

## CHAPTER IX

### THE PROBLEM OF THREE BODIES

#### 1. Introductory remarks.

The problem of three or more bodies is one of the most celebrated in mathematics, and justly so. Nevertheless until recently the interest in it was directed toward the formal side, and in particular toward the formal solution by means of series.

It was Poincaré who first obtained brilliant qualitative results, especially with reference to the very special limiting '*restricted problem of three bodies*' treated first by Hill. As far as the general problem is concerned, the main achievements of Poincaré were the following:

- (1) he established the existence of various *types of periodic motions* by the method of analytic continuation;
- (2) he proved that, by the very structure of the differential equations, *complete trigonometric series* would be available; and
- (3) he pointed out the *asymptotic validity* of these series.

All of these results hold for any Hamiltonian system as well as for the problem of three bodies. Unfortunately an accessory parameter  $\mu$  is present always in his researches, where for  $\mu = 0$  the system is of a special integrable type. Thus the difficulties which arise are partly due to the special nature of the integrable limiting case when two of the three bodies are of mass 0, rather than inherent in the problem itself.

It is not too much to say that the recent work of Sundman is one of the most remarkable contributions to the problem of three bodies which has ever been made. He proves that, at least if the angular momentum of the bodies is not about every axis through the center of gravity, *the least of the three mutual distances will always exceed a specifiable constant* depending on the initial configuration; thus *triple collision is proved to be impossible*, while it is shown that the *singularity at double collision is of removable type*. In this way a conjecture of Weierstrass as to the impossibility of triple collision is established, and convergent series valid for all the motion are found for the coordinates and the time. By obtaining such series Sundman 'solved' the problem of three bodies in the sense specified by Painlevé. As a matter of fact, however, the existence of such series is merely a reflection of the physical fact that triple collision can not occur, and signifies nothing else as to the qualitative nature of the solution.

In the present chapter I propose to take up the problem of three or more bodies, and to endeavor to apply as far as possible the points of view developed in the earlier chapters, and in particular to show what seems to be the real significance of Sundman's results.

## 2. The equations of motion and the classical integrals.

Let us suppose the three bodies under consideration (taken as particles) to be at the points  $P_0, P_1, P_2$  in space, and to have masses  $m_0, m_1, m_2$ , respectively. We denote the distance  $P_0P_1$  by  $r_2$ ,  $P_0P_2$  by  $r_1$  and  $P_1P_2$  by  $r_0$ . If we write

$$(1) \quad U = \frac{m_0 m_1}{r_2} + \frac{m_0 m_2}{r_1} + \frac{m_1 m_2}{r_0},$$

and if we let  $x_i, y_i, z_i$  ( $i = 0, 1, 2$ ) be the rectangular coordinates of the corresponding bodies  $P_i$ , while  $x'_i, y'_i, z'_i$  stand for the components of velocity, the **equations of motion** may be written as 9 equations of the second order

$$(2) \quad m_i \frac{d^2 x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad m_i \frac{d^2 y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i},$$

$$(i = 0, 1, 2),$$

which are evidently of **Lagrangian form**; or as 18 equations of the first order

$$(3) \quad \begin{cases} \frac{dx_i}{dt} = x'_i & \frac{dy_i}{dt} = y'_i & \frac{dz_i}{dt} = z'_i \\ m_i \frac{dx'_i}{dt} = \frac{\partial U}{\partial x_i} & m_i \frac{dy'_i}{dt} = \frac{\partial U}{\partial y_i} & m_i \frac{dz'_i}{dt} = \frac{\partial U}{\partial z_i} \end{cases}$$

$$(i = 0, 1, 2)$$

which are of course easily converted by slight modification into **Hamiltonian form**. We shall not effect this modification which may be done in the usual way, nor shall we state the usual principles of variation applicable to this case (see Chapter II).

The integral expressing the **conservation of energy** is seen to be

$$(4) \quad \frac{1}{2} \sum m_i (x_i'^2 + y_i'^2 + z_i'^2) = U - K$$

where  $K$  is a constant of integration.

Besides this integral there are of course the 6 integrals of **linear momentum** expressing the fact that the center of gravity moves with uniform velocity in a straight line; if we take a reference system in which the center of gravity is fixed and at the origin, these integrals

reduce to

$$(5) \quad \begin{aligned} \sum m_i x_i &= \sum m_i y_i = \sum m_i z_i = 0, \\ \sum m_i x'_i &= \sum m_i y'_i = \sum m_i z'_i = 0. \end{aligned}$$

There are also 3 integrals which express the *constancy of the total angular momentum* about any axis fixed in space. If we take the axes as the coordinate axes, these integrals become

$$(6) \quad \begin{aligned} \sum m_i (y_i z'_i - z_i y'_i) &= a, \quad \sum m_i (z_i x'_i - x_i z'_i) = b \\ \sum m_i (x_i y'_i - y_i x'_i) &= c, \end{aligned}$$

where  $a, b, c$  are constants of integration.

These 10 integrals are all the essentially independent integrals which are known.

### 3. Reduction to the 12th order.

The reduction of the system of differential equations (3) to the 12th order may be at once accomplished by use of the integrals of linear momentum as, for instance, by the following method due to Lagrange. Let the coordinates of  $P1$  with reference to  $P0$  be  $(x, y, z)$  and let the coordinates of  $P2$ , with reference to the center of gravity of  $P0$  and  $P1$  be  $(\xi, \eta, \zeta)$ . If we write for convenience

$$(7) \quad \begin{aligned} p &= \frac{m_1}{m_0 + m_1}, \quad q = \frac{m_0}{m_0 + m_1}, \\ M &= m_0 + m_1 + m_2, \quad m = \frac{m_0 m_1}{m_0 + m_1}, \quad \mu = \frac{(m_0 + m_1)m_2}{m_0 + m_1 + m_2}, \end{aligned}$$

we obtain the explicit formulas of transformation

$$(8) \quad \begin{aligned} x &= x_1 - x_0, & y &= y_1 - y_0, & z &= z_1 - z_0, \\ \xi &= x_2 - px_1 - qx_0, & \eta &= y_2 - py_1 - qy_0, & \zeta &= z_2 - pz_1 - qz_0, \end{aligned}$$

together with the inverse formulas,

$$(9) \quad \begin{cases} x_0 = -\frac{m_2}{M}\xi - px, & y_0 = -\frac{m_2}{M}\eta - py, & z_0 = -\frac{m_2}{M}\zeta - pz, \\ x_1 = -\frac{m_2}{M}\xi + qx, & y_1 = -\frac{m_2}{M}\eta + qy, & z_1 = -\frac{m_2}{M}\zeta + qz, \\ x_2 = \frac{m_0 + m_1}{M}\xi, & y_2 = \frac{m_0 + m_1}{M}\eta, & z_2 = \frac{m_0 + m_1}{M}\zeta, \end{cases}$$

which follow with the aid of (5).

The system of the 12th order so obtained may be written in the elegant form

$$(10) \quad \begin{cases} \frac{dx}{dt} = x' & \frac{dy}{dt} = y' & \frac{dz}{dt} = z' \\ \frac{d\xi}{dt} = \xi' & \frac{d\eta}{dt} = \eta' & \frac{d\zeta}{dt} = \zeta' \\ m \frac{dx'}{dt} = \frac{\partial U}{\partial x} & m \frac{dy'}{dt} = \frac{\partial U}{\partial y} & m \frac{dz'}{dt} = \frac{\partial U}{\partial z} \\ \mu \frac{d\xi'}{dt} = \frac{\partial U}{\partial \xi} & \mu \frac{d\eta'}{dt} = \frac{\partial U}{\partial \eta} & \mu \frac{d\zeta'}{dt} = \frac{\partial U}{\partial \zeta} \end{cases} .$$

With these variables the equations (5) may be regarded as satisfied identically while the *integrals of angular momentum* take the form

$$(11) \quad \begin{cases} m(yz' - zy') + \mu(\eta\zeta' - \zeta\eta') = a, \\ m(zx' - xz') + \mu(\zeta\xi' - \xi\zeta') = b, \\ m(xy' - yx') + \mu(\xi\eta' - \eta\xi') = c, \end{cases}$$

and the *integral of energy* is

$$(12) \quad \frac{1}{2}m(x'^2 + y'^2 + z'^2) + \frac{1}{2}\mu(\xi'^2 + \eta'^2 + \zeta'^2) = U - K .$$

It will be seen that equations (10) may be looked upon as the *equations of motion of two particles* in space at  $(x,y,z)$  and  $(\xi,\eta,\zeta)$ , with masses  $m$  and  $\mu$  respectively, and in a conservative field of force with potential energy  $U$ . These equations can also be derived from either the Lagrangian or Hamiltonian form by use of the variational principles (chapter II).

#### 4. Lagrange's equality.

Let us write

$$(13) \quad R^2 = \frac{m_0 m_1 r_2^2 + m_0 m_2 r_1^2 + m_1 m_2 r_0^2}{M} = mr^2 + \mu\rho^2 ,$$

where

$$(14) \quad r^2 = x^2 + y^2 + z^2, \quad \rho^2 = \xi^2 + \eta^2 + \zeta^2 .$$

If now we substitute in (13) the explicit values of  $r^2$  and  $\rho^2$  obtained from (14), and differentiate twice, there results an equality due to Lagrange,

$$(15) \quad \frac{d^2 R^2}{dt^2} = 2(U - 2K)$$

when use is made of (10) and (12); it is to be observed that  $U$  is homogeneous of dimensions  $-1$  in  $x, y, z, \xi, \eta, \zeta$ , so that

$$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} + \zeta \frac{\partial}{\partial \zeta} = -U .$$

#### 5. Sundman's inequality.

In order to arrive at Sundman's inequality, we propose to seek an upper bound for  $(dR/dt)^2$  when  $x, y, z, \xi, \eta, \zeta$  are regarded as given quantities while  $x', y', z', \xi', \eta', \zeta'$  are to vary at pleasure except that they are to yield the given values of the constant  $K$  of energy and of the constants  $a, b, c$  of angular momentum. This is a purely algebraic problem.

We have

$$RR' = mrr' + \mu\rho\rho'$$

whence

$$R^2 R'^2 = (mrr'^2 + \mu\rho'^2)(mr'^2 + \mu\rho'^2) - m\mu(m\rho' - \rho r')^2,$$

which may be written

$$R'^2 = mr'^2 + \mu\rho'^2 - \frac{m\mu}{R^2}(r\rho' - \rho r')^2.$$

Furthermore we have the obvious identities

$$x'^2 + y'^2 + z'^2 = r'^2 + \frac{1}{r^2}[(yz' - zy')^2 + (zx' - xz')^2 + (xy' - yx')^2],$$

$$\xi'^2 + \eta'^2 + \zeta'^2 = \rho'^2 + \frac{1}{\rho^2}[(\eta\zeta' - \zeta\eta')^2 + (\zeta\xi' - \xi\zeta')^2 + (\xi\eta' - \eta\xi')^2]$$

Multiplying these last two equations through by  $m$  and  $\mu$  respectively, and subtracting them, member for member, from the preceding equation, there results the equation

$$(16) \quad R'^2 + P = 2(U - K)$$

where  $P$  (to be minimized) is a sum of seven squares,

$$(17) \quad P = \frac{m}{r^2}[(yz' - zy')^2 + (zx' - xz')^2 + (xy' - yx')^2] \\ + \frac{\mu}{\rho^2}[(\eta\zeta' - \zeta\eta')^2 + (\zeta\xi' - \xi\zeta')^2 + (\xi\eta' - \eta\xi')^2] \\ + \frac{m\mu}{R^2}(r\rho' - \rho r')^2$$

Here the energy integral (12) has been made use of.

From this relation due to Sundman we may derive the *inequality* which plays a fundamental part in his work and in the present chapter.

If we write

$$U = yz' - zy', \quad V = \eta\zeta' - \zeta\eta',$$

it will be observed that there are two terms in  $P$  of the form

$$S = \frac{m}{r^2}U^2 + \frac{\mu}{\rho^2}V^2,$$

while the first integral of angular momentum yields

$$mU + \mu V = a.$$

It is easily found that the minimum value of  $S$  when  $U$  and  $V$  vary subject to the restriction just written, while  $r$  and  $\rho$  remain fixed, is  $a^2 / R^2$ . Similarly there are two other analogous pairs of terms with minimum values  $b^2 / R^2$ ,  $c^2 / R^2$ , respectively. Hence we conclude that we have

$$(18) \quad P \geq f^2 / R^2 ,$$

$$(19) \quad f^2 = a^2 + b^2 + c^2 .$$

Suppose now that we eliminate  $U$  between Sundman's equality (16) and Lagrange's equality (15). This gives us

$$2RR'' + R'^2 + 2K = P ,$$

whence, by using (18), we obtain the inequality referred to:

$$(20) \quad 2RR'' + R'^2 + 2K \geq \frac{f^2}{R^2} .$$

If we define the **auxiliary function of Sundman**,

$$(21) \quad H = RR'^2 + 2KR + \frac{f^2}{R} ,$$

the inequality (20) enables us to infer the relation

$$(22) \quad H' = FR' , \quad (F \geq 0) .$$

Hence  $H$  increases (or at least does not decrease) as  $R$  increases, and decreases (or at least does not increase) as  $R$  decreases. This is the consequence which is of fundamental importance in what follows.

## 6. The possibility of collision.

Thus far we have been taking for granted the existence of solutions in the ordinary sense. In fact, inspection of the differential equations shows the **existence** of a unique analytic solution for which the coordinates and velocities have assigned values at  $t = t_0$ , provided that the bodies  $P_0, P_1, P_2$ , are geometrically distinct. In the case of the coincidence of two or three of these bodies, the right-hand members of the differential equations are no longer analytic, or even defined, so that the existence theorems of chapter I fail to apply.

But, according to the results there obtained, either these solutions can be continued for all values of the time, or (for example), as  $t$  increases, continuation is only possible up to  $t$ .

Let us consider this possibility in the light of the elementary **existence theorems**.

In the 18-dimensional manifold of states of motion associated with the 18 dependent variables

$$x_i, y_i, z_i, x'_i, y'_i, z'_i, \quad (i = 0, 1, 2),$$

we need to exclude the three 15-dimensional analytic manifolds

$$r_i = 0, \quad (i = 0, 1, 2).$$

The remaining region is open towards infinity and along these excluded boundary manifolds.

According to the results obtained, indefinite analytic extension of a particular motion will be possible unless as  $t$  approaches a certain critical value  $\bar{t}$ , the corresponding point  $P$  approaches the boundary of the open region specified.

Now suppose if possible that the least of the three mutual distances does not approach 0 as  $t$  approaches  $\bar{t}$ ; here it is not implied that a specific mutual distance such as  $POP$  remains least near to  $\bar{t}$ . We can then find positions of the three bodies for  $t$  arbitrarily near to  $\bar{t}$ , for which the three mutual distances exceed a definite positive constant  $d$ . But by the energy integral relation (4), in which

$$U < (m_0 m_1 + m_0 m_2 + m_1 m_2) / d,$$

it is clear that the velocities  $x'_i, y'_i, z'_i$  are uniformly limited. It is physically obvious that for such an initial condition, continuation of the motion is possible for an interval of time independent of the particular mutual distances or velocities, because of the character of the forces which enter; we shall not stop to obtain an explicit expression for such an interval on the basis of our first existence theorem. Thus a contradiction results.

***Analytic continuation of a particular motion in the problem of three bodies will be possible unless as  $t$  approaches a certain value  $\bar{t}$ , the least of the three mutual distances approaches 0.***

At this stage it is desirable to revert to Lagrange's equality (15). As  $t$  approaches  $\bar{t}$ ,  $U$  becomes positively infinite of course. Hence if we represent  $R^2$  as a function of  $t$  in the plane by taking  $t$  and  $R^2$  as rectangular coordinates, the corresponding curve will be concave upwards for  $t$  sufficiently near  $\bar{t}$ . Therefore  $R^2$  either becomes infinite, or tends toward a finite positive value, or approaches 0.

The **first** case is manifestly impossible, since one of the bodies would then recede indefinitely far from the two which approach coincidence as  $t$  approaches  $\bar{t}$  and such a state of affairs is impossible because of the fact that the forces on the distant body are bounded in magnitude.

In the **second** case it is clear that a particular distance approaches 0,

for instance  $r_2$ , while the other two approach definite equal limiting values. This is the case of double collision. Since the forces on the non-colliding body are finite near collision, it approaches a definite limiting position; and thus the other two colliding bodies approach a corresponding limiting position, since the center of gravity may be taken fixed and at the origin in the space of the three bodies.

In the **third** case we have triple collision of course, and this takes place at the origin. However if the constant  $f$  is not 0, triple collision cannot take place, as follows from (22) immediately. For it is seen that  $dR^2/dt$  will be negative for  $t$  near  $\bar{t}$  in the case of triple collision, since  $d^2R^2/dt^2$  is positive by Lagrange's equality (15). Hence  $H$  will decrease with  $R$  (or at least not increase) as  $t$  approaches  $\bar{t}$ . But inspection of  $H$  shows that  $H$  becomes positively infinite as  $R$  approaches 0. Thus a contradiction, is reached.

*As  $t$  approaches  $\bar{t}$ , there is either double collision between a definite pair of the bodies at a definite point, while the third body approaches a definite distinct point, or there is triple collision at the common center of gravity. If, however,  $f$  is not 0, i.e., if the angular momentum of the three bodies about every axis in space is not constantly 0, triple collision can never take place at  $\bar{t}$ .*

Henceforth we shall make the assumption  $f > 0$ , thereby eliminating the possibility of triple collision in the sense above specified.

This assumption may be looked upon as merely confining attention to the general case. In fact it is readily proved that in the case  $f=0$ , the motion is essentially in a fixed plane. Thus immediate reduction of the problem is possible. Moreover in the case  $f=0$  the angular momentum about a perpendicular to the plane of motion at the center of gravity vanishes. Thus we are only excluding a special case of motion in a plane. The case excluded is of great interest and should be given thorough consideration on its own account.

## 7. Indefinite continuation of the motions.

In the general case under consideration it is thus plain that any motion can be continued up to a double collision.

We propose now to take up briefly the case of double collision in order to render it physically plausible that the motion admits of continuation beyond such a double collision in a certain definite



manner. Analytic weapons sufficiently powerful to deal with the singularity of double collision were first developed by Sundman (loc. cit.). A different method of attack, not going outside of the domain of equations of usual dynamical type, has since been obtained by Levi-Civita. A rigorous treatment of the question will not be attempted here, but the analytic details can be supplied without difficulty on the basis of the researches of Sundman or Levi-Civita.

Let us suppose that the bodies  $P_0$  and  $P_1$  collide for instance, while  $P_2$  is at a distance away. The motion of  $P_0$  and  $P_1$  near collision will clearly be essentially as in the two body problem. What we propose to do is to ignore the disturbing forces due to  $P_2$  during the near approach of  $P_0$  and  $P_1$  to collision, i.e., to replace  $U$  by its single component  $m_0 m_1 / r_2$ , and then to take it for granted that the situation is of essentially the same nature in the actual case.

But if the motion of  $P_0$  and  $P_1$  were just as in the two body problem, their center of gravity would move with uniform velocity in a straight line, while, relative to this point,  $P_0$  and  $P_1$  would move in a fixed straight line until they collide. More precisely,  $P_0$  and  $P_1$  will be at distances inversely proportional to their masses from the center of gravity, while their squared relative velocity is  $2(m_0 + m_1)/r_2$  increased by a certain constant whose value depends on the total energy relative to the center of gravity. The motion relative to the center of gravity will be thought of as merely reversed in direction after collision. In the original reference system the bodies  $P_0$  and  $P_1$  will describe two **cusped curves**, and will collide at the common cusp; the cuspidal tangents of the two curves are of course oppositely directed, and it would be easy to specify the precise motion near collision by giving the explicit formulas.

Evidently such a motion of collision in the two body problem is completely characterized by the following quantities:

- (1) the three coordinates of the point of collision;
- (2) the three velocity components of the center of gravity at collision;
- (3) the two angular coordinates  $\theta, \varphi$  fixing the direction in space of the axis of the cusp described by  $P_1$ , which is the same direction as that of the line of motion relative to their center of gravity;
- (4) the energy constant.

Thus 9 coordinates in all are required to characterize uniquely a state of collision in the two body problem. But to specify any state of

motion before or after collision it is necessary to give the time  $\tau$  that has elapsed since collision.

Furthermore, any motion in which the two bodies almost collide can be characterized in a similar way. Here it is supposed that the initial conditions are slightly modified at some time before collision. In the modified motion it is easy to generalize the above coordinates as follows:

- (1) instead of the coordinates of the point of collision, we may take the coordinates of the center of gravity when the bodies are nearest to one another;
- (2) the corresponding velocity components of the center of gravity may be used as before;
- (3) the angular coordinates  $\theta, \varphi$  may refer to the direction of the transverse axis of the conies described relative to the center of gravity;
- (4) the constant of total energy may be used as before. When the motion is modified slightly in this manner, these 9 coordinates will be only slightly modified.

In addition to these 9 coordinates, the plane of the relative motion must be fixed by a further angular coordinate  $\psi$ , and the perihelion distance  $p$  must be specified. This gives 11 coordinates to fix upon a particular motion of the two bodies in general position. In order to specify a particular state of motion it is sufficient to specify the time  $\tau$  measured from perihelion passage.

The coordinate  $p$  is not available in the special case of circular motion relative to the center of gravity but this possibility does not arise near a state of collision of the type under consideration.

Hence we find **12 appropriate coordinates** in all, corresponding of course to the fact that we have a system of differential equations of the 12th order in the two-body problem.

Let us consider the coordinates in the two-body problem somewhat more attentively. The 6 coordinates determining the position of the center of gravity at nearest approach are obviously unrestricted coordinates. In other words, these sets of 6 coordinates are in one-to-one correspondence with the neighborhood of a point in 6-dimensional space. Similarly the 2 coordinates fixing the axial direction are in one-to-one correspondence with the neighborhood of a

point of the  $\theta, \varphi$  sphere and are thus unrestricted in the same sense; and so are the total energy and the time  $\tau$  of course. On the other hand, the perihelion distance  $p$  is always positive, and as  $p$  approaches 0, the motion approaches that of a definite motion of collision, independently of the coordinate  $\psi$  which fixes the plane of the motion. Suppose then that we introduce the following coordinates

$$\alpha = p \cos \psi, \quad \beta = p \sin \psi,$$

as coordinates serving to replace  $p$  and  $\psi$ . Collision is then characterized by the conditions

$$\alpha = \beta = 0.$$

The new coordinates  $\alpha, \beta$  are, however, unrestricted.

Consequently in the problem of two bodies, the states of motions near a particular state of collision are in one-to-one, continuous correspondence with the neighborhood of a point in a 12-dimensional space. With this representation the states of motions at collision constitute a 9-dimensional surface through the point.

It is obvious that in a certain sense the singularity of collision is removed by the use of the above coordinates.

Let us return now to the problem of three bodies in the case under consideration when two and only two of the bodies, say  $P_0$  and  $P_1$ , collide. For the motion of collision, we must have as before a definite point of collision, a definite vector velocity of their center of gravity at collision, a cuspidal direction in which collision takes place, and finally a limiting total energy. Furthermore any state of motion before or after collision is characterized by the elapsed time  $\tau$ .

For motions near a motion of collision, these 9 coordinates admit of simple generalization. For example the instant of 'perihelion' passage can be fixed as that at which the distance  $P_0P_1$  is a minimum, and in this way the position and velocity coordinates of the center of gravity, the axial coordinates, and the perihelion distance can be defined at once, and also the energy constant. The angular coordinate  $\psi$  can be taken as that given by the plane which bisects the small dihedral angle defined by the two planes through  $P_0P_1$  and the velocity vectors at  $P_0, P_1$ , respectively relative to their center of gravity. The time  $\tau$  is defined as before. The coordinates  $p, \psi$  may be replaced by  $\alpha, \beta$  of course.

***Thus on the basis of physical reasoning it appears certain that***

*the singularity of double collision is of removable type, and that the states of motion at double collision form three 15-dimensional (analytic) sub-manifolds in the 18-dimensional manifold  $M_{18}$  of states of motion, corresponding to the collisions of  $P_0$  and  $P_1$ , of  $P_0$  and  $P_2$ , and of  $P_1$  and  $P_2$ , respectively.*

When the manifold of states of motion is augmented by the adjunction of the parts of the boundary corresponding to double collision, it is obvious that indefinite analytic continuation of a motion is possible unless, as  $t$  approaches a certain value  $\bar{t}$  ( $t < \bar{t}$  say), there are an infinite number of double collisions. Let us eliminate this possibility for the case  $f > 0$ , which is under consideration.

In the first place we observe that not only  $R$  but also  $R'$  must be continuous at double collision. In fact the differential equations themselves show that  $\frac{d^2\xi}{dt^2}$ ,  $\frac{d^2\eta}{dt^2}$ ,  $\frac{d^2\zeta}{dt^2}$  are continuous at collision so that  $\rho'$  as well as  $\rho$  must be continuous. On the other hand  $r'$  will not be; but, since we have

$$r^2 r'^2 = (xx' + yy' + zz')^2 \leq (x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) \leq \frac{r^2}{2m}(U + |K|)$$

on account of the energy integral (12), it is clear that  $rr'$  is continuous and vanishes at collision. Hence  $R'$  is continuous at collision, having the value  $\mu \frac{\rho\rho'}{R}$ , as follows from (13).

Secondly, as  $t$  approaches  $\bar{t}$ , the least  $r_i$  must approach 0. Otherwise we should have  $r_i > d > 0$  ( $i = 0, 1, 2$ ) indefinitely near  $\bar{t}$ . We have already seen that, because of the energy integral, this would require  $x', y', z', \xi', \eta', \zeta'$  to be limited, so that continuation of the motion during a definite interval of time, dependent only on  $d$ , would be possible without collision. This is absurd.

Thirdly,  $R$  must approach a finite limit as  $t$  approaches  $\bar{t}$ , as follows from Lagrange's equality (15), just as in the case of approach to double collision, inasmuch as  $R'$  and  $R$  are both continuous at double collision. Reasoning on the basis of Sundman's inequality (22) in the same way as before, we infer also that  $R$  cannot approach as  $t$  approaches  $\bar{t}$ .

Hence we conclude that as  $t$  approaches  $\bar{t}$ , the body  $P_2$  approaches a definite limiting position distinct from the corresponding definite limiting coincident position of  $P_0$  and  $P_1$ . But it is

physically obvious, and might readily be established analytically, that there can only be a finite number of collisions for  $t < \bar{t}$  in such a case. Thus a contradiction arises.

***In the augmented manifold of states of motion  $M_{18}$ , indefinite continuation of every motion for which  $f > 0$  is possible in either sense of time. In the case  $f = 0$ , continuation can only be terminated by triple collision.***

Hitherto we have dealt with only the 18-dimensional manifold  $M_{18}$ . It is easy to modify the above results so as to apply to the manifold  $M_{12}$ , obtained when only those motions are considered for which the center of gravity of  $P_0, P_1, P_2$  lies at the origin. In this case the six coordinates fixing the position and velocity of the center of gravity of  $P_0$  and  $P_1$ , for instance, determine these coordinates for  $P_2$ .

***Entirely similar results obtain in the 12-dimensional manifold  $M_{12}$  obtained by fixing upon those motions for which the center of gravity of the three bodies lies at the origin.***

As remarked earlier, these results can be fully established by use of the explicit regularizations effected by Sundman or Levi-Civita. An inspection of the formulas leads to the following additional conclusion:

***In the augmented manifold  $M_{18}$  not only are the states of motion at collision to be regarded as constituted by three 15-dimensional analytic manifolds, but the curves of motion are also to be regarded as analytic and as varying analytically with the initial point and interval, provided this interval be measured by such a parameter as  $u$  where***

$$t = \int r_0 r_1 r_2 du .$$

## 8. Further properties of the motions.

The case  $K < 0$  is immediately disposed of, so far as the general qualitative character of the motions are concerned. Lagrange's equality

(15) insures that  $\frac{d^2 R^2}{dt^2}$  will then exceed  $4|K|$ . Hence  $R^2$ , when

plotted as a function of  $t$  in the  $t, R^2$  plane of rectangular coordinates, yields a curve with a single minimum which is everywhere concave upwards and rises indefinitely.

Evidently the same conclusion holds for  $K = 0$ , at least unless  $U$  approaches 0. But this can only happen if all three mutual distances increase indefinitely.

***In the case  $K \leq 0$ ,  $f > 0$ , at least two, if not all three, of the mutual distances increase indefinitely as time increases and decreases. In the case  $K \leq 0$ ,  $f = 0$ , the same is true unless the motion terminates in triple collision in one direction of the time.***

A fuller qualitative consideration of the motions  $K \leq 0$  is obviously desirable. But on account of the results just stated it seems proper to consider this case as 'solved' in the qualitative sense.

Henceforth we shall confine attention to the case  $f > 0$ ,  $K > 0$ , i.e., to the case when the angular momentum of the three bodies about every line through the center of gravity is not constantly 0, and the potential energy is insufficient to allow all three mutual distances to increase indefinitely.

The case  $f = 0$ ,  $K > 0$  thus remains. Here the motion is essentially in one plane, and it may be possible to obtain results similar to those here obtained in the case  $f > 0$ ,  $K > 0$  by suitable refinement of Sundraan's inequality.

We proceed to develop some of the simple and important properties of the motion in the case  $f > 0$ ,  $K > 0$ .

***In the case  $f > 0$ ,  $K > 0$  the least of the three mutual distances cannot exceed  $\frac{M^2}{3K}$ .***

The proof is immediate. By the energy integral (12),  $U$  is at least as great as  $K$ . But  $r_0, r_1, r_2$  are at least as great as  $r$ , the least distance. Hence we obtain

$$\frac{m_0 m_1 + m_0 m_2 + m_1 m_2}{r} \geq K.$$

The numerator on the left is not more than  $\frac{M^2}{3}$ , whence the stated inequality follows at once.

***In the case  $f > 0$ ,  $K > 0$ , the largest distance  $r_i$  will necessarily exceed  $k$  times the smallest distance  $r_j$ , provided that***

$$R \leq \frac{2m^{*1/2}f^2}{k^2M^3} \quad \text{or} \quad R \geq \frac{kM^{5/2}}{3K},$$

***where  $m^*$  denotes the least of the three masses  $m_0, m_1, m_2$ .***

To establish this fact, let  $k_1$  denote the actual ratio of the largest

to the smallest distance. Then we have at once

$$R^2 \leq \frac{(m_0 m_1 + m_0 m_2 + m_1 m_2) k_1^2 r^2}{M} \leq \frac{M k_1^2 r^2}{3}$$

where  $r$  denotes the smallest distance. Likewise we find by a similar calculation

$$U \leq \frac{m_0 m_1 + m_0 m_2 + m_1 m_2}{r} \leq \frac{M^2}{3r}.$$

But Sandman's equality (16) together with (18) gives

$$\frac{f^2}{R^2} < 2U.$$

If we employ the inequalities for  $R^2$  and  $U$  derived above, this gives readily

$$r > \frac{4f^2}{k_1^2 M^3}.$$

But inasmuch as  $R$  is at least  $m^{1/2} r$ , while  $m$  in turn is at least half of the least mass  $m^*$  (see (7)), we find

$$R > \frac{2m^{*1/2} f^2}{k_1^2 M^3}.$$

Consequently if  $R$  is at most of the first stated value, we infer at once that  $k_1$  exceeds  $k$ . This proves the first of the two results.

In order to prove the second result, let  $\bar{r}$  denote the greatest distance. We then obtain

$$R^2 \leq \frac{(m_0 m_1 + m_0 m_2 + m_1 m_2) \bar{r}^2}{M} \leq \frac{M \bar{r}^2}{3},$$

whence there results

$$\bar{r} > \frac{R}{M^{1/2}}.$$

If we use the inequality already derived for the least distance  $r$ , in combination with the one just written, we find

$$k_1 > \frac{3KR}{M^{5/2}}.$$

Hence if  $R$  is at least of the second value,  $k_1$  will exceed  $k$ . This is the second result to be proved.

***In the case  $f > 0$ ,  $K > 0$ , any part of the curve  $R = R(t)$ , ( $t$ ,  $R$ , rectangular coordinates) for which  $R < \frac{f}{2^{1/2} K^{1/2}}$  consists of a finite arc, concave upwards and with a single minimum. If  $R = R_0$  gives this minimum, the curve rises on either side until***

$$R > \frac{f^2}{2KR_0},$$

with corresponding slope  $R'$  at least as great as demanded by the inequality

$$R'^2 \geq \frac{R - R_0}{R} \left[ \frac{f^2}{R_0 R} - 2K \right]$$

at every intermediate stage.

To prove this statement, we observe first that when  $R$  is restricted as in the first part,  $R$  cannot be a constant. In fact if it were, Lagrange's equality (15) would yield  $U = 2K$ . But the combination of Sundman's equality (16) and of (18) with the equation  $U = 2K$  would give

$$\frac{f^2}{R^2} \leq 2K,$$

in contradiction with the limitation imposed upon  $R$ . The same kind of argument shows that if  $R'$  vanishes when  $R$  is so restricted, then  $R''$  must be positive. For otherwise, by using Lagrange's equality, we find  $U < 2K$ , and thence by using Sundman's equality (16) and (18) we are led to the contradictory conclusion written above.

If there is a point  $R' = 0$  along the arc under consideration, it corresponds to a proper minimum. On either side of it  $H$  (section 5) will increase (or at least not decrease) with  $R$ , until a second point  $R' = 0$  reached for  $R = R_1$ . Hence we obtain

$$2KR_1 + \frac{f^2}{R_1} > 2KR_0 + \frac{f^2}{R_0}$$

whence, since  $R_1 > R_0$ ,

$$2K > \frac{f^2}{R_0 R_1}.$$

In this case  $R$  does increase until the specified value is passed. Furthermore until this happens,  $H$  is as great as  $H_0$ . This fact demonstrates that  $R'^2$  is as great at every stage as stated, so that  $R$  must finally so increase.

The case when  $R' \neq 0$  anywhere along the arc can be eliminated. Here  $H$  must decrease (or at least not increase) with decreasing  $R$ . Consequently  $R$  cannot approach 0, since  $H$  then becomes infinite. As  $R$  approaches its lower limit  $R_0$ ,  $R'$  will approach 0. Consequently we infer that the inequality of the statement for  $R'^2$  continues to hold if  $R_0$  be defined in this manner.

But this kind of asymptotic approach to  $R = R_0$  as  $t$  increases (or



decreases) indefinitely is impossible. This impossibility may be made evident as follows. In the inequality  $H \geq H_0$  we may replace the inequality sign by the equality sign. Thereby we define a new curve  $R = R(t)$  whose slope for any  $R$  is not greater in numerical value than that along the actual curve under consideration. Hence the new curve so defined approaches the  $t$  axis less rapidly, and must also approach  $R = R_0$  asymptotically as follows from the equation  $H = H_0$ . But, by differentiation of this equation as to  $t$ , there results

$$2RR'' + R'^2 + 2K - \frac{f^2}{R^2} = 0.$$

Hence as  $t$  approaches infinity, and  $R, R'$  approach  $R_0, 0$ , it is clear that  $R''$  would approach a definite positive quantity, which is absurd.

The results thus far obtained may be regarded as concerned with motions in which the three bodies are all near together at some instant  $t = t_0$ , the amount of separation being measured by  $R$ . The bodies will separate in such a way that  $R$  increases, and very rapidly as long as  $R$  is not too large or small, until  $R$  has become very large.

We turn next to derive somewhat analogous results when at least one of the three mutual distances is large. Here it is convenient to use the quantity  $\rho$  instead of  $R$ , but it is to be borne in mind that  $r$  denotes the smallest of the three distances in what follows.

*In the case  $f > 0, K > 0$  as long as  $\rho \geq \frac{2M^2}{3K}$ , one and the same distance  $r_i$  is the least distance.*

Under this condition it follows that  $\rho$  is at least twice the least of the distances  $r = r_2$ . Hence  $r_0$  and  $r_1$  exceed  $r$ , since  $\rho$  is the distance from  $P_2$  to the center of gravity of  $P_0$  and  $P_1$ . But when  $r_0$  and  $r_1$  are greater than  $r_2$ , one and the same distance  $r_2$  remains least.

*In the case  $f > 0, K > 0$ , for  $\rho \geq \frac{2M^2}{3K}$ , the inequality*

$$\rho'' > -\frac{8M}{\rho^2}$$

*obtains. If for any such value of  $\rho$ , we have*

$$\rho' \geq \frac{4M^{1/2}}{\rho^{1/2}},$$

$\rho$  will constantly increase without bound.

We begin with the identity

$$\rho\rho'' + \rho'^2 = \xi\xi'' + \eta\eta'' + \zeta\zeta'' + \xi'^2 + \eta'^2 + \zeta'^2.$$

The last three terms on the right give the square of the velocity, of the point  $(\xi, \eta, \zeta)$ , while  $\rho'^2$  is the square of the radial velocity and is therefore not greater. By virtue of this fact and the differential equations (10) we obtain

$$\rho\rho'' \geq \frac{1}{\mu} \left( \xi \frac{\partial U}{\partial \xi} + \eta \frac{\partial U}{\partial \eta} + \zeta \frac{\partial U}{\partial \zeta} \right).$$

But the terms in parenthesis on the right are precisely  $\rho \frac{\partial U}{\partial n}$  where

$P_2$  is taken to vary by a distance  $n$  along the straight line which joins  $P_2$  to the center of gravity of  $P_0$  and  $P_1$ . Clearly the rate of change of  $r_0$  and  $r_1$  with respect to  $n$  cannot exceed 1 in absolute value, and we infer

$$\rho\rho'' \geq -\frac{\rho}{\mu} \left( \frac{m_1 m_2}{r_0^2} + \frac{m_0 m_2}{r_1^2} \right) > -M\rho \left( \frac{1}{r_0^2} + \frac{1}{r_1^2} \right)$$

(see (7)). Now in the case under consideration  $r_0$  and  $r_1$  exceed  $\rho - r$  and therefore  $\rho/2$ . This leads to the first inequality to be proved.

Instead of continuing analytically we need simply observe that this inequality may be looked upon as requiring that a particle moves along a  $\rho$  axis acted upon by a force towards the origin which does not exceed the gravitational force due to a mass  $8M$ . But in this case it is obvious that the particle will recede indefinitely provided that the initial velocity outward is as great as the velocity of fall from infinity under the attraction of such a mass. This is precisely the fact stated.

It should be noted that since the initial value of  $\rho$  is as great as

$$\frac{2M^2}{3K}, \quad \rho \text{ continues greater than this quantity, and accordingly one}$$

and the same distance  $r$  is the least of the three distances always.

We propose next to combine these results in order to show that, for the minimum  $R_0$  sufficiently small,  $R$  and  $\rho$  increase indefinitely. The qualitative basis of the reasoning is obvious. According to what has been proved, for  $R^*$  and  $R^{**}$  arbitrarily large a positive  $R_0$  can be chosen so small that all motions for which the minimum  $R$  is not more than  $R_0$  correspond to an  $R$  which increases from the

minimum to  $R^*$  and has, for  $R = R^*$ , a derivative  $R'$  which is at least as great as  $R^{*'}.$  This means of course that  $\rho^*$  is arbitrarily large since

$$\lim_{R \rightarrow \infty} \frac{R}{\rho} = (m_0 m_2 + m_1 m_2)^{1/2}$$

uniformly. Furthermore since the relation

$$RR' = mrr' + \mu\rho\rho'$$

obtains, it is clear that  $|\rho\rho'|$  must be large, and in particular  $|\rho'|$  must be large, provided that  $|rr'|$  is uniformly bounded. But we have

$$r'^2 \leq x'^2 + y'^2 + z'^2 < \frac{2U}{m}$$

by the energy integral (12). Hence

$$r'^2 r'^2 < \frac{2(m_0 m_1 + m_0 m_2 + m_1 m_2)r}{m} < \frac{2M^2 r}{m^*}$$

since  $m$  exceeds one half of the least mass  $m^*.$  Thus we find

$$|rr'| < \frac{M^2}{K^{1/2} m^{*1/2}},$$

and thereby establish the fact that  $|rr'|$  is uniformly bounded.

***For  $f > 0$ ,  $K > 0$ , if  $R_0$  is taken sufficiently small, every motion for which the three bodies approach so closely that  $R \leq R_0$  at some instant is such that two of the distances  $r_0, r_1$  become infinite with  $t$  while  $r_2$  remains less than  $\frac{M^2}{3K}.$***

We shall not pause to develop an analytic formula which yields a suitable  $R_0$ , although the specific results found above would supply the basis for such a computation.

There is an interesting question to which we wish to refer briefly in conclusion. Which one of the three bodies will recede indefinitely from the other two nearby bodies, in the case of a near approach to triple collision? The answer is to be found in the following statement:

***Any motion of the above type is characterized by the property that one and the same body  $P_2$  remains relatively remote from the two nearest bodies  $P_0, P_1$  throughout the entire motion.***

The truth of this fact is readily inferred. At the beginning of this section it was shown that, for  $R$  greater or less than fixed values, the ratio of the largest to the smallest distance would be arbitrarily large. Hence we need only consider this intermediate range of values of  $R$ . But in such a range, if the ratio of the largest to the smallest side did

not remain large for  $R_0$  sufficiently small, there would be configurations of the three bodies in which the distances  $r_i$  and the ratios  $\frac{r_i}{r_j}$  lie between fixed bounds, no matter how small  $R_0$  is chosen. However, the value of  $U$  does not exceed an assignable quantity in such configurations, and thus, by the energy integral (12), the same would be true of the velocities  $x', y', z', \xi', \eta', \zeta'$ . Finally it is clear that  $RR'$  would not exceed an assignable quantity. But we have established that  $R'$  becomes arbitrarily large in such a definite range of values of  $R$ , so that this conclusion is absurd.

Evidently there is further work to be done in the more precise determination of the motions on the quantitative side, but the facts developed above are sufficient to show that the only possibility of simultaneous near approach of the three bodies for given  $f > 0, K > 0$ , is that in which the three bodies act as a pair of bodies, one member of which corresponds to a close double pair  $P_0, P_1$ , while the second is  $P_2$ . The motions of  $P_2$  and the center of gravity of  $P_0, P_1$  are then along nearly hyperbolic paths, while  $P_0, P_1$  move in nearly elliptic paths relative to their center of gravity.

#### 9. On a result of Sundman.

Sundman established (loc. cit.) that for given initial coordinates and velocities with  $f > 0, K > 0$ , the quantity  $R(t)$  for the corresponding motion will always exceed a specifiable positive constant. This fact is at once evident from the analysis of section 8. In the contrary case we should have indefinitely near approach to triple collision, and thus a motion for which  $R'$  is arbitrarily large for the given initial value of  $R$ , which is of course absurd.

#### 10. The reduced manifold $M_7$ of states of motion.

Let us turn next to the consideration of the problem of three bodies after use has been made of the 10 known integrals to reduce the system of differential equations from the 18th to the 8th order. In other words the 10 corresponding constants of integration are given fixed values, and attention is directed towards the  $\infty^7$  motions which correspond to the given set of constants. In what follows we shall suppose that not all the constants of angular momentum vanish, and that the constant of energy is positive, i.e., we take  $f > 0, K > 0$ .

The angular momentum vector with components  $a, b, c$  will define a spatial direction which plays an important role in the sequel. Evidently two motions which correspond to the same configuration of positions and velocities at some instant, aside from mere angular orientation relative to this axis of angular momentum, will continue to differ merely in this respect. In other words, if  $\varphi$  denotes any angular coordinate which fixes the orientation about the axis of angular momentum, while  $u_1, \dots, u_7$  are any set of relative coordinates which do not involve  $\varphi$ , the differential equations defining the  $\infty^7$  motions take the form

$$\begin{cases} \frac{du_i}{dt} = U_i(u_1, \dots, u_7) \\ \frac{d\varphi}{dt} = \Phi(u_1, \dots, u_7) \end{cases} \quad (i=1, \dots, 7)$$

The first set of equations constitutes a system of the 7th order in the coordinates  $u_1, \dots, u_7$ , while the last equation enables one to determine  $\varphi$  by a further integration. If it be desired, the time  $t$  can be eliminated, and the system becomes of the 6th order,

$$\frac{du_i}{du_1} = \frac{U_i}{U_1} \quad (i=2, 3, \dots, 7).$$

Thus from the purely formal standpoint the system of the 18th order can be 'reduced' to one of the 6th order.

From the point of view which we shall adopt, there is no essential gain in actually carrying through such a reduction which can be accomplished without affecting the Hamiltonian form.

Let us consider the augmented manifold  $M_{18}$  of states of motion, in which the singularities corresponding to double collision have been removed by the method indicated in section 7.

The boundary of  $M_{18}$  is to be regarded as made up of states of motion specified by one of the following possibilities: one of the coordinates  $x_i, y_i, z_i$  increases indefinitely in absolute value; the quantity  $R$  approaches 0; the energy constant of some pair  $P_i, P_j$  of the bodies relative to their center of gravity at the instant increases indefinitely in absolute value. It is clear that points away from the boundary in the specific sense of these three possibilities will have limited coordinates, with not all three distances small; the condition of energy imposed insures that the energy constant relative to the center of gravity of all three bodies is not large in absolute value, while the

fact that the relative energy constants are not large means that the nearest pair of bodies must shortly separate to a considerable distance. Thus either all coordinates and velocity components are limited, and none of the mutual distances are small, or else the motion is near such a state in time, and therefore not near to the boundary of  $M_{18}$ .

In  $M_{18}$  the totality of motions is represented as a steady fluid motion, in which the stream lines correspond to the possible types of motion. When the 10 constants of integration are specified, we are directing attention to the corresponding fluid motion of the sub-manifold  $M_8$  into itself in which the stream lines represent the  $\infty^7$  motions under consideration.

Motions which differ merely in orientation with respect to the axis of angular momentum yield a closed one parameter family of stream lines, corresponding states of which give closed curves ; in other words  $u_1, \dots, u_7$  are the same along such a curve, while  $\varphi$  varies from to  $2\pi$ . In the special case of the Lagrangian equilateral triangle and straight line solutions when the mutual distances are inalterable, the corresponding closed curve is itself a stream line.

The 'reduced manifold  $M_7$  of states of motion' corresponds to the  $\infty^7$  set of states of motion given by sets of coordinates such as  $u_1, \dots, u_7$ , which are distinct except in orientation about the axis of angular momentum.

It is evident that in the original  $M_{18}$  the closed curves which give the states of motion differing only in orientation will give  $\infty^{17}$  analytic curves, one and only one through each point. Hence if we desire to obtain more precise information as to the possible singularities of  $M_7$ , it is only necessary to determine the singularities of  $M_8$ . We propose to investigate the singularities of  $M_8$ , and thus of  $M_7$ , sufficiently to establish the following result:

***For general values  $f > 0$ ,  $K > 0$ , the analytic reduced manifold  $M_7$  of states of motion is without singularity, and has a boundary specified by the fact that either  $R$  approaches 0 or  $\infty$ , or that the energy constant of some pair of the bodies relatively to their center of gravity become indefinitely large and negative.***

Let us first justify briefly the statement made about the boundary of  $M_7$ . At some distance from the boundary none of the coordinates can be large since none of the distances  $r_i$  are large, and the center of gravity is at the origin. Since the energy constant for the three

bodies is given, the partial energy constants cannot be large and positive. Consequently unless one of these partial constants is large and negative, the state of motion is not near the boundary of  $M_7$ .

In dealing with the analytic character of  $M_8$ , and so of  $M_7$ , we can assume that the state of motion under consideration is not a state of double collision. In fact the 'molecule' of states of motion in  $M_{18}$  near a state of double collision is carried analytically into a molecule about a modified position, not corresponding to a state of double collision. The invariant sub-manifold  $M_7$ , will thus be analytic all along a particular stream line or nowhere along it.

Let us then employ the coordinates  $x, y, z, \xi, \eta, \zeta, x', y', z', \xi', \eta', \zeta'$  which are available in  $M_{12}$ , within which we may take  $M_8$  to lie. The sets of these 12 coordinates which satisfy the remaining angular momentum and energy conditions (11) and (12), furnish uniquely the states of motion of  $M_8$  near to the particular motion of  $M_8$  under consideration. It is evident that in general these 4 equations may be solved analytically for any 4 of the 12 variables; i.e.,  $M_8$  analytic at the corresponding point.

We can show, however, that for general values of  $f > 0$  and  $K > 0$  there can be no singularities whatsoever in  $M_8$ . Let us choose coordinate axes so that  $x = y = \eta = 0$  at the instant under consideration, i.e., the particle  $P_1$  lies in the  $z$  direction from  $P_0$ , while the line from  $P_2$  to the center of gravity of  $P_0$  and  $P_1$  lies in the  $x, z$  plane. Let us attempt to solve the 4 equations for  $x', y', z', \eta'$  as functions of the other variables. The condition that this be possible will be satisfied if the corresponding Jacobian determinant

$$\begin{vmatrix} 0 & 0 & 0 & \xi \\ 0 & -z & 0 & -\zeta \\ z & 0 & 0 & 0 \\ x' & y' & z' & \eta' \end{vmatrix}$$

does not vanish; here we have removed an obvious factor  $m$  from the first three columns, and a factor  $\mu$  from the last column. Thus is analytic at this point provided that the inequality

$$-\xi^2 z' \neq 0$$

hold. But it has been pointed out that  $z$  is not 0. Furthermore, we can take  $\xi \neq 0$  unless  $P_2$  is on the straight line  $P_0 P_1$  constantly. And we can take  $z' \neq 0$  unless the distance  $P_0 P_1$  (and similarly any other distance  $P_i P_j$ ) is a constant. Hence we infer that either  $M_8$  is

analytic along the particular stream line under consideration, or the three bodies lie upon a straight line, or at a constant distance from each other, but not in the same straight line.

In the latter case the bodies  $P_0$ ,  $P_1$ ,  $P_2$  are known to lie at the vertices of an equilateral triangle in a plane perpendicular to the angular momentum vector; this triangle rotates at a constant angular velocity about its center of gravity. Furthermore it is known that there is one and only one size of triangle of this kind for an assigned angular velocity. Thus there will be in general no such motion for which  $f$  and  $K$  have the preassigned values.

Similarly in the first case further examination shows that the distances are inalterable. It is known that there are three solutions for an assigned angular velocity, and thus in general no solution for general values of  $f$  and  $K$ .

***In any case the manifold  $M_7$  can only have a singularity at a point corresponding to an equilateral triangle solution or to a straight line solution at constant mutual distances. These possibilities will only arise when certain analytic relations between  $f$  and  $K$  are satisfied. It is only as  $f$  and  $K$  vary through these critical values that the nature of  $M_7$  from the standpoint of analysis situs can change.***

The manifold  $M_7$  has fundamental importance for the problem of three bodies, but, so far as I know, it has nowhere been studied even with respect to the elementary question of connectivity. The work of Poincaré refers to the existence of certain periodic motions, i.e., of certain closed stream lines in  $M_7$ , obtained by the method of analytic continuation from a limiting integrable case of the problem of three bodies; nearby motions, i.e., stream lines in the torus-shaped neighborhood of such a closed stream line, are also considered in relation to the formal series; but he does not consider  $M_7$  in the large.

In conclusion it may be observed that the states of motion in which the three bodies move constantly in a plane through the center of gravity perpendicular to the angular momentum vector, correspond to an invariant sub-manifold  $M_5$  within  $M_7$ , which contains the exceptional singularities when these exist. So far as dimensionality is concerned, this manifold  $M_5$  would be suited to form the complete



boundary of a surface of section (chapter V) of properly extended type.

### 11. Types of motion in $M_7$ .

The problem of three bodies is distinguished from the type of non-singular problem which we have considered earlier, in that the manifold of states of motion is not closed. The singularity along the boundary cannot be removed by any exercise of analytic ingenuity. In fact consider a tube of stream lines in  $M_7$  described by a 'molecule' of states of motion near triple collision at  $t = 0$ . It is clear that the molecule tends toward the boundary of  $M_7$  as  $t$  increases, since we have then line  $R = \infty$  according to the results deduced above (section 8). The half tube so generated is then carried into part of itself, and would have to correspond to an infinite value of the invariant 7-dimensional volume integral. This situation does not arise when the manifold of states of motion is closed and non-singular.

More precisely, the stream lines corresponding to motions of near approach to triple collision not only lie wholly near the boundary of  $M_7$ , and approach it as  $t$  increases or decreases indefinitely, but they fill out three entirely distinct regions of  $M_7$ , since for every such motion there is a particular one of the three bodies which recedes indefinitely from the other two bodies.

*The stream lines corresponding to near approach to triple collision thus fill three distinct 7-dimensional continua of  $M_7$ , corresponding to the fact that  $P_0$ ,  $P_1$ , or  $P_2$  may be the relatively distant body during such a motion. These continua lie near to the boundary of  $M_7$ , and every stream line in them approaches the boundary in either sense of time.*

Of course these continua are not precisely defined until the degree to which triple collision is approached is precisely specified.

It is natural to believe that in this case of indefinite recession, the two nearby bodies have a definite limiting energy constant, orientation of plane of motion, eccentricity, and a limiting linear and angular momentum with reference to the center of gravity of the three bodies. In any case these motions may properly be regarded as to a large extent 'known'.

The very interesting question now arises: Do the motions for

which  $\lim R = \infty$  in one or both directions of the time fill  $M_7$  densely or only in part? It is important to understand the nature of the difficulty inherent in this question. By actual computation of the motions, it can doubtless be established whether or not a specific motion belongs to one of these continua or not. Certainly, for  $|K|$  small, almost all of  $M_7$  would be filled by these continua in consequence of the results obtained in the case  $K \leq 0$ . Nevertheless when there exists a single periodic motion in  $M_7$  of stable type, it will not be possible to determine whether or not nearby motions belong to these continua without solving the fundamental problem of stability in this particular case. We have already alluded to the highly difficult character of the problem of stability (chapter VIII), which arises precisely because in a dynamical problem, such as the problem of three bodies, formal stability of the first order insures the satisfaction of all the infinitely many further more delicate conditions for complete formal stability.

The question can, however, be put in a very suggestive form, which in my opinion renders it probable that the motions for which  $\lim R = \infty$  for  $\lim t = +\infty$  fill up  $M$  densely, as do those for which  $\lim R = \infty$  for  $\lim t = -\infty$ ; because of the reversibility of the system of differential equations, both conjectures must be either true or false.

The manifold  $M_7$  has already been conceived of as a 7-dimensional fluid in steady motion. This fluid must be thought of as having infinite extent and as incompressible, in consequence of the existence of a 7-dimensional volume invariant integral. The three types of motion with near approach to triple collision correspond to three streams which enter  $M_7$  from the infinite region and leave it there.

What is likely to happen to an arbitrary point of the fluid? It seems to me probable that in general such a point will move about until it is caught up by one of these streams and carried away. It may, however, be anticipated that there will be found certain points which remain at rest or move in closed stream lines, and so are not carried off. In conformity with the results of chapter VII, there must then necessarily exist other stream lines which remain near to the closed stream line as time increases or as time decreases. More generally, there will exist recurrent types of stream lines corresponding to recurrent motions, and various other stream lines which remain in their vicinity as time

increases or decreases. The stream lines corresponding to such recurrent motions and nearby motions cannot of course approach the boundary of  $M_7$ .

For the determination of the distribution of such periodic motions, recurrent motions, and motions in their vicinity, it is obvious that elaborate detailed analysis would be necessary. In conclusion we shall merely effect an obvious classification based on the function  $R(t)$ :

*An arbitrary motion in the problem of three bodies for the case  $f > 0, K > 0$  is of one of the following types as  $t$  increases:*

- (1)  *$R$  increases toward  $+\infty$ , in which case one body recedes indefinitely from the other two, while the near pair remain within finite distance of one another;*
- (2)  *$R$  tends toward a value  $\bar{R}$  while  $U$  approaches  $2K$ , in which case the limiting motion is of special determinable type as in Lagrange's equilateral triangle solution;*
- (3)  *$R(t)$  is uniformly bounded as in case (2) but oscillatory. Here the motion is wholly one of finite distances and velocities except possibly for occasional double collisions or approach to such collisions, and there necessarily exist periodic or other recurrent motions among the limit motions;*
- (4)  *$R(t)$  is oscillatory with upper bound  $+\infty$  and a positive lower bound. This is an intermediate case in which the motion is one with finite velocities except near occasional double collision or approach to double but not triple collision, while from time to time one of the three bodies recedes arbitrarily far from the near pair only to approach them again later.*

*Similar results obviously hold as  $t$  decreases.*

The only part of this statement calling for any explanation is that if  $R$  approaches  $\bar{R}$ ,  $U$  approaches  $2K$ . But this can be proved to follow from Lagrange's equality (15).

## **12. Extension to $n > 3$ bodies and more general laws of force.**

In indicating the possibility of generalizing the above results, both in respect to the number of bodies and the law of force, we shall entirely put to one side the question of collision. It would suffice for our purpose, however, if any kind of continuation after multiple

collision were possible in which the constants of linear and angular momentum as well as of energy are the same after as before collision, and if also  $R'$ , where

$$R^2 = \frac{1}{2M} \sum m_i m_j r_{ij}^2,$$

may be regarded as continuous at collision; here the masses of  $P_1, \dots, P_n$  are  $m_1, \dots, m_n$ , respectively, while  $M$  is the sum of these masses, and  $r_{ij}$  denotes the distance  $P_i P_j$ .

Let the function  $U$  of forces be any function of the mutual distances  $r_{ij}$ , of dimensions - 1 in these distances. For a function  $U$  of this type, the original form of differential equations, of the 10 integrals, and of Lagrange's equality (15) and of the inequality (20) due to Sundman will subsist, provided that  $f$  denotes the total angular momentum of the system about the center of gravity. Our main reasoning above was essentially based upon this analytical framework. Hence we can state the following result:

*Let  $U$  be any analytic function depending on the mutual distances between  $n$  bodies  $P_i$  ( $i = 1, \dots, n$ ), with coordinates  $(x_i, y_i, z_i)$  and masses  $m_i$ , respectively; let  $U$  be furthermore homogeneous of dimensions - 1 in these distances. If the  $n$  bodies are sufficiently near together, with assigned positive values of the total angular momentum  $f$  and the constant of energy  $K$ , at least two of the mutual distances will become very large in either sense of the time.*

*Further consideration shows that the condition of homogeneity upon  $U$  can be lightened to the form of an inequality*

$$\sum \left( x_i \frac{\partial U}{\partial x_i} + y_i \frac{\partial U}{\partial y_i} + z_i \frac{\partial U}{\partial z_i} \right) \geq -dU$$

*where  $0 < d < 2$ , without affecting the argument that at least two of the mutual distances become very large.*

In this argument the function  $H$  has to be generalized to the form

$$H = R^d \left[ R'^2 + \frac{f^2}{(2-d)R^2} + 2K \right].$$

I have not attempted to ascertain conditions under which at least two of the mutual distances become infinite.