Impact Waves and Detonation

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CONTENTS

Introduction

As Riemann (Uber die Fortpflanzung ebener Luftwellen von endlicher Sohwingungsweite, Gött. Ges. d. Wiss. **8** (1880), und Riemann's ges. Werke, 2 Aufl. S. 156. Vergl. auch Riemann-Weber, Partielle differentialgleichungen, 5 Aufl. Bd. II, S. 507) was carrying out the integration of partial differential equations for a one-dimensional flow of an ideal gas, he made the discovery that a state of flow marked by constant distribution of density and velocity could pass over to a state of flow in which certain surfaces would form within the gases at which the constant magnitudes – density and velocity – mentioned above would vary within finite limits. A discussion concerning the further course of these disturbances can only follow after the differential equations have been affected by such conditions as will satisfy the equations of state for the gas on both sides of the unstable surface. These conditions lead to the statement that the laws of the conservation of mass end of energy as well as the impact law must not be violated by the passage of the gas through the unstable surface. Riemann in his treatment of the subject made the error of considering

the energy equation unnecessary and introduced in its stead the assumption that the changes of state suffered by the gases in passing the unstable surface was adiabatic. In consequence, as Lord Rayleigh (Theory of Sound, vol. II, p. 41) has pointed out, his equations do not satisfy the energy laws. Later, Hugoniot (Journ. de l'ecole polytech., Paris, **57**, **58**, (1887), (1889)) without knowledge of the work of Riemann, gave an extended mathematical analysis of one-dimensional air movement in which the relationship with the energy laws was clearly brought out. His treatment of the unstable surfaces (which hereafter will be designated "**impact waves**" or **concentration impulse**) revealed the fact that by taking into account the energy laws the changes of state suffered by the gases in passing the surfaces of instability did not follow the law of (static) adiabatics but another law which he called "**dynamic adiabatics**" and which will be referred to in what follows as the "**Hugoniot equation**."

Later an extended treatment of the mathematical side of our problem will be given, following the work of Hadamard (Propagation des ondes. Paris, (1903) and of Zemplen (Unstetege Bewegungen in Flussigkeitem, (Enzykl. d. math. Wissen. Bd. IV, 2 Tell, I Halfte). In the mathematical nomenclature we shall refer to a surface whose two sides differ in density and velocity by finite amounts, as unstable surfaces of the "first order." Unstable surfaces of the second, third, etc., orders are those whose first, second, etc. derivatives of those magnitudes are instable in reference to space and time. **Our impulse wave is therefore an instability of the first order**.

An important deduction of the theory is the consequence that concentration waves of finite over-pressure spontaneously pass into steep compression impulses (sound waves) whose rate of propagation is the normal rate of sound propagation in the gases only for the limiting case of infinitely small compression; but with increasing intensity the velocity of propagation may increase indefinitely. The fact that sound waves may travel with velocity greater than the ordinary speed of sound, was first demonstrated by Mach (Wiener Ber. **72** (1875), **75** (1877), **77** (1878), and his co-workers. He produced the sound waves studied either by an electrical spark or by a fulminate. Martin (Z. f. d. ges. 8chiess. u. Sprengstoffwesen, **12**, 39 (1917)) likewise worked with a number of explosives for the production of the sound waves studied by him. He succeeded in establishing a quantitative relation between the **brisanz** of the explosive and the velocity of propagation of its sound wave. Further, we have Wolff (Ann. d. Phys. **69**, 329 (1899)) to thank for extensive measurements of sound waves generated by heavy explosions.

All of these measurements have to do with the case of the free, spatial propagation of sound waves whose theoretical treatment has so far been **unsuccessful**. With the view of testing out the theory of one-dimensional movement in gases, Vielle (Memorial des poudres et salpetres, **10**, 177, (1899-1900)) carried out a great number of experiments. He prevented the spatial expansion of the sound waves by producing the sound within a steel tube. By this means he was able to observe the increasing "**steepness**" of the wave front and to increase its velocity of propagation threefold above the normal velocity of sound.

Technical practice has presented us with two groups of phenomena whose relationship to the theory of compressional impulses has only become known and made clear after long and arduous experimental effort. The first group is concerned with the flow of gases and vapors from openings of different forms and is of special importance for the construction of steam turbines. Extended analyses of these processes and the problems they present will be found by Stodola (Die Dampfturbinen, Berlin, 1905), Prandtl (Handworlerbuch d. Naturwissenschaften, Bd. 4, Jene, 1913), Schroter and Prandtl (Enzykl. d. math. Wissen. Bd. V, Teil 1 Heft 2).

The second group of phenomena connected with the theory of compressional impulses arises from the rapid chemical transformations of explosive material. That the effect of such an explosive transformation on the surrounding air is to produce a disturbance of the nature of a sound wave, has already been referred to. But the spatial propagation of the area of explosive transformation within the explosive gases (the **detonation wave**) is in itself only a special case of a compressional impulse.

The "**detonation wave**" was first observed and measured by Berthelot (Sur la force des matieres explosives, Paris, (1883) C.R. **93**, 18, (1881)). Its close relationship with Riemann's theory of compressional impact was recognized by Schuster (Philos. Trans. London (1893) p. 152); while Chapman (Phil. Mag. **47**, 90 (1890)) was the first to deduce from the principles enumerated by Riemann the complete fundamental equations leading to the determination of the rate of propagation of the "detonation wave." An extended analysis and discussion of these equations accompanied by numerical experimental values was later carried out by Jouguet (Jour. d. Math. **1**, 347 (1905) **2**, 5, (1906)) and by Crussard (Bull. de la soc. d l'ind, minerale, Saint-Etienne, **6**, 109 (1907)). Their results showed satisfactory and far-reaching agreement between the experimental values obtained by Dixon (Phil. Trans. London (1893) and (1903)) and the values calculated by them. An investigation carried out by

Taffenel and Dautrische (C.R. **155**, 1221 (1912)) in which they sought to demonstrate the theory of compressional impulses numerically as applied to solid explosives, came to grief through their error in using an approximated form of van der Waal's equation of state as an expression representing the real condition of gases at any concentration. In a short communication (Becket, Z. f. Elektrochem. **23**, 40 (1917), Z. f. Physik **4**, 393 (1921) I brought together a few considerations which in the simplest way and without any assumptions concerning the state of the reacting components led directly to the equations for detonation. I was able to show by the use of an equation of state based on the experimental values obtained by Amagat (Becket, l.c) that these equations led to reasonable values for the rate of propagation of the detonation wave even in the case of solid explosives.

The theory of compressional impulses therefore seems to rest upon a well established mathematical basis which is further supported by extensive experimental results. But in spite of this, from a purely physical standpoint, its present form is unsatisfactory. The initial given conditions required for an expression of state (density, pressure, velocity) existing on both sides of the surface of instability are indeed sufficient for a thorough macroscopic description of the phenomena; nevertheless they give us no insight into the actual processes involved in the transformation, It is for instance not made clear why in a detonation wave the compression no longer remains adiabatic but follows the Hugoniot equation instead. In order to arrive at a purely physical theory some insight is required of the macroscopic structure of the wave front. In what follows I shall show in Section I by simple means and by figures, in Section 2 by mathematical treatment of the same processes how the surfaces of instability originate if it is assumed that the fluid is free from friction and heat conduction. When, however, it is recognized and taken into account (Section 3) that no substance exists free from friction and heat conduction it must follow that a sharply defined surface of instability cannot arise. The impact wave must have a finite thickness. This statement was first made by Prandtl (Z. f. d. ges. Turbinenwesen **3**, 241, (1906)). If the differential equations for one-dimensional movement are affected with terms expressing the effect of friction and heat conduction (Section 4), there is obtained by integration without particular difficulty not only the Riemann-Hugoniot equations for the macroscopic characteristics of impulse waves (Section 5), but/also the equations lend some insight into their microscopic structure (Section 6). The computation of the thickness of impulse waves will be illustrated by numerical examples.

A knowledge of the processes taking place within the wave front is also a necessary preliminary to a real knowledge of the detonation wave; by carrying out the consequences of the theory of instability one is led by compelling and unmistakable ways to values of detonation velocity (Section 8), and detonation pressure (Section 9); yet it remains entirely unexplained how the initial components against the wave front are brought to a condition of activation. By application of the knowledge on concerning compressional impulses an understanding of this process is somewhat assisted although much yet remains to be satisfactorily explained (Section 10).

A. The Formation of Compression Impulses

1. A simple method of treatment

In order to represent in a simple way how compression impulses may be formed, imagine the device represented in Figure 1 - a long tube closed at the left by a piston *a*, and filled with air. A small velocity *dw*, is imparted to the piston. This movement produces in the gases a weak compression wave that travels from left to right with the velocity of sound $c = \sqrt{\gamma RT}$. At a given instant (Fig. 1, b), the gas to the right of the wave front remains unchanged and at rest, while the air between the wave front and the piston is adiabatically compressed by an amount $d\rho$, and has the velocity *dw*. The velocity of the piston is now increased by the amount *dw* whereby a second compression wave is produced in the gas and is propagated along the tube behind the first (Fig. 1, c). By repeating this process the velocity of the piston is finally

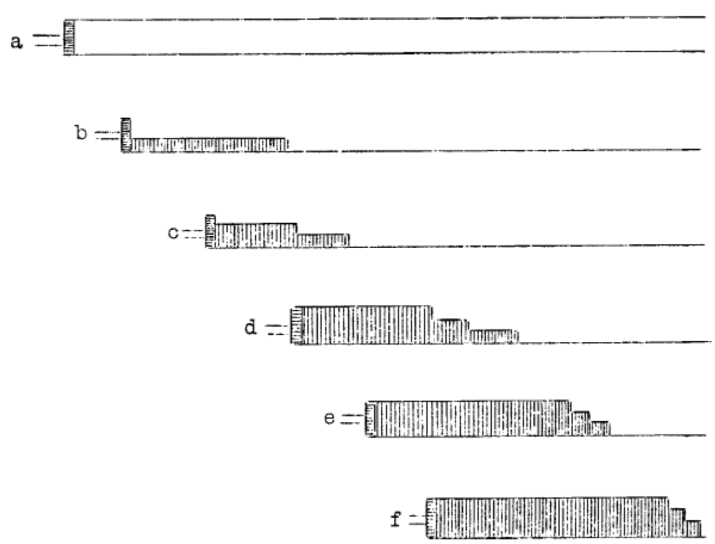

Fig.1

brought to the velocity w. There is thus produced within the mass of gas in the tube a terrassed form of wave whose particles to the left move with the velocity w. What is the further history and fate of this wave? In the first place it is plain that the stratum of the terrass to the left has a greater velocity relative to the tube than the strata to the right. Besides, the temperature and hence the sound velocity is greater in the strata to the left than to the right. As a consequence the strata draw together and the **wave front becomes steeper**, (Figure l, e and l, f). It must not be overlooked what will happen when the steepness of the wave front becomes infinite (a condition to be considered in Section 2).

If, on the contrary, the piston is given a velocity to the left a **rarefaction** wave will be produced in the tube as may be easily realized from analogy to what has been stated. The rarefaction wave will, contrary to the compression wave, become ever flatter and flatter the further it advances in the tube.

In conventional expositions of the subject (for example, that of Riemann-Weber, vol. 2) as also in Section 9 of this "Arbeit," a consideration of rarefaction waves will be excluded because they involve a loss of entropy and because from the second law of thermodynamics they are impossible of propagation. It will be shown here that from the standpoint of pure mechanics they cannot develop. At the end of the next paragraph, also in Section 9, it will be shown that both conditions (the thermodynamic and mechanic) are really identical.

2. A mathematical treatment of the same processes.

Anticipating applications to be made later, the differential equations describing the unidimensional gas movement will be so written as to include the effect of friction and heat conductivity.

ξ represents the very small thickness of any cross section of the tube; x the spatial coordinate measured along the length of the tube; t the time; u the velocity; ρ the density; p the pressure.[1] Then, as is customary, the change in a characteristic G of a material particle with time may be written

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + u\frac{\partial G}{\partial x}, \qquad (1)$$

also

$$\frac{d\xi}{dt} = \xi\frac{\partial u}{\partial x}. \qquad (2)$$

The mass of the cross section layer ξ is $\rho\xi$. The momentum

[1] All computations to *follow* refer to a column of cross section unity.

$$\rho\xi E + \frac{u^2}{2} p_{11}$$

is the effective pressure in the direction of the axis of the tube and perpendicular to the surface of the layer ξ; λ the heat conductivity; μ a friction coefficient. Then, from elementary laws,

$$\frac{d}{dt}(\rho\xi) = 0\,,$$

$$\frac{d}{dt}(u\rho\xi) = -\frac{\partial p_{11}}{\partial x}\xi\,,$$

$$\frac{d}{dt}\rho\xi(E + \frac{u^2}{2}) = \left[-\frac{\partial(p_{11}u)}{\partial x} + \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)\right]\xi\,,$$

in which

$$p_{11} = p - \mu\frac{\partial u}{\partial x}\,,$$

where μ is related to viscosity, η as indicated by the equation[2]

$$\mu = \frac{4}{3}\eta\,, \qquad (*)$$

which follows from the symmetry characteristics of pressure tensors p_{ik}. The three equations may then be written

$$\frac{d\rho}{dt} = -\rho\frac{\partial u}{\partial x}\,, \qquad (3a)$$

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(p - \mu\frac{\partial u}{\partial x}\right), \qquad (3b)$$

$$\frac{dE}{dt} = \left(p - \mu\frac{\partial u}{\partial x}\right)\frac{1}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right). \qquad (3c)$$

$$\boxed{\begin{aligned} &\frac{d}{dx}\rho u = 0 \\ &2\rho u\frac{du}{dx} + \frac{dp}{dx} - \frac{d}{dx}\left(\frac{4}{3}\mu\frac{du}{dx}\right) = 0 \\ &\rho uT\frac{dS}{dx} = \frac{4}{3}\mu\left(\frac{du}{dx}\right)^2 - \frac{dq}{dx} \end{aligned}}$$

Introducing the entropy S, by the relation

$$TdS = dE - p\frac{d\rho}{\rho^2}$$

(3c) may be written

$$\rho T\frac{dS}{dt} = \mu\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right), \qquad (3c')$$

in which the change of entropy with time is given as affected by friction and heat conductivity.

But **for the present we will neglect the effect of friction and heat conduction**. Equation (3c') will then read simply

$$S = \text{const.}$$

That is, compression in the waves takes place adiabatically and for the case of

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[2] Weber and Gans, Report d. Phys. I, 1, p.349.

an ideal gas,

$$p = a^2 \rho^k \, , \tag{4}$$

where a^2 is a constant and $k = c_p / c_v$ the ratio of specific heats. With reference to equations (1) and (4) and with $\mu = 0$ and $\lambda = 0$,

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0 \, , \tag{5a}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{dp}{d\rho} \frac{\partial \rho}{\partial x} = 0 \, . \tag{5b}$$

The integrals $u\,(x,t)$ and $\rho(x,t)$ corresponding to the simple treatment of the process carried out in Section 1, permit of a much simpler derivation, with the aid of the theory of characteristics, than that given by Riemann, Hadamard. To this end consider a linear element (dx,dt) drawn in the plane x,t (Fig. 2). Its direction is indicated by the equation

$$dx = \phi dt \, .$$

Any function whatever as $G(x,t)$ changes along this line by the value $dG = \left(\frac{\partial G}{\partial x} \phi + \frac{\partial G}{\partial t} \right) dt$. From the expressions for u and ρ in equations (5) we will select as function of G, $u = f(\rho)$ where f, primarily an undetermined function of ρ, gives f'. Then,

$$d\left[u + f(\rho)\right] = \left(\frac{\partial u}{\partial x} \phi + f' \frac{\partial \rho}{\partial x} \phi + \frac{\partial u}{\partial t} + f' \frac{\partial \rho}{\partial t} \right) dt \, .$$

By addition and subtraction of the expression

$$u \frac{\partial u}{\partial x} + f' u \frac{\partial \rho}{\partial x} \, ,$$

the expression within the parentheses becomes

$$d(u+f) = \left[\left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} f'(\phi - u) \right\} + f' \left\{ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} \frac{\phi - u}{f'} \right\} \right] dt \, .$$

From (5a) and (5b) the right side of the above equation vanishes when

$$f'(\phi - u) = \frac{1}{\rho} \frac{dp}{d\rho} \quad \text{and} \quad \frac{\phi - u}{f'} = \rho \, ,$$

that is, if

$$f' = \frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} \quad \text{and} \quad \phi = u + \sqrt{\frac{dp}{d\rho}} \, ,$$

or

$$f' = -\frac{1}{\rho} \sqrt{\frac{dp}{d\rho}} \quad \text{and} \quad \phi = u - \sqrt{\frac{dp}{d\rho}} \, .$$

But this means, in reference to the problem in hand, that the curve

$$\frac{dx}{dt} = u + \sqrt{\frac{dp}{d\rho}}$$

the expression

$$u + \int \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}} d\rho = const. \tag{6a}$$

and along curve

$$\frac{dx}{dt} = u - \sqrt{\frac{dp}{d\rho}}$$

the expression

$$u - \int \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}} d\rho = const. \tag{6b}$$

The application of this result to the problem as simply discussed in Section 1 is self-evident: In the tube of infinite length, the position of the piston at $t = 0$ is $x = 0$ and it is at rest (Figure 2). Its position in succeeding interval is

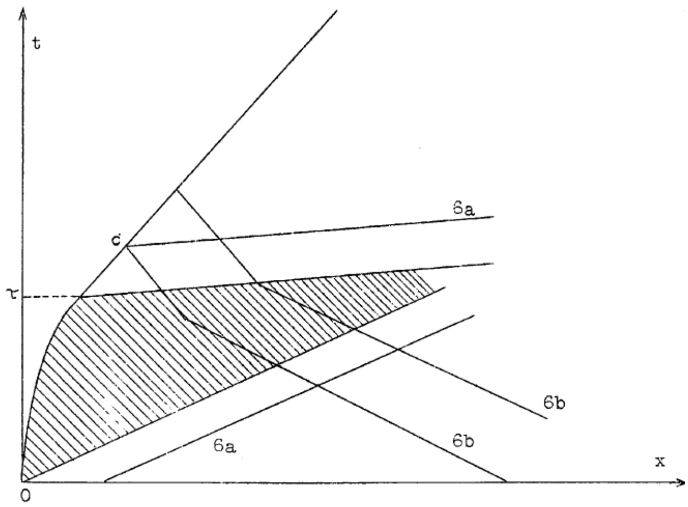

Fig.2

indicated by the curve C in the x,t coordinate figure, as its velocity constantly changes between the instant t = 0 and $t = \tau$, and from then on it proceeds at a constant velocity u_1. If we indicate by the index s values referring to the piston, then, for

$$0 < t_s < \tau\,; \;\; x_s = \frac{g}{2}t_s{}^2 \;\text{ and }\; u_s = gt_s$$

for

$$t > \tau\,; \;\; x_s = g\tau t_s - \frac{g}{2}\tau^2 \;\text{ and }\; u_s = g\tau = u_1 \tag{7}$$

Further, throughout the tube, let t = 0, then u = 0 and $\rho = \rho_0$ and the curves constructed from (6b) fill the entire space between the x-axis and the

curve C. Since, now, for $t = 0$, $u - \int \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}}d\rho$ has the same value throughout the entire range,

$$u - \int \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}}d\rho = const.$$

and besides, since for the curve (6a)

$$u + \int \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}}d\rho = const.,$$

so must it also follow that along the (6a) curve u and ρ remain constant. On the x-axis itself $u = 0$. Therefore, throughout the entire range the relationship between u and ρ will be

$$u = \int_{\rho_0}^{\rho} \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}}d\rho. \tag{8}$$

At the piston and hence along the curve C, u_s (according to (7)) is given; and from (8) ρ_s may also be known. We can therefore draw through every point x_s, t_s the straight line

$$x - x_s = (t - t_s)\left[u_s + \left(\sqrt{\frac{dp}{d\rho}}\right)_s\right] \tag{9}$$

along which u and ρ have constant values u_s and ρ_s.

In the case of the piston motion (7) the portion of the coordinate figure enclosed by the x-axis and the curve C will be divided into three parts by the two lines drawn according to (9) from the points 0 and τ. For the lower portion $u = 0$. The middle portion u varies between $u = 0$ and $u = u_1$. In the upper portion u is finally constant = u_1.

In gaseous media according to (4):

$$\sqrt{\frac{dp}{d\rho}} = a\sqrt{k}\rho^{(k-1)/2} \quad \text{and} \quad \int \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}}d\rho = \frac{2a\sqrt{k}}{k-1}\rho^{(k-1)/2}.$$

If the velocity of sound at initial conditions be given as

$$c_0 = a\sqrt{k}{\rho_0}^{(k-1)/2},$$

then according to (8)

$$u = \frac{2}{k-1}\left(\sqrt{\frac{dp}{d\rho}} - c_0\right)$$

or

$$\left(\frac{\rho}{\rho_0}\right)^{(k-1)/2} = \left(\frac{p}{p_0}\right)^{(k-1)/(2k)} = 1 + \frac{u}{c_0}\frac{k-1}{2} \tag{10}$$

Finally the slope of the curves (6a) and (6b) is given by

$$\left.\begin{aligned} u+\sqrt{\frac{dp}{d\rho}} &= c_0 + u\frac{k+1}{2} \\ u-\sqrt{\frac{dp}{d\rho}} &= -c_0 + u\frac{3-k}{2} \end{aligned}\right\}. \tag{11}$$

This solution denies that u may possess at the instant of crossing of any two curves of the (9) group, two different values. The intersection of two curves of the (9) group is the complete analytical counterpart of the conditions referred to in Section 1, where one wave overtakes another. Position X and time T of this coincidence are given by the values of x and t calculated from (9) together with the equation obtained by differentiating with respect to t_s:

$$-gt_s = \tan\frac{k+1}{2} - t_s g(k+1) - c_0 \,,$$

where, by the help of (11) and (7) the magnitudes x_s, u_s, $\left(\frac{dp}{d\rho}\right)_s$ are expressed as functions of t_s. In this way there is obtained

$$T = \frac{2}{k+1}\left(t_s k + \frac{c_0}{g}\right),$$

$$X = \frac{1}{2}kgt_s^{\,2} + c_0 T \,.$$

The first position of instability occurs from the coordinate point of reference, $t_s = 0$ at the instant $T_0 = \frac{c_0}{g}\frac{2}{k+1}$ and at the point $X_0 = \frac{c_0^{\,2}}{g}\frac{2}{k+1}$.

If the piston in one-half second is moved from rest to a velocity of 100 m/s and then proceeds at that constant rate,

$g = 200 \ \ m/s^2$,

ρ_0 = 330 m/s,

k = 1.4,

τ = 0.5 sec,

u_1 = 100 m/s,

so that the time and place of the first surface of instability will be

X = 453 m,

T = 1.38 sec.

For this example the pressure increase calculated from (10)

$$\frac{p_1}{p_0} = 1.51 \,,$$

and the increase in density

$$\frac{\rho_1}{\rho_0} = 1.34 \,.$$

In Figure 3 the example just given is represented graphically. The course of the velocity u of the wave along the axis of the tube x, is drawn for the intervals 0.2, 0.6, 1.0, and 1.4 sec. The figure plainly shows the increasing steepness of the wave form.

The mechanical production of a compression impulse according to the above, depends upon the condition that within an adiabatic wave train those regions of greater density strive to become more dense at the expense of the less dense regions. That is the velocity expressed by (6a),

$$\frac{dx}{dt} = u + \left(\sqrt{\frac{dp}{d\rho}}\right)_{adiab.}$$

must increase with increasing density. If we substitute for u its value in (8) we

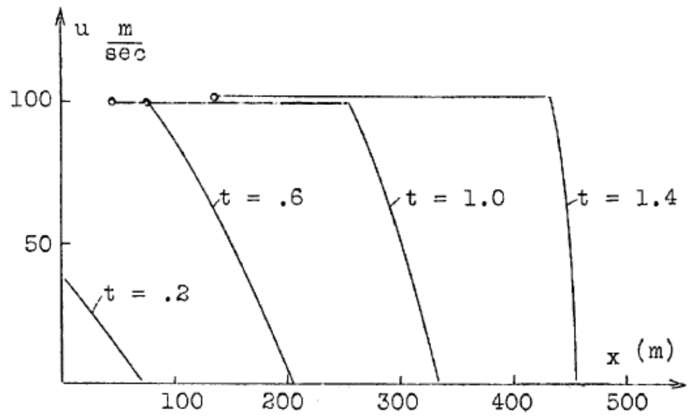

Fig.3

have the condition

$$\frac{d}{d\rho}\left(\int \frac{1}{\rho}\sqrt{\frac{dp}{d\rho}}d\rho + \sqrt{\frac{dp}{d\rho}}\right) > 0 \,,$$

or

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\sqrt{\frac{dp}{d\rho}}\right) > 0 \,.$$

If we substitute for ρ, 1/v we obtain

$$v\frac{3d}{dv}\sqrt{-\frac{dp}{dv}} > 0 \,,$$

or, finally,

$$\left(\frac{d^2 p}{dv^2}\right)_{adiab.} > 0 \,.$$

It is possible then to make the following generalization: *In any given medium it is mechanically possible to produce only compression or rarefied*

impulses according as the value of $\left(\dfrac{d^2 p}{dv^2}\right)_{adiab.}$ *is positive or negative*. Exactly this same criterion will be met with (Section 9) in discussing the thermodynamic possibility of producing compressional impulses.

3. The necessity of taking into account the effect of friction and heat conductivity.

The considerations set forth in Sections 1 and 2 gave a solution of the problem only to the instant at which instability in the gases appeared. A further consideration of the processes is made possible if there be added to the Riemann-Hugoniot line of analysis three equations involving the magnitudes u, and p on both sides of the instable surface. This extension of the analysis of the processes is made necessary if we are to secure the reasoning against any possible violation of the laws of the conservation of mass and of energy, also the impact law. These equations are identical with equations (14). They will later on receive extended consideration.

This procedure is free from objection - indeed, it seems the only possible one - in so far as equations (5) are axiomatically accepted as describing what actually takes place. But from the standpoint of physics, this objection may be made: Equations (5) hold only so long as friction and heat conductivity may be considered negligible. But since no substances are known to exist free from these characteristics, these equations must give results that are in error as soon as the temperature decrease or the rate of change of volume exceeds a certain limit. These values according to the above considerations would appear to be toe significant to be neglected. The application of equation (5) are not admissible at this point.

If we refer for a moment to the simple exposition of the process as given in Section l, we will be led to expect the following: When the wave front has reached a certain steepness, the counter forces of friction and heat conduction oppose the tendency to further compression. A condition will be reached where these two tendencies compensate each other and from this point on a quasi-stationary wave form will be propagated along the tube.

Before seeking in this sense an integration of the general equation (3) we shall attempt to show in a wholly qualitative way how the course of temperature change is influenced by heat conduction. Let the line *ABCD* represent the course of temperature change in the neighborhood of a compression wave (Fig. 4). Assume the increase of pressure to be such that due to adiabatic compression, the absolute temperature is increased threefold;

for example, from 300° to 900° absolute. The role of heat conductivity will be the most significant among the gas molecules at *B* and *C* - the positions of greatest change in the temperature gradient. The gases flowing from *D* may

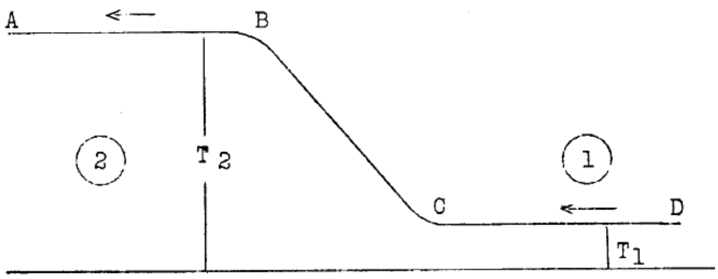

Fig.4

gain in temperature about 200° and at *B* be cooled by a like amount. At 500° they are affected by adiabatic compression that increases the temperature threefold, that is, to 1500°. By conduction they lose at *B* 200°, thus proceeding toward *A* at a temperature of 1300°. At first sight the paradoxical result would seem to be that in consequence of heat conduction an initial temperature difference of 600° has been increased to 1000°! But in truth, with the change in temperature difference there has followed a change in pressure and density difference which are in themselves a source of wave formation thrown back from the original wave front toward the piston.[3] In this way the actual processes in the formation of compression impulses are seen to be so complicated that at present a complete theoretical treatment of their formation seems out of the question. Only after the impulse wave has become quasi-stationary do we again find conditions more satisfactory for theoretical analysis.

From a consideration of the above roughly qualitative discussion it is not to be wondered at if we meet with surprising temperature difference in impact waves of high compression.

B. The Stationary Compression Impulse

4. Differential equations.

In this paragraph we shall investigate the characteristics of compression impulses after they have assumed the form of a quasi-stationary wave. We shall imagine that the coordinate system of reference moves synchronously with the compression wave. In this way the wave may be treated as actually stationary. We shall therefore integrate equations (3) for the case that the

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[3] These waves find their analogue in detonation in the "retonation waves" of Dixon and le

partial derivatives vanish with the time. Accordingly, we substitute for $\frac{d}{dt}$, $u\frac{\partial}{\partial x}$ and write

$$u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0\,,$$

$$\rho u\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(p-\mu\frac{\partial u}{\partial x}\right)=0\,,$$

$$\rho u\frac{\partial E}{\partial x}=\left(p-\mu\frac{\partial u}{\partial x}\right)\frac{u}{\rho}\frac{\partial \rho}{\partial x}+\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right).$$

The first equation may be integrated at once and by that the second. If we substitute from the solution of the first and second equations ρu and $p-\frac{\partial u}{\partial x}$ in the third equation, it may also be integrated. By the aid of the three integration constants *M*, *J*, and *F* and by the substitution of the density ρ, the reciprocal specific volume 1/*v*, there is obtained the differential equations for the stationary compression impulse.

$$u = Mv\,, \tag{12a}$$

$$M^2v+p-J=\mu M\frac{dv}{dx}, \tag{12b}$$

$$E+Jv-\frac{1}{2}M^2v^2-F=\frac{\lambda}{M}\frac{dT}{dx}. \tag{12c}$$

From these equations energy *E* and temperature *T* are seen to be given functions of pressure and volume. A second integration of these equations gives the desired continuous transfer of the magnitudes p_1, v_1, u_1 in front of the concentration impulse, to their magnitudes p_2, v_2, u_2 behind it. The relations that prevail between these six magnitudes are at once manifest by observing that only within the wave front itself do the expressions $\frac{dv}{dx}$ and $\frac{dT}{dx}$ differ appreciably from 0. For any point outside the wave front we may therefore write

$$\frac{u}{v}=M\,,$$

$$\frac{u^2}{v}+p=J\,,$$

Chatelier.

$$E + \frac{u^2}{2} + pv = F\ . \qquad (13)$$

If we compare any two such positions with each other, we must have

$$\frac{u_1}{v_1} = \frac{u_2}{v_2}\ , \qquad (14a)$$

$$\frac{u_1^{\ 2}}{v_1} + p_1 = \frac{u_2^{\ 2}}{v_2} + p_2\ , \qquad (14b)$$

$$E_1 + \frac{u_1^{\ 2}}{2} + p_1 v_1 = E_2 + \frac{u_2^{\ 2}}{2} + p_2 v_2\ . \qquad (14c)$$

These fundamental equations expressing the macroscopic characteristics of impulse waves are, as given, independent of the magnitude of friction μ, and of heat conductivity λ. They are identical with the stipulations made in the introductory treatment for the conditions on both sides of the layer of instability, and could, in fact, be directly written there if it is also specified that for the case of a stationary wave the transport per second of mass, impulse and energy through any two cross sections of the tube are the same.

5. The macroscopic characteristics of compressional impulses.

Before carrying out the integration of equations (12) we will gather some conception of the significance of equations (14). To this end we solve (14a) and (14b) for u_1 and u_2 and substitute the values in (14c). We then have

$$u_1^{\ 2} = v_1^{\ 2} \frac{p_2 - p_1}{v_1 - v_2}\ , \qquad (15a)$$

$$u_2^{\ 2} = v_1 \frac{p_2 - p_1}{v_1 - v_2}\ , \qquad (15b)$$

$$E_2 - E_1 = \frac{1}{2}(p_1 + p_2)(v_1 - v_2)\ . \qquad (15c)$$

Equation (15c) is the **Hugoniot equation** which in the case of impact waves – detonation - takes the place of the adiabatic relation, $dS = 0$.

For small differences $E_2 - E_1$ and $v_1 - v_2$, (15c) becomes $dE - pdv = 0$, - an expression identical with the adiabatic.

The velocity of propagation *D*, of the impact (detonation) wave in a medium at rest and the flow velocity *W* set up in the medium behind the detonation wave are expressed by

$$\left.\begin{aligned} D &= u_1 = v_1\sqrt{\frac{p_2 - p_1}{v_1 - v_2}} \\ W &= u_1 - u_2 = (v_1 - v_2)\sqrt{\frac{p_2 - p_1}{v_1 - v_2}} \end{aligned}\right\}. \qquad (16)$$

The impulse (detonation) wave is determined by the initial condition of the medium (p_1 and v_1) as well as the pressure p_2, within the wave. Further, it is desired to find the factors (*D*, *W*, T_2, etc.).

First, we shall carry out the calculation for a **perfect gas** where

$$pv = RT\ , \qquad (17a)$$

$$E_2 - E_1 = \overline{c_v}(T_2 - T_1)\ , \qquad (17b)$$

where $\overline{c_v}$ is the average specific heat between T_1 and T_2 absolute. Let

$$\zeta_1 = \frac{2\overline{c_v}}{R} + 1\ ,$$

and

$$\pi = \frac{p_2}{p_1}. \qquad (18)$$

Then

$$\frac{T_2}{T_1} = \pi\frac{\pi + \zeta_1}{\pi\zeta_1 + 1}, \qquad (19a)$$

$$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{\pi\zeta_1 + 1}{\pi + \zeta_1}, \qquad (19b)$$

hence

$$D^2 = p_1 v_1 \frac{\pi\zeta_1 + 1}{\zeta_1 - 1}, \qquad (19c)$$

$$W^2 = p_1 v_1 (\zeta_1 - 1)\frac{(\pi - 1)^2}{\pi\zeta_2 + 1}. \qquad (19d)$$

If the dependence of temperature on c_v be neglected then $\zeta_1 = \dfrac{k+1}{k-1}$ = 6 (for diatomic gases). Hence, as soon as the value of π becomes large as compared to 6, temperature *T* becomes proportional to pressure *p*. It is therefore necessary that ζ_1 be taken as a function of *T*.[4] According to the results of Pier (Z f Elektrochem. **15**, 536 (1909), also **16**, 897 (1910)) and Siegel (Z. f. physik. Chem. **87**, 641, (1914)) the specific heat of oxygen and nitrogen carried out experimentally to 3000 ° abs. is

$$\overline{c_v}^{273,T} = 4.78 + 0.45 \times 10^{-3}\,T\,\frac{cal}{mol \cdot grad}\ ,$$

[4] Rudenberg, Artill. Monatshefte (1916), p.237, has carried through a computation assuming c_v constant.

from which we find

$$\zeta_1 = 5.82 + 0.46 \times 10^{-3} T_2 .$$

Since the values given in the following table are carried out for temperatures much above 3000° abs., the results given can be taken as representing only the order of the magnitudes to be expected. With the value given above for ζ_1 (19a) becomes a quadratic equation for T. Using this calculated value the other equations under (19) give the numerical results sought for the fluid air.

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6. The structure of the compression impulse.

7. The thickness of the impulse wave.

C. Applications to Detonation

8. The general fundamental equations for detonation.

9. Normal detonation.

10. The processes within the detonation wave.

Summary of Results

When proper consideration is paid to friction and heat conduction that must be present with all bodies, it is shorn in the analysis made of compression impulses that the theory of instable surfaces may be dispensed with.

The actual thickness of the wave front has been numerically determined for a gas and for a liquid.

By the introduction of a chemical transformation within the impact wave a complete general conception of detonation and normal burning is secured.

While the complete analysis and calculation of the rate of normal burning has not yet been effected, the present consideration, by making use of the principle of stability, has been able to set a definitive value for normal detonation velocity. This value is in excellent agreement with observed results in the case of gases and gives fair approximations for the case of solids.

By application of the deductions here presented, the possibility is offered of following the physical characteristics and chemical transformations experimentally to an order of 100,000 atmospheres pressure.

An important difficulty in understanding detonation phenomena is overcome by a consideration of heat conductivity.