Chaptet 8 Symplectic manifolds

A symplectic structure on a manifold is a closed nondegenerate differential 2-form. The phase space of a mechanical system has a natural symplectic structure.

On a symplectic manifold, as on a riemannian manifold, there is a natural isomorphism between vector fields and 1-forms. A vector field on a symplectic manifold corresponding to the differential of a function is called a ***hamiltonian vector field***. A vector field on a manifold determines a phase flow, i.e., a one-parameter group of diffeomorphisms. The phase flow of a hamiltonian vector field on a symplectic manifold preserves the symplectic structure of phase space.

The vector fields on a manifold form a Lie algebra. The hamiltonian vector fields on a symplectic manifold also form a Lie algebra. The operation in this algebra is called the ***Poisson bracket***.

37. Symplectic structures on manifolds

We define here symplectic manifolds, hamiltonian vector fields, and the standard symplectic structure on the cotangent bundle.

A. *Definition*

Let M^{2n} be an even-dimensional manifold. A symplectic structure on M^{2n} is a closed nondegenerate differential 2-form ω^2 on M^{2n}:

$$d\omega^2 = 0 \quad \text{and} \quad \forall \xi \neq 0 \quad \exists \boldsymbol{\eta} : \omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) \neq 0 \quad (\boldsymbol{\xi}, \boldsymbol{\eta} \in TM_x)$$

The pair (M^{2n}, ω^2) is called a ***symplectic manifold***.

EXAMPLE. Consider the vector space R^{2n} with coordinates p_i, q_i and let $\omega^2 = \sum dp_i \wedge dq_i$.

PROBLEM. Verify that (R^{2n}, ω^2) is a symplectic manifold. For n =1 the pair (R^{2n}, ω^2) is the pair (the plane, area).

The following example explains the appearance of symplectic manifolds in dynamics. Along with the tangent bundle of a differentiable manifold, it is often useful to look at its dual - the

cotangent bundle.

B ***The cotangent bundle and its symplectic structure***

Let V be an n-dimensional differentiable manifold. A 1-form on the tangent space to V at a point $\boldsymbol{x}$ is called a ***cotangent vector*** to V at $\boldsymbol{x}$. The set of all cotangent vectors to V at $\boldsymbol{x}$ forms an n-dimensional vector space, dual to the tangent space TV_x. We will denote this vector space of cotangent vectors by T^*V_x and call it the ***cotangent space*** to V at $\boldsymbol{x}$.

The union of the cotangent spaces to the manifold at all of its points is called the ***cotangent bundle*** of V and is denoted by T^*V. The set T^*V has a natural structure of a differentiable manifold of dimension $2n$. A point of T^*V is a 1-form on the tangent space to V at some point of V. If $\boldsymbol{q}$ is a choice of n local coordinates for points in V, then such a form is given by its n components $\boldsymbol{p}$. Together, the $2n$ numbers $\boldsymbol{p}$, $\boldsymbol{q}$ form a collection of local coordinates for points in T^*V.

There is a natural projection $f: T^*V \to V$ (sending every 1-form on TV_x to the point $\boldsymbol{x}$). The projection f is differentiable and surjective. The pre-image of a point $\boldsymbol{x} \in V$ under is the cotangent space T^*V_x.

Theorem. *The cotangent bundle T*V has a natural symplectic structure. In the local coordinates described above, this symplectic structure is given by the formula*

$$\omega^2 = d\boldsymbol{p} \wedge d\boldsymbol{q} = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n .$$

PROOF. First, we define a distinguished 1-form on T^*V. Let

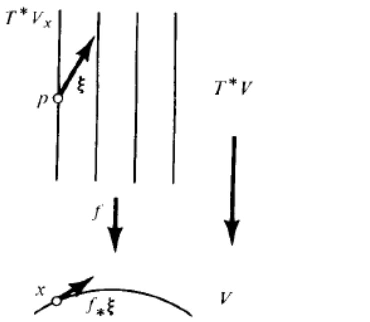

Figure 166 The 1-form **p** d**q** on the cotangent bundle

$\xi \in T(T^*V)_P$ be a vector tangent to the cotangent bundle at the point

$p \in T^*V_x$ (Figure 166). The derivative $f_*: T(T^*V) \to TV$ of the natural projection $f: T^*V \to V$ takes $\boldsymbol{\xi}$ to a vector $f_*\boldsymbol{\xi}$ tangent to V at $\boldsymbol{x}$. We define a 1-form ω^1 on T^*V by the relation $\omega^1(\xi) = p(f_*\xi)$. In the local coordinates described above, this form is $\omega^1 = \boldsymbol{p}d\boldsymbol{q}$. By the example in A, the closed 2-form $\omega^2 = d\omega^1$ is nondegenetate.

Remark. Consider a lagrangian mechanical system with configuration manifold V and function L. It is easy to see that the lagrangian "generalized velocity" $\dot{\boldsymbol{q}}$ is a tangent vector to the configuration manifold V, and the "generalized momentum" $\boldsymbol{p} = \partial L / \partial \dot{\boldsymbol{q}}$ is a cotangent vector. Therefore, the "***p***, ***q***" phase space of the lagrangian system is the cotangent bundle of the configuration manifold. The theorem above shows that the phase space of a mechanical problem has a natural symplectic manifold structure.

PROBLEM. Show that the Legendre transform does not depend on the coordinate system: it takes a function $L: TV \to R$ on the tangent bundle to a function $H: T^*V \to R$ on the cotangent bundle.

C *Hamiltonian vector fields*

A riemannian structure on a manifold establishes an isomorphism between the spaces of tangent vectors and 1-forms. A symplectic structure establishes a similar isomorphism.

Definition. To each vector $\boldsymbol{\xi}$ tangent to a symplectic manifold (M^{2n}, ω^2) at the point $\boldsymbol{x}$, we associate a 1-form ω^1_ξ on TM_x by the formula

$$\omega^1_\xi(\boldsymbol{\eta}) = \omega^2(\boldsymbol{\eta}, \xi) \quad \forall \boldsymbol{\eta} \in TM_x.$$

PROBLEM. Show that the correspondence $\boldsymbol{\xi} \to \omega^1_\xi$ is an isomorphism between the $2n$-dimensional vector spaces of vectors and of 1-forms.

EXAMPLE. In $R^{2n} = \{(\boldsymbol{p}, \boldsymbol{q})\}$ we will identify vectors and 1 -forms by using the euclidean structure $(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{p}^2 + \boldsymbol{q}^2$. Then the correspondence $\xi \to \omega^1_\xi$ determines a transformation $R^{2n} \to R^{2n}$.

PROBLEM. Calculate the matrix of this transformation in the basis ***p***, ***q***.

ANSWER.

$$\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

We will denote by I the isomorphism $I: T^*M_x \to TM_x$ constructed above. Now let H be a function on a symplectic manifold M^{2n}. Then dH is a differentia 1-form on M, and at every point there is a tangent vector to M associated to it. In this way we obtain a vector field IdH on M.

Definition. The vector field IdH is called a ***hamiltonian vector field***; H is called the ***hamiltonian function***.

EXAMPLE. If $M^{2n} = R^{2n} = \{(\boldsymbol{p},\boldsymbol{q})\}$, then we obtain the phase velocity vector field of Hamilton's canonical equations:

$$\dot{\boldsymbol{x}} = IdH(\boldsymbol{x}) \Leftrightarrow \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} \text{ and } \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}.$$

38 Hamiltonian phase flows and their integral invariants

Liouville's theorem asserts that the phase flow preserves volume. Poincaré found a whole series of differential forms which are preserved by the hamiltonian phase flow.

A *Hamiltonian phase flows preserve the symplectic structure*

Let (M^{2n},ω^2) be a symplectic manifold and $H: M^{2n} \to R$ a function. Assume that the vector field IdH corresponding to H gives a 1-parameter group of diffeomorphisms $g^t: M^{2n} \to M^{2n}$:

$$\left.\frac{d}{dt}\right|_{t=0} g^t\boldsymbol{x} = IdH(\boldsymbol{x}).$$

The group g^t is called the ***hamiltonian phase flow*** with hamiltonian function H.

Theorem. *A hamiltonian phase flow preserves the symplectic structure*:

$$(g^t)^*\omega^2 = \omega^2.$$

In the case $n = 1$, $M^{2n} = R^2$, this theorem says that the phase flow g^t preserves area (***Liouville's theorem***).

For the proof of this theorem, it is useful to introduce the following

notation (Figure 167).

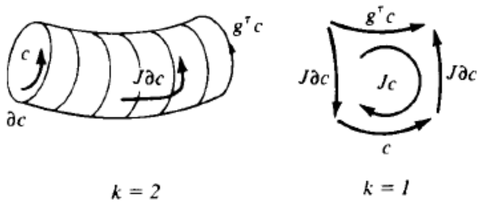

Figure 167 Track of a cycle under homotopy

Let M be an arbitrary manifold, c a k-chain on M and $g^t : M \to M$ a one-parameter family of differentiable mappings. We will construct a (k+1)-chain Jc on M, which we will call the ***track of the chain c under the homotopy*** g^t, $0 \le t \le \tau$.

Let (D,f,Or) be one of the cells in the chain c. To this cell will be associated a cell (D',f',Or') in the chain Jc, where $D' = I \times D$ is the direct product of a the interval $0 \le t \le \tau$ and D; the mapping $f' : D' \to M$ is obtained from $f : D \to M$ by the formula $f'(t,x) = g^t f(x)$; and the orientation Or' of the space R^{k+1} containing D' is given by the frame $\boldsymbol{e_0}, \boldsymbol{e_1}, \ldots, \boldsymbol{e_k}$, where $\boldsymbol{e_0}$ is the unit vector of the t axis, and $\boldsymbol{e_1}, \ldots, \boldsymbol{e_k}$ is an oriented frame for D.

We could say that Jc is the chain swept out by c under the homotopy g^t, $0 \le t \le \tau$. The boundary of the chain Jc consists of "end-walls" made up of the initial and final positions of c, and "side surfaces" filled in by the boundary of c.

It is easy to verify that under the choice of orientation made above,

(1) $\qquad \partial(Jc_k) = g^t c_k - c_k - J\partial c_k .$

Lemma. *Let* γ *be a 1-chain in the symplectic manifold* (M^{2n}, ω^2). *Let* g^t *be a phase flow on M with hamiltonian function H. Then*

$$\frac{d}{dt}\int_{J\gamma} \omega^2 = \int_{g^t \gamma} dH .$$

PROOF. It is sufficient to consider a chain γ with one cell $f : [0,1] \to M$. We introduce the notation

$$f'(s,t) = g^t f(s), \quad \xi = \frac{\partial f'}{\partial s} \quad \text{and} \quad \boldsymbol{\eta} = \frac{\partial f'}{\partial t} \in TM_{f'(s,t)} .$$

By the definition of the integral

$$\int_{J\gamma} \omega^2 = \int_0^1 \int_0^\tau \omega^2(\xi, \boldsymbol{\eta}) dt ds .$$

But by the definition of the phase flow, $\boldsymbol{\eta}$ is a vector (at the point $f(s, t)$ of the hamiltonian field with hamiltonian function H. By definition of a hamiltonian field, $\omega^2(\boldsymbol{\xi},\boldsymbol{\eta}) = dH(\xi)$. Thus

$$\int_{J\gamma}\omega^2 = \int_0^{\tau}\left(\int_{g^t\gamma} dH\right)dt\,.$$

Corollary. *If the chain* γ *is closed (*$\partial\gamma = 0$*), then* $\int_{J\gamma}\omega^2 = 0$.

PROOF.

$$\int_{\gamma} dH = \int_{\partial\gamma} H = 0\,.$$

PROOF OF THE THEOREM. We consider any 2-chain c. We have

$$0 = \int_{Jc} d\omega^2 = \int_{\partial Jc}\omega^2 = \left(\int_{g^t} - \int_{c} - \int_{J\partial c}\right)\omega^2 = \int_{g^t c}\omega^2 - \int_{c}\omega^2$$

(1 since ω^2 is closed, 2 by Stokes' formula, 3 by formula (1), 4 by the corollary above with $\gamma = \partial c$). Thus the integrals of the form ω^2 on any chain c and on its image $g^t c$ are the same.

PROBLEM. Is every one-parameter group of diffeomorphisms of M^{2n} which preserves the symplectic structure a hamiltonian phase flow?

Hint. Cf. Section 40.

B *Integral invariants*

Let $g : M \to M$ be a differentiable map.

Definition. A differential k-form ω is called an ***integral invariant*** of the map g if the integrals of ω on any k-chain c and on its image under g are the same:

$$\int_{gc}\omega = \int_{c}\omega\,.$$

EXAMPLE. If $M = R^2$ and $\omega^2 = dp \wedge dq$ is the area element, then ω^2 is an integral invariant of any map g with jacobian 1.

PROBLEM. Show that a form ω^k is an integral invariant of a map g if and only if $g^*\omega^k = \omega^k$.

PROBLEM. Show that if the forms ω^k and ω^l are integral invariants of the map g, then the form $\omega^k \wedge \omega^l$ is also an integral invariant of g.

The theorem in subsection *A* can be formulated as follows:

Theorem. *The form* ω^2 *giving the symplectic structure is an integral invariant of a hamiltonian phase flow.*

We now consider the exterior powers of ω^2,

$$(\omega^2)^2 = \omega^2 \wedge \omega^2, \ (\omega^2)^3 = \omega^2 \wedge \omega^2 \wedge \omega^2, \ldots.$$

Corollary. *Each of the forms* $(\omega^2)^2$, $(\omega^2)^3$, $(\omega^2)^4$, ... *is an* ***integral invariant*** *of a hamiltonian phase flow.*

PROBLEM. Suppose that the dimension of the symplectic manifold (M^{2n}, ω^2) is $2n$. Show that $(\omega^2)^k = 0$ for k>n, and that $(\omega^2)^n$ is a nondegenerate $2n$-form on M^{2n}.

We define a volume element on M^{2n} using $(\omega^2)^n$. Then, a hamiltonian phase flow preserves volume, and we obtain ***Liouville's theorem*** from the corollary above.

EXAMPLE. Consider the symplectic coordinate space $M^{2n} = R^{2n} = \{(\boldsymbol{p}, \boldsymbol{q})\}$, $\omega^2 = d\boldsymbol{p} \wedge d\boldsymbol{q} = \sum dp_i \wedge dq_i$. In this case the form $(\omega^2)^k$ is proportional to the form

$$\omega^{2k} = \sum dp_{i_1} \wedge \cdots \wedge dp_{i_k} \wedge dq_{i_1} \wedge \cdots \wedge dq_{i_k} .$$

The integral of ω^{2k} is equal to the sum of the oriented volumes of projections onto the coordinate planes $(p_{i_1}, \ldots, p_{i_k}, q_{i_1}, \ldots, q_{i_k})$.

A map $g : R^{2n} \to R^{2n}$ is called ***canonical*** if it has ω^2 as an integral invariant. A canonical map is generally called a ***canonical transformation***. Each of the forms $\omega^4, \omega^6, \ldots, \omega^{2n}$ is an integral invariant of every canonical transformation. Therefore, under a canonical transformation, the **sum of the oriented areas of projections** onto the coordinate planes $(p_{i_1}, \ldots, p_{i_k}, q_{i_1}, \ldots, q_{i_k})$, $1 \le k \le n$, is preserved. In particular, **canonical transformations preserve volume**.

The hamiltonian phase flow given by the equations $\dot{\pi} = -\partial H / \partial \boldsymbol{q}$, $\dot{\boldsymbol{q}} = \partial H / \partial \boldsymbol{p}$ consists of canonical transformations g^t.

The integral invariants considered above are also called ***absolute integral invariants***.

Definition. A differential k-form ω is called a ***relative integral invariant*** of the map $g: M \to M$ if $\int_{gc} \omega = \int_c \omega$ for every closed k-chain c.

Theorem. *Let ω be a relative integral invariant of a map g. Then $d\omega$ is an absolute integral invariant of g.*

PROOF. Let c be a k + 1-chain. Then

$$\int_c d\omega = \int_{\partial c} \omega = \int_{gc} \omega = \int_{\partial gc} \omega = \int_{gc} d\omega .$$

(1 and 4 are by Stokes' formula, 2 by the definition of relative invariant, and 3 by the definition of boundary).

EXAMPLE. A canonical map $g: R^{2n} \to R^{2n}$ has the 1-form

$$\omega^1 = \boldsymbol{p}d\boldsymbol{q} = \sum_{i=1}^{n} p_i dq_i \quad \text{as a relative integral invariant.}$$

In fact, every closed chain c on R^{2n} is the boundary of some chain σ, and we find

$$\int_{gc} \omega^1 = \int_{g\partial\sigma} \omega^1 = \int_{\partial g\sigma} \omega^1 = \int_{g\sigma} d\omega^1 = \int_{\sigma} d\omega^1 = \int_{\partial\sigma} \omega^1 = \int_c \omega^1 ;$$

(1 and 6 are by definition of σ, 2 by definition of ∂, 3 and 5 by Stokes' formula, and 4 since g is canonical and $d\omega^1 = d(pdq) = dq \wedge dq = \omega^2$).

PROBLEM. Let $d\omega^k$ be an absolute integral invariant of the map $g: M \to M$. Does it follow that ω^k is a relative integral invariant?

ANSWER. No, if there is a closed k-chain on M which is not a boundary.

C *The law of conservation of energy*

Theorem. *The function H is a first integral of the hamiltonian phase flow with hamiltonian function H.*

PROOF. The derivative of H in the direction of a vector $\boldsymbol{\eta}$ is equal

to the value of dH on $\boldsymbol{\eta}$. By definition of the hamiltonian field $\boldsymbol{\eta} = IdH$ we find

$$dH(\boldsymbol{\eta}) = \omega^2(\boldsymbol{\eta}, IdH) = \varepsilon^2(\boldsymbol{\eta}, \boldsymbol{\eta}) = 0 .$$

PROBLEM. Show that the 1-form dH is an integral invariant of the phase flow with hamiltonian function H.

39 *The Lie algebra of vector fields*

Every pair of vector fields on a manifold determines a new vector field, called their ***Poisson bracket***. The Poisson bracket operation makes the vector space of infinitely differentiable vector fields on a manifold into a Lie algebra.

A *Lie algebras*

One example of a Lie algebra is a three-dimensional oriented euclidean vector space equipped with the operation of vector multiplication. The vector product is bilinear, skew-symmetric, and satisfies the ***Jacobi identity***

$$[[A,B],C]+[B,C],A]+[C,A],B]=0 .$$

Definition. A ***Lie algebra*** is a vector space L, together with a bilinear skew-symmetric operation $L \times L \to L$ which satisfies the Jacobi identity.

The operation is usually denoted by square brackets and called the ***commutator***.

PROBLEM. Show that the set of $n \times n$ matrices becomes a Lie algebra if we define the commutator by $[A,B] = AB - BA$.

B *Vector fields and differential operators*

Let M be a smooth manifold and $\boldsymbol{A}$ a smooth vector field on M: at every point $x \in M$ we are given a tangent vector $\boldsymbol{A}(\boldsymbol{x}) \in TM_x$. With every such vector field we associate the following two objects:

1. The ***one-parameter group of diffeomorphisms*** or ***flow***

$A^t : M \to M$ for which A is the velocity vector field (Figure 168):[1]

$$\left.\frac{d}{dt}\right|_{t=0} \boldsymbol{A}^t \boldsymbol{x} = \boldsymbol{A}(\boldsymbol{x}) .$$

2. The first-order differential operator L_A. We refer here to the differentiation of functions in the direction of the field $\boldsymbol{A}$: for any function $\phi : M \to R$ the ***derivative in the direction of A*** is a new function $L_A \phi$, whose value at a point $\boldsymbol{x}$ is

$$(L_A \phi)(\boldsymbol{x}) = \left.\frac{d}{dt}\right|_{t=0} \phi(\boldsymbol{A}^t \boldsymbol{x}) .$$

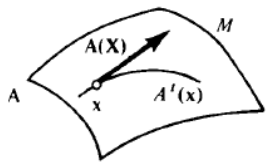

Figure 168 The group of diffeomorphisms given by a vector field

PROBLEM. Show that the operator L_A is linear:

$$L_A(\lambda_1\varphi_1 + \lambda_1\varphi_1) = \lambda_1 L_A \varphi_1 + \lambda_2 L_A \varphi_2 \quad (\lambda_1, \lambda_2 \in R).$$

Also, prove Leibniz's formula

$$L_A(\varphi_1\varphi_2) = \varphi_1 L_A \varphi_2 + \varphi_2 L_A \varphi_1 .$$

EXAMPLE. Let $(x_1, \ldots, x_n)$ be local coordinates on *M*. In this coordinate system the vector $\boldsymbol{A}(\boldsymbol{x})$ is given by its components $(A_1(\boldsymbol{x}), \ldots, A_n(\boldsymbol{x}))$; the flow A^t is given by the system of differential equations

$$\begin{cases} \dot{x}_1 = A_1(\boldsymbol{x}) \\ \quad \cdots \\ \dot{x}_n = A_n(\boldsymbol{x}) \end{cases},$$

and, therefore, the derivative of $\varphi = \varphi(x_1, \ldots, x_n)$ in the direction **A** is

$$L_A \varphi = A_1 \frac{\partial \varphi}{\partial x_1} + \cdots + A_n \frac{\partial \varphi}{\partial x_n} .$$

[1] By theorems of existence, uniqueness, and differentiability in the theory of ordinary differential equations, the group A^t is defined if the manifold *M* is compact. In the general case the maps A^t are defined only in a neighborhood of $\boldsymbol{x}$ and only for small *t*; this is enough for the following constructions.

We could say that in the coordinates $(x_1,\dots,x_n)$ the operator L_A has the form

$$L_A = A_1\frac{\partial}{\partial x_1}+\cdots+A_n\frac{\partial}{\partial x_n};$$

this is the general form of a first-order linear differential operator on coordinate space.

PROBLEM. Show that the correspondences between vector fields ***A***, flows A^t, and differentiations L_A are one-to-one.

C *The Poisson bracket of vector fields*

Suppose that we are given two vector fields ***A*** and ***B*** on a manifold *M*. The corresponding flows A^t and B^s do not, in general, commute: $A^tB^s \neq B^sA^t$ (Figure 169).

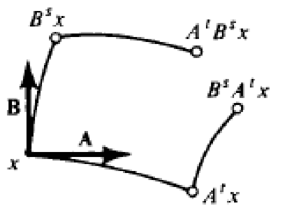

Figure 169 Non-commutative flows

PROBLEM. Find an example.

Solution. The fields $\boldsymbol{A}=\boldsymbol{e}_1$, $B = x_1\boldsymbol{e}_2$ on the (x_1,x_2)-plane.

To measure the degree of noncommutativity of the two flows A^t and B^s we consider the points A^tB^sx and B^sA^tx. In order to estimate the difference between these points, we compare the value at them of some smooth function φ on the manifold *M*. The difference

$$\Delta(t;s;x)=\varphi(A^tB^sx)-\varphi(B^sA^tx)$$

is clearly a differentiable function which is zero for s = 0 and for t = 0. Therefore, the first term different from 0 in the Taylor series in *s* and *t* of Δ at 0 contains *st*, and the other terms of second order vanish. We will calculate this principal bilinear term of Δ at 0.

Lemma 1. *The mixed partial derivative* $\partial^2\Delta/\partial s\partial t$ *at 0 is equal to the commutator of differentiation in the directions **A** and **B***:

$$\left.\frac{\partial^2}{\partial s\partial t}\right|_{s=t=0}\{\varphi(A^tB^sx)-\varphi(B^sA^tx)\}=(L_BL_A\varphi-L_AL_B\varphi)(x)$$

PROOF. By the definition of L_A,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \varphi(A^t B^s x) = (L_A\varphi)(B^s x).$$

If we denote the function $L_A\varphi$ by ψ, then by the definition of L_B

$$\left.\frac{\partial}{\partial t}\right|_{s=0} \psi(B^s x) = (L_B\psi)x.$$

Thus,

$$\left.\frac{\partial^2}{\partial s\partial t}\right|_{s=t=0} \varphi(A^t B^s x = (L_B L_A\varphi)x.$$

We now consider the commutator of differentiation operators $L_B L_A - L_A L_B$. At first glance this is a second-order differential operator.

Lemma 2. *The operator* $L_B L_A - L_A L_B$ *is a first-order linear differential operator.*

PROOF. Let $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$ be the components of the fields $\boldsymbol{A}$ and $\boldsymbol{B}$ in the local coordinate system $(x_1,\dots,x_n)$ on *M*. Then

$$L_B L_A\varphi = \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i}\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\varphi = \sum_{i,j=1}^{n} B_i \frac{\partial A_j}{\partial x_i}\frac{\partial}{\partial x_j}\varphi + \sum_{i,=1}^{n} B_i A_j \frac{\partial^2\varphi}{\partial x_i \partial x_j}$$

If we subtract $L_A L_B\varphi$, the term with the second derivatives of φ vanishes, and we obtain

$$(L_B L_A - L_A L_B)\varphi = \sum_{i,j=1}^{n}\left(B_i \frac{\partial A_j}{\partial x_i} - A_i \frac{\partial B_j}{\partial x_i}\right)\frac{\partial\varphi}{\partial x_j}.$$

Since every first-order linear differential operator is given by a vector field, our operator $L_B L_A - L_A L_B$ also corresponds to some vector field *C*.

Definition. The ***Poisson bracket*** or ***commutator*** of two vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$ on a manifold *M*[2] is the vector field *C* for which

$$L_C = L_B L_A - L_A L_B.$$

The Poisson bracket of two vector fields is denoted by

$$\boldsymbol{C} = [\boldsymbol{A}, \boldsymbol{B}].$$

PROBLEM. Suppose that the vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by their

[2] In many books the bracket is given the opposite sign. Our sign agrees with the sign of the commutator in the theory of Lie groups (cf. subsection F).

components A_i, B_i in coordinates x_i. Find the components of the Poisson bracket.

Solution. In the proof of Lemma 2 we proved the formula

$$[\boldsymbol{A},\boldsymbol{B}]_j = \sum_{i=1}^{n}\left(B_i \frac{\partial A_j}{\partial x_i} - A_i \frac{\partial B_j}{\partial x_i} \right).$$

PROBLEM. Let A_1 be the linear vector field of velocities of a rigid body rotating with angular velocity ω_1 around 0, and A_2 the same thing with angular velocity ω_2. Find the Poisson bracket $[A_1, A_2]$.

D ***The Jacobi identity***

Theorem. *The Poisson bracket makes the vector space of vector fields on a manifold M into a Lie algebra.*

PROOF. Linearity and skew-symmetry of the Poisson bracket are clear. We will prove the Jacobi identity. By definition of Poisson bracket, we have

$$\begin{aligned} L_{[[A,B],C]} &= L_C L_{[A,B]} - L_{[A,B]} L_C \\ &= L_C L_B L_A - L_C L_A L_B + L_A L_B L_C - L_B L_A L_C \end{aligned}.$$

There will be 12 terms in all in the sum

$$L_{[[A,B],C]} + L_{[[B,C],A]} + L_{[[C,A],B]}.$$

Each term appears in the sum twice, with opposite signs.

E ***A condition for the commutativity of flows***

Let ***A*** and ***B*** be vector fields on a manifold *M*.

Theorem. *The two flows A^t and B^s commute if and only if the Poisson bracket of the corresponding vector fields* [***A***, ***B***] *is equal to zero.*

PROOF. If $A^t B^s = B^s A^t$, then [***A***, ***B***] = 0 by Lemma 1. If [***A***, ***B***] = 0, then, by Lemma 1,

$$\varphi(A^t B^s x) - \varphi(B^s A^t x) = o(s^2 + t^2), \quad s \to 0 \quad \text{and} \quad t \to 0$$

for any function φ at any point *x*. We will show that this implies $\varphi(A^t B^s x) = \varphi(B^s A^t x)$ for sufficiently small *s* and *t*. If we apply this to the local coordinates ($\varphi = x_1, \ldots, \varphi = x_n$), we obtain $A^t B^s = B^s A^t$.

Consider the rectangle $0 \le t \le t_0$, $0 \le s \le s_0$ (Figure 170) in the *t*, *s*-plane. To every path going from (0, 0) to (t_0, s_0) and consisting of a finite number of intervals in the coordinate directions, we associate a

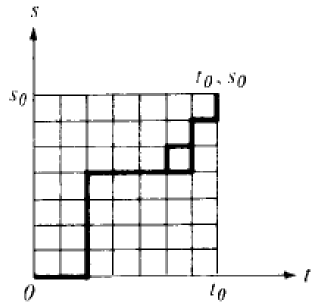

Figure 170 Proof of the commutativity of flows

product of transformations of the flows A^t and B^s. Namely, to each interval $t_1 \le t \le t_2$ we associate $A^{t_2 - t_1}$, and to each interval $s_1 \le s \le s_2$ we associate $B^{s_2 - s_1}$; the transformations are applied in the order in which the intervals occur in the path, beginning at (0, 0). For example, the sides ($0 \le t \le t_0$, $s = 0$) and ($t = t_0$, $0 \le s \le s_0$) corresponds to the product $B^{s_0} A^{t_0}$, and the sides ($t = 0$, $0 \le s \le s_0$) and ($s = s_0$, $0 \le t \le t_0$) to the product $A^{t_0} B^{s_0}$.

In addition, we associate to each such path in the (*t*, *s*)-plane a path on the manifold *M* starting at the point *x* and composed of trajectories of the flows A^t and B^s (Figure 171). If a path in the (*t*, *s*)-plane

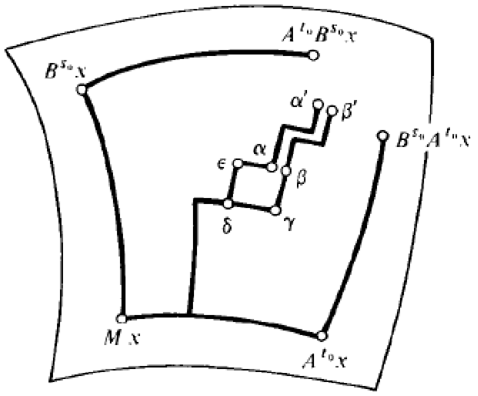

Figure 171 Curvilinear quadrilateral $\beta\gamma\delta\varepsilon\alpha$

corresponds to the product $A^{t_1} B^{s_1} \cdots A^{t_n} B^{s_n}$, then on the manifold *M* the corresponding path ends at the point $A^{t_1} B^{s_1} \cdots A^{t_n} B^{s_n} x$. Our goal will be to show that all these paths actually terminate at the one point $A^{t_0} B^{s_0} x = B^{s_0} A^{t_0} x$.

We partition the intervals $0 \le t \le t_0$ and $0 \le s \le s_0$ into *N* equal parts, so that the whole rectangle is divided into N^2 small rectangles. The passage from the sides (0,0) - $(t_0,0)$ - (t_0, s_0) to the sides (0, 0) - (0, s_0) - (t_0, s_0)can be accomplished in N^2 steps, in each of

which a pair of neighboring sides of a small rectangle is exchanged for the other pair (Figure 172). In general, this small rectangle corresponds to a non-closed curvilinear quadrilateral $\beta\gamma\delta\varepsilon\alpha$

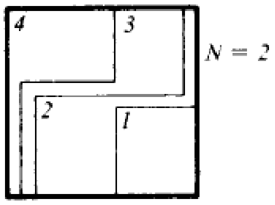

Figure 172 Going from one pair of sides to the other

corresponding to the largest values of s and t. As we saw earlier, $\rho(\alpha,\beta) \le C_1 N^{-3}$ (where the constant $C_1 > 0$ does not depend on N). Using the theorem of the differentiability of solutions of differential equations with respect to the initial data, it is not difficult to derive from this a bound on the distance between the ends α' and β' of the paths $x\delta\gamma\beta\beta'$ and $x\delta\varepsilon\alpha\alpha'$ on $M: \rho(\alpha',\beta') < C_2 N^{-3}$, where the constant $C_2 > 0$ again does not depend on N. But we broke up the whole journey from $B^{s_0}A^{t_0}x$ to $A^{t_0}B^{s_0}$ into N^2 such pieces. Thus $\rho(A^{t_0}B^{s_0}x, B^{s_0}A^{t_0}x) \le N^2 C_2 N^{-3} \sim 1/N$. Therefore, $A^{t_0}B^{s_0}x = B^{s_0}A^{t_0}x$.

F ***Appendix*****:** ***Lie algebras and Lie groups***

A ***Lie group*** is a group G which is a differentiable manifold, and for which the operations (product and inverse) are differentiable maps $G \times G \to G$ and $G \to G$.

The tangent space, TG_e, to a Lie group G at the identity has a natural Lie algebra structure; it is defined as follows:

For each tangent vector $\boldsymbol{A} \in TG_e$ there is a one-parameter subgroup $A^t \subset G$ with velocity vector $\boldsymbol{A} = (d/dt)\big|_{t=0} A^t$.

The degree of non-commutativity of two subgroups A^t and B^t is measured by the product $A^t B^s A^{-t} B^{-s}$. It turns out that there is one and only one subgroup C^r for which

$$\rho(A^t B^s A^{-t} B^{-s}, C^{st}) = o(s^2 + t^2) \quad \text{as s and } t \to 0 .$$

The corresponding vector $\boldsymbol{C} = (d/dt)\big|_{r=0} C^r$ is called the ***Lie bracket*** $\boldsymbol{C}$ = [$\boldsymbol{A}$, $\boldsymbol{B}$] of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. It can be verified that the

operation of Lie bracket introduced in this way makes the space TG_e into a Lie algebra (i.e., the operation is bilinear, skew-symmetric, and satisfies the Jacobi identity). This algebra is called the ***Lie algebra of the Lie group*** G.

PROBLEM. Compute the bracket operation in the Lie algebra of the group $SO(3)$ of rotations in three-dimensional euclidean space.

Lemma 1 shows that the Poisson bracket of vector fields can be defined as the Lie bracket for the "infinite-dimensional Lie group" of all diffeomorphisms of the manifold M.

On the other hand, the Lie bracket can be defined using the Poisson bracket of vector fields on a Lie group G. Let $g \in G$. Right translation R_g is the map $R_g : G \to G$, $R_g h = hg$. The differential of R_g at the point e maps TG_e into TG_g. In this way, every vector $A \in TG_e$ corresponds to a vector field on the group: it consists of the right translations $(R_g)_* A$ and is called a ***right-invariant vector field***. Clearly, a right-invariant vector field on a group is uniquely determined by its value at the identity.

PROBLEM. Show that the Poisson bracket of right-invariant vector fields on a Lie group G is a right-invariant vector field, and its value at the identity of the group is equal to the Lie bracket of the values of the original vector fields at the identity.

40 The Lie algebra of hamiltonian functions

The hamiltonian vector fields on a symplectic manifold form a subalgebra of the Lie algebra of all fields. The hamiltonian functions also form a Lie algebra: the operation in this algebra is called the ***Poisson bracket*** of functions. The **first integrals** of a hamiltonian phase flow form a subalgebra of the Lie algebra of hamiltonian functions.

A ***The Poisson bracket of two functions***

Let (M^{2n}, ω^2) be a symplectic manifold. To a given function $H : M^{2n} \to R$ on the symplectic manifold there corresponds a one-parameter group $g_H^t : M^{2n} \to M^{2n}$ of canonical transformations of M^{2n} — the phase flow of the hamiltonian

function equal to H. Let $F: M^{2n} \to R$ be another function on M^{2n}.

Definition. The ***Poisson bracket*** (F, H) of functions F and H given on a symplectic manifold (M^{2n}, ω^2) is the derivative of the function F in the direction of the phase flow with hamiltonian function H:

$$(F,H)(x) = \left.\frac{d}{dt}\right|_{t=0} F(g_H^t(x)) .$$

Thus, the Poisson bracket of two functions on M is again a function on M.

Corollary 1. *A function F is a first integral of the phase flow with hamiltonian function H if and only if its Poisson bracket with H is identically zero*: $(F,H) \equiv 0$.

We can give the definition of Poisson bracket in a slightly different form if we use the isomorphism I between 1-forms and vector fields on a symplectic manifold (M^{2n}, ω^2). This isomorphism is defined by the relation (cf. Section 37)

$$\omega^2(\boldsymbol{\eta}, I\omega^1) = \omega^1(\boldsymbol{\eta}) .$$

The velocity vector of the phase flow g_H^t is IdH. This implies

Corollary 2. *The Poisson bracket of the functions F and H is equal to the value of the 1-form dF on the velocity vector IdH of the phase flow with hamiltonian function H*:

$$(F,H) = dF(IdH) .$$

Using the preceding formula again, we obtain

Corollary 3. *The Poisson bracket of the functions F and H is equal to the "skew scalar product" of the velocity vectors of the phase flows with hamiltonian functions H and F*:

$$(F,H) = \omega^2(IdH, IdF) .$$

It is now clear that

Corollary 4. *The Poisson bracket of the functions F and H is a skew-symmetric bilinear function of F and H*:

$$(F,H) = -(H,F) ,$$

and

$$(H, \lambda_1 F_1 + \lambda_2 F_2) = \lambda_1 (H, F_1) + \lambda_2 (H, F_2) \qquad (\lambda_i \in R) .$$

Although the arguments above are obvious, they lead to nontrivial deductions, including the following generalization of a ***theorem of E. Noether***.

Theorem. *If a hamiltonian function H on a symplectic manifold* (M^{2n}, ω^2) *admits the one-parameter group of canonical transformations given by a hamiltonian F, then F is a first integral of the system with hamiltonian function H.*

PROOF. Since H is a first integral of the flow $g_F^t (H,F) = 0$ (Corollary 1). Therefore, (F, H) = 0 (Corollary 4) and F is a first integral (Corollary 1).

PROBLEM 1. Compute the Poisson bracket of two functions F and H in the canonical coordinate space $R^{2n} = \{(\boldsymbol{p}, \boldsymbol{q})\}$, $\omega^2(\boldsymbol{\xi}, \boldsymbol{\eta}) = (I\boldsymbol{\xi}, \boldsymbol{\eta})$.

Solution. By Corollary 3 we have

$$(F,H) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \right)$$

(we use the fact that I is symplectic and has the form

$$I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

in the basis ($\boldsymbol{p}$, $\boldsymbol{q}$)).

PROBLEM 2. Compute the Poisson brackets of the basic functions p_i , and q_j .

Solution. The gradients of the basic functions form a "symplectic basis": their skew-scalar products are

$$(p_i p_j) = (p_i q_j) = (q_i, q_j) = 0 \quad \text{if} \quad i \neq j ,$$

$$(q_i p_i) = -(p_i, q_i) = 1 .$$

PROBLEM 3. Show that the map $A : R^{2n} \to R^{2n}$ sending ($\boldsymbol{p}$, $\boldsymbol{q}$) $\to$ ($\boldsymbol{P}$($\boldsymbol{p}$,$\boldsymbol{q}$),$\boldsymbol{Q}$($\boldsymbol{p}$,$\boldsymbol{q}$)) is canonical if and only if the Poisson brackets of any two functions in the variables ($\boldsymbol{p}$, $\boldsymbol{q}$) and ($\boldsymbol{P}$, $\boldsymbol{Q}$) coincide:

$$(F,H)_{\boldsymbol{p},\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}} \frac{\partial F}{\partial \boldsymbol{q}} - \frac{\partial H}{\partial \boldsymbol{q}} \frac{\partial F}{\partial \boldsymbol{p}} = \frac{\partial H}{\partial \boldsymbol{P}} \frac{\partial F}{\partial \boldsymbol{Q}} - \frac{\partial H}{\partial \boldsymbol{Q}} \frac{\partial F}{\partial \boldsymbol{P}} = (F,H)_{\boldsymbol{P},\boldsymbol{Q}} .$$

Solution. Let A be canonical. Then the symplectic structures $d\boldsymbol{p} \wedge d\boldsymbol{q}$ and $d\boldsymbol{P} \wedge d\boldsymbol{Q}$ coincide. But the definition of the Poisson bracket (F,H) was given invariantly in terms of the symplectic structure; it did not involve the coordinates. Therefore,

$$(F,H)_{\boldsymbol{p},\boldsymbol{q}} = (F,H) = (F,H)_{\boldsymbol{P},\boldsymbol{Q}} .$$

Conversely, suppose that the Poisson brackets $(P_i, Q_j)_{\boldsymbol{p},\boldsymbol{q}}$ have the standard form of Problem 2. Then, clearly, $d\boldsymbol{P} \wedge d\boldsymbol{Q} = d\boldsymbol{p} \wedge d\boldsymbol{q}$, i.e., the map A is canonical.

PROBLEM 4. Show that the Poisson bracket of a product can be calculated by Leibniz's rule:

$$(F_1F_2,H)=F_1(F_2,H)+F_2(F_1,H).$$

Hint. The Poisson bracket (F_1F_2,H) is the derivative of the product F_1F_2 in the direction of the field *IdH*.

B ***The Jacobi identity***

Theorem. *The Poisson bracket of three functions A, B and C satisfies the Jacobi identity*:

$$((A,B),C)+((B,C),A)+((C,A),B) = 0.$$

Corollary (**Poisson's theorem**). *The Poisson bracket of two first integrals of a system with hamiltonian function H is again a first integral.*

PROOF OF THE COROLLARY. By the Jacobi identity,

$$((F_1,F_2),H)=(F_1,(F_2,H))+(F_2,(H,F_1))=0+0,$$

as was to be shown.

In this way, by knowing two first integrals we can find a third, fourth, etc. by a simple computation. Of course, not all the integrals we get will be essentially new, since there cannot be more than $2n$ independent functions on M^{2n}. Sometimes we may get functions of old integrals or constants, which may be zero. But sometimes we do obtain new integrals.

PROBLEM. Calculate the Poisson brackets of the components p_1,p_2,p_3, M_1,M_2,M_3 of the linear and angular momentum vectors of a mechanical system.

ANSWER. $(M_1,M_2)=M_3$, $(M_1,p_1)=0$, $(M_1p_2)=p_3$, $(M_1,p_3)=-p_2$. This implies:

Theorem. *If two components, M_1 and M_2, of the angular momentum of some mechanical problem are conserved, then the third component is also conserved.*

PROOF OF THE JACOBI IDENTITY. Consider the sum

$$((A,B),C)+((B,C),A)+((C,A),B).$$

This sum is a "linear combination of second partial derivatives" of the functions A, B, and C. We will compute the terms in the second derivatives of A:

$$((A,B),C)+((C,A),B)=(L_CL_B-L_BL_C)A,$$

where L_ξ is differentiation in the direction of ξ and $\boldsymbol{F}$ is the hamiltonian field with hamiltonian function *F*.

But, by Lemma 2, Section 39, the commutator of the differentiations $L_C L_B - L_B L_C$ is a *first*-order differential operator. This means that none of the second derivatives of *A* are contained in our sum. The same thing is true for the second derivatives of *B* and *C*. Therefore, the sum is zero.

Corollary 5. *Let B and C be hamiltonian fields with hamiltonian functions B and C. Consider the Poisson bracket* [*B*, *C*] *of the vector fields. This vector field is hamiltonian, and its hamiltonian function is equal to the Poisson bracket of the hamiltonian functions* (*B*, *C*).

PROOF. Set (*B*,*C*)=*D*. The Jacobi identity can be rewritten in the form

$$(A,D)=((A,B),C)-((A,C),B),$$

$$L_D = L_C L_B - L_B L_C\,,\quad L_D = L_{[B,C]}\,,$$

as was to be shown.

C *The Lie algebras of hamiltonian fields, hamiltonian functions, and first integrals*

A linear subspace of a Lie algebra is called a ***subalgebra*** if the commutator of any two elements of the subspace belongs to it. A subalgebra of a Lie algebra is itself a Lie algebra. The preceding corollary implies, in particular,

Corollary 6. *The hamiltonian vector fields on a symplectic manifold form a subalgebra of the Lie algebra of all vector fields*.

Poisson's theorem on first integrals can be re-formulated as

Corollary 7. *The first integrals of a hamiltonian phase flow form a subalgebra of the Lie algebra of all functions*.

The Lie algebra of hamiltonian functions can be mapped naturally onto the Lie algebra of hamiltonian vector fields. To do this, to every function *H* we associate the hamiltonian vector field **H** with hamiltonian function *H*.

Corollary 8. *The map of the Lie algebra of functions onto the Lie algebra of hamiltonian fields is an algebra homomorphism. Its kernel consists of the locally constant functions. If* M^{2n} *is connected, the kernel is one-dimensional and consists of constants*.

PROOF. Our map is linear. Corollary 5 says that our map carries the Poisson bracket of functions into the Poisson bracket of vector fields. The kernel consists of functions H for which IdH=0. Since I is an isomorphism, dH=0 and H=constant.

Corollary 9. The phase flows with hamiltonian functions H_1 and H_2 commute if and only if the Poisson bracket of the functions H_1 and H_2 is (locally) constant.

PROOF. By the theorem in Section 39, E, it is necessary and sufficient that $[\boldsymbol{H}_1, \boldsymbol{H}_2] \equiv 0$, and by Corollary 8 this condition is equivalent to $d(H_1, H_2) \equiv 0$.

We obtain yet another ***generalization of E. Noether's theorem***: *given a flow which commutes with the one under consideration, one can construct a first integral.*

D ***Locally hamiltonian vector fields***

Let (M^{2n}, ω^2) be a symplectic manifold and $g^t : M^{2n} \to M^{2n}$ a one-parameter group of diffeomorphisms preserving the symplectic structure. Will g^t be a hamiltonian flow?

EXAMPLE. Let M^{2n} be a two-dimensional torus T^2, a point of which is given by a pair of coordinates (p, q)mod 1. Let ω^2 be the usual area element $dp \wedge dq$. Consider the family of translations $g^t(p,q) = (p+t, q)$ (Figure 173). The maps g^t preserve the

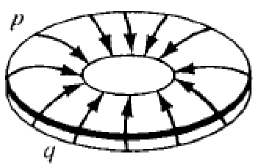

Figure 173 A locally hamiltonial field on the torus

symplectic structure (i.e., area). Can we find a hamiltonian function corresponding to the vector field ($\dot{p} = 1$, $\dot{q} = 0$)? If $\dot{p} = -\partial H / \partial q$ and $\dot{q} = \partial H / \partial p$, we would have $\partial H / \partial p = 0$ and $\partial H / \partial q = -1$, i.e., $H = -q + C$. But q is only a *local* coordinate on T^2; there is no map $H : T^2 \to R$ for which $\partial H / \partial p = 0$ and $\partial H / \partial q = 1$. Thus g^t is not a hamiltonian phase flow.

Definition. A ***locally hamiltonian vector field*** on a symplectic

manifold (M^{2n}, ω^2) is the vector $I\omega^1$, where ω^1 is a closed 1-form on M^{2n}.

Locally, a closed 1-form is the differential of a function, $\omega^1 = dH$. However, in attempting to extend the function H to the whole manifold M^{2n} we may obtain a "many-valued hamiltonian function, " since a closed 1-form on a non-simply-connected manifold may not be a differential (for example, the form dq on T^2). A phase flow given by a locally hamiltonian vector field is called a ***locally hamiltonian flow.***

PROBLEM. Show that a one-parameter group of diffeomorphisms of a symplectic manifold preserves the symplectic structure if and only if it is a locally hamiltonian phase flow.

Hint. Cf. Section 38A.

PROBLEM. Show that in the symplectic space R^{2n}, every one-parameter group of canonical diffeomorphisms is a hamiltonian (preserving $d\mathbf{p} \wedge d\mathbf{q}$) is a hamiltonian flow.

Hint. Every closed 1-form on R^{2n} is the differential of a function.

PROBLEM. Show that the locally hamiltonian vector fields form a sub-algebra of the Lie algebra of all vector fields. In addition, the Poisson bracket of two locally hamiltonian fields is actually a hamiltonian field, with a hamiltonian function uniquely determined by the given fields ξ and η by the formula $H = \omega^2(\xi, \eta)$. Thus, the hamiltonian fields form an ideal in the Lie algebra of locally hamiltonian fields.

41 Symplectic geometry

A euclidean structure on a vector space is given by a symmetric bilinear form, and a symplectic structure by a skew-symmetric one. The geometry of a symplectic space is different from that of a euclidean space, although there are many similarities.

A *Symplectic vector spaces*

Let R^{2n} be an even-dimensional vector space.

Definition. A ***symplectic linear structure*** on R^{2n} is a nondegenerate bilinear skew-symmetric 2-form given in R^{2n}. This form is called the s***kew-scalar product*** and is denoted by $[\xi, \eta] = -[\eta, \xi]$. The space

R^{2n}, together with the symplectic structure [,], is called a ***symplectic vector space***.

EXAMPLE. Let $(p_1,\ldots,p_n,q_1,\ldots,q_n)$ be coordinate functions on R^{2n}, and ω^2 the form

$$\omega^2 = p_1 \wedge q_1 + \cdots + p_n \wedge q_n.$$

Since this form is nondegenerate and skew-symmetric, it can be taken for a skew-scalar product: $[\xi,\eta] = \omega^2(\eta,\xi)$. In this way the coordinate space $R^{2n} = \{(\boldsymbol{p},\boldsymbol{q})\}$ receives a symplectic structure. This structure is called the ***standard symplectic structure***. In the standard symplectic structure the and skew-scalar product of two vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is equal to the sum of the oriented areas of the parallelogram (ξ,η) on the n coordinate planes (p_i,q_i).

Two vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ in a symplectic space are called ***skew-orthogonal*** $(\xi \perp \eta)$ if their skew-scalar product is equal to zero.

PROBLEM. Show that $\xi \perp \xi$: every vector is skew-orthogonal to itself.

The set of all vectors skew-orthogonal to a given vector $\boldsymbol{\eta}$ is called the ***skew-orthogonal complement*** to $\boldsymbol{\eta}$.

PROBLEM. Show that the skew-orthogonal complement to $\boldsymbol{\eta}$ is (2n-1)-dimensional hyperplane containing $\boldsymbol{\eta}$.

Hint. If all vectors were skew-orthogonal to $\boldsymbol{\eta}$, then the form [,] would be degenerate.

B *The symplectic basis*

A euclidean structure under a suitable choice of basis (it must be orthonormal) is given by a scalar product in a particular standard form. In exactly the same way, a symplectic structure takes the standard form indicated above in a suitable basis.

PROBLEM. Find the skew-scalar product of the basis vectors $\boldsymbol{e}_p$, and $\boldsymbol{e}_q$ (i = 1, ..., n) in the example presented above.

Solution. The relations

(1) $\qquad [\boldsymbol{e}_{p_i},\boldsymbol{e}_{p_j}] = [\boldsymbol{e}_{p_i},\boldsymbol{e}_{q_j}] = [\boldsymbol{e}_{q_i},\boldsymbol{e}_{q_j}] = 0,\ \ [e_{p_i},e_{q_i}] = 1$

follow from the definition of $p_1 \wedge q_1 + \cdots + p_n \wedge q_n$.

We now return to the general symplectic space.

Definition. A ***symplectic basis*** is a set of $2n$ vectors, $\boldsymbol{e}_{p_i}, \boldsymbol{e}_{q_i}$ (i = 1,…,n) whose scalar products have the form (1).

In other words, every basis vector is skew-orthogonal to all the basis vectors except one, associated to it; its product with the associated vector is equal to ± 1.

Theorem. *Every symplectic space has a symplectic basis. Furthermore, we can take any nonzero vector* ***e*** *for the first basis vector.*

PROOF. This theorem is entirely analogous to the corresponding theorem in euclidean geometry and is proved in almost the same way.

Since the vector ***e*** is not zero, there is a vector ***f*** not skew-orthogonal to it (the form [,] is nondegenerate). By choosing the length of this vector, we can insure that its skew-scalar product with ***e*** is equal to 1. In the case n = 1, the theorem is proved.

If n > 1, consider the skew-orthogonal complement D (Figure 174)

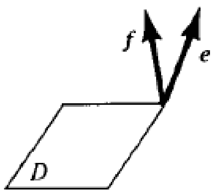

Figure 174 Skew-orthogonal complement

to the pair of vectors ***e*** and ***f***. D is the intersection of the skew-orthogonal complements to ***e*** and ***f***. These two ($2n$-1)-dimensional spaces do not coincide, since ***e*** is not in the skew-orthogonal complement to ***f***. Therefore, their intersection has even dimension $2n$ -2.

We will show that D is a symplectic subspace of R^{2n}, i.e., that the skew-scalar product [,] restricted to D is nondegenerate. If a vector $\xi \in D$ were skew-orthogonal to the whole subspace D, then since it would also be skew-orthogonal to ***e*** and to ***f***, ξ would be skew-orthogonal to R^{2n}, which contradicts the nondegeneracy of [,] on R^{2n}. Thus D^{2n-2} is symplectic.

Now if we adjoin the vectors ***e*** and ***f*** to a symplectic basis for D^{2n-2} we get a sympletic basis for D^{2n-2}, and the theorem is proved by induction on n.

Corollary. *All symplectic spaces of the same dimension are isomorphic*.

If we take the vectors of a symplectic basis as coordinate unit vectors, we obtain a coordinate system p_i, q_i in which [,] takes the standard form $p_1 \wedge q_1 + \cdots + p_n \wedge q_n$. Such a coordinate system is called ***symplectic***.

C *The symplectic group*

To a euclidean structure we associated the orthogonal group of linear mappings which preserved the euclidean structure. In a symplectic space the symplectic group plays an analogous role.

Definition. A linear transformation $S: R^{2n} \to R^{2n}$ of the symplectic space R^{2n} to itself is called ***symplectic*** if it preserves the skew-scalar product:

$$[S\xi, S\boldsymbol{\eta}] = [\xi, \boldsymbol{\eta}], \quad \forall \xi, \boldsymbol{\eta} \in R^{2n}.$$

The set of all symplectic transformations of R^{2n} is called the ***symplectic group*** and is denoted by $Sp(2n)$.

It is clear that the composition of two symplectic transformations is symplectic. To justify the term symplectic group, we must only show that a symplectic transformation is nonsingular; it is then clear that the inverse is also symplectic.

PROBLEM. Show that the group $Sp(2)$ is isomorphic to the group of real two-by-two matrices with determinant 1 and is homeomorphic to the interior of a solid three-dimensional torus.

Theorem. *A transformation* $S: R^{2n} \to R^{2n}$ *of the standard symplectic space* ($\boldsymbol{p}$, $\boldsymbol{q}$) *is symplectic if and only if it is linear and canonical, i.e., preserves the differential 2-form*

$$\omega^2 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

PROOF. Under the natural identification of the tangent space to R^{2n} with R^{2n}, the 2-form ω^2 goes to [,].

Corollary. *The determinant of any symplectic transformation is equal to* 1.

PROOF. We already know (Section 38B) that canonical maps preserve the exterior powers of the form ω^2. But its n-th exterior power is (up to a constant multiple) the volume element on R^{2n}. This means that symplectic transformations S of the standard $R^{2n} = \{(\boldsymbol{p}, \boldsymbol{q})\}$

preserve the volume element, so det $S = 1$. But since every symplectic linear structure can be written down in standard form in a symplectic coordinate system, the determinant of a symplectic transformation of any symplectic space is equal to 1.

Theorem. *A linear transformation* $S: R^{2n} \to R^{2n}$ *is symplectic if and only if it takes some (and therefore any) symplectic basis into a symplectic basis.*

PROOF. The skew-scalar product of any two linear combinations of basis vectors can be expressed in terms of skew-scalar products of basis vectors. If the transformation does not change the skew-scalar products of basis vectors, then it does not change the skew-scalar products of any vectors.

D *Planes in symplectic space*

In a euclidean space all planes are equivalent: each of them can be carried into any other one by a motion. We will now look at a symplectic vector space from this point of view.

PROBLEM. Show that a nonzero vector in a symplectic space can be carried into any other nonzero vector by a symplectic transformation.

PROBLEM. Show that not every two-dimensional plane of the symplectic space R^{2n} can be obtained from a given 2-plane by a symplectic transformation.

Hint. Consider the planes (p_1, p_2) and (p_1, q_1).

Definition. A k-dimensional plane (i.e., subspace) of a symplectic space is called ***null*** (or ***isotropic***) if it is skew-orthogonal to itself, i.e., if the skew-scalar product of any two vectors of the plane is equal to zero.

EXAMPLE. The coordinate plane $(p_1, \ldots, p_n)$ in the symplectic coordinate system $\boldsymbol{p}$, $\boldsymbol{q}$ is null. (Prove it!)

PROBLEM. Show that any non-null two-dimensional plane can be carried into any other non-null two-plane by a symplectic transformation.

For calculations in symplectic geometry it may be useful to impose some euclidean structure on the symplectic space. We fix a symplectic coordinate system $\boldsymbol{p}$, $\boldsymbol{q}$ and introduce a euclidean structure using the coordinate scalar product

$$(\boldsymbol{x},\boldsymbol{x}) = \sum p_i^2 + q_i^2, \text{ where } \boldsymbol{x} = \sum p_i \boldsymbol{e}_{p_i} + q_i \boldsymbol{e}_{q_i} .$$

The symplectic basis $\boldsymbol{e}_{\boldsymbol{p}}, \boldsymbol{e}_{\boldsymbol{q}}$ is orthonormal in this euclidean structure. The skew-scalar product, like every bilinear form, can be expressed in terms of the scalar product by

(2) $$[\boldsymbol{\xi},\boldsymbol{\eta}] = (I\boldsymbol{\xi},\boldsymbol{\eta})$$

where $I : R^{2n} \to R^{2n}$ is some operator. It follows from the skew-symmetry of the skew-scalar product that the operator I is skew-symmetric.

PROBLEM. Compute the matrix of the operator I in the symplectic basis $\boldsymbol{e}_{\boldsymbol{p}}, \boldsymbol{e}_{\boldsymbol{q}}$.

ANSWER.

$$\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

where E is the $n \times n$ identity matrix.

Thus, for $n = 1$ (in the p, q-plane), I is simply rotation by 90°, and in the general case I is rotation by 90° in each of the n planes p_i, q_i.

PROBLEM. Show that the operator I is symplectic and that $I^2 = -E_{2n}$.

Although the euclidean structures and the operator I are not invariantly associated to a symplectic space, they are often convenient.

The following theorem follows directly from (2).

Theorem. *A plane π of a symplectic space is null if and only if the plane $I\pi$ is orthogonal to π.*

Notice that the dimensions of the planes π and $I\pi$ are the same, since I is nonsingular. Hence

Corollary. *The dimension of a null plane in R^{2n} is less than or equal to n.*

This follows since the two k-dimensional planes π and $I\pi$ cannot be orthogonal if $k > n$.

We consider more carefully the n-dimensional null planes in the symplectic coordinate space R^{2n}. An example of such a plane is the coordinate $\boldsymbol{p}$-plane. There are in all C_{2n}^n n-dimensional coordinate planes in $R^{2n} = \{(\boldsymbol{p}, \boldsymbol{q})\}$.

PROBLEM. Show that there are 2^n null planes among the C_{2n}^n

n-dimensional coordinate planes: to each of the 2^n partitions of the set (1, ...,n) into two parts $(i_1,\ldots,i_k)$, $(j_1,\ldots,j_{n-k})$ we associate the null coordinate plane $p_1,\ldots,p_{i_k},q_{j_1},\ldots,q_{j_{n-k}}$.

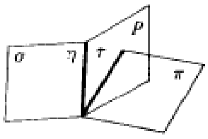

Figure 175 Construction of a coordinate plane σ transversal to a given plane π.

In order to study the **generating functions** of canonical transformations we need

Theorem. *Every n-dimensional null plane* π *in the symplectic coordinate space* R^{2n} *is transverse to at least one of the* 2^n *coordinate null planes*.

PROOF. Let P be the null plane $p_1,\ldots,p_n$ (Figure 175). Consider the intersection $\tau=\pi\cap P$. Suppose that the dimension of τ is equal to k, $0\le k\le n$. Like every k-dimensional subspace of the n-dimensional space, the plane τ is transverse to at least one (n - k)-dimensional coordinate plane in P, let us say the plane

$$\eta=(p_{i_1},\ldots,p_{i_{n-k}})\,;\ \ \tau+\eta=P\,,\ \ \tau\cap\eta=0\,.$$

We now consider the null n-dimensional coordinate plane

$$\sigma=(p_{i_1},\ldots,p_{i_{n-k}},q_{j_1},\ldots,q_{j_k})\,,\ \ \eta=\sigma\cap P\,,$$

and show that our plane π is transverse to σ:

$$\pi\cap\sigma=0\,.$$

We have

$$\left.\begin{matrix}\tau\subset\pi,\pi\perp\pi\Rightarrow\tau\perp\pi\\ \eta\subset\sigma,\sigma\perp\sigma\eta\tau\perp\sigma\end{matrix}\right\}\Rightarrow(\tau+\eta)\perp(\pi\cap\sigma)\Rightarrow P\perp(\pi\bigcap\sigma)\,.$$

But P is an n-dimensional null plane. Therefore, every vector skew-orthogonal to P belongs to P (cf. the corollary above). Thus ($\pi\cap\sigma$)$\subset P$. Finally,

$$\pi\cap\sigma=(\pi\cap P)\cap(\sigma\cap P)=\tau\bigcap\eta=0\,,$$

as was to be shown.

PROBLEM. Let π_1 and π_2 be two k-dimensional planes in symplectic R^{2n}. Is it always possible to carry π_1 to π_2 by a symplectic transformation? How many classes of planes are there which cannot be carried one into another?

ANSWER. $[k/2]+1$, if $k\le n$; $[(2n-k)/2]+1$ if $k\ge n$.

E *Symplectic structure and complex structure*

Since $I^2 = -E$ we can introduce into our space R^{2n} not only a symplectic structure [,] and euclidean structure (,), but also a complex structure, by defining multiplication by $i = \sqrt{-1}$ to the action of *I*. The space R^{2n} is identified in this way with a complex space C^n (the coordinate space with coordinates $z_k = p_k + iq_k$). The linear transformations of R^{2n} which preserve the euclidean structure form the orthogonal group *O*(2*n*); those preserving the complex structure form the complex linear group *GL*(*n*,*C*).

PROBLEM. Show that transformations which are both orthogonal and symplectic are complex, that those which are both complex and orthogonal are symplectic, and that those which are both symplectic and complex are orthogonal; thus that the intersection of two of the three groups is equal to the intersection of all three:

$$O(2n) \cap Sp(2n) = Sp(2n) \cap GL(n,C) = GL(n,C) \cap O)2n) .$$

This intersection is called the ***unitary group*** *U*(*n*).

Unitary transformations preserve the hermitian scalar product $(\xi,\eta) + i[\xi,\eta]$; scalar and skew-scalar products on R^{2n} are its real and imaginary parts.

42 Parametric resonance in systems with many degrees of freedom

During our investigation of oscillating systems with periodically varying parameters (cf. Section 25), we explained that **parametric resonance** depends on the behavior of the **eigenvalues** of a certain linear transformation ("the mapping at a period"). The dependence consists of the fact that an equilibrium position of a system with periodically varying parameters is **stable** if the eigenvalues of the mapping at a period have modulus less than 1, and **unstable** if at least one of the eigenvalues has modulus greater than 1.

The mapping at a period obtained from a system of Hamilton's equations with periodic coefficients is symplectic. The investigation in Section 25 of parametric resonance in a system with one degree of freedom relied on our analysis of the behavior of the eigenvalues of symplectic transformations of the plane. In this paragraph we will

analyze, in an analogous way, the **behavior of the eigenvalues** of symplectic transformations in a phase space of any dimension. The results of this analysis (due to M. G. Krein) can be applied to the study of conditions for the appearance of parametric resonance in mechanical systems with many degrees of freedom.

A ***Symplectic matrices***

Consider a linear transformation of a symplectic space, $S: R^{2n} \to R^{2n}$. Let $p_1, \ldots, p_n; q_1, \ldots, q_n$ be a symplectic coordinate system. In this coordinate system, the transformation is given by a matrix *S*.

Theorem. *A transformation is symplectic if and only if its matrix S in the symplectic coordinate system* (***p***, ***q***) *satisfies the relation*

$$S' IS = I,$$

where

$$I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

and S' is the transpose of S.

PROOF. The condition for being symplectic ($[S\xi, S\boldsymbol{\eta}] = [\boldsymbol{\xi}, \boldsymbol{\eta}]$ for all ξ and $\boldsymbol{\eta}$) can be written in terms of the scalar product by using the operator *I*, as follows:

$$(IS\xi, S\boldsymbol{\eta}) = (I\xi, \boldsymbol{\eta}), \quad \forall \xi, \boldsymbol{\eta}$$

or

$$(S' IS\xi, \boldsymbol{\eta}) = (I\xi, \boldsymbol{\eta}), \quad \forall \xi, \boldsymbol{\eta},$$

as was to be shown.

B ***Symmetry of the spectrum of a symplectic transformation***

Theorem. *The characteristic polynomial of a symplectic transformation*

$$p(\lambda) = \det(S - \lambda E)$$

is reflexive, i.e., $p(\lambda) = \lambda^{2n} p(1/\lambda)$.

PROOF. We will use the facts that det *S* = det *I* =1, $I^2 = -E$, and det A' = det *A*. By the theorem above, S = $-IS'^{-1}I$. Therefore,

$$\begin{aligned} p(\lambda) = \det(S - \lambda E) &= \det(-IS'^{-1} - \lambda E) = \det(-S'^{-1} + \lambda E) \\ &= \det(-E + \lambda S) \\ &= \lambda^2 \det\left(S - \frac{1}{\lambda} E\right) = \lambda^{2n} p\left(\frac{1}{\lambda}\right) \end{aligned}.$$

Corollary. *If* λ *is an eigenvalue of a symplectic transformation, then* $1/\lambda$ *is also an eigenvalue.*

On the other hand, the characteristic polynomial is real; therefore, if λ is a complex eigenvalue, then $\bar{\lambda}$ is an eigenvalue different from λ. It follows that the roots λ of the characteristic polynomial lie symmetrically with respect to the real axis and to the unit circle (Figure 176). They come in 4-tuples,

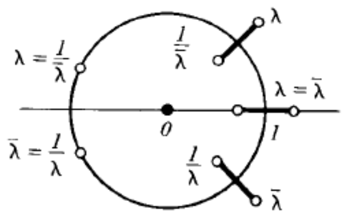

Figure 176 Distribution of the eigenvalues of a symplectic transformation

$$\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}} \quad (|\lambda| \neq 1, \operatorname{Im} \lambda \neq 0),$$

and pairs lying on the real axis,

$$\lambda = \bar{\lambda}, \quad \frac{1}{\lambda} = \frac{1}{\bar{\lambda}}$$

or on the unit circle,

$$\lambda = \frac{1}{\bar{\lambda}}, \quad \bar{\lambda} = \frac{1}{\lambda}.$$

It is not hard to verify that the multiplicities of all four points of a 4-tuple (or both points of a pair) are the same.

C *Stability*

Definition. A transformation S is called ***stable*** if

$$\forall \varepsilon > 0, \exists \delta > 0 : |\boldsymbol{x}| < \delta \Rightarrow |S^N \boldsymbol{x}| < \varepsilon, \quad \forall N > 0.$$

PROBLEM. Show that if at least one of the eigenvalues of a symplectic transformation S does not lie on the unit circle, then S is unstable.

Hint. In view of the demonstrated symmetry, if one of the eigenvalues does not lie on the unit circle, then there exists an eigenvalue outside the unit circle $|\lambda| > 1$; in the corresponding invariant subspace, S is an "expansion with a rotation."

PROBLEM. Show that if all the eigenvalues of a linear transformation are distinct and lie on the unit circle, then the transformation is stable.

Hint. Change to a basis of eigenvectors.

Definition. A symplectic transformation S is called ***strongly stable*** if every symplectic transformation sufficiently close to S is stable.

In Section 25 we established that $S : R^2 \to R^2$ is strongly stable if $\lambda_{1,2} = e^{\pm i\alpha}$ and $\lambda_1 \neq \lambda_2$.

Theorem. *If all 2n eigenvalues of a symplectic transformation S are distinct and lie on the unit circle, then S is strongly stable.*

PROOF. We enclose the $2n$ eigenvalues λ in $2n$ non-intersecting neighborhoods, symmetric with respect to the unit circle and the real axis (Figure 177). The $2n$ roots of the characteristic polynomial

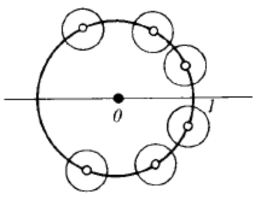

Figure 177 Behavior of simple eigenvalues under a small change of the symplectic transformation

depend continuously on the elements of the matrix of S. Therefore, if the matrix S_1 is sufficiently close to S, exactly one eigenvalue λ_1 of the matrix of S_1 will lie in each of the $2n$ neighborhoods of the $2n$ points of λ. But if one of the points λ_1 did not lie on the unit circle, for example, if it lay outside the unit circle, then by the theorem in subsection B, there would be another point λ_2, $|\lambda_2 < 1|$ in the same neighborhood, and the total number of roots would be greater than $2n$, which is not possible.

Thus all the roots of S_1 lie on the unit circle and are distinct, so S_1 is stable.

We might say that an eigenvalue λ of a symplectic transformation can leave the unit circle only by colliding with another eigenvalue (Figure 178); at the same time, the complex-conjugate

eigenvalues will collide, and from the two pairs of roots on the unit circle we obtain one 4-tuple (or pair of real λ).

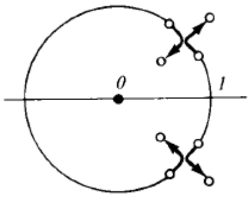

Figure 178 Behavior of multiple eigenvalues under a small change of the symplectic transformation

It follows from the results of Section 25 that the condition for parametric resonance to arise in a linear canonical system with a periodically changing hamilton function is precisely that the corresponding symplectic transformation of phase space should cease to be stable. It is clear from the theorem above that this can happen only after a collision of eigenvalues on the unit circle. In fact, as M. G. Krein noticed, not every such collision is dangerous.

It turns out that the eigenvalues λ with $|\lambda| = 1$ are divided into two classes: ***positive*** and ***negative***. When two roots with the same sign collide, the roots "go through one another," and cannot leave the unit circle. On the other hand, when two roots with different signs collide, they generally leave the unit circle.

M.G. Krein's theory goes beyond the limits of this book; we will formulate the basic results here in the form of problems.

PROBLEM. Let λ and $\overline{\lambda}$ be simple (multiplicity 1) eigenvalues of a symplectic transformation S with $|\lambda| = 1$. Show that the two-dimensional invariant plane π_λ corresponding to λ and $\overline{\lambda}$ is non-null.

Hint. Let ξ_1 and ξ_2 be complex eigenvectors of S with eigenvalues λ_1 and λ_2. Then if $\lambda_1, \lambda_2 \neq 1$, the vectors ξ_1 and ξ_2 are skew-orthogonal: $[\xi_1, \xi_2] = 0$.

Let ξ be a real vector of the plane π_λ, where $\operatorname{Im}\lambda > 0$ and $|\lambda| = 1$. The eigenvalue λ is called ***positive*** if $[S\xi, \xi] > 0$.

PROBLEM. Show that this definition is correct, i.e., it does not depend on the choice of $\xi \neq 0$ in the plane π_λ.

Hint. If the plane π_λ contained two non-collinear skew-orthogonal

vectors, it would be null.

In the same way, an eigenvalue λ of multiplicity k with $|\lambda| = 1$ is of definite sign if the quadratic form $[S\xi, \xi]$ is (positive or negative) definite on the invariant $2k$-dimensional subspace corresponding to $\lambda, \bar{\lambda}$.

PROBLEM. Show that S is strongly stable if and only if all the eigenvalues λ lie on the unit circle and are of definite sign.
Hint. The quadratic form $[S\xi, \xi]$ is invariant with respect to S.

43 A symplectic atlas

In this paragraph we prove ***Darboux's theorem***, according to which every symplectic manifold has local coordinates $\boldsymbol{p}$, $\boldsymbol{q}$ in which the symplectic structure can be written in the simplest way: $\omega^2 = d\boldsymbol{p} \wedge d\boldsymbol{q}$.

A *Symplectic coordinates*

Recall that the definition of manifold includes a **compatibility condition** for the charts of an atlas. This is a condition on the maps $\varphi_i^{-1}\varphi_j$ going from one chart to another. The maps $\varphi_i^{-1}\varphi_j$ are maps of a region of coordinate space.

Definition. An atlas of a manifold M^{2n} is called ***symplectic*** if the standard symplectic structure $\omega^2 = d\boldsymbol{p} \wedge d\boldsymbol{q}$ is introduced into the coordinate space $R^{2n} = \{(\boldsymbol{p}, \boldsymbol{q})\}$, and the transfer from one chart to another is realized by a canonical (i.e., ω^2 -preserving) transformation $\varphi_i^{-1}\varphi_j$.

PROBLEM. Show that a symplectic atlas defines a symplectic structure on M^{2n}.

The converse is also true: every symplectic manifold has a symplectic atlas. This follows from the following theorem.

B *Darboux's theorem*

Theorem. *Let* ω^2 *be a closed nondegenerate differential 2-form in a neighborhood of a point* $\boldsymbol{x}$ *in the space* R^{2n}. *Then in some neighborhood of* $\boldsymbol{x}$ *one can choose a coordinate system* $(p_1,\ldots,p_n;\ q_1,\ldots,q_n)$ *such that the form has the standard form*:

$$\omega^2 = \sum_{i=1}^{n} dp_i \wedge dq_i \ .$$

This theorem allows us to extend to all symplectic manifolds any assertion of a local character which is invariant with respect to canonical transformations and is proven for the standard phase space (R^{2n}, $\omega^2 = d\boldsymbol{p} \wedge d\boldsymbol{q}$).

C *Construction of the coordinates* p_1 ***and*** q_1

For the first coordinate p_1 we take a non-constant linear function (we could have taken any differentiable function whose differential is not zero at the point $\boldsymbol{x}$). For simplicity we will assume that $p_1(\boldsymbol{x}) = 0$.

Let $\boldsymbol{P}_1 = Idp_1$ denote the hamiltonian field corresponding to the function p_1 (Figure 179). Note that $\boldsymbol{P}_1(\boldsymbol{x}) \neq 0$; therefore, we can draw a hyperplane N^{2n-1} through the point $\boldsymbol{x}$ which does not contain the vector $\boldsymbol{P}_1(\boldsymbol{x})$ (we could have taken any surface transverse to $\boldsymbol{P}_1(\boldsymbol{x})$ as N^{2n-1}).

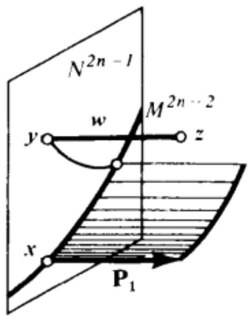

Figure 179 Construction of symplectic coordinates

Consider the hamiltonian flow P_1^t with hamiltonian function p_1. We consider the time *t* necessary to go from *N* to the point *z* = $\boldsymbol{P}_1^t(\boldsymbol{y})$ ($\boldsymbol{y} \in N$) under the action of P_1^t as a function of the point *z*. By the usual theorems in the theory of ordinary differential equations, this function is defined and differentiable in a neighborhood of the point $\boldsymbol{x} \in R^{2n}$. Note that $q_1 = 0$ on N and that the derivative of q_1 in the direction of the field $\boldsymbol{P}_1$ is equal to 1:

$$(q_1, p_1) = 1 .$$

D ***Construction of symplectic coordinates by induction on n***

If n = 1, the construction is finished. Let n > 1. We will assume that Darboux's theorem is already proved for R^{2n-2}. Consider the set M given by the equations $p_1 = q_1 = 0$. The differentials dp_1 and dq_1 are linearly independent at $\boldsymbol{x}$ since $\omega^2(Idp_1, Idq_1) = (q_1, p_1) = 1$. Thus, by the implicit function theorem, the set M is a manifold of dimension 2n - 2 in a neighborhood of x; we will denote it by M^{2n-2}.

Lemma. *The symplectic structure* ω^2 *on* R^{2n} *induces a symplectic structure on some neighborhood of the point* $\boldsymbol{x}$ *on* M^{2n-2}.

PROOF. For the proof we need only the nondegeneracy of ω^2 on TM_x. Consider the symplectic vector space TR_x^{2n}. The vectors $\boldsymbol{P}_1(\boldsymbol{x})$ and $\boldsymbol{Q}_1(\boldsymbol{x})$ of the hamiltonian vector fields with hamiltonian functions p_1 and q_1 belong to TR_x^{2n}. Let $\xi \in TM_x$. The derivatives of p_1 and q_1 in the direction ξ are equal to zero. This means that $dp_1(\xi) = \omega^2(\xi, \boldsymbol{P}_1) = 0$ and $dq_1(\xi) = \omega^2(\xi, Q_1) = 0$. Thus TM_x is the skew-orthogonal complement to $\boldsymbol{P}_1(\boldsymbol{x})$, $\boldsymbol{Q}_1(\boldsymbol{x})$. By Section 41B, the form ω^2 on TM_x is nondegenerate.

By the induction hypothesis there are symplectic coordinates in a neighborhood of the point $\boldsymbol{x}$ on the symplectic manifold (M^{2n-2}, $\omega^2\big|_M$). Denote them by p_i, q_i (i=2,…,n). We extend the functions $p_2, \ldots, q_n$ to a neighborhood of $\boldsymbol{x}$ in R^{2n} in the following way. Every point $\boldsymbol{z}$ in a neighborhood of $\boldsymbol{x}$ in R^{2n} can be uniquely represented in the form $\boldsymbol{z} = P_1^t Q_1^s \boldsymbol{w}$, where $\boldsymbol{w} \in M^{2n-2}$, and s and t are small numbers. We set the values of the coordinates $p_2, \ldots, q_n$ at $\boldsymbol{z}$ equal to their values at the point $\boldsymbol{w}$ (Figure 179). The 2n functions $p_1, \ldots, p_n, q_1, \ldots, q_n$ thus constructed form a local coordinate system in a neighborhood of $\boldsymbol{x}$ in R^{2n}.

E ***Proof that the coordinates constructed are symplectic***

Denote by P_i^t and Q_i^t (i =1,...,n) the hamiltonian flows with hamiltonian functions p_i and q_i, and by $\boldsymbol{P}_i$ and $\boldsymbol{Q}_i$ the corresponding vector fields. We will compute the Poisson brackets of

the functions $p_1,\dots,q_n$. We already saw in C that $(q_1,p_1)=1$. Therefore, the flows P_i^t and Q_i^t commute: $P_i^t Q_1^s = Q_1^s P_i^t$.

Recalling the definitions of $p_2,\dots,q_n$ we see that each of these functions is invariant with respect to the flows P_i^t and Q_i^t. Thus the Poisson brackets of p_1 and q_1 with all $2n$ - 2 functions p_i, q_i (i > 1) are equal to zero.

The map $P_i^t Q_1^s$ therefore commutes with all $2n$ -2 vector fields P_i^t, Q_i^t (i > 1). Consequently, it leaves each of the $2n$ - 2 vector fields $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$ (i > 1) fixed. $P_i^t Q_1^s$ preserves the symplectic structure ω^2 since the flows P_i^t and Q_i^t are hamiltonian; therefore, the values of the form ω^2 on the vectors of any two of the $2n$ - 2 fields $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$ (i > 1) are the same at the points $\boldsymbol{z}$ = $P_1^t Q_1^s \boldsymbol{w} \in R^{2n}$ and $\boldsymbol{w} \in M^{2n-2}$. But these values are equal to the values of the Poisson brackets of the corresponding hamiltonian functions. Thus, the values of the Poisson bracket of any two of the $2n$ -2 coordinates p_i, q_i (i > 1) at the points $\boldsymbol{z}$ and $\boldsymbol{w}$ are the same if $\boldsymbol{z}$ = $P_1^t Q_1^s \boldsymbol{w}$.

The functions p_1 and q_1 are ***first integrals*** of each of the $2n$ -2 flows P_i^t, Q_i^t (i > 1). Therefore, each of the $2n$ - 2 fields $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$ is tangent to the level manifold $p_1 = q_1 = 0$. But this manifold is M^{2n-2}. Therefore, each of the $2n$ - 2 fields $\boldsymbol{P}_i$, $\boldsymbol{Q}_i$ (i > 1) is tangent to M^{2n-2}**.** Consequently, these fields are hamiltonian fields on the symplectic manifold (M^{2n-2} , $\omega^2\big|_M$), and the corresponding hamiltonian functions are $p_i\big|_M$, $q_i\big|_M$ (i > 1). Thus, in the whole space (R^{2n}, ω^2), the Poisson bracket of any two of the $2n$ - 2 coordinates p_i, q_i (i > 1) considered on M^{2n-2} is the same as the Poisson bracket of these coordinates in the symplectic space (M^{2n-2}, $\omega^2\big|_M$).

But, by our induction hypothesis, the coordinates on M^{2n-2} ($p_i\big|_M$, $q_i\big|_M$ (i > 1)) are symplectic. Therefore, in the whole space R^{2n}, the Poisson brackets of the constructed coordinates have the standard values

$$(p_i,p_j)=(p_i,q_j)=(q_i,q_j)=0 \quad \text{and} \quad (q_i,p_i)=1 .$$

The Poisson brackets of the coordinates $\boldsymbol{p}$, $\boldsymbol{q}$ on R^{2n} have the same

form if $\omega^2 = \sum dp_i \wedge dq_i$. But a bilinear form ω^2 is determined by its values on pairs of basis vectors. Therefore, the Poisson brackets of the coordinate functions determine the shape of ω^2 uniquely. Thus

$$\omega^2 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n ,$$

and Darboux's theorem is proved.

(End of Chapter 8)