$\dot{B}B^t + B\dot{B}^t = 0$$

$$\dot{B}B^{-1} + (\dot{B}B^{-1})^t = 0$$

Lemma *2. Every skew-symmetric operator A on a three-dimensional oriented euclidean space is the operator of vector multiplication by a fixed vector:*

$$A\boldsymbol{q} = [\boldsymbol{\omega}, \boldsymbol{q}] \text{ for all } \boldsymbol{q} \in R^3$$

PROOF. The skew-symmetric operators from R^3 to R^3 form a linear space. Its dimension is 3, since a skew-symmetric 3x3 matrix is determined by its three elements below the diagonal. The operator of vector

multiplication by ω is linear and skew-symmetric. The operators of vector multiplication by all possible vectors ω in three-space form a linear subspace of the space of all skew-symmetric operators. The dimension of this subspace is equal to 3. Therefore, the subspace of vector multiplications is the space of all skew-symmetric operators.

CONCLUSION OF THE PROOF OF THE THEOREM. By Lemmas 1 and 2,

$$\dot{\boldsymbol{q}} = A\boldsymbol{q} = [\boldsymbol{\omega}, \boldsymbol{q}]$$

In cartesian coordinates the operator A is given by an antisymmetric matrix; we denote its elements by $\pm\omega_{1,2,3}$:

$$A = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

In this notation the vector $\boldsymbol{\omega} = \omega_1 \boldsymbol{e_1} + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3$ will be an eigenvector with eigenvalue 0. By applying A to the vector $\boldsymbol{q} = q_1 \boldsymbol{e_1} + q_2 \boldsymbol{e}_2 + q_3 \boldsymbol{e}_3$, we obtain by a direct calculation

$$A\boldsymbol{q} = [\boldsymbol{\omega}, \boldsymbol{q}].$$

E *Transferred velocity*

The case of purely rotational motion

Suppose now that the system K rotates (**r** = 0), and that a point in K is moving ($\dot{\boldsymbol{Q}} \neq 0$). From (2) we find (Figure 106)

$$\dot{\boldsymbol{q}} = \dot{B}\boldsymbol{Q} + B\dot{\boldsymbol{Q}} = [\boldsymbol{\omega}, \boldsymbol{q}] + \boldsymbol{v}'.$$

In other words, we have shown:

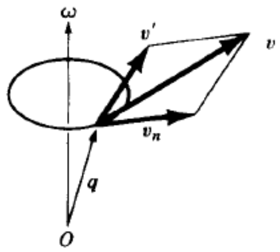

Figure 106 Addition of velocities

Theorem. *If a moving system K rotates relative to* $0 \in k$*, then the absolute velocity is equal to the sum of the relative velocity and the transferred velocity:*

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v_n},$$

where

(5) $v = \dot{q} \in k$ *is the absolute velocity,*

$v' = B\dot{\boldsymbol{Q}} \in k$ *is the relative velocity,*

$v_n = \dot{B}\boldsymbol{Q} = [\boldsymbol{\omega}, \boldsymbol{q}] \in k$ *is the transferred velocity of rotation.*

Finally, the general case can be reduced to the two cases above, if we consider an auxiliary system K_1 which moves by translation with respect to k and with respect to which K moves by rotating around $0 \in K_1$. From formula (2) one can see that

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v_n} + \boldsymbol{v_0},$$

where

$v = \dot{q} \in k$ *is the absolute velocity,*

$v' = B\dot{\boldsymbol{Q}} \in k$ *is the relative velocity,*

$v_n = \dot{B}\boldsymbol{Q} = [\boldsymbol{\omega}, \boldsymbol{q} - \boldsymbol{r}] \in k$ *is the transferred velocity of rotation.*

$\boldsymbol{v_0} = \dot{\boldsymbol{r}} \in k$ *is the velocity of motion of the moving coordinate system.*

PROBLEM. Show that the angular velocity of a rigid body does not depend on the choice of origin of the moving system K in the body.

PROBLEM. Show that the most general movement of a rigid body is a helical movement, i.e., the composition of a rotation through angle φ around some axis and a translation by h along it.

PROBLEM. A watch lies on a table. Find the angular velocity of the hands of the watch: (a) relative to the earth, (b) relative to an inertial coordinate system.

Hint. If we are fiven three coordinate system k, K1, and K2, then the angular velocity of K2 relative to k is equal to the sum of the angular volocities of K1 relative to k and of K2 relative to K1, since

$$(E + A_1 t + \cdots)(E + A2t + \cdots) = E + (A_1 + A_2)t + \cdots .$$

27 **Inertial forces and the Coriolis force**

The equations of motion in a non-inertial coordinate system differ from the equations of motion in an inertial system by additional terms called inertial forces. This allows us to detect experimentally the non-inertial nature of a system (for example, the rotation of the earth around its axis).

A *Coordinate systems moving by translation*

Theorem. *In a coordinate system K which moves by translation relative to an inertial system k, the motion of a mechanical system takes place as if the coordinate system were inertial, but on every point of mass m an additional "inertial force" acted:* $\boldsymbol{F} = -m\ddot{\boldsymbol{r}}$ *, where* $\ddot{\boldsymbol{r}}$ *is the acceleration of the system K.*

PROOF. If $\boldsymbol{Q} = \boldsymbol{q} - \boldsymbol{r}(t)$, then $m\ddot{\boldsymbol{Q}} = m\ddot{\boldsymbol{q}} - m\ddot{\boldsymbol{r}}$. The effect of the translation of the coordinate system is reduced in this way to the appearance of an additional homogeneous force field - mW, where **W** is the acceleration of the origin.

EXAMPLE 1. At the moment of takeoff, a rocket has acceleration $\ddot{\boldsymbol{r}}$ directed upward (Figure 107). Thus, the coordinate system *K* connected to the rocket is not inertial. and an observer inside can detect the existence of a force field *mW* and measure the inertial force, for example, by means of weighted springs. In this case the inertial force is called *overload.*

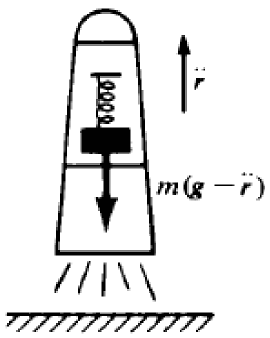

Figure 107 Overload

EXAMPLE 2. When jumping from a loft, a person has acceleration g, directed downwards. Thus, the sum of the inertial force and the force of gravity is equal to zero: weighted springs show that the weight of any object is equal to zero, so such a state is called *weightlessness.* In exactly the same way, weightlessness is observed in the free ballistic flight of a satellite since the force of inertia is opposite to the gravitational force of the earth.

EXAMPLE 3. If the point of suspension of a pendulum moves with acceleration W(t), then the pendulum moves

as if the force of gravity g were variable and equal to g – W(t).

B *Rotating coordinate systems*

Let $B_t : K \to k$ be a rotation of the coordinate system *K* relative to the stationary coordinate system *k*. We will denote by $\boldsymbol{Q}(t) \in K$ the radius vector of a moving point in the moving coordinate system, and by $\boldsymbol{q}(t) = B_t\boldsymbol{Q}(t) \in k$ the radius vector in the stationary system. The vector of angular velocity in the moving coordinate system is denoted, as in Section 26, by $\boldsymbol{\Omega}$. We assume that the motion of the point q in *k* is subject to Newton's equation $m\ddot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.

Theorem. *Motion in a rotating coordinate system takes place as if three additional inertial forces acted on every moving point* $\boldsymbol{Q}$ *of mass m:*

1. *the inertial force of rotation:* $m[\dot{\boldsymbol{\Omega}}, \boldsymbol{Q}]$,
2. *the Coriolis force:* $2m[\boldsymbol{\Omega}, \dot{\boldsymbol{Q}}]$
3. *the centrifugal force:* $m[\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \boldsymbol{Q}]]$.

Thus

$$m\ddot{\boldsymbol{Q}} = \boldsymbol{F} - m[\dot{\boldsymbol{\Omega}}, \boldsymbol{Q}] - 2m[\boldsymbol{\Omega}, \dot{\boldsymbol{Q}}] - m[\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \boldsymbol{Q}]]$$

where

$$B\boldsymbol{F}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) = f(B\boldsymbol{Q}, (B\boldsymbol{Q}))\,.$$

The first of the inertial forces is observed only in nonuniform rotation. The second and third are present even in uniform rotation.

The centrifugal force (Figure 108) is always directed outward from the instantaneous axis of rotation $\boldsymbol{\Omega}$; it has magnitude $|\boldsymbol{\Omega}|^2 r$, where *r* is the distance to this axis. This force does not depend on the velocity of the relative motion, and acts even on a body at rest in the coordinate system *K*.

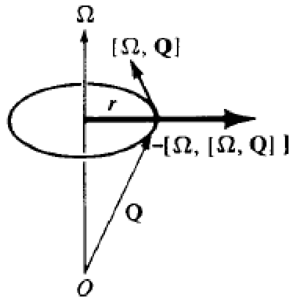

Figure 108 Centrifugal force of inertia

The Coriolis force depends on the velocity $\dot{\boldsymbol{Q}}$. In the northern hemisphere of the earth it deflects every body moving along the earth to the right, and every falling body eastward.

PROOF OF THE THEOREM. We notice that for any vector $\boldsymbol{X} \in K$ we have $\dot{B}\boldsymbol{X} = B[\boldsymbol{\Omega}, \boldsymbol{X}]$. In fact, by Section 26, $\dot{B}\boldsymbol{X} = [\boldsymbol{\omega}, \boldsymbol{x}] = [B\boldsymbol{\Omega}, B\boldsymbol{X}]$. This is equal to $B[\boldsymbol{\Omega}, \boldsymbol{X}]$ since the operator B preserves the metric and orientation, and therefore the vector product. Since $\boldsymbol{q} = B\boldsymbol{Q}$ we see that $\dot{\boldsymbol{q}} = \dot{B}\boldsymbol{Q} + B\dot{\boldsymbol{Q}} = B(\dot{\boldsymbol{Q}} + [\boldsymbol{\Omega}, \boldsymbol{Q}])$. Differentiating once more, we obtain

$$\ddot{q} = \dot{B}(\dot{Q} + [\Omega, Q]) + B(\ddot{Q} + [\dot{\Omega}, Q] + [\Omega, \dot{Q}])$$
$$= B([\Omega, (\dot{Q} + [\Omega, Q])] + \ddot{Q} + [\dot{\Omega}, Q] + [\Omega, \dot{Q}])$$
$$= B(\ddot{Q} + 2[\Omega, \dot{Q}] + [\Omega, [\Omega, Q]] + [\dot{\Omega}, Q])$$

(We again used the relationship $\dot{B}X = B[\Omega, X]$; this time $X = \dot{Q} + [\Omega, Q]$)

We will consider in more detail the effect of the earth's rotation on laboratory experiments. Since the earth rotates practically uniformly, we can take $\dot{\Omega} = 0$. The centrifugal force has its largest value at the equator, where it attains $\Omega^2 \rho / g \approx (7.3\times10^{-5})^2 \cdot 6.4\times10^6 / 9.8 \approx 3/1000$ the weight. Within the limits of a laboratory it changes little, so to observe it one must travel some distance. Thus, within the limits of a laboratory the rotation of the earth appears only in the form of the Coriolis force: in the coordinate system **Q** associated to the earth, we have, with good accuracy,

$$\frac{d}{dt} m\dot{Q} = mg + 2m[\dot{Q}, \Omega]$$

(the centrifugal force is taken into account in g).

EXAMPLE 1 A stone is thrown (without initial velocity) into a 250 m deep mine shaft at the latitude of Leningrad. How far does it deviate from the vertical? We solve the equation

$$\ddot{Q} = g + [\dot{Q}, \Omega]$$

by the following approach, taking $\Omega << 1$. We set (Figure 109)

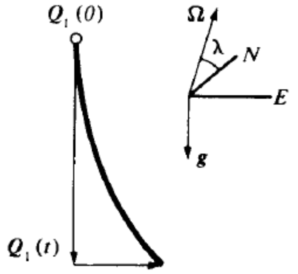

Figure 109 Displacement of a falling stone by Coriolis force

$$Q = Q_1 + Q_2,$$

where $\dot{Q}_2(0) = Q_2(0) = 0$ and $Q_1 = Q_1(0) + gt^2/2$. For Q_2, we then get

$$\ddot{Q}_2 = 2[gt, \Omega] + O(\Omega^2), \quad Q_2 \approx \frac{t^3}{3}[g, \Omega] \approx \frac{2t}{3}[h, \Omega], \quad h = \frac{gt^2}{2}$$

From this it is apparent that the stone lands about

$$\frac{2t}{3}|h||\Omega|\cos\lambda \approx \frac{2\cdot7}{3}\times250\times7\times10^{-5}\times\frac{1}{2}m \approx 4cm$$

to the east.

PROBLEM. By how much would the Coriolis force displace a missile fired vertically upwards at Leningrad from falling back onto its launching pad, if the missile rose 1 kilometer?

EXAMPLE 2 *(The Foucault pendulum).* Consider small oscillations of an ideal pendulum, taking into account the Coriolis force. Let e_x, e_y, and e_z be the axes of a coordinate system associated to the earth, with e_z directed upwards, and e_x and e_y in the horizontal plane (Figure 110). In the approximation

of small oscillations, $\dot{z}=0$ (in comparison with $\dot{x}$ and $\dot{y}$); therefore, the horizontal component of the Coriolis force will be $2m\dot{y}\Omega_z \boldsymbol{e}_x - 2m\dot{x}\Omega_z \boldsymbol{e}_y$. From this we get the equations of motion

$$\begin{cases} \ddot{x} = -\omega^2 x + 2\dot{y}\Omega_z \\ \ddot{y} = -\omega^2 y - 2\dot{x}\Omega_z \end{cases}$$

($\Omega_z = |\Omega| \sin\lambda_0$, where λ_0 is the latitude)

If we set $x+iy=w$, then $\dot{w}=\dot{x}+i\dot{y}$, $\ddot{w}=\ddot{x}+i\ddot{y}$, and the two equations reduce to one complex equation

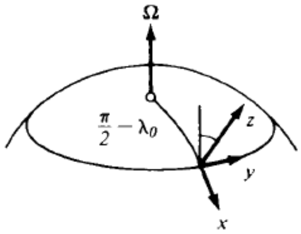

Figure 110 Coordinate system for studying the motion of a Foucault pendulum

$$\ddot{w} + i2\Omega_z \dot{w} + \omega^2 w = 0 .$$

We solve it: $w=e^{\lambda t}$, $\lambda^2 + 2i\Omega_z\lambda + \omega^2 = 0$, $\lambda = -i\Omega_z \pm i\sqrt{\Omega_z^2+\omega^2}$. But $\Omega_z^2 << \omega^2$. Therefore,

$\sqrt{\Omega_z^2+\omega^2} = \omega + O(\Omega_z^2)$, from which it follows, by disregarding Ω_z^2, that

$$\lambda \approx -i\Omega_z \pm i\omega$$

or, to the same accuracy,

$$w = e^{-i\Omega_z t}(c_1 e^{i\omega t} + c_2 e^{-i\omega t}) .$$

For $\Omega_z = 0$ we get the usual harmonic oscillations of a spherical pendulum. We see that the effect of the Coriolis force reduces to a rotation of the whole picture with angular velocity $-\Omega_z$, where $|\Omega_z| = |\boldsymbol{\Omega}|\sin\lambda_0$.

In particular, if the initial conditions correspond to a planar motion ($y(0)=\dot{y}(0)=0$), then the plane of oscillation will be rotating with angular velocity $-\Omega_z$ with respect to the earth's coordinate system (Figure 111).

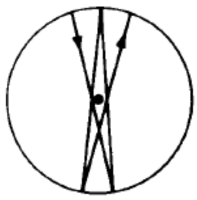
Figure 111 Trajectory of a Foucault pendulum

At a pole, the plane of oscillation makes one turn in a twenty-four-hour day (and is fixed with respect to a coordinate system not rotating with the earth). At the latitude of Moscow (56 deg) the plane of oscillation turns 0.83 of a rotation in a twenty-four-hour day, i.e., 12.5 deg in an hour.

PROBLEM. A river flows with velocity 3 km/hr. For what radius of curvature of a river bend is the Coriolis force from the earth's rotation greater than the centrifugal force determined by the flow of the river?

ANSWER. The radius of curvature must be least on the order of 10 km for a river of medium width. The solution of this problem explains why a large river in the northern hemisphere (for example, the Volga in the middle of its course), undermines the base of its right bank, while a river like the Moscow River, with its abrupt bends of small radius, undermines either the left or right (whichever is outward from the bend) bank.

28 Rigid bodies

In this paragraph we define a rigid body and its inertia tensor, inertia ellipsoid, moments of inertia, and axes of inertia.

A *The configuration manifold of a rigid body*

Definition. A *rigid body* is a system of point masses, constrained by holonomic relations expressed by the fact that the distance between points is constant:

(1) $$\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right| = r_{ij} = const\,.$$

Theorem. *The configuration manifold of a rigid body is a six-dimensional manifold, namely,* $R^3 \times SO(3)$ *(the direct product of a three-dimensional space* R^3 *and the group* S0(3) *of its rotations), as long as there are three points in the body not in a straight line.*

PROOF. Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ be three points of the body which do not lie in a straight line. Consider the right-handed orthonormal frame whose first vector is in the direction of $\boldsymbol{x}_2 - \boldsymbol{x}_1$, and whose second is on the $\boldsymbol{x}_3$ side in the $\boldsymbol{x}_1\,\boldsymbol{x}_2\,\boldsymbol{x}_3$-plane (Figure 112). It follows from the conditions $\left|\boldsymbol{x}_i - \boldsymbol{x}_j\right| = r_{ij}$ $(i = 1,2,3)$, that the positions of all the points of the body are uniquely determined by the positions of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$, which are given by the position of the frame. Finally, the space of frames in R^3 is $R^3 \times SO(3)$, since every frame is obtained from a fixed one by a rotation and a translation.

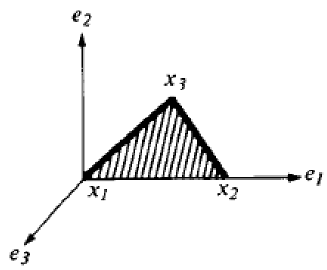

Figure 112 Configuration manifold of a rigid body

PROBLEM. Find the configuration space of a rigid body, all of whose points lie on a line.

ANSWER. $R^3 \times S^2$.

Definition. A rigid body with a fixed point *0* is a system of point masses constrained by the condition $x_1 = 0$ in addition to conditions (1).

Clearly, its configuration manifold is the three-dimensional rotation group SO(3).

B *Conservation laws*

Consider the problem of the motion of a free rigid body under its own inertia, outside of any force field. For an (approximate) example we can use the rolling of a spaceship. The system admits all translational displacements: they do not change the lagrangian function. By Noether's theorem there exist three first integrals: the three components of the vector of momentum. Therefore, we have shown:

Theorem. *Under the free motion of a rigid body, its center of mass moves uniformly and linearly.*

Now we can look at an inertial coordinate system in which the center of inertia is stationary. Then we have:

Corollary. *A free rigid body rotates about its center of mass as if the center of mass were fixed at a stationary point 0.*

In this way, the problem is reduced to the problem, with three degrees of freedom, of the motion of a rigid body around a fixed point *0*. We will study this problem in more detail (not necessarily assuming that *0* is the center

of mass of the body).

The lagrangian function admits all rotations around *0.* By Noether's theorem there exist three corresponding first integrals: the three components of the vector of angular momentum. The total energy of the system, E=T, is also conserved (here it is equal to the kinetic energy). Therefore, we have shown:

Theorem. *In the problem of the motion of a rigid body around a stationary point 0,* in *the absence of outside forces, there are four first integrals: Mx, My, Mz, and E.*

From this theorem we can get qualitative conclusions about the motion without any calculation.

The position and velocity of the body are determined by a point in the six-dimensional manifold TSO(3) - the tangent bundle of the configuration manifold SO(3). The first integrals *Mx, My, Mz.,* and *E* are four functions on *TSO(3).* One can verify that in the general case (if the body does not have any particular symmetry) these four functions are independent. Therefore, the four equations

$$M_x C_1, \quad M_y C_2, \quad M_z C_3, \quad E = C_4 > 0$$

define a two-dimensional submanifold V_c in the six-dimensional manifold *TSO(3).*

This manifold is invariant: if the initial conditions of motion give a point on V_c, then for all time of the motion, the point in TSO(3) corresponding to the position and velocity of the body remains in V_c.

Therefore, V_c admits a tangent vector field (namely, the field of velocities of the motion on *TSO(3));* for $C_4 > 0$ this field cannot have singular points. Furthermore, it is easy to verify that is compact (using *E)* and orientable (since *TSO(3)* is orientable).

In topology it is proved that the only connected orientable compact two-dimensional manifolds are the spheres with n handles, $n \geq 0$ (Figure 113). Of these, only the torus *(n* = 1) admits a tangent vector field without singular points. Therefore, the invariant manifold V_c is a two-dimensional torus (or several tori).

Figure 113 Two-dimensional compact connected orientable manifolds

We will see later that one can choose angular coordinates $\varphi_1, \varphi_2, (\mathrm{mod}\, 2\pi)$ on this torus such that a motion represented by a point of V_c is given by the equations $\dot{\varphi}_1 = \omega_1(c)$, $\dot{\varphi}_2 = \omega_2(c)$.

In other words, a rotation of a rigid body is represented by the superposition of two periodic motions with (usually) different periods: if the frequencies ω_1 and ω_2 are non-commensurable, then the body never returns to its original state of motion. The magnitudes of the frequencies ω_1 and ω_2 depend on the initial conditions C.

C *The inertia operator*

We now go on to the quantitative theory and introduce the following notation. Let *k* be a stationary

coordinate system and K a coordinate system rotating together with the body around the point *0:* in *K* the body is at rest.

Every vector in K is carried over to k by an operator *B.* Corresponding vectors in K and k will be denoted by the same letter; capital for K and lower case for *k.* So, for example (Figure 114),

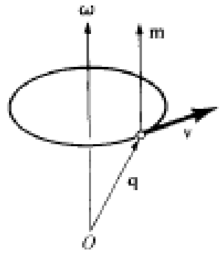

Figure 114 Radius vector and vectors of velocity, angular velocity and angular momentum of a point of the body in space

$\boldsymbol{q} \in k$ is the radius vector of a point in space;

$\boldsymbol{Q} \in K$ is its radius vector in the body, $\boldsymbol{q} = B\boldsymbol{Q}$ q = *BQ;*

$\boldsymbol{v} = \dot{\boldsymbol{q}} \in k$ is the velocity vector of a point in space;

$\boldsymbol{V} \in K$ is the same vector in the body, $\boldsymbol{v} = B\boldsymbol{V}$;

$\omega \in k$ is the angular velocity in space;

$\boldsymbol{\Omega} \in K$ is the angular velocity in the body, $\boldsymbol{\omega} = B\boldsymbol{\Omega}$ *;*

$\boldsymbol{m} \in k$ is the angular momentum in space;

$\boldsymbol{M} \in K$ is the angular momentum in the body, $\boldsymbol{m} = B\boldsymbol{M}$.

Since the operator $B : K \to k$ preserves the metric and orientation, it preserves the scalar and vector products.

By definition of angular velocity (Section 26),

$$\boldsymbol{v} = [\boldsymbol{\omega}, \boldsymbol{q}].$$

By definition of the angular momentum of a point of mass *m* with respect to O,

$$\boldsymbol{m} = [\boldsymbol{q}, m\boldsymbol{v}] = m[\boldsymbol{q}, [\boldsymbol{\omega}, \boldsymbol{q}]].$$

Therefore,

$$\boldsymbol{M} = m[\boldsymbol{Q}, [\boldsymbol{\Omega}, \boldsymbol{Q}]].$$

Hence, there is a linear operator transforming $\boldsymbol{\Omega}$ to $\boldsymbol{M}$:

$$A : K \to K\,, \quad A\boldsymbol{\Omega} = \boldsymbol{M}\ .$$

This operator still depends on a point of the body (**Q**) and its mass (m).

Lemma. *The operatoe A is symmetric.*

PROOF. In view of the relation $([a,b],c) = ([c,a],b)$ we have, for any **X** and **Y** in K,

$$(A\boldsymbol{X}, \boldsymbol{Y}) = m([\boldsymbol{Q}, [\boldsymbol{X}, \boldsymbol{Q}]], \boldsymbol{Y}) = m([\boldsymbol{Y}, \boldsymbol{Q}], [\boldsymbol{X}, \boldsymbol{Q}])\ ,$$

and the last expression is symmetric in **X** and **Y**.

By substituting the vector of angular velocity $\boldsymbol{\Omega}$ for **X** and **Y** and noticing that $[\boldsymbol{\Omega}, \boldsymbol{Q}]^2 = \boldsymbol{V}^2 = \boldsymbol{v}^2$, we obtain:

Corollary. *The kinetic energy of a point of a body is a quadratic form with respect to the vector of angular velocity* $\boldsymbol{\Omega}$ *, namely:*

$$T = \frac{1}{2}(A\boldsymbol{\Omega}, \boldsymbol{\Omega}) = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\Omega}) .$$

The symmetric operator A is called the *inertia operator* (or tensor) *of the point* Q. If a body consists of many points Q_i with masses m_i, then by summing we obtain:

Theorem. *The angular momentum* M *of a rigid body with respect to a stationary point 0 depends linearly on the angular velocity* $\boldsymbol{\Omega}$ *, i.e., there exists a linear operator* $A: K \to K$, $A\boldsymbol{\Omega} = \boldsymbol{M}$ *. The operator* A *is symmetric. he kinetic energy of a body is a quadratic form with respect to the angular velocity* $\boldsymbol{\Omega}$,

$$T = \frac{1}{2}(A\boldsymbol{\Omega}, \boldsymbol{\Omega}) = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\Omega}) .$$

PROOF. By definition, the angular momentum of a body is equal to the sum of the angular momenta of its points:

$$\boldsymbol{M} = \sum_i \boldsymbol{M_i} = \sum_i A_i \boldsymbol{\Omega} = A\boldsymbol{\Omega}, \quad \text{where } \sum_i A_i .$$

Since by the lemma the inertia operator *Ai* of every point is symmetric, the operator A is also symmetric. For kinetic energy we obtain, by definition,

$$T = \sum_i T_i = \sum_i \frac{1}{2}(\boldsymbol{M}_i, \boldsymbol{\Omega}) = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\Omega}) = \frac{1}{2}(A\boldsymbol{\Omega}, \boldsymbol{\Omega}) .$$

D *Principal axes*

Like every symmetric operator, A has three mutually orthogonal characteristic directions. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3 \in K$ be their unit vectors and I_1, I_2, and I_3 their eigenvalues. In the basis $\boldsymbol{e}_i$, the inertia operator and the kinetic energy have a particularly simple form:

$$M_i = I_i \Omega_i , \quad T = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$$

The axes $\boldsymbol{e}_i$ are called the *principal axes* of the body at the point *0*.

Finally, if the numbers I_1, I_2, and I_3 are not all different, then the axes $\boldsymbol{e}_i$ are not uniquely defined. We will further clarify the meaning of the eigenvalues I_1, I_2, and I_3 .

Theorem. For a rotation of a rigid body fixed at a point 0,with angular velocity $\boldsymbol{\Omega} = \Omega \boldsymbol{e}$ ($\Omega = |\boldsymbol{\Omega}|$) around the $\boldsymbol{e}$ axis, the kinetic energy is equal to

$$T = \frac{1}{2} I_e \Omega^2 , \quad \text{where } I_e = \sum_i m_i r_i^2$$

and r_i is the distance of the *i*-th point to the $\boldsymbol{e}$ axis (Figure 115).

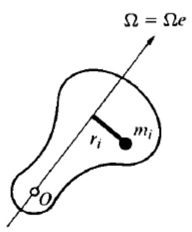

Figure 115 Kinetic energy of a body rotating around an axis

PROOF. By definition $T = 1/2 \sum m_i \boldsymbol{v}_i^2$; but $|\boldsymbol{v}_i| = \boldsymbol{\Omega} r_i$, so $T = 1/2 \left(\sum m_i r_i^2\right)\Omega^2$.
The number I_e depends on the direction **e** of the axis of rotation $\boldsymbol{\Omega}$ in the body.

Definition. I_e is called the *moment of inertia of the body with respect to the* **e** *axis:*

$$I_e = \sum_i m_i r_i^2 .$$

By comparing the two expressions for T we obtain:

Corollary. *The eigenvalues I_i of the inertia operator A are the moments of inertia of the body with respect to the principal axes* $\boldsymbol{e}_i$.

E ***The inertia ellipsoid***

In order to study the dependence of the moment of inertia I_e upon the direction of the axis **e** in a body, we consider the vectors $\boldsymbol{e}/\sqrt{I_e}$, where the unit vector **e** runs over the unit sphere.

Theorem. *The vectors $\boldsymbol{e}/\sqrt{I_e}$ form an ellipsoid in K.*

PROOF. If $\boldsymbol{\Omega} = \boldsymbol{e}/\sqrt{I_e}$, then the quadratic form $T = \frac{1}{2}(A\boldsymbol{\Omega}, \boldsymbol{\Omega})$ is equal to 1/2. Therefore, $\{\boldsymbol{\Omega}\}$ is the level set of a positive definite quadratic form, i.e., an ellipsoid.

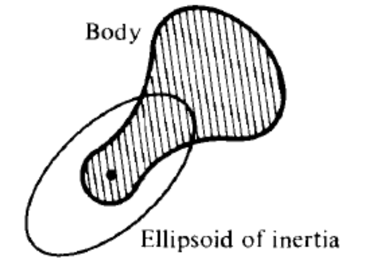

Figure 116 Ellipsoid of inertia

One could say that this ellipsoid consists of those angular velocity vectors $\boldsymbol{\Omega}$ whose kinetic energy is equal to 1/2.

Definition. The ellipsoid $\{\boldsymbol{\Omega} : (A\boldsymbol{\Omega}, \boldsymbol{\Omega}) = 1\}$ is called the *inertia ellipsoid of the body* at the point 0 (Figure 116).

In terms of the principal axes e_i, the equation of the inertia ellipsoid has the form

$$I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 = 1$$

Therefore *the principal axes of the inertia ellipsoid are directed along the principal axes of the inertia tensor, and their lengths are inversely proportional to* $\sqrt{I_e}$.

Remark. If a body is stretched out along some axis, then the moment of inertia with respect to this axis is small, and consequently, the inertia ellipsoid is also stretched out along this axis; thus, the inertia ellipsoid may resemble the shape of the body. If a body has an axis of symmetry of order k passing through 0 (so that it coincides with itself after rotation by $2\pi/k$ around the axis), then the inertia ellipsoid also has the same symmetry with respect to this axis. But a triaxial ellipsoid does not have axes of symmetry of order $k > 2$. Therefore, every axis of symmetry of a body of order $k > 2$ is an axis of rotation of the inertia ellipsoid and, therefore, a principal axis.

EXAMPLE. The inertia ellipsoid of three points of mass m at the vertices of an equilateral triangle with center 0 is an ellipsoid of revolution around an axis normal to the plane of the triangle (Figure 117).

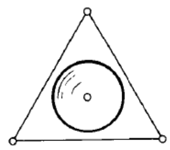
Figure 117 Ellipsoid of inertia of an equilateral triangle

If there are several such axes, then the inertia ellipsoid is a sphere, and any axis is principal.

PROBLEM. Draw the line through the center of a cube such that the sum of the squares of its distances from the vertices of the cube is: (a) largest, (b) smallest.

We now remark that the inertia ellipsoid (or the inertia operator or the moments of inertia I_1, I_2, and I_3)

completely determines the rotational characteristics of our body: if we consider two bodies with identical inertia ellipsoids, then for identical initial conditions they will move identically (since they have the same lagrangian function $L = T$).

Therefore, from the point of view of the dynamics of rotation around 0, *the space of all rigid bodies is three-dimensional,* however many points com- pose the body.

We can even consider the "solid rigid body of density $\rho(\boldsymbol{Q})$," having in mind the limit as $\Delta\boldsymbol{Q} \to 0$ of the sequence of bodies with a finite number of points $\boldsymbol{Q}_i$ with masses $\rho(\boldsymbol{Q}_i)\Delta\boldsymbol{Q}_i$ (Figure 118) or, what amounts to the same thing, any body with moments of inertia

$$I_e = \iiint \rho(\boldsymbol{Q}) r^2(\boldsymbol{Q}) d\boldsymbol{Q},$$

where r is the distance from $\boldsymbol{Q}$ to the **e** axis.

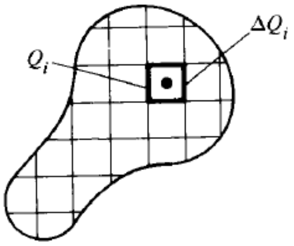

Figure 118 Continuous solid rigid body

EXAMPLE. Find the principal axes and moments of inertia of the uniform planar plate $|x| \le a$, $|y| \le b$, $z = 0$ with respect to 0.

Solution. Since the plate has three planes of symmetry, the inertia ellipsoid has the same planes of symmetry and, therefore, principal axes *x, y,* and *z.* Furthermore,

$$I_y = \int_{-a}^{a} \int_{-b}^{b} x^2 \rho dx dy = \frac{ma^2}{3}$$

In the same way

$$I_x = \frac{mb^2}{3}$$

Clearly, $I_z = I_x + I_y$.

PROBLEM. Show that the moments of inertia of any body satisfy the triangle inequalities

$$I_3 \le I_2 + I_1, \quad I_2 \le I_1 + I_3, \text{ and } I_1 \le I_2 + I_3,$$

and that equality holds only for a planar body.

PROBLEM. Find the axes and moments of inertia of a homogeneous ellipsoid of mass *m* with semiaxes *a, b,* and *c* relative to the center 0.

Hint. First look at the sphere.

PROBLEM. Prove Steiner's theorem: The moments of inertia of any rigid body relative to two parallel axes, one of which passes through the center of mass, are related by the equation

$$I = I_0 + mr^2,$$

where *m* is the mass of the body, *r* is the distance between the axes, and I_0 is the moment of inertia relative to the axis passing through the center of mass.

Thus the moment of inertia relative to an axis passing through the center of mass is less than the moment of inertia relative to any parallel axis.

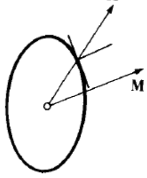

Figure 119 Angular velocity, ellipsoid of inertia and angular momentum

PROBLEM. Find the principal axes and moments of inertia of a uniform tetrahedron relative to its vertices.

PROBLEM. Draw the angular momentum vector **M** for a body with a given inertia ellipsoid rotating

with a given angular velocity $\boldsymbol{\Omega}$.

ANSWER. **M** is in the direction normal to the inertia ellipsoid at a point on the $\boldsymbol{\Omega}$ axis (Figure 119)

PROBLEM. A piece is cut off a rigid body fixed at the stationary point *O*. How are the principal moments of inertia changed? (Figure 120).

ANSWER. All three principal moments are decreased.

Hint. Cf. Section 24.

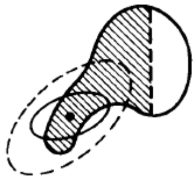

Figure 120 Behavior of moments of inertia as the body becomes smaller

PROBLEM. A small mass ε is added to a rigid body with moments of inertia $I_1 > I_2 > I_3$ at the point $\boldsymbol{Q} = x_1\boldsymbol{e_1} + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e_3}$. Find the change in I_1 and $\boldsymbol{e_1}$ with error O(ε^2).

Solution. The center of mass is displaced by a distance of order ε. Therefore, the moments of inertia of the old body with respect to the parallel axes passing through the old and new centers of mass differ in magnitude of order ε^2. At the same time, the addition of mass changes the moment of inertia relative to any fixed axis by order ε.Therefore, we can disregard the displacement of the center of mass for calculations with error O(ε^2). Thus, after addition of a small mass the kinetic energy takes the form

$$T = T_0 + \frac{1}{2}\varepsilon[\Omega, Q]^2 + O(\varepsilon^2),$$

where $T_0 = \frac{1}{2}(I_1\Omega_1^2 + I_3\Omega_3^2 + I_3\Omega_3^2)$ is the kinetic energy of the original body. We look for the eigenvalue $I_1(\varepsilon)$ and eigenvector $\boldsymbol{e_1}(\varepsilon)$ of the inertia operator in the form of a Taylor series in ε. By equating coefficients of ε in the relation $A(\varepsilon)\boldsymbol{e_1}(\varepsilon) = I_1(\varepsilon)\boldsymbol{e_1}(\varepsilon)$, we find that, within error O(ε^2);

$$I_1(\varepsilon) \approx I_1 + \varepsilon(x_2^2 + x_3^2), \text{ and } \boldsymbol{e}_1(\varepsilon) \approx \boldsymbol{e}_1 + \varepsilon\left(\frac{x_1x_2}{I_2 - I_1}\boldsymbol{e}_2 + \frac{x_1x_3}{I_3 - I_1}\boldsymbol{e}_3\right).$$

From the formula for $I_1(\varepsilon)$ it is clear that the change in the principal moments of inertia (to the first approximation in ε) is as if neither the center of mass nor the principal axes changed. The formula for $\boldsymbol{e}_1(\varepsilon)$ demonstrates how the directions of the principal axes change: the largest principal axis of the inertia ellipsoid approaches the added point, and the smallest recedes from it. Furthermore, the addition of a small mass on one of the principal planes of the inertia ellipsoid rotates the two axes lying in this plane and does not change the direction of the third axis. The appearance of the differences of moments of inertia in the denominator is connected with the fact that the major axes of an ellipsoid of revolution are not defined. If the inertia ellipsoid is nearly an ellipsoid of revolution (i.e., $I_1 \approx I_2$) then the addition of a small mass could strongly turn the axes $\boldsymbol{e_1}$ and $\boldsymbol{e}_2$ in the plane spanned by them.

29 Euler's equations. Poinsot's description of the motion

Here we study the motion of a rigid body around a stationary point in the absence of outside forces and the similar motion of a free rigid body. The motion turns out to have two frequencies.

A *Euler's equations*

Consider the motion of a rigid body around a stationary point O. Let **M** be the angular momentum vector of the body relative to *0* in the body, $\boldsymbol{\Omega}$ the angular velocity vector in the body, and *A* the inertia operator ($A\boldsymbol{\Omega} = \boldsymbol{M}$); the vectors $\boldsymbol{\Omega}$ and **M** belong to the moving coordinate system *K* (Section 26). The angular momentum vector of the body relative to O in space, $\boldsymbol{m} = B\boldsymbol{M}$, is preserved under the motion (Section 28B). Therefore, the vector **M** in the body ($\boldsymbol{M} \in K$) must move so that $\boldsymbol{m} = B_t \boldsymbol{M}(t)$ does not change when *t* changes.

Theorem.

(1) $$\frac{d\boldsymbol{M}}{dt} = [\boldsymbol{M}, \boldsymbol{\Omega}].$$

PROOF. We apply formula (5), Section 26 for the velocity of the motion of the "point" $\boldsymbol{M}(t) \in K$ with respect to the stationary space *k*. We get

$$\dot{\boldsymbol{m}} = B\dot{\boldsymbol{M}} + [\boldsymbol{\omega}, \boldsymbol{m}] = B(\dot{\boldsymbol{M}} + [\boldsymbol{\Omega}, \boldsymbol{M}]).$$

But since the angular momentum **m** with respect to the space is preserved ($\dot{m} = 0$),

$$\dot{\boldsymbol{M}} + [\boldsymbol{\Omega}, \boldsymbol{M}] = 0.$$

Relation (1) is called the *Euler equations.* Since $\boldsymbol{M} = A\boldsymbol{\Omega}$, (1) can be viewed as a differential equation for **M** (or for $\boldsymbol{\Omega}$). If

$$\boldsymbol{\Omega} = \Omega_1 \boldsymbol{e_1} + \Omega_2 \boldsymbol{e}_2 + \Omega_3 \boldsymbol{e_3} \quad \text{and} \quad \boldsymbol{M} = M_1 \boldsymbol{e_1} + M_2 \boldsymbol{e}_2 + M_3 \boldsymbol{e_3}$$

are the decompositions of $\boldsymbol{\Omega}$ and **M** with respect to the principal axes at O, then $M_i = I_i \Omega_i$ and (1) becomes the system of three equations

(2) $$\frac{dM_1}{dt} = a_1 M_2 M_3, \quad \frac{dM_2}{dt} = a_2 M_3 M_1, \quad \frac{dM_3}{dt} = a_3 M_1 M_2,$$

where $a_1 = (I_2 - I_3)/I_2 I_3$, $a_2 = (I_3 - I_1)/I_3 I_1$, and $a_3 = (I_1 - I_2)/I_1 I_2$, or, in the form of a system of three equations for the three components of the angular velocity,

$$I_1 \frac{d\Omega_1}{dt} = (I_2 - I_3)\Omega_2 \Omega_3,$$

$$I_2 \frac{d\Omega_2}{dt} = (I_3 - I_1)\Omega_3 \Omega_1,$$

$$I_3 \frac{d\Omega_3}{dt} = (I_1 - I_2)\Omega_1 \Omega_1.$$

Remark. Suppose that outside forces act on the body, the sum of whose moments with respect to O is equal to **n** in the stationary coordinate system and **N** in the moving system (**n** = B**N**). Then

$$\dot{\boldsymbol{m}} = \boldsymbol{n}$$

and the Euler equations take the form

$$\frac{d\boldsymbol{M}}{dt} = [\boldsymbol{M}, \boldsymbol{\Omega}] + \boldsymbol{N}.$$

B *Solutions of the Euler equations*

Lemma. *The Euler equations* (2) *have two quadratic first integrals*

$$2E = \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}, \text{ and } M^2 = M_1^2 + M_2^2 + M_3^2.$$

PROOF. E is preserved by the law of conservation of energy, and M^2 by the law of conservation of angular momentum **m,** since $\boldsymbol{m}^2 = \boldsymbol{M}^2 = M^2$.

Thus, **M** lies in the intersection of an ellipsoid and a sphere. In order to study the structure of the curves of intersection we will fix the ellipsoid $E > 0$ and change the radius M of the sphere (Figure 121).

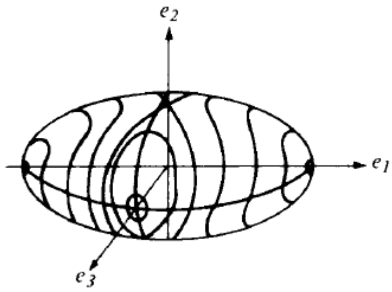

Figure 121 Trajectories of Euler's equation on an energy level surface

We assume that $I_1 > I_2 > I_3$. The semiaxes of the ellipsoid will be $\sqrt{2EI_1} > \sqrt{2EI_2} > \sqrt{2EI_3}$. If the radius M of the sphere is less than the smallest semiaxes or larger than the largest ($M < \sqrt{2EI_3}$ or $M > \sqrt{2EI_1}$), then the intersection is empty, and no actual motion corresponds to such values of E and M. If the radius of the sphere is equal to the smallest semiaxes, then the intersection consists of two points. Increasing the radius, so that $\sqrt{2EI_3} < M < \sqrt{2EI_2}$, we get two curves around the ends of the smallest semiaxes. In exactly the same way, if the radius of the sphere is equal to the largest semiaxes we get their ends, and if it is a little smaller we get two closed curves close to the ends of the largest semiaxes. Finally, if $M = \sqrt{2EI_2}$, the intersection consists of two circles. Each of the six ends of the semiaxes of the ellipsoid is a separate trajectory of the Euler equations (2) - a stationary position of the vector **M.** It corresponds to a fixed value of the vector of angular velocity directed along one of the principal axes $\boldsymbol{e}_i$; during such a motion, $\boldsymbol{\Omega}$ remains collinear with **M.** Therefore, the vector of angular velocity retains its position $\boldsymbol{\omega}$ in space collinear with **m:** the body simply rotates with fixed angular velocity around the principal axis of inertia $\boldsymbol{e}_i$, which is stationary in space.

Definition. A motion of a body, under which its angular velocity remains constant ($\boldsymbol{\omega} = const$, $\boldsymbol{\Omega} = const$) is called a *stationary rotation.*

We have proved:

Theorem. *A rigid body fixed at a point O admits a stationary rotation around any of the three principal axes* e_1,

e_2, *and* e_3.

If, as we assumed, $/_1 > /_2 > /_3$, then the right-hand side of the Euler equations does not become 0 anywhere else, i.e., there are no other stationary rotations.

We will now investigate the stability (in the sense of Liapunov) of solutions to the Euler equations.

Theorem. *The stationary solutions* $\mathbf{M}=M_1e_1$ *and* $\mathbf{M}=M_3e_3$ *of the Euler equations corresponding to the largest and smallest principal axes are stable, while the solution corresponding to the middle axis* ($\mathbf{M}=M_2e_2$) *is unstable.*

PROOF. For a small deviation of the initial condition from M_1e_1 or M_3e_3, the trajectory will be a small closed curve, while for a small deviation from M_2e_2 it will be a large one.

PROBLEM. Are stationary *rotations of the body* around the largest and smallest principal axes Liapunov stable?

ANSWER. No.

C *Poinsot's description of the motion*

It is easy to visualize the motion of the angular momentum and angular velocity vectors in a body ($\mathbf{M}$ and $\boldsymbol{\Omega}$) - they are periodic if $M \neq \sqrt{2EI_i}$.

In order to see how a body rotates *in space*, we look at its inertia ellipsoid.

$$E = \{\boldsymbol{\Omega} : (A\boldsymbol{\Omega}, \boldsymbol{\Omega}) = 1\} \subset K,$$

where $A : \boldsymbol{\Omega} \to \boldsymbol{M}$ is the symmetric operator of inertia of the body fixed at O.

At every moment of time the ellipsoid E occupies a position B_tE in the stationary space *k*.

Theorem (Poinsot). *The inertia ellipsoid rolls without slipping along a stationary plane perpendicular to the angular momentum vector* **m** *(Figure 122).*

PROOF. Consider a plane π perpendicular to the momentum vector **m** and tangent to the inertia ellipsoid B_tE. There are two such planes, and at the point of tangency the normal to the ellipsoid is parallel to **m.**

But the inertia ellipsoid E has normal $grad(A\boldsymbol{\Omega}, \boldsymbol{\Omega}) = 2A\boldsymbol{\Omega} = 2\boldsymbol{M}$ at the point $\boldsymbol{\Omega}$. Therefore, at the points $\pm\xi = \boldsymbol{\omega}/\sqrt{2T}$ of the $\boldsymbol{\omega}$ axis, the normal to B_tE is collinear with **m.**

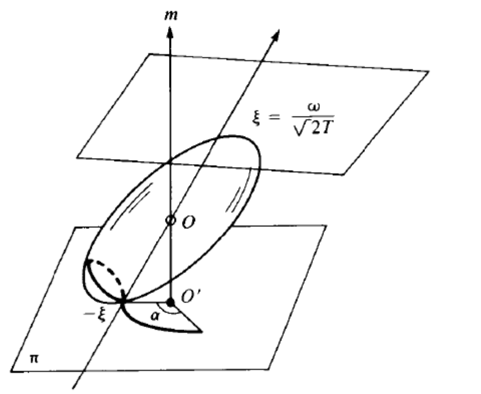

Figure 122 Rolling of the ellipsoid of inertia on the invariable plane

So the plane π is tangent to B_tE at the points $\pm\xi$ on the instantaneous axis of rotation. But the

scalar product of ξ with the stationary vector **m** is equal to $\pm(1/\sqrt{2T}(\boldsymbol{m},\boldsymbol{\omega}) = \pm\sqrt{2T}$, and is therefore constant. So the distance of the plane π from O does not change, i.e., π is stationary.

Since the point of tangency lies on the instantaneous axis of rotation, its velocity is equal to zero. This implies that the ellipsoid B_tE rolls without slipping along π . (The plane π is sometimes called the *invariable plane.*)

Corollary. *Under initial conditions close to a stationary rotation around the large (or small) axis of inertia, the angular velocity always remains close to its initial position, not only in the body $\boldsymbol{\Omega}$ but also in space $\boldsymbol{\omega}$* **.**

We now consider the trajectory of the point of tangency in the stationary plane π . When the point of tangency makes an entire revolution on the ellipsoid, the initial conditions are repeated except that the body has turned through some angle α around the **m** axis. The second revolution will be exactly like the first; if $\alpha = 2\pi(p/q)$, the motion is completely periodic; if the angle is not commensurable with 2π , the body will never return to its initial state.

In this case the trajectory of the point of tangency is dense in an annulus with center *0'* in the plane (Figure 123).

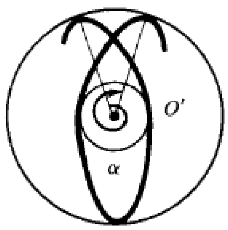

Figure 123 Trajectory of the point of contact on the invariable plane

PROBLEM. Show that the connected components of the invariant two- dimensional manifold V_c (Section 28B) in the six-dimensional space *TS0(3)* are tori, and that one can choose coordinates φ_1 and $\varphi_2 \bmod 2\pi$ on them so that $\dot{\varphi}_1 = \omega_1(c)$ and $\dot{\varphi}_2 = \omega_2(c)$.

Hint. Take the phase of the periodic variation of M as φ_1 .

We now look at the important special case when the inertia ellipsoid is an ellipsoid of revolution:

$$I_2 = I_3 \neq I_1 .$$

In this case the axis of the ellipsoid $B_t\boldsymbol{e_1}$ the instantaneous axis of rotation $\boldsymbol{\omega}$, and the vector **m** always lie in one plane. The angles between them and the length of the vector $\boldsymbol{\omega}$ are preserved; the axes of rotation ($\boldsymbol{\omega}$) and symmetry ($B_t\boldsymbol{e_1}$) sweep out cones around the angular momentum vector **m** with the same angular velocity (Figure 124). This motion around **m** is called *precession.*

PROBLEM. Find the angular velocity of precession.

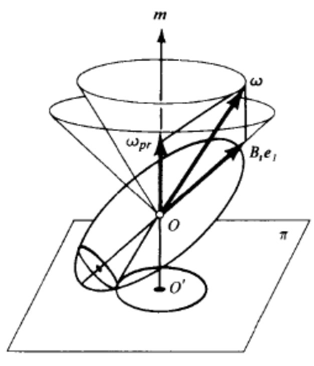

Figure 124 Rolling of an ellipsoid of revolution on the invariable plane

ANSWER. Decompose the angular velocity vector $\boldsymbol{\omega}$ into components in the directions of the angular momentum vector **m** and the axis of the body $B_t\boldsymbol{e_1}$. The first component gives the angular velocity of precession,

Hint. Represent the motion of the body as the product of a rotation around the axis of momentum and a subsequent rotation around the axis of the body. The sum of the angular velocity vectors of these rotations is equal to the angular velocity vector of the product.

Remark. In the absence of outside forces, a rigid body fixed at a point *0* is represented by a lagrangian system whose configuration space is a group, namely SO(3), and the lagrangian function is invariant under left translations. One can show that a significant part of Euler's theory of rigid body motion uses only this property and therefore holds for an arbitrary left-invariant lagrangian system on an arbitrary Lie group. In particular, by applying this theory to the group of volume-preserving diffeomorphisms of a domain Din a riemannian manifold, one can obtain the basic theorems of the hydrodynamics of an ideal fluid. (See Appendix 2.)

30 **Lagrange's top**

We consider here the motion of an axially symmetric rigid body fixed at a stationary point in a uniform force field. This motion is composed of three periodic processes: rotation, precession, and nutation.

A ***Euler angles***

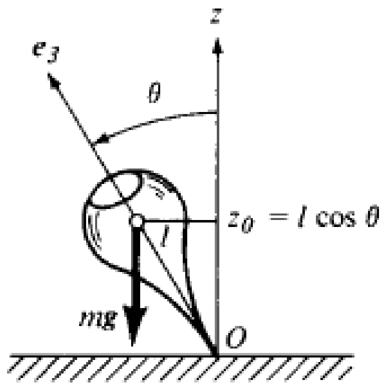

Figure 125 Lagrangian top

Consider a rigid body fixed at a stationary point *O* and subject to the action of the gravitational force m**g**. The problem of the motion of such a "heavy rigid body" has not yet been solved in the general case and in some sense is unsolvable.

In this problem with three degrees of freedom, only two first integrals are known: the total energy E=T+U, and the projection Mz of the angular momentum on the vertical. There is an important special case in which the problem can be completely solved-the case of a *symmetric top.* A symmetric or lagrangian top is a rigid body fixed at a stationary point *O* whose inertia ellipsoid at *O* is an ellipsoid of revolution and whose center of gravity lies on the axis of symmetry $\mathbf{e}_3$ (Figure 125). In this case, a rotation around the $\mathbf{e}_3$ axis does not change the lagrangian function, and by Noether's theorem there must exist a first integral in addition to *E* and Mz (as we will see, it turns out to be the projection M_3 of the angular momentum vector on the $\mathbf{e}_3$ axis).

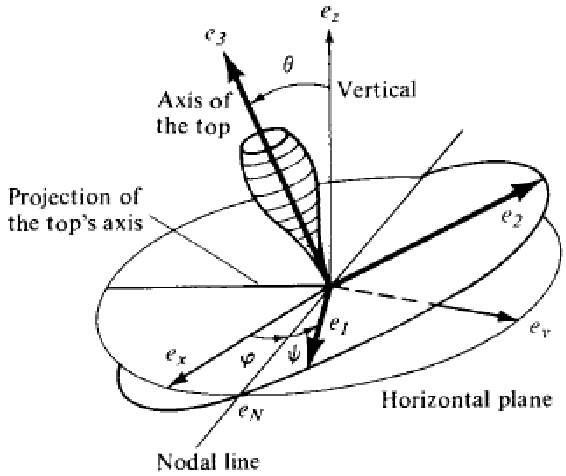

Figure 126 Euler angles

If we can introduce three coordinates so that the angles of rotation around the *z* axis and around the axis of the top are among them, then these coordinates will be cyclic, and the problem with three degrees of freedom will reduce to a problem with one degree of freedom (for the third coordinate).

Such a choice of coordinates on the configuration space SO(3) is possible; these coordinates φ, ψ, θ are called the

Euler angles and form a local coordinate system in SO(3) similar to geographical coordinates on the sphere: they exclude the poles and are multiple-valued on one meridian.

We introduce the following notation (Figure 126):

$\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$*:* the unit vectors of a right-handed cartesian stationary coordinate system at the stationary point O*;*

$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$*:* the unit vectors of a right moving coordinate system connected to the body, directed along the principal axes at O*;*

$I_2 = I_3 \neq I_1$: are the moments of inertia of the body at O*;*

$\boldsymbol{e}_N$ *:* the unit vector of the axis $[\boldsymbol{e}_z, \boldsymbol{e}_3]$, called the "line of nodes" (all vectors are in the "stationary space" k).

In order to carry the stationary frame $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ into the moving frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, we must perform three rotations:

1. Through an angle φ around the $\boldsymbol{e}_z$ axis. Under this rotation, $\boldsymbol{e}_z$ remains fixed, and $\boldsymbol{e}_x$ goes to $\boldsymbol{e}_N$.

2. Through an angle θ around the $\boldsymbol{e}_N$ axis. Under this rotation, $\boldsymbol{e}_z$ goes to $\boldsymbol{e}_3$, and $\boldsymbol{e}_N$ remains fixed.

3. Through an angle ψ around the $\boldsymbol{e}_3$ axis. Under this rotation, $\boldsymbol{e}_N$ goes to $\boldsymbol{e}_1$, and $\boldsymbol{e}_3$ stays fixed.

After all three rotations, $\boldsymbol{e}_x$ has gone to $\boldsymbol{e}_1$ and $\boldsymbol{e}_z$ to $\boldsymbol{e}_3$, therefore, $\boldsymbol{e}_y$ *goes to* $\boldsymbol{e}_2$

The angles φ, ψ and θ are called the *Euler angles.* It is easy to prove:

Theorem. *To every triple of numbers* φ, θ, ψ *the construction above associates a rotation of three-dimensional space,* $B(\varphi, \theta, \psi) \in SO(3)$, *taking the frame* $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ *into the frame* $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. *In addition, the mapping* $(\varphi, \theta, \psi) \to B(\varphi, \theta, \psi)$ *gives local coordinates*

$$0 < \varphi < 2\pi\,, \quad 0 < \psi < 2\pi\,, \quad 0 < \theta < \pi$$

on SO(3), *the configuration space of the top. Like geographical longitude,* φ *and* ψ *can be considered as angles* $\bmod 2\pi$ *; for* $\theta = 0$ *or* $\theta = \pi$ *the map* $(\varphi, \theta, \psi) \to B$ *has a pole-type singularity.*

B *Calculation of the lagrang ian function*

We will express the lagrangian function in terms of the coordinates φ, θ, ψ and their derivatives.

The potential energy, clearly, is equal to

$$U = \iiint zg\,dm = mgz_0 = mgl\cos\theta$$

where z_0 is the height of the center of gravity above 0 (Figure 125).

We now calculate the kinetic energy. A small trick is useful here: we consider the *particular case* when $\varphi = \psi = 0$.

Lemma. *The angular velocity of a top* is *expressed in terms of the derivatives of the Euler angles by theformula*

$$\omega = \dot{\theta}\boldsymbol{e_1} + (\dot{\varphi}\sin\theta)\boldsymbol{e_2} + (\dot{\psi} + \dot{\varphi}\cos\theta)\boldsymbol{e_3}$$

if $\varphi = \psi = 0$.

PROOF. We look at the velocity of a point of the top occupying the position rat time *t*. After time *dt* this point takes the position (within *(dt)*^{2}*)*

$$B(\varphi + d\varphi, \theta + d\theta, \psi + d\psi)B^{-1}(\varphi, \theta, \psi)\boldsymbol{r}$$

where $d\varphi = \dot{\varphi}dt$, $d\theta = \dot{\theta}dt$ and $d\psi = \dot{\psi}dt$.

Consequently, to the same accuracy the displacement vector is the sum of the three terms

$$B(\varphi + d\varphi, \theta, \psi)B^{-1}(\varphi, \theta, \psi)\boldsymbol{r} - \boldsymbol{r} = [\boldsymbol{\omega}_\varphi, \boldsymbol{r}]dt$$

$$B(\varphi, \theta + d\theta, \psi)B^{-1}(\varphi, \theta, \psi)\boldsymbol{r} - \boldsymbol{r} = [\boldsymbol{\omega}_\theta, \boldsymbol{r}]dt\ ,$$

$$B(\varphi, \theta, \psi + d\psi)B^{-1}(\varphi, \theta, \psi)\boldsymbol{r} - \boldsymbol{r} = [\boldsymbol{\omega}_\psi, \boldsymbol{r}]dt$$

(the angular velocities $\boldsymbol{\omega}_\varphi$, $\boldsymbol{\omega}_\theta$, and $\boldsymbol{\omega}_\psi$ are defined by these formulas).

Therefore, the velocity of the point **r** is $\boldsymbol{v} = [\boldsymbol{\omega}_\varphi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi, \boldsymbol{r}]$, so the angular velocity of the body is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\varphi + \boldsymbol{\omega}_\theta + \boldsymbol{\omega}_\psi$$

where the terms are defined by the formulas above.

It remains to decompose the vectors $\boldsymbol{\omega}_\varphi$, $\boldsymbol{\omega}_\theta$, and $\boldsymbol{\omega}_\psi$ with respect to $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. We have not yet used the fact that $\varphi = \psi = 0$. If $\varphi = \psi = 0$, then

$$B(\varphi + d\varphi, \theta, \psi)B^{-1}(\varphi, \theta, \psi)$$

is simply a rotation around the axis $\boldsymbol{e}_z$ through an angle $d\varphi$, so

$$\boldsymbol{\omega}_\varphi = \dot{\varphi}\boldsymbol{e}_z$$

Furthermore, $B(\varphi, \theta + d\theta, \psi)B^{-1}(\varphi, \theta, \psi)$ is simply a rotation around the axis $\boldsymbol{e}_N = \boldsymbol{e}_x = e_1$ through an angled $d\theta$ in the case $\varphi = \psi = 0$, so

$$\boldsymbol{\omega}_\theta = \dot{\theta}e_1.$$

Finally, $B(\varphi, \theta, \psi + d\psi)B^{-1}(\varphi, \theta, \psi)$ is a rotation through an angle $d\psi$ around the axis $\boldsymbol{e}_3$, so

$$\boldsymbol{\omega}_\psi = \dot\psi e_3 .$$

In short, for $\varphi = \psi = 0$ we have

$$\omega = \dot\varphi e_z + \dot\theta e_1 + \dot\psi e_3$$

But, clearly, for $\varphi = \psi = 0$

$$\boldsymbol{e}_z = \boldsymbol{e_3} \cos\theta + \boldsymbol{e_2} \sin\theta$$

So the components of the angular velocity along the principal axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, *and* $\boldsymbol{e}_3$ *are*

$$\omega_1 = \dot\theta , \quad \omega_2 = \dot\varphi \sin\theta , \quad \omega_3 = \dot\psi + \dot\varphi \cos\theta .$$

Since $T = 1/2(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$, the kinetic energy for $\varphi = \psi = 0$ is given by the formula

$$T = \frac{I_1}{2}(\dot\theta^2 + \dot\varphi^2 \sin^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\varphi\cos\theta)^2 .$$

But the kinetic energy cannot depend on φ and ψ: these are cyclic coordinates, and by a choice of origin of reference for φ and ψ which does not change T we can always make $\varphi = 0$ and $\varphi = \psi = 0$. Thus the formula we got for the kinetic energy is true for all φ and ψ.

In this way we obtain the lagrangian function

$$L = \frac{I_1}{2}(\dot\theta^2 + \dot\varphi^2 \sin^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\varphi\cos\theta)^2 - mgl\cos\theta .$$

C *Investigation of the motion*

To the cyclic coordinates φ and ψ there correspond the first integrals

$$\frac{\partial L}{\partial\dot\varphi} = M_z = \dot\varphi(I_1\sin^2\theta + I_3\cos^2\theta) + \dot\psi I_3\cos\theta ,$$

$$\frac{\partial L}{\partial\dot\psi} = M_3 = \dot\varphi I_3\cos\theta + \dot\psi I_3 .$$

Theorem. *The inclination* θ *of the axis of the top to the vertical changes with time in the same way as in the one-dimensional system with energy*

$$E' = \frac{I_1}{2}\dot\theta^2 + U_{eff}(\theta) ,$$

where the effective potential energy is given by the formula

$$U_{eff} = \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta .$$

PROOF. Following the general theory, we express $\dot\varphi$ and $\dot\psi$ in terms of M_3 and M_z. We get the total energy of the system as

$$E = \frac{I_1}{2}\theta'^2 + \frac{M_3^2}{2I_3} + mgl\cos\theta + \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta}$$

and

$$\dot{\varphi}=\frac{M_z-M_3\cos\theta}{I_1\sin^2\theta}.$$

The number $M_3^2/2I_3=E-E'$, independent θ , does not affect the equation for θ.

In order to study the one-dimensional system above it is convenient to make the substitution $\cos\theta=u$, $-1\le u\le 1$.

We also write

$$\frac{M_z}{I_1}=a,\ \frac{M_3}{I_1}=b,\ \frac{2E'}{I_1}=\alpha .$$

Then we can rewrote the law of conservation of energy E' as

$$\dot{u}^2=f(u),$$

where $f(u)=(\alpha-\beta u)(1-u^2)-(a-bu)^2$, and the law of variation of the azimuth φ as

$$\dot{\varphi}=\frac{a-bu}{1-u^2}.$$

We notice that $f(u)$ is a polynomial of degree 3, $f(+\infty)=+\infty$, and $f(\pm1)=-(a\mp b)<0$ if $a\neq\pm b$. On the other hand, actual motions correspond to constants a,b,α and β for which $f(u)\ge 0$ for some $-1\le u\le 1$. Thus $f(u)$ has exactly two real roots u_1 and u_2 on the interval $-1\le u\le 1$ (and one for $u>1$, Figure 127). Therefore, the inclination θ of the axis of the top changes periodically between two limit values θ_1 and θ_2 (Figure 128). This periodic change in inclination is called *nutation.*

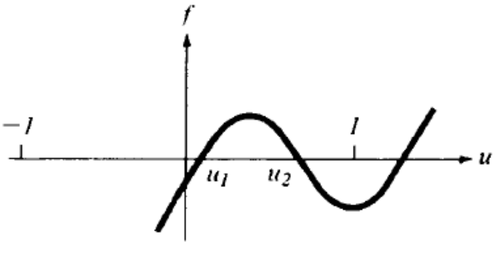

Figure 127 Graph of the function $f(u)$

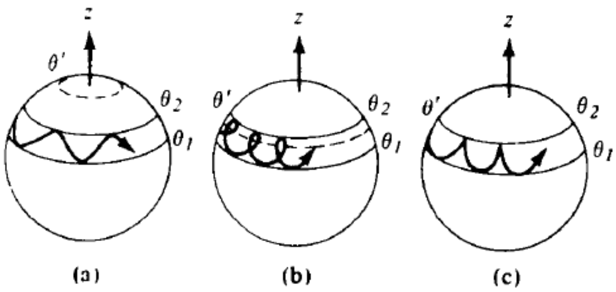

Figure 128 Path of the top's axis on the unit sphere

We now consider the motion of the azimuth of the axis of the top. The point of intersection of the axis with the unit sphere moves in the ring between the parallels θ_1 and θ_2. The variation of the azimuth of the axis is determined by the equation

$$\dot{\varphi} = \frac{a - bu}{1 - u^2}.$$

If the root u' of the equation $a = bu$ lies outside of (u_1, u_2), then the angle φ varies monotonically and the axis traces a curve like a sinusoid on the unit sphere (Figure l28(a)). If the root u' of the equation $a = bu$ lies inside (u_1, u_2), then the rate of change of φ is in opposite directions on the parallels θ_1 and θ_2 ,and the axis traces a looping curve in the sphere (Figure 128(b)).
If the root u' of $a = bu$ lies on the boundary (e.g., $u' = u_2$), then the axis traces a curve with cusps (Figure 128(c)).
The last case, although exceptional, is observed every time we release the axis of a top launched at inclination θ_2 without initial velocity; the top first falls, but then rises again.

The azimuthal motion of the top is called *precession.* The complete motion of the top consists of rotation around its own axis, nutation, and precession. Each of the three motions has its own frequency. If he frequencies are incommensurable, the top never returns to its initial position, although it approaches it arbitrarily closely.

31 Seeping tops and fast tops

The formulas obtained in Section 30 reduce the solution of the equations of motion of a top to elliptic integrals. However, qualitative information about the motion is usually easy to obtain without turning to quadrature. In this paragraph we investigate the stability of a vertical top and give approximate formulas for the motion of a rapidly spinning top.

A *Sleeping tops*

We consider first the particular solution of the equations of motion in which the axis of the top is always vertical ($\theta = 0$) and the angular velocity is constant (a "sleeping" top). In this case, clearly, $M_z = M_3 = I_3\omega_3$ (Figure 129).

We will look at the motion of the *axis of the top,* and not of the top itself. Will the axis of the top stably remain close to the vertical, i.e., will θ remain small? Expressing the effective potential energy of the system

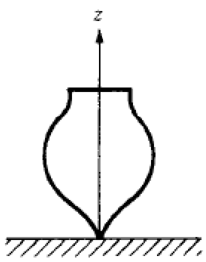

Figure 129 Sleeping top

$$U_{eff} = \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} + mgl\cos\theta$$

as a power series in θ, we find

$$U_{eff} = \frac{I_3{}^2\omega_3^2(\theta^4/4)}{2I_1\theta^2} + \cdots - mgl\frac{\theta^2}{2} + \cdots = C + A\theta^2 + \cdots,$$

$$A = \frac{\omega_3^2 I_3^2}{8I_1} - \frac{mgl}{2}.$$

If $A > 0$, the equilibrium position $\theta = 0$ of the one-dimensional system is stable, and if $A < 0$ it is

unstable. Thus, the *condition for stability has the form*

$$\omega_3^2 > \frac{4mglI_1}{I_3^2}.$$

When friction reduces the velocity of a sleeping top to below this limit, the top wakes up.

PROBLEM. Show that, for $\omega_3^2 > 4mglI_1 / I_3^2$, the axis of a sleeping top is stable with respect to perturbations which change the values of M_z, and M_3, as well as θ.

B *Fast tops*

A top is called *fast* if the kinetic energy of its rotation is large in comparison with its potential energy:

$$\frac{1}{2} I_3\omega_3^2 >> mgl$$

It is clear from a similarity argument that multiplying the angular velocity by N is exactly equivalent to dividing the weight by N^2.

Theorem. *If, while the initial position of a top is preserved, the angular velocity is multiplied by N, then the trajectory of the top will be exactly the same as if the angular velocity remained as it was and the acceleration of gravity g were divided by N^2. In the case of large angular velocity the trajectory clearly goes N times faster.*

In this way we can study the case $g \to 0$ and apply the results to study the case $\omega \to \infty$.

To begin, we consider the case $g = 0$, i.e., the motion of a symmetric top in the absence of gravity. We compare two descriptions of this motion: Lagrange's (Section 30C) and Poinsot's (Section 29C).

We first consider Lagrange's equation for the variation of the angle of inclination θ of the top's axis.

Lemma. *In the absence of gravity, the angle θ_0 satisfying $M_z = M_3 \cos\theta_0$ is a stable equilibrium position of the equation of motion of the top's axis. The frequency of small oscillations of θ near this equilibrium position is equal to*

$$\omega_{nut} = \frac{I_3\omega_3}{I_1}.$$

PROOF. In the absence of gravity the effective potential energy reduces to

$$U_{eff} = \frac{(M_z - M_3\cos\theta)^2}{2I_1 \sin^2\theta}$$

This nonnegative function has the minimum value of zero for the angle $\theta = \theta_0$ determined by the condition $M_z = M_3\cos\theta_0$ (Figure 130). Thus, the angle of

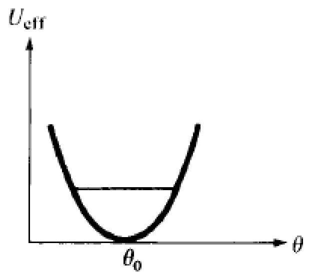

Figure 130 Effective potential energy of a top

inclination θ_0 of the top's axis to the vertical is stably stationary: for small deviations of the initial angle θ from θ_0, there will be periodic oscillations of θ near θ_0 (nutation). The frequency of these oscillations is easily determined by the following general formula: the frequency ω of small oscillations in a one-dimensional system with energy

$$E = \frac{a\dot{x}^2}{2} + U(x)\,, \quad U(x_0) = \min U(x)$$

is given (Section 22D) by the formula

$$\omega^2 = \frac{U''(x_0)}{a}\,.$$

The energy of the one-dimensional system describing oscillations of the inclination of the top's axis is

$$\frac{I_1}{2}\dot{\theta}^2 + U_{eff}$$

For $\theta = \theta_0 + x$ we find $M_z - M_3\cos\theta = M_3(\cos\theta_0 - \cos(\theta_0 + x)) = M_3 x \sin\theta_0 + O(x^2)$

$$U_{eff} = \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta} + o(x^2) = \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta}\,,$$

from which we obtain the expression for the frequency of nutation

$$\omega_{nut} = \frac{I_3\omega_3}{I_1}\,.$$

From the formula $\dot{\varphi} = (M_z - M_3\cos\theta)/I_1\sin^2\theta$ it is clear that, for $\theta = \theta_0$, the azimuth of the axis does not change with time: the axis is stationary. The azimuthal motion of the axis under small deviations of θ from θ_0 could also be studied with the help of this formula, but we will deal with it differently.

The motion of a top in the absence of gravity can be considered in Poinsot's description. Then the axis of the top rotates uniformly around the angular momentum vector, preserving its position in space. Thus, the axis of the top describes a circle on the sphere whose center corresponds to the angular momentum vector (Figure 131).

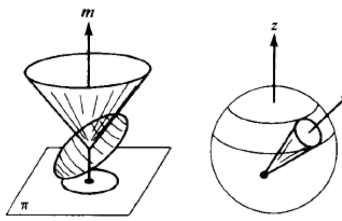

Figure 131 Comparison of the descriptions of the motion of a top according to Lagrange and Poinsot

Remark. Now the motion of the top's axis, which according to Lagrange was called *nutation,* is called *precession* in Poinsot's description of motion.

This means that the formula obtained above for the frequency of a small nutation, $\omega_{nut} = I_3\omega_3/I_1$ agrees with the formula for the frequency of precession $\omega = M/I_1$ in Poinsot's description: when the amplitude of

nutation approaches zero, $I_3\omega_3 \to M$.

C *A top in a weak field*

We go now to the case when the force of gravity is not absent, but is very small (the values of Mz and M_3 are fixed). In this case a term $mgl\cos\theta$, small together with its derivatives, is added to the effective potential energy. We will show that this term slightly changes the frequency of nutation.

Lemma. *Suppose that the function* f(x) *has a minimum at* $x = 0$ *and Taylor expansion* $f(x) = Ax^2/2+\cdots$, $A > 0$. *Suppose that the function* h(x) *has Taylor expansion* $h(x) = B + Cx + \cdots$. *Then, for sufficiently small* ε *, the function* $f_c(x) = f(x) + \varepsilon h(x)$ *has a minimum at the point (Figure 132)*

$$x_\varepsilon = -\frac{C\varepsilon}{A} + O(\varepsilon^2),$$

which is close to zero. In addition, $f''_\varepsilon(x_\varepsilon) = A + O(\varepsilon)$.

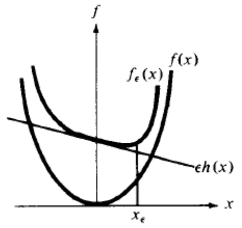

Figure 132 Displacement of the minimum under a small change of the function

PROOF. We have $f'_\varepsilon(c) = Ax + C\varepsilon + O(x^2) + O(\varepsilon x)$, and the result is obtained by applying the implicit function theorem to $f'_\varepsilon(x)$.

By the lemma, the effective potential energy for small g has a minimum θ_g close to θ_0, and at this point U'' differs slightly from $U''(\theta_0)$. Therefore, the frequency of a small nutation near θ_0 is close to that obtained for $g = 0$:

$$\lim_{g\to 0}\omega_{nut} = \frac{I_3}{I_1}\omega_3.$$

D *A rapidly thrown top*

We now consider the special initial conditions when we release the axis of the top without an initial push from a position with inclination θ_0 to the vertical.

Theorem. *If the axis of the top is stationary at the initial moment* ($\dot\varphi = \dot\theta = 0$) *and the top is rotating rapidly around its axis* ($\omega_3 \to \infty$), *which is inclined from the vertical with angle* θ_0 ($M_z = M_3\cos\theta_0$), *then asymptotically, as* $\omega_3 \to \infty$,

1. the nutation frequency is proportional to the angular velocity;

2. the amplitude of nutation is inversely proportional to the square of the angular velocity;

3. the frequency of precession is inversely proportional to the angular velocity;

4. the following asymptotic formulas hold (as $\omega_3 \to \infty$*):*

$$\omega_{nut} \approx \frac{I_3}{I_1}\omega_3, \quad a_{nut} \approx \frac{I_1 mgl}{I_3^2\omega_3^2}\sin\theta_0, \quad \omega_{prec} \approx \frac{mgl}{I_3\omega_3}$$

(here $f(\omega_3) \approx g(\omega_3)$ if $\lim\limits_{\omega_3 \to \infty} (f/g) = 1$).

For the proof, we look at the case when the initial angular velocity is fixed, but $g \to \infty$. Then by interpreting the formulas with the aid of a similarity argument (cf. Section B), we obtain the theorem.

We already know from Section 30C that under our initial conditions the axis of the top traces a curve with cusps on the sphere.

We apply the lemma to locate the minimum point θ_g of the effective potential energy. We set (Figure 133)

$$\theta = \theta_0 + x, \quad \cos\theta = \cos\theta_0 - x\sin\theta_0 + \cdots.$$

Then we obtain, as above, the Taylor expansion in x at θ_0:

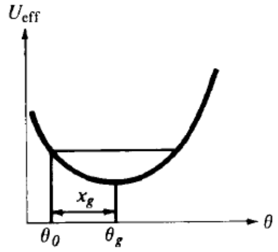

Figure 133 Definition of the amplitude of nutation

$$U_{eff}\Big|_{g=0} = \frac{I_3^2\omega_3^2}{2I_1}x^2 + \cdots, \quad mgl\cos\theta = mgl\cos\theta_0 - xmgl\sin\theta_0 + \cdots.$$

Applying the lemma to $f = U_{eff}\Big|_{g=0}$, $g = \varepsilon$, $h = ml\cos(\theta_0 + x)$, we find that the minimum of the effective potential energy U_{eff} is attained at angle of inclination:

$$\theta_g = \theta_0 + x_g, \quad x_g = \frac{I_1 ml\sin\theta_0}{I_3^2\omega_3^2}g + O(g^2).$$

Thus the inclination θ of the top's axis will oscillate near θ_g (Figure 134). But, at the initial moment, $\theta = \theta_0$ and $\dot{\theta} = 0$. This means that θ_0 corresponds to the highest position of the axis of the top. Thus, for small g, the amplitude of nutation is asymptotically equal to

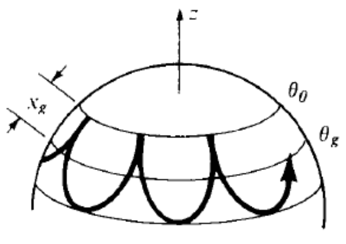

Figure 134 Motion of a top's axis

$$a_{nut} \approx x_g \approx \frac{I_1 ml\sin\theta_0}{I_3^2\omega_3^2}g \quad (g \to 0).$$

We now find the precession at motion of the axis. From the general formula

$$\dot{\varphi} = \frac{M_z - M_3\cos\theta}{I_1\sin^2\theta}$$

for $M_z = M_3\cos\theta_0$ and $\theta = \theta_0 + x$, we find that $M_z = M_3\cos\theta = M_3 x\sin\theta_0 + \cdots$, so

$$\dot{\varphi} = \frac{M_3}{I_1\sin\theta_0 x} + \cdots.$$

But x oscillates harmonically between 0 and $2x_g$ (up to $O(g^2)$). Therefore, the average value of the velocity of precession over the period of nutation is asymptotically equal to

$$\overline{\dot{\varphi}} \approx \frac{M_3}{I_1 \sin\theta_0} x_g \approx \frac{mgl}{I_3\omega_3} = 1\,.$$

PROBLEM. Show that

$$\lim_{g\to 0}\lim_{t\to\infty} \frac{\varphi(t)-\varphi(0)}{tmgl/I_3\omega_3} = 1\,.$$

END OF CHAPTER 6