TOPOLOGICAL METHODS IN HYDRODYNAMICS

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INTRODUCTION

A group theoretical approach to hydrodynamics considers hydrodynamics to be the differential geometry of diffeomorphism groups. The principle of least action implies that the motion of a fluid is described by the geodesics on the group in the right-invariant Riemannian metric given by the kinetic energy. Investigation of the geometry and structure of such groups turns out to be useful for describing the global behavior of fluids for large time intervals.

We begin with a survey of conservation laws for incompressible and barotropic fluid flows and superconductivity. These laws are determined by the infinitesimal structure of the corresponding diffeomorphism groups (i.e., the structure of their Lie algebras and coalgebras). For example, the equations of an inviscid incompressible fluid are Hamiltonian on the coadjoint orbits of the group of volume-preserving diffeomorphisms (Arnold 1966, 1969a, 1989). It is well known that for a two-dimensional flow there is an infinite number of enstrophy-type integrals [$\int f(\operatorname{curl} \boldsymbol{v}) d^2 x$], and in a three-dimensional case there is the total helicity integral [$\int (\operatorname{curl} \boldsymbol{v}, \boldsymbol{v}) d^3 x$]. It turns out that these ideal hydrodynamics equations (as well as barotropic fluid and superconductivity equations) have an infinite number of invariants for flows on an arbitrary even-dimensional manifold (“generalized enstrophies”). In an odd-dimensional case, they have at least one first integral (“generalized helicity") [we follow the paper of Serre (1984) and Khesin & Chekanov (1989)].

The second Section is devoted to the ergodic interpretation of

hydrodynamical invariants. For three-dimensional manifolds, the helicity invariant coincides with the average linking number of the trajectories of corresponding curl vector fields (Arnold 1974). We present a survey of recent works extending this theme: new energy estimates for nontrivial linked fields (Freedman 1988; Freedman & He 1991a, b), ergodic interpretation of multidimensional hydrodynamic invariants as the average multi-linking number of certain surfaces, and interpretation of Novikov invariants (Khesin 1991) and of Godbillon-Vey-type characteristic classes (Tabachnikov 1990).

Further results concern the geometry and curvatures of the different diffeomorphism groups themselves. Shnirelman's theorem (1985) states that the group of volume-preserving diffeomorphisms of a three- (or higher-) dimensional ball has a finite diameter if it is considered to be a Riemannian manifold. Contrary to this case, it was recently shown that the symplectomorphism group of any even-dimensional manifold has an infinite diameter (Eliashberg & Ratiu 1991b). In the conclusion, we discuss some results on the curvatures of diffeomorphism groups. The negativeness of these curvatures is responsible for hydrodynamical instability and for unreliable forecasts (Arnold 1966, 1989).

I. INVARIANTS OF MOTION FOR FLUID FLOWS

1.1 Hydrodynamics on Riemannian Manifolds

Let M^n denote a compact Riemannian manifold (without boundary) and μ a volume form [which, in general, has no relation to the volume form induced by the metric (. , .)]. The equation of the incompressible fluid on M is:

$$\dot{\boldsymbol{v}} = -(\boldsymbol{v}, \nabla)\boldsymbol{v} + \nabla p \qquad (1)$$

where $\boldsymbol{v}$ and p are a time-dependent vector field and a function on M, respectively, the flow of $\boldsymbol{v}$ preserves the form μ (i.e., $L_v \mu = 0$ or div $\boldsymbol{v}$ = 0), and $(\boldsymbol{v}, \nabla)\boldsymbol{v}$ denotes the covariant derivative $\nabla_v \boldsymbol{v}$ for the Riemannian connection.

Theorem 1 Equation (1) has

(i) the integral

$$I(\boldsymbol{v}) = \int_M u \wedge (du)^m \qquad (2a)$$

in the case of an arbitrary odd-dimensional manifold M ($n = 2m + 1$);

(ii) an infinite series of integrals

$$I_f(v) = \int_M f\left(\frac{(du)^m}{\mu}\right)\mu \qquad (2b)$$

in the case of an arbitrary even-dimensional manifold M ($n = 2m$), where u is the 1-form induced from $\boldsymbol{v}$ by the "lifting of indices" defined by the metric $u(\xi) = (v, \xi)$, $\forall \xi \in T_x M$, and f is an arbitrary function of one variable.

The statement above was settled for the standard R^n by explicit coordinate calculations by Tartar and Serre (Serre 1984). In our summary, we follow the works by Khesin & Chekanov (1989) and Ovsienko et a1. (1989) in which the general theorem was proved. This generalization differs considerably from that proposed by Dezin (1983) in which the odd-dimensional integral is obtained. Symmetries of the two- and three-dimensional equations were investigated by Olver (1982).

Proof. This theorem is based primarily on the investigation of the structure of the Lie algebra $\mathcal{G}$ of divergence-free vector fields and its dual space.

There exists a natural isomorphism between the dual space $\mathcal{G}$* and the quotient space of differential l-forms over M modulo differentials of functions: $\Omega_1 / d\Omega_0$ (Marsden & Weinstein 1983). The corresponding pairing is $<[u], \boldsymbol{v}> = \int_M u(\boldsymbol{v})\mu$ where $v \in \mathcal{G}$ and the form $u \in \Omega_1$ is an arbitrary representative of $[u] \in \Omega_1 / d\Omega_0$.

Let G = SDiff (M) be the group of all volume-preserving diffeomorphisms. Then the definition of the change of variables in the integral and the invariance of μ imply the coincidence of the coadjoint G-action with the G-action on the space of l-forms.

The crucial point of the proof is the following

Proposition. The integrals I (u) and $I_f(u)$ given by {2a, b) are well-defined functionals on $\mathcal{G}$* (i.e., they don't depend on the choice of the representative u in the class [u]) and are invariants of the coadjoint action.

Proof. Since the coadjoint action is push-forward, the statement follows from the change of variables formula and the coordinate-free definition of the corresponding integrals.

Let (. , .) be a Riemannian metric on M (whose volume form differs, in general, from the given volume μ). It defines a nondegenerate scalar

product on $\mathcal{G}$: $(\boldsymbol{v},\boldsymbol{w})_{\mathcal{G}}=\int(\boldsymbol{v},\boldsymbol{w})\mu$ and, hence, an invertible operator $A:\mathcal{G}\to\mathcal{G}^*$, called the inertia operator [see Arnold (1989)].

This operator takes Equation (1) on $\mathcal{G}$ into the following equation on $\mathcal{G}$ *: $[\dot{u}]=-L_v[u]$, where $[u]$ = A$\boldsymbol{v}$ and the operator L_v is the Lie derivative determined by the vector field $\boldsymbol{v}$ on M. [Indeed, for a particular representative u of this class $[u]$, this equation has a more recognizable form: $\dot{u}=-L_v u+d\psi$, cf (1).] So, in fact, the equation for an ideal fluid is the equation on $\mathcal{G}$ *; moreover, this is Hamiltonian under the canonical Lie-Poisson linear structure on the coalgebra $\mathcal{G}$ * [the Hamiltonian function is the kinetic energy $H(\boldsymbol{v})$ = $<A\boldsymbol{v}, \boldsymbol{v}>$, see Arnold (1966, 1989)]. This implies that trajectories of the Euler equation are tangent to the orbits of the coadjoint G-action on $\mathcal{G}$ *, hence, $I_f([u])$ and $I([u])$ are its integrals [for details see Khesin & Chekanov (1989)].

Example: In the standard metric of R^3 the integral (2a) coincides with

$$I(\boldsymbol{v})=\int(\boldsymbol{v},\operatorname{curl}\boldsymbol{v})\mu\,,$$

and for R^2 the integral (2b) coincides with

$$I_f(\boldsymbol{v})=\int f(\operatorname{curl}\boldsymbol{v})\mu=\int f(\Delta h_v)\mu\,,$$

where h_v is the "stream function" of the vector field $\boldsymbol{v}$, i.e., the Hamiltonian for $\boldsymbol{v}$, relative to the symplectic form μ [see Arnold (1966)].

Remark 1. The invariant (2b) of the plane-parallel 2m-dimensional flow induced by any (2m- 1)-dimensiona1 flow is trivial, since $(du)^m=0$. Therefore, the reduction of dimension gives no integrals different from (2a).

Remark 2. The integrals $I([u])$ and $I_f([u])$ do not form a complete set of continuous invariants of coadjoint orbits. By analogy with two- and three-dimensional cases (Arnold 1974), it is possible to construct parametrized families of orbits with equal values of these functionals. A precise description of the coadjoint orbits for the diffeomorphism groups still remains an unsolved and intriguing problem. In particular, it is unknown whether there exists a dense orbit in each level set of the integral $I(\boldsymbol{v})$ for n = 3.

Remark 3. The manifold M may be multi-connected. In the

nonsimply-connected case, the cohomological class of $[u]$ is also an invariant [compare Arnold (1969b)]. Other examples of (discrete) invariants of the Euler equation are the number of points on M where du is degenerate and the order of its degeneracy (here $[u] = A\boldsymbol{v}$).

Remark 4. The manifold M may be noncompact or may have a boundary (we may consider $M = R^n$). In general, we should consider vector fields tangent to the boundary.

The rest of this section is devoted to the case of fluid motion on odd-dimensional manifolds. In this case, Theorem 1 provides the existence of one invariant. The geometrical approach to its proof allows us to obtain the following statements.

Corollary 1. On an odd-dimensional manifold (n=2m+1), define the curl vector field ξ by $i_\xi \mu = (du)^m$, where $i_\xi *$ denotes the inner product of the vector field ξ with a differential form. Then ξ is "frozen in the fluid."

Proof. The class $[u]$ [and therefore, $(du)^m$] is Lie transported by the flow; the volume μ is invariant and, hence, the vector field ξ is also Lie transported geometrically, i.e., is "frozen in the fluid."

Corollary 2. [for $n = 3$ see Arnold (1974)]. The incompressible fluid equations on an odd-dimensional manifold have a set of integrals

$$I_c(\boldsymbol{v}) = \int_c u \wedge (du)^m ,$$

where the integral is taken over any ergodic component c of the momentary curl vector field.

The proof follows immediately from the Stokes formula and from the observation that the restriction of $(du)^m$ to the boundary of any ergodic component vanishes.

Remark 5. All the integrals in question are invariants of the coadjoint representation of the corresponding Lie groups (the so-called Casimir elements), i.e., they do not depend on the particular choice of the Hamiltonian. This opens the way to the investigation of the nonlinear stability problems by Routh or by the energy-momentum method (Arnold

1967, Holm et a1 1985, Marsden et a1 1989). We also think that information about the orbits can be of help in the study of the Cauchy problem of multi-dimensional hydrodynamics.

Note that the existence of an infinite series of integrals for the flow of an even-dimensional fluid has nothing to do with the complete integrability of the corresponding hydrodynamic equations. The invariants considered only define the coadjoint orbits (generally speaking, infinite-dimensional) on which the evolution takes place. For the equations on this orbit, there is a unique energy integral, while the integrability requires an infinite number of integrals.

Nevertheless, in the recent paper by Murometz & Razboynich (1990), it is shown that the hydrodynamics equations on a plane admit finite-dimensional truncations of arbitrarily large size that turn out to be Hamiltonian integrable systems. It would be very interesting to find nontrivial integrable subsystems for multidimensional hydrodynamics.

1.2 Generalized Superconductivity and Barotropic Fluid Equations

It turns out that multidimensional generalizations of superconductivity and barotropic fluid equations have conservation laws analogous to the hydrodynamic invariants. In both cases, the equations have an infinite series of integrals if the dimension n of the manifold is even, and at least one integral if this dimension is odd [see Holm & Kuperschmidt (1983) and Holm et al (l985) for n = 3 and Khesin & Chekanov (1989) for the general cases]. The reasons for these similarities are different.

(a) The relation of superconductivity to the equations of incompressible fluid is due to the fact that, at a high density, an electronic gas is similar to a fluid. Indeed, the repelling of the particles having equal charges in the electronic clusters makes the gas incompressible.

The equation of (nonrelativistic) superconductivity in R^3 is

$$\dot{\boldsymbol{v}} = -(\boldsymbol{v}, \nabla)\boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{B} + \nabla p \,, \tag{3}$$

where $\boldsymbol{v}$ denotes a divergence-free field of the electronic gas velocity, $\boldsymbol{B}$ is a constant external divergence-free magnetic field, and the symbol $\times$ is the vector product for the standard metric (Feynman 1972).

We define the analog of this equation on an arbitrary manifold. Let M^n be a manifold with volume μ and Riemannian metric g. We suppose that $\boldsymbol{v}$

is a divergence-free vector field with respect to μ, and $\boldsymbol{B}$ is a "strictly divergence-free" (n - 2):vector field with respect to the volume $\sqrt{g}$, i.e., the substitution of $\boldsymbol{B}$ in $\sqrt{g}$ is exact: $i_B\sqrt{g} = d\alpha$ (for example, if $H_2(M) = 0$, the condition $di_B\sqrt{g} = 0$ is sufficient). We define "the vector product" of the field $\boldsymbol{v}$ and (n - 2)-vector field $\boldsymbol{B}$ in the standard way: $\boldsymbol{v} \times \boldsymbol{B} = *(\boldsymbol{v} \wedge \boldsymbol{B})$, where $*$ is the Hodge isomorphism of k- and (n - k)-vector fields induced by the metric g. We call (3) the generalized superconductivity equation.

Theorem 2. The multidimensional superconductivity Equation (3) has the integrals $I(\boldsymbol{v})$ and $I_f(\boldsymbol{v})$ given by (2a,b) with the replacement of u by $u+\alpha$, where α is a 1-form satisfying the relation $d\alpha = i_B\sqrt{g}$. The 1-form u is obtained by the "index lifting" action of the metric g on the vector field $\boldsymbol{v}$.

In fact, the Euler form of these equations differs from the standard hydrodynamics equation by a Coriolis force-type term. As in the ideal hydrodynamics case, the superconductivity equation is Hamiltonian on $\mathcal{G}$ *, but the corresponding Hamiltonian function differs from the quadratic form by the shift of the origin of $\mathcal{G}$ *.

(b) The configuration space of the barotropic fluid on a manifold is a semi-direct product of the diffeomorphism group and the space of all functions on the manifold considered; see Marsden et a1 (1984) for a derivation via reduction for the Lagrangian representation. The similarity of the barotropic fluid to the ideal one can be explained by its "incompressibility" in the coordinates connected with the density.

More precisely, a barotropic fluid (the pressure of which depends only on the density) on a manifold M with metric g is described by the following system of equations for the velocity $\boldsymbol{v}$ and density ρ:

$$\begin{cases} \dot{\boldsymbol{v}} = -(\boldsymbol{v}, \nabla)\boldsymbol{v} + \nabla p(\rho) \\ \dot{\rho} + \operatorname{div}(\rho\boldsymbol{v}) = 0 \end{cases}. \qquad (4)$$

We use the n-form $\theta = \rho\mu \in \Omega(M)$, where ρ is the fluid density

function and μ is the standard volume form induced by the metric, i.e., $\mu = \sqrt{g}$.

Theorem 3. The barotropic fluid Equations (4) admit the integrals $I(\boldsymbol{v})$ and $I_f(\boldsymbol{v})$ given by (2a,b) with the replacement of n-form μ by $\theta = \rho\mu$.

A heuristic proof of the theorem is based on the fact that the density ρ is transported by the flow, and the fluid is incompressible with respect to the new volume θ (depending on time and on the initial conditions). Thus, we can apply Theorem 1, the assumptions of which require no connection between the metric and the volume form.

1.3 Topology of Steady Flows

The investigation of steady flows is one of the most advanced areas in hydrodynamics. For the three-dimensional case the complete description of analytic steady ideal flows is given by the following theorem.

Theorem 4. (Arnold 1966) Assume that the region D is bounded by a compact analytic surface and that the field of velocities is analytic and not everywhere collinear with its curl. Then the region of the flow can be partitioned by an analytic submanifold into a finite number of cells, in each of which the flow is constructed in a standard way. Namely, the cells are of two types: those fibered into tori invariant under the flow and those fibered into surfaces diffeomorphic to the annulus $R \times S^1$, invariant under the flow. On each of these tori the flow lines are either all closed or all dense, and on each annulus all flow lines are closed.

In this theorem, it is important that the velocity and vorticity vector fields are not collinear. Computer experiments conducted by Henon (1966) show a more complicated behavior for the flow lines for a velocity field satisfying $\mathrm{curl}\,\boldsymbol{v} = \lambda \boldsymbol{v}$ (Beltrami flow) than described in the theorem for the general case. The results of Henon's calculations suggest that some flow lines densely fill a three-dimensional region [see further research in Dombre et a1 (1986)].

Another approach to the investigation of steady flows (or rather, another side of the same approach) is connected with the description of the flows as

extremals of the energy functional $\int v^2 d^n x$. For vector fields "frozen in the fluid" (for example, magnetic fields), the corresponding variational problem goes as follows: minimize the energy ($\int v^2 d^n x$) of a divergence-free vector field $\boldsymbol{v}$ by the action of volume-preserving diffeomorphisms on $\boldsymbol{v}$. The energy of such a frozen field is closely related to the topology of its flow lines (see Section 2), and the extremal field is stationary (Arnold 1974).

For the two-dimensional case, the problem is to minimize the Dirichlet integral, $\int (\nabla h)^2 d^2 x$, among all functions h "equal-valued" with the given function h_0 (i.e., among all functions obtained from the given one by area-preserving diffeomorphisms). If the initial function on a disk has only one critical point, then the extremum in the problem is accessible on the symmetrized function, the value of which depends only on the distance to the center of the disk (Mostow 1968). This is the only case of a satisfactory description of an extremum. It should be mentioned that this extremal vector field not only has the energy minimum among all diffeomorphic fields, but also has the energy maximum among all isovortical fields (Kop'ev & Leont'ev 1988). The last result implies the acoustic instability of such rotations if any weak dissipation of energy exists.

So far the extremals are unknown if the topological type of the initial function is more complicated. Perhaps they should have some singularities (Moffatt 1986). Further investigations of this and related variational problems are contained in the recent papers by Laurence & Stredulinsky (1990) and Laurence & Avallaneda (1991).

In conclusion, we recall two results concerning the Dirichlet problem in 2-dimensional domains. For ideal and barotropic steady flows there is only one solution for the boundary problem in a rectangle in the analytic category, while there is an infinite number of solutions in the C^∞-category (Troshkin 1988). As far as we know, analogous results for higher dimensions have not been proved yet.

2. ERGODIC INTERPRETATION OF HYDRODYNAMIC INVARIANTS

The “frozenness” of a vorticity vector field into an ideal fluid and the helicity conservation law are crucial points of three-dimensional hydrodynamics. A description of the relation between these two concepts and the ergodic interpretation of total helicity as the average linking number of trajectories of a curl field (Arnold 1974) stimulated interest and recent progress in this subject.

2.1 Main Definitions for the Three-Dimensional Case

Let M^3 be a simply-connected manifold with volume μ, and ξ and η two divergence-free vector fields on *M*; let g_ξ^t and g_η^t denote their phase flows. Given two points $x, y \in M$, we define the “asymptotic linking number“ of the trajectories of g_ξ^t and g_η^t starting at *x* and *y*, respectively. For this purpose, we first connect any two points on *M* by a “short” path $\mathit{\Delta}$ (the conditions imposed on the short paths will be described below and are satisfied by almost any choice of the “short“ paths $\mathit{\Delta}$),

We select two large numbers, *T* and *S*, and close the segment $g_\xi^t x$ ($0 \le t \le T$) and $g_\eta^t y$ ($0 \le t \le S$) of the trajectories issuing from *x* and *y* by short paths $\mathit{\Delta}(g_\xi^T x, x)$ and $\mathit{\Delta}(g_\eta^S y, y)$, so that we obtain two closed curves $\mathit{\Gamma} = \mathit{\Gamma}_T(x)$ and $\mathit{\Gamma}' = \mathit{\Gamma}(y)$. We assume that these curves are non-intersecting (this is true for almost all pairs of *x* and *y*, and for almost all *T* and *S*). Then, the linking number $N_{T,S}(x, y)$ of $\mathit{\Gamma}$ and $\mathit{\Gamma}'$ is well-defined.

Definition 1: The asymptotic linking number of the pair of trajectories $g_\xi^t x$ and $g_\eta^t y$ is defined as the limit

$$\lambda(x, y) = \lim_{T,S\to\infty} \frac{N_{T,S}(x, y)}{T \cdot S}$$

(*T* and *S* are to be varied so that $\mathit{\Gamma}$ does not meet $\mathit{\Gamma}'$). It turns out that this limit exists almost everywhere and is independent of the system of short paths $\mathit{\Delta}$ [as an element of $L_1(M \times M)$].

Definition 2: The average linking number λ of two divergence-free vector fields ξ and η is

$$\lambda = \iint_{M\times M} \lambda(x_1, x_2)\mu_1\mu_2 \,.$$

Theorem 5. (Arnold 1974) The average linking number λ of two divergence-free vector fields ξ and η on a simply connected

three-dimensional manifold M with volume μ coincides with

$$\int_M i_\xi \mu \wedge d^{-1}(i_\xi \mu).$$

The condition of the vanishing of the divergence for the vector field η on M is equivalent to the condition $di_\eta \mu = 0$ or $i_\eta \mu = dv$ by simple connectivity of M. Note that the integral $\int i_\xi \mu \wedge v$ evidently does not depend on the choice of $v \in \Omega^1(M)$ for a fixed dv.

Remark. If $\xi = \eta$, the integral $\int_M i_\xi \mu \wedge d^{-1}(i_\xi \mu)$ coincides with the helicity invariant for vector field $\boldsymbol{v}$, the curl of which is equal to $\xi = \operatorname{curl} \boldsymbol{v}$.

Indeed, by definition, $i_{\operatorname{curl} \boldsymbol{v}} \mu = du$, where u is a l-form induced from $\boldsymbol{v}$ by "index lifting" defined by the metric (see Section 1.1); therefore,

$$\int_M i_\xi \mu \wedge d^{-1}(i_\xi \mu) = \int du \wedge u.$$

2.2 Linking Numbers in Magnetohydrodynamics

ln magnetohydrodynamics, we assume that the magnetic field B is "frozen" in the ideal fluid of infinite conductivity, filling a manifold M. The fluid flow preserves the volume it on M induced by the metric g. The velocity field $\boldsymbol{v}$ and the frozen magnetic field $\boldsymbol{B}$ (div$\boldsymbol{v}$ = div$\boldsymbol{B}$ = 0) satisfy the so-called ideal magnetohydrodynamic equations:

$$\dot{\boldsymbol{v}} = -(\boldsymbol{v}, \nabla)\boldsymbol{v} + \operatorname{curl}\boldsymbol{B} \times \boldsymbol{B} + \nabla p$$
$$\dot{\boldsymbol{B}} = [\boldsymbol{v}, \boldsymbol{B}]. \tag{5}$$

(The second equation is the definition of the "frozenness" of the field $\boldsymbol{B}$, [.,.] denotes the Jacobi-Lie bracket of two vector fields.)

Notice that the frozenness of the field $\boldsymbol{B}$ immediately implies the conservation law for the average linking number of the trajectories of this field. (This means that the integral $\int i_{\boldsymbol{B}} \mu \wedge d^{-1}(i_{\boldsymbol{B}} \mu)$ is the invariant of the motion by the above theorem.)

It turns out that there is also another invariant for these equations:

Theorem 6. (Vishik & Dolzanskii 1978) The value of $\int (\boldsymbol{v}, \boldsymbol{B})\mu$ is preserved by the solutions of the magnetohydrodynamic Equations (5).

Corollary 3. (Khesin & Chekanov 1989) The magnetohydrodynamic invariant $\int (\boldsymbol{v}, \boldsymbol{B})\mu$ on a simply connected three-dimensional manifold coincides with the average linking number of vector fields curl$\boldsymbol{v}$ and $\boldsymbol{B}$.

Proof. Indeed, applying our theorem to vector fields $\boldsymbol{\xi} = \boldsymbol{B}$ and $\boldsymbol{\eta} = \mathrm{curl}\boldsymbol{v}$ (and using the relation $[u] = A\boldsymbol{v}$, $du = i_{\mathrm{curl}\boldsymbol{v}}\mu$), we obtain

$$\int i_{\boldsymbol{B}}\mu \wedge d^{-1}(i_{\mathrm{curl}\boldsymbol{v}}\mu) = \int i_{\boldsymbol{B}}\mu \wedge d^{-1}(du) = \int i_{\boldsymbol{B}}\mu \wedge u = \int u(\boldsymbol{B})\mu \, .$$

QED

Note that, in spite of the dependence of $\boldsymbol{v} = A^{-1}[u]$ on the choice of metric, the field curl$\boldsymbol{v}$ is defined unambiguously by $i_{\mathrm{curl}\boldsymbol{v}}(\mu) = d[u]$.

The ergodic interpretation of $\int (\boldsymbol{v}, \boldsymbol{B})\mu$ as an average linking number of curl$\boldsymbol{v}$ and $\boldsymbol{B}$ is somehow unexpected since curl$\boldsymbol{v}$ (in contrast to $\boldsymbol{B}$) is not "frozen" [see Equations (5)]. The evolution changes the field $[u]$ (and, hence. $d[u]$ as well) by some additive summand, which depends on $\boldsymbol{B}$, but it turns out that the average linking number of the kernel field curl$\boldsymbol{v}$ and the vector field $\boldsymbol{B}$ is preserved.

Notice that the Lie algebra $\tilde{\mathcal{G}}$ associated with these equations is the semidirect product of the Lie algebra $\mathcal{G}$ of all divergence-free vector fields on M (with volume μ) and of its dual space $\mathcal{G}^* : \tilde{\mathcal{G}} = \mathcal{G} \perp \mathcal{G}^*$, see Holm & Kuperschmidt (1983) and Marsden et a1 (1984).

Moffatt (1990a) reviews the applications of flow invariants for obtaining solutions of nontrivial topological structure for two- and three-dimensional magnetohydrodynamic equations.

2.3 Estimates of Energy and Helicity of Vector Fields

Natural magnetohydrodynamical systems (for example, stars) are usually nonconservative and dissipate magnetic energy $E(\boldsymbol{B}) = \int (\boldsymbol{B}, \boldsymbol{B})\mu$. This

diffusion corresponds to an additional dissipative term $\nu\Delta\boldsymbol{v}$ in the first equation of (5). Nevertheless. the field $\boldsymbol{B}$ remains frozen in the fluid (due to the second equation) as long as the fluid's evolution follows this system. The question is whether the topology of the divergence-free vector field $\boldsymbol{B}$ determines a lower boundary for its energy E.

A quadratic form of helicity $I(\boldsymbol{B}) = \int (\mathrm{curl}^{-1}\boldsymbol{B}, \boldsymbol{B})\mu$ turns out to be a very useful tool in the solution of this problem. Unlike the magnetic energy, this form is invariant under the group of volume-preserving diffeomorphisms due to its coordinate-free definition as $I(\boldsymbol{B}) = \int d^{-1}(i_{\boldsymbol{B}}\mu) \wedge i_{\boldsymbol{B}}\mu$. It turns out that complicated topology of flow lines (and so nontrivial helicity) is an obstacle to the full dissipation of the energy. The precise result is as follows. Consider the curl operator on divergence-free vector fields homological to zero. This operator is invertible and the corresponding inverse operator curl^{-1} has a spectrum accumulating at zero on both sides.

Theorem 7. (Arnold 1974) An eigenvector field of the operator curl^{-1} corresponding to the eigenvalue of largest modulus λ has minimum energy in the class of divergence-free fields obtained from the eigenfield under the action of volume-preserving diffeomorphisms.

For example, the standard Hopf vector field on $S^3[[\boldsymbol{v}_H(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$ on $S^3\{(x_1, x_2, x_3, x_4) \in R^4$, $\sum x_i^2 = 1\}]$ corresponds to the maximal eigenvalue (=1) on S^3 with the standard metric and has the minimal energy among the diffeomorphic fields [notice that $I(\boldsymbol{v}_H) = 1$].

The existence of the vector field with minimal energy on the given orbit implies, of course, that the energy must be separated from zero on this orbit. If all trajectories of a vector field are closed and unlinked (for example, the rotation field of a 3-dimensional ball around a one-dimensional axis), this field can be deformed to make its energy arbitrarily small. This fact supposed by Zeldovich (see Arnold 1974) was recently proved by Freedman.

To the contrary, now let a divergence-free vector field $\boldsymbol{v}$ on a manifold M

have a trajectory (or a set of trajectories) realizing an essential link L and, moreover, let this field $\boldsymbol{v}$ be "modeled on L" (i.e. there is a tubular neighborhood of $L \subset M$ that carries a foliation by circles that are integral curves of $\boldsymbol{v}$ near the link L). Freedman (1988) proved that in such a case there is a positive lower bound of the energy $E(g^{*}\boldsymbol{v})$ over the orbit, with $g \in S\mathrm{Diff}(M)$. In fact he proved that there exist no diffeomorphisms that simultaneously transform all long linked trajectories together into trajectories that are short enough.

Moffatt (l990b) suggested using these lower boundaries of the energy as invariants of knots or links (or, more precisely, invariants of tubular neighborhoods of knots or links). Namely, for any knot. consider a satellite flux-tube of volume V carrying an "unlinked" vector field of flux Q (across any meridian section of the tube), and look at the associated energy of this vector field. This energy…….

(continued).

2.4 Ergodic Meaning of Multidimensional Invariants

Theorem 8.

Theorem 9.

2.5 Interpretation of Godbillon-Vey-type Characteristic Classes

Theorem 10. (Tabachnikov 1990)

3. DIFFERENTIAL GEOMETRY OF DIFFEPMORPHISM GROUPS

3.1 Finiteness of the Diameter for the Group of Volume-Preserving Diffeomorphisms

Theorem 11. (Shnirelman 1985)

3.2 Infinite Diameter of the Symplectomorphism Group

3.3 Curvatures of Diffeomorphism Groups

Theorem 12. (Bao & Ratiu 1990)

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