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ACOUSTICS OF A NONHOMOGENEOUS MOVING MEDIUM

Translation "Akustiks Neodnorodnoi Dvizhushcheisya Sredy". Ogiz, Gosudarstvennoe Izdatel'stvo, Tekhniko-Teoreticheskoi Literatury, Moskva, 1946, Leningrad.

NACA TM 1399, 1956

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CHAPTER I

ACOUSTICS EQUATIONS OF A NONHOMOCENEOUS MOVING MEDIUM

1. Outline of Dynamics of a Compressible Fluid

The medium in which sound is propagated, whether it is a gas, a liquid, or a solid body, has an atomic structure. If, however, the frequency of the sound vibrations is not too large, this atomic character of the medium may be ignored.

For a gas it may be shown (ref. 1) that if $f << 1/\tau$, where f is the frequency of the vibrations and τ the time taken to traverse the free path between collisions, the gas may be considered as a dense medium characterized by certain constants. This method of considering the problem is assumed in **aerodynamics** and in the **theory of elasticity**. Since the atomic character of the medium is ignored, the phenomenon of the **dispersion** of sound cannot, in all strictness, be taken into account. Fortunately, in the majority of practical problems, the dispersion of sound does not have great significance. For this reason, phenomena which require consideration of the atomic nature of the medium will not be considered, and the **aerodynamic equations of a compressible gas** will be used as the basis of the theoretical analysis of the acoustics of a moving medium.

These equations are first considered without the assumption of any specific restrictions for the acoustics (such as large frequency and small amplitude of vibrations). The equations of the dynamics of a compressible gas express the three fundamental laws of conservation: (1) conservation of **matter**, (2) conservation of **momentum**, and (3) conservation of **energy**. In order to formulate these laws, a certain system of coordinates x, y, and z fixed relative to the undisturbed medium, is chosen. Further, t is the time, $\boldsymbol{v}$ is the velocity of the gas in this system., $v_1 = v_x$, $v_2 = v_y$, and $v_3 = v_z$ are the components of $\boldsymbol{v}$ along the x, y and z axes, respectively, and ρ is the density of the gas. In these notations, the law of the conservation of matter, mathematically expressed by the equation of continuity, assumes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_k) = 0 , \qquad (1.1)$$

where the summation is carried out for k = 1, 2, and 3. The vector $\rho \boldsymbol{v}$ is the **flow density** vector of the substance. This equation states that the

change in amount of substance in any small volume is equal to the flow of the substance through the surface enclosing this volume.

The vector $\rho\boldsymbol{v}$ may be considered also as the vector of the **momentum density**. The change of momentum in any small volume should be equal to the momentum transported by the motion of the fluid through the surface enclosing this volume plus the force applied to the volume.

The momentum flow due to the transport of momentum is a tensor with the components: $\rho v_i v_k$ ($i, k = 1, 2, 3$). The assumption is made that there are no volume forces. Hence the force applied to the volume is equal to the resultant of the stresses applied to the surface of the volume. The tensor of these stresses will be denoted by T_{ik} and is composed of the scalar pressure p and the viscous components s_{ik}

$$T_{ik} = p \cdot \delta_{ik} - s_{ik}\,, \qquad (1.2)$$

where $\delta_{ik} = 1$ if $i = k$, and $\delta_{ik} = 0$ if $i \neq k$.

When applied to a small volume, the law of the conservation of momentum can be written in the form

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_k}(T_{ik} + \rho v_i v_k) = 0\,, \qquad (1.3)$$

i and $k = 1, 2$ and 3 and again is summed for $k = 1, 2$, and 3. The equation of the conservation of energy should express the fact that the change in the total energy in a small volume, made up of the kinetic energy and the internal energy of a unit volume of the gas, is equal to the flow of the kinetic and internal energy through the surface enclosing this volume, the heat flow through this surface plus the work performed by the stresses acting on this volume. The part of the energy flow vector due to the transport of the kinetic energy $\rho \cdot \frac{v^2}{2}$ and the internal energy ρE (E is the energy of unit mass of the gas) is $(\rho\frac{v^2}{2} + \rho E)\boldsymbol{v}$. If the heat flow vector is denoted by $\boldsymbol{S}(S_1, S_2, S_3)$ and the conservation law is applied to a small volume,

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \rho E\right) + \frac{\partial}{\partial x_k}\left[\left(\rho\frac{v^2}{2} + \rho E\right)v_k + S_k\right] + \frac{\partial}{\partial x_k}(v_i T_{ik}) = 0$$

$$(1.4)$$

where the summation is for i and $k = 1, 2$ and 3. The last term gives the work of the stresses on a unit volume. For an isotropic, homogeneous

liquid (or gas), the stresses S_{ik} are connected with the deformations v_{ik} according to the Newtonian relation[1]

$$S_{ii} = 2\mu v_{ii} + \gamma \cdot \operatorname{div} \boldsymbol{v}\,; \quad S_{ik} = 2\mu \cdot v_{ik}\,, \qquad (1.5)$$

where μ is the viscosity of the gas and v_{ik} is the tensor of the deformations

$$v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \qquad (1.6)$$

The magnitude γ can be written in the form $\gamma = \mu' - 2\mu/3$, where μ' is the so-called **second coefficient of viscosity** (see ref. (i)). With this coefficient, account is taken of the conversion of the energy of the macroscopic motion of a gas into the energy of the internal degrees of freedom of the molecules (the rotation of the molecules), a fact which is of appreciable significance only for ultrasonic frequencies. For this reason, in the majority of cases the assumption may be made that $\mu' = 0$ and $\gamma = -2\mu/3$ (the value assumed in the theory of Stokes).

The flow of heat ***S*** expressed in terms of the gradient of the absolute temperature *T* is

$$S_k = \lambda \cdot \frac{\partial T}{\partial x_k}\,; \quad \lambda = \rho \cdot c_v \kappa\,, \qquad (1.7)$$

whore κ is the **coefficient of the heat conductivity** of the gas and c_v is the **specific heat of the gas at constant volume**.

To the three fundamental hydrodynamic equations, (1.1), (1.3) and (1.4), the **equation of state** of the gas (or liquid) connecting the pressure *p*, the density ρ, and the temperature *T* is added

$$p = Z(\rho, T)\,. \qquad (1.8)$$

Equations (1.1), (1.3) and (1.4) permit a rational determination of the flow of substance ***L***, the flow of momentum represented by the tensor M_{ik}, and the flow of energy ***N***, which, like the flow of substance, can be written in vector form. This determination will be such that the divergence of the flow, taken with inverse sign, is equal to the derivative with respect to the time of the density of the corresponding magnitude. In this manner from equation (1.1) for the flow of substance (equal to the flow of momentum) the following is obtained:

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[1] This form for v_{ik} follows from the assumption of the isotropic character and homogeneity of the gas or liquid if a linear relation is assumed between the stress tensor S_{ik} and the deformation tensor v_{ik}.

$$\boldsymbol{L} = \rho \boldsymbol{v} \ . \qquad (1.9)$$

From equation (1.3), substitution of the value of S_{ik} from equation (1.5), gives the tensor of the momentum flow

$$M_{ii} = \rho v_i^{\,2} + p + \gamma \cdot \operatorname{div} \boldsymbol{v} - 2\mu \cdot v_{ii}$$
$$M_{ik} = \rho v_i v_k - 2\mu v_{ik} = M_{ki} \ ; \ \ i \neq k \ . \qquad (1.10)$$

where, as before, *i* and *k* = 1, 2 and 3.

The terms of the form $\rho v_i^{\,2}$, $\rho v_i v_k$ give the momentum flow due to the transport of momentum by the motion of the fluid, and the terms containing ρ, μ and γ give the flow of momentum due to the action of the pressure forces and the viscous stresses.

Finally, from equation (1.4), substitution of S_{ik} from equation (1.5) yields the energy flow

$$\boldsymbol{N} = \left(\rho \frac{v^2}{2} + \rho E \right) \boldsymbol{v} + \boldsymbol{S} + \rho \boldsymbol{v} + \mu \left\{ \nabla v^2 + (\operatorname{rot} \boldsymbol{v} \times \boldsymbol{v}) \right\} + \gamma \cdot div \boldsymbol{v} \cdot \boldsymbol{v}$$

$$(1.11)$$

The first term gives the energy flow due to the transport of energy by the fluid, the second (***S***) gives the heat flow, and the term[2] $\rho \boldsymbol{v}$ and the terms with μ and γ give the part of the energy flow due to the work of the pressure forces and the viscous stresses.

The fundamental equations can also be written in vector from, by substitution of the value of the tensor T_{ik} from equations (1.2) and (1.5) in equations (1.3) and (1.4). Equation (1.1) may, however, be as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 \, . \qquad (1.12)$$

If use is made of (1.12) equation (1.3) can be written in the form

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \mu \Delta \boldsymbol{v} + \frac{1}{3} \mu \nabla \operatorname{div} \boldsymbol{v} \ , \qquad (1.13)$$

where ∇ is the symbol for the gradient and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$. The magnitude $\frac{d\boldsymbol{v}}{dt}$ is the total derivative of the velocity with respect to time and is equal to

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}, \nabla)\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \frac{v^2}{2} + (\operatorname{rot} \boldsymbol{v} \times \boldsymbol{v}) \, . \qquad (1.14)$$

The energy equation (eq. (1.4)), with the aid of equation (1.12), assumes

[2] The vector $N = (\rho v^2/2 + \rho)\boldsymbol{v}$, representing the flow of energy for an ideal incompressible liquid, is called the N. Umov vector (ref. 3)

the form

$$\rho \frac{dE}{dt} = \lambda \cdot \Delta T + Q - p \cdot \operatorname{div} \boldsymbol{v}\,, \qquad (1.15)$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + (\boldsymbol{v}, \nabla)E\,, \qquad (1.15')$$

where Q is the dissipative function

$$Q = \sum_{i,k=1}^{3} S_{ik} \cdot v_{ik}\ . \qquad (1.16)$$

If this equation is divided by ρ, it may be interpreted so that a change of energy of unit mass dE/dt is equal to the heat flow $\lambda \Delta T / \rho$, the amount of heat divided by the work of the viscous forces Q / ρ, and the work of the pressure forces $-\rho \operatorname{div} \boldsymbol{v} / \rho$ ..

This equation may also be interpreted in terms of thermodynamics. The first law of thermodynamics for unit mass of substance yields

$$dE = TdS - p \cdot dV\ , \qquad (1.17)$$

where E is the energy of unit mass; S, its entropy; p, the pressure, and V, the specific volume ($V = 1/\rho$). Thus

$$\frac{dE}{dt} = T\frac{dS}{dt} - p\frac{dV}{dt} = T\frac{dS}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt}\ . \qquad (1.18)$$

On the other hand,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v}, \nabla)\rho = -\rho \cdot \operatorname{div} \boldsymbol{v}\,, \qquad (1.19)$$

so that

$$p \cdot \frac{\operatorname{div} \boldsymbol{v}}{\rho} = -\frac{p}{\rho^2} \cdot \frac{d\rho}{dt}\,. \qquad (1.20)$$

For adiabatic processes

$$\frac{dE}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt}\,, \qquad (1.21)$$

from which

$$E = \int^p \frac{dp}{\rho} - \frac{p}{\rho}\,. \qquad (1.22)$$

The magnitude

$$w = E + \frac{p}{\rho} = \int^p \frac{dp}{\rho} \qquad (1.23)$$

is termed the ***heat function***. If the process is nonadiabatic, equation (1.18) holds. From equations (1.15) and (1.18) the following is obtained:

$$T\frac{dS}{dt} = \frac{\lambda}{\rho}\Delta T + \frac{Q}{\rho}\ . \qquad (1.24)$$

The magnitude $T(dS/dt)$ is the increase of heat of unit mass of the gas, which is determined exclusively by the heat conductivity and the work of the friction forces. If λ and μ are neglected since the effects produced by them in the over-all energy balance are usually small corrections, the following results:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + (\boldsymbol{v}, \nabla S) = 0 , \tag{1.25}$$

that is, the adiabatic motion of the fluid. The **Bernoulli theorem** holds for this motion if it is also irrotational (rot $\boldsymbol{v} = 0$).

If

$$\boldsymbol{v} = -\nabla \Phi , \tag{1.26}$$

where Φ is the **velocity potential**, from equations (1.13) and (1.14)

$$\nabla \left[-\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 \right] = -\frac{\nabla p}{\rho} , \tag{1.27}$$

and since, on the basis of equation (1.23), $p / \rho = \nabla w$, integration of equation (1.27) gives

$$w = \int^{p} \frac{dp}{\rho} = \frac{\partial \Phi}{\partial t} - \frac{1}{2} (\nabla \Phi)^2 . \tag{1.27'}$$

If the compressibility of the fluid is neglected,

$$w = \frac{p}{\rho_0} + \text{constant} , \tag{1.28}$$

so that

$$p = \rho_0 \frac{\partial \Phi}{\partial t} - \frac{\rho_0}{2} (\nabla \Phi)^2 + \text{constant} \tag{1.29}$$

and in the case of steady flows ($\partial \Phi / \partial t = 0$)

$$p = \text{constant} - \frac{\rho_0}{2} (\nabla \Phi)^2 = \text{constant} - \frac{\rho_0 v^2}{2} . \tag{1.30}$$

Because the entropy remains constant during the motion for an ideal fluid ($\lambda = \mu = 0$) introduction of the variables ρ and S in the equation of state, equation (1.8), in place of the variables ρ and T, is expedient since with such a choice of variables one of the variables (S) remains constant, whereas the temperature T varies even for an ideal fluid (for adiabatic compressions and expansions of the fluid). The following may be written in place of equation (1.8)

$$p = Z'(\rho, S) . \tag{1.8'}$$

2. Equations of Acoustics in Absence of Wind

The equations which determine the propagation of sound in a

motionless medium can now be considered. The vibrations of the medium are called ***sonic vibrations*** or simply ***sound*** if the amplitude of the vibrations is so small that it is possible to neglect all the changes in state of the gas in any small volume are produced in it by the transport (convection) of mass, momentum and energy. This situation is the condition of **linearity** of the vibrations. Further, these vibrations are assumed to occur with frequencies in the hearing range (the region of classical acoustics) or near this range (infra and ultra sound). Mathematically the above assumption reduces to the neglect of the terms in the aerodynamic equations of a compressible gas which contain second powers or the products of small magnitudes which determine the deviations of the state of the gas from equilibrium. Where π is the deviation of the pressure from the equilibrium value p_0, p is set equal to $p_0 + \pi$, $\rho = \rho_0 + \delta$, where ρ_0 is the value of the density for $p = p_0$ and $T = T_0$, and finally $\boldsymbol{v} = \xi$ (ξ is a small velocity). Similarly, for the temperature, entropy, and energy;

$$T = T_0 + \theta,$$

$$S = S_0 + \sigma,$$

$$E = E_0 + \varepsilon.$$

In place of equations (1.12) and (1.13), the following is obtained:

$$\rho_0 \cdot \frac{\partial \xi}{\partial t} = -\nabla \pi + \mu \cdot \Delta \xi + \frac{1}{3} \mu \nabla \operatorname{div} \xi, \qquad (1.31)$$

$$\frac{\partial \delta}{\partial t} + \rho_0 \operatorname{div} \xi = 0 \qquad (1.32)$$

The equation of state of the gas, for an ideal gas in the variables ρ and T is

$$p = \rho \cdot rT, \qquad (1.33)$$

where r is the gas constant for unit mass; and in the variables ρ and S

$$p = \rho\gamma \cdot \frac{p_0}{\rho_0{}^{\gamma}} \cdot e^{(S - S_0)/c_v}, \qquad (1.34)$$

where c_v is the specific heat at constant volume ($c_v = r/(\gamma - 1)$), and $\gamma = c_p / c_v$ is the ratio of the specific heats at constant pressure and constant volume. For small changes of state the following is obtained from equation (1.34):

$$\pi = \gamma \frac{p_0}{\rho_0} \delta + \frac{p_0}{c_v} \sigma + \cdots = c^2 \delta + h\sigma + \cdots; \quad h = \frac{p_0}{c_v}.$$

For $\sigma = 0$, only the first term representing small changes in pressure for small adiabatic compression or expansion of the gas remains. The

magnitude

$$c = \sqrt{\gamma \frac{p_0}{\rho_0}} \qquad (1.35)$$

is the **adiabatic velocity of sound**. The second term gives the change in pressure produced by the addition or decrease of heat. The changes of entropy σ obey equation (1.24) which is written by neglecting magnitudes of the second order of smallness as follows:

$$T_0 \frac{\partial \sigma}{\partial t} = \frac{\lambda}{\rho_0} \Delta\theta\,; \;\; \lambda = \rho c_v \kappa\,. \qquad (1.36)$$

The changes in temperature θ may be expressed in terms of the changes in density and entropy. From equation (1.17)

$$T = \left(\frac{\partial E}{\partial S}\right)_\rho . \qquad (1.37)$$

The energy of an ideal gas is equal to

$$E = c_v T = \frac{p}{(\gamma-1)\rho} = \frac{p_0}{{\rho_0}^\gamma} \cdot \frac{\rho^\gamma}{\gamma-1} \cdot e^{\sigma/c_v}\,, \qquad (1.38)$$

from which $\dfrac{\partial E}{\partial S} = \dfrac{\partial E}{\partial \sigma}$ is obtained in the form

$$T = \frac{p_0}{{\rho_0}^\gamma} \cdot \frac{\rho^{\gamma-1}}{(\gamma-1)c_v} \cdot e^{\sigma' c_v} = \frac{p}{(\gamma-1)\rho c_v}\,, \qquad (1.37')$$

that is, for small values of ρ and S

$$\theta = \frac{p_0}{{\rho_0}^2 c_v}\delta + \frac{p_0}{\rho_0(\gamma-1)c_v}\sigma + \cdots, \qquad (1.39)$$

where the first term represents the change in temperature during adiabatic compression or expansion of the gas and the second term represents the change in temperature due to the change in entropy of the gas.

Substitution in equation (1.36) yields

$$\frac{\partial \sigma}{\partial t} = \kappa \Delta\sigma + \kappa_1 \Delta\delta\,; \;\; \kappa_1 = \kappa \frac{(\gamma-1)c_v}{\rho_0}\,. \qquad (1.40)$$

Equations (1.31), (1.32), and (1.40) together with the equation of state (1.34) determine the propagation of sound in a motionless medium when account is taken of the viscosity and heat conductivity of the medium.

The effects arising from the presence of viscosity and heat conductivity reduce, in a first approximation, to the absorption of the sound by the medium. This **absorption** is generally not large and its magnitude for a plane wave can be determined without difficulty. If its direction of propagation is along the *ox* axis, the frequency of the sound

equals ω, and the wave number vector is equal to *k*,

$$\left.\begin{array}{l} \xi = \xi_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \\ \delta = \delta_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \\ \sigma = \sigma_0 e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \end{array}\right\}, \qquad (1.41)$$

where $\xi_0, \delta_0, \sigma_0$ are the amplitudes of vibration of the corresponding magnitudes. Substitution of equations (1.41) in equations (1.31), (1.32) and (1.40) yields

$$i\omega\rho_0\xi_0 = ik(c^2\delta_0 + h\sigma_0) - \frac{4}{3}\mu k^2\xi_0, \qquad (1.31')$$

$$i\omega\delta_0 - ik\rho_0\xi_0 = 0, \qquad (1.32')$$

$$i\omega\sigma_0 = -\kappa k^2\sigma_0 - \kappa_1 k^2\delta_0. \qquad (1.40')$$

Elimination of the amplitudes gives the relation between *k* and

$$\omega\rho_0 = k \cdot \left[c^2 \cdot \frac{k\rho_0}{\omega} - \frac{h \cdot \kappa_1 k^2}{(i\omega + \kappa k^2)} \right] + \frac{4}{3} i\mu k^2. \qquad (1.42)$$

If *k* is set equal to $\omega / c - i\alpha$, where α is the coefficient of damping of the wave, the velocity of propagation *c*' in the first approximation is equal to *c*, and the damping coefficient α is equal to

$$\alpha = \frac{2}{3}\frac{\mu \cdot \omega^2}{\rho c^3} + \frac{\kappa}{2\rho}\left(1 - \frac{a^2}{c^2}\right)\frac{\omega^2}{c^3}, \qquad (1.43)$$

where $a^2 = p_0 / \rho_0$ is the square of the **isothermal velocity of sound**. For air $\alpha = 1.1 \times 10^{-13} f^2 cm^{-1}$, where $f = \omega / 2\pi$ is the frequency of sound in Hz (1 Hertz = 1 cycle/sec). Hence in many cases the effect of the ***viscosity and heat conductivity maybe neglected*** or their effect taken into account by introduction of the absorption coefficient in the final results. The smallness of the effect of viscosity and heat conductivity of the air on the propagation of sound is determined not only by the smallness of the coefficients μ and κ but also by the smallness of the gradients of all magnitudes which vary in the sound propagation.

Equations (1.31) and (1.40) show that these gradients enter the equation in the form of second derivatives of ξ, σ and so forth (for example, $\mu\Delta\xi$ and $\kappa\Delta\sigma$). In the propagation of a wave in free space these derivatives are in order of magnitude equal to ξ / λ^2, σ / λ^2, …, and so forth, and become appreciable only for very short wave lengths (as the final equation for the absorption coefficient α shows since α increases proportionally to the square of the frequency.

Near the boundaries of solid or fluid bodies which maybe considered

as stationary, the losses by viscosity and heat conductivity increase. In these cases sharper changes of state of the gas in space occur and the second derivatives of ξ, σ and δ are determined not by the length of the wave but either by the **dimensions of the body** *l* so that $\Delta\xi \cong \xi / l^2$ and $\Delta\sigma \cong \sigma / l^2$ or by the 'natural" length $d' = \sqrt{\nu / \omega}$ (this length is in addition to the lengths λ and *l*, and is determined from dimensional considerations), where ν is the kinematic viscosity ($\nu = \mu / \rho$), or by the length $d'' = \sqrt{\kappa / \omega}$. In these cases the order of the magnitudes is given by $\Delta\xi \cong \xi / d^2$ and $\Delta\sigma \cong \sigma / d^2$.

In general, the losses by viscosity and heat conductivity near the boundary of a solid or fluid body are determined by the least of the three lengths λ, *l* and *d* (*d* ', *d*").

Despite the increase in the losses near walls and stationary boundaries, the losses remain small and can be considered a correction to the motion which occurs without losses (except for the case of the propagation of sound in very narrow channels). An example of the approximate computation of the effects of viscosity and heat conductivity may be found in the work of the author (ref. 4).

In addition to the absorption of sound associated with the heat conductivity and the viscosity of the medium still another **molecular absorption** of sound exists which was discovered by V. Knudsen (ref. 5) and explained by G. Kneser (ref. 6). The physical character of this absorption lies in the conversion of the energy of the sound vibrations into the energy of inner molecular motion (energy of rotation of the molecules). This absorption likewise increases with the frequency and is of special significance for the ultrasonic range.

As the consideration of these problems deviates from the present subject, discussion is limited to references given.

In all those cased where the losses of the sound energy are not of interest, the viscosity and heat conductivity of the air may be ignored. If λ and μ are set equal to 0 in equations (1.3') and (1.40), $\sigma = 0$, that is, adiabatic propagation of sound is obtained and the equations describing this propagation assume the form

$$\rho_0 \cdot \frac{\partial \xi}{\partial t} = -\nabla \pi \,, \tag{1.44}$$

$$\frac{\partial \delta}{\partial t} + \rho_0 \cdot \operatorname{div} \xi = 0 \,, \tag{1.45}$$

$$\pi = c^2 \delta \,. \tag{1.46}$$

These equations may be solved with the aid of the single function ϕ which is termed the ***velocity potential*** (or simply the ***potential***). The first three equations (1.44) are satisfied by setting

$$\left.\begin{aligned} \pi &= \rho_0 \cdot \frac{\partial \phi}{\partial t} \\ \xi &= -\nabla \phi \end{aligned}\right\}. \tag{1.47}$$

The **wave equation** for the potential from equations (1.46) and (1.45) is obtained:

$$\Delta \phi - \frac{1}{c^2} \cdot \frac{\partial^2 \phi}{\partial t^2} = 0 \,, \tag{1.48}$$

which, in the presence of bodies, must be solved with the **boundary condition**

$$-\left(\frac{\partial \phi}{\partial n}\right) = \xi_{0n} \quad \text{(on the surface of the body)} \tag{1.49}$$

where $\dfrac{\partial}{\partial n}$ is the derivative along the normal to the surface of the body and ξ_{0n} is the normal velocity of the surface assumed as small. In place of equation (1.49), for stationary bodies

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{(on the surface of the body)} \tag{1.49'}$$

For a unique solution of the problem of the sonic field described by equation (1.48) the **initial conditions** for ϕ and $\dfrac{\partial \phi}{\partial t}$ must be formulated in addition to the boundary conditions of equations (1.49) or (1.49').

3. Energy and Energy Flow in Acoustics

For linear acoustics all magnitudes referring to the sound are computed with an accuracy up to the first degree of the amplitude A, which may, for example, be the amplitude of a piston which excites sound vibrations. Achievement of more accurate solutions of the equations of hydrodynamics will yield the succeeding approximation containing terms proportional to A^2, and so forth (when account is taken of nonlinear

phenomena). For the pressure p, the density ρ, and the velocity of motion $\boldsymbol{v}$, the following series is written:

$$p = p_0 + \pi_1 + \pi_2 + \cdots,$$
$$\rho = \rho_0 + \delta_1 + \delta_2 + \cdots,$$
$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \cdots. \quad (1.50)$$

The magnitudes p_0, ρ_0 and $\boldsymbol{v}_0$ refer to the motion undisturbed by the sound; the magnitudes π_1, δ_1 and ξ_1 are proportional to A, the magnitudes π_2, δ_2 and $\boldsymbol{\xi}_2$ are proportional to A^2, and so forth. The energy and energy flow contain the squares of the magnitudes δ_1, $\boldsymbol{\xi}_1$ and π_1. For this reason caution must be used when the energy and energy flow are computed in linear acoustics, as was pointed out by I. Bronshtein and B. Konstantinov (ref. 7) and also by N. N. Andreev (ref. 8), since these magnitudes, being of the order of A^2, may also contain the first degrees of the succeeding approximations π_2, δ_2 and ξ_2 while their contribution will be of the same order as the contribution from the squares of π_1, δ_1 and $\boldsymbol{\xi}_1$.

The general expression for the ***energy density*** of a compressible medium is

$$U = \frac{\rho v^2}{2} + \rho E\,, \quad (1.51)$$

where E is the internal energy of unit mass of the medium The ***energy flow*** $\boldsymbol{N}$, computed on the basis of equation (1.11) with the viscosity and heat conductivity neglected, is equal to

$$\boldsymbol{N} = U\boldsymbol{v} + p\boldsymbol{v}\,. \quad (1.52)$$

From the law of the **conservation of energy**,

$$\frac{\partial U}{\partial t} + \operatorname{div} \boldsymbol{N} = 0\,. \quad (1.53)$$

This equation is one of the fundamental equations of hydrodynamics, that is, equation (1.4) for the case of an ideal fluid ($\mu = \lambda = \gamma = 0$).

For an ***ideal gas*** $\rho E = \dfrac{p}{\gamma - 1}$ (equation (1.38)); hence

$$\boldsymbol{N} = \frac{\rho v^2}{2}\boldsymbol{v} + \frac{\gamma p \boldsymbol{v}}{\gamma - 1}\,. \quad (1.52')$$

For acoustics the initial medium is considered motionless ($\boldsymbol{v}_0 = 0$). The energy of the sound $\varepsilon_2 = U_2 - \rho_0 E_0$ and the flow of sonic energy $\boldsymbol{N}_2$ is obtained with an accuracy up to the order of magnitude A^2. Terms of the order of A^3 rejected,

$$\frac{\partial \varepsilon_2}{\partial t} + \operatorname{div} \boldsymbol{N}_2 = 0, \tag{1.53}$$

where

$$\varepsilon_2 = \frac{\rho_0 \xi_1^2}{2} + \frac{\pi_1 + \pi_2}{\gamma - 1},$$

$$\boldsymbol{N}_2 = \frac{\gamma p_0}{\gamma - 1}(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2) + \frac{\gamma \pi_1}{\gamma - 1}\xi_1 . \tag{1.54}$$

Inasmuch as

$$\begin{aligned} p &= p_0 + \left(\frac{dp}{d\rho}\right)_0 (\delta_1 + \delta_2) + \frac{1}{2}\left(\frac{d^2 p}{d\rho^2}\right)_0 \delta_1{}^2 + \cdots \\ &= p_0 + c_0^2(\delta_1 + \delta_2) + \frac{1}{2}(\gamma - 1)c_0^2 \delta_1{}^2 + \cdots \qquad , \\ &= p_0 + \pi_1 + \pi_2 + \cdots \end{aligned} \tag{1.55}$$

($c_0^2 = \frac{dp}{dr} = \gamma \frac{p_0}{\rho_0}$ is the square of the adiabatic velocity) and $\pi_1 = c_0^2 \delta_1$, equation (1.54) may be rewritten in the form (1.54')

$$\varepsilon_2 = \frac{\rho_0 \xi_1{}^2}{2} + \frac{\pi_1^2}{2\rho_0 c_0^2} + \frac{c_0^2}{\gamma - 1}(\delta_1 + \delta_2),$$

$$\boldsymbol{N}_2 = \frac{c_0^2 \rho_0}{\gamma - 1}(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2) + \frac{\gamma \pi_1}{\gamma - 1}\xi_1 . \tag{1.54'}$$

For a ***homogeneous medium*** at rest ($\boldsymbol{v}_0 = 0$, $c_0 = \text{constant}$, and $\rho_0 = \text{constant}$), a new form of the conservation law follows from equation (1.53) in which the energy of the sound and its flow are expressed only in terms of the magnitudes characteristic of linear acoustics (π_1, δ_1 and $\boldsymbol{\xi}_1$), not containing the second approximations (π_2, δ_2 and $\boldsymbol{\xi}_2$). The **equation of continuity** expressing the law of the conservation of matter (equation (1.12)), when written with an accuracy up to terms of the order of A^2 is

$$\frac{\partial(\delta_1 + \delta_2)}{\partial t} + \rho_0 \operatorname{div}(\xi_1 + \xi_2) + \operatorname{div}(\delta_1 \xi_1) = 0 . \tag{1.56}$$

This equation Is multiplied by $\frac{c_0^2}{\gamma - 1}$ and the result is subtracted from equation (1.53). Inasmuch as $\delta_1 = \frac{\pi_1}{c_1^2}$, equation (1.54) yields

$$\frac{\partial \varepsilon_1}{\partial t} + \operatorname{div} \boldsymbol{N}_1 = 0, \tag{1.57}$$

where

$$\varepsilon_1 = \frac{\rho_0 \xi_1^2}{2} + \frac{\pi_1^2}{2\rho_0 c_0^2} ,$$

$$\boldsymbol{N}_1 = \pi_1 \boldsymbol{\xi}_1 . \tag{1.58}$$

The new expressions obtained for the energy of sound and the energy flow ε_1 are precisely those which are applied in acoustics. In particular, if the potential ϕ ($\xi_1 = -\nabla\phi$, $\pi_1 = \rho_0 \frac{\partial\phi}{\partial t}$, see equation (1.47)) of the sound wave is introduced, then

$$\varepsilon_1 = \frac{\rho_0}{2}(\nabla\phi)^2 + \frac{1}{2\rho_0^2 c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2 ,$$

$$\boldsymbol{N}_1 = -\rho_0 \frac{\partial\phi}{\partial t}\nabla\phi . \tag{1.59}$$

If, as is often the case, the potential ϕ depends harmonically on the time and is given in complex form (ϕ is proportional to $e^{i\omega t}$), the mean energy in time and the mean flow in time are equal to

$$\varepsilon_1 = \frac{\rho_0}{4}\nabla\phi \cdot \nabla\phi^* + \frac{\omega^2}{4\rho_0^2 c_0^2}\phi\phi^* ,$$

$$\boldsymbol{N}_1 = \frac{i\omega\rho_0}{4}\left\{\phi^* \cdot \nabla\phi - \phi \cdot \nabla\phi^*\right\}, \tag{1.60}$$

where the sign * indicates that the conjugate complex magnitude should be taken.

The expressions for the **energy** and **energy flow** equations, (1.54) and (1.58), are physically equivalent because the medium is supposedly ***homogeneous*** (in a nonhomogeneous medium equations (1.59) are not valid). In order to show the equivalence of the two forms of the conservation laws, one of which is a consequence of the other (under the given conditions) the radiation of sound is considered. In figure 1 is

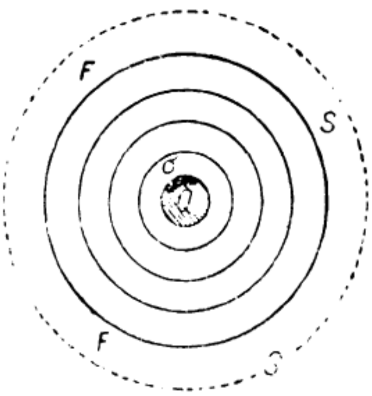

Figure 1.

shown a sound source Q (solid body), a certain part of whose surface σ executes vibrations which excite sound waves. If the vibration started at the time instant $t = 0$, at the moment t the surface of the wave front will be the surface F (see fig. 1). The entire space between this surface and the source Q will be filled with energy radiated by the sound. With an arbitrary control surface S enclosing the sound source, the conservation theorem (1.53) is applied in integral form to the volume V included between S and Q. In order to do this, equation (1.53) must be integrated over the volume and then, the theorem of Gauss is used in transforming the integral of $\operatorname{div} \boldsymbol{N}_2$ to a surface integral. This integral will consist of the integral over the surface S and the surface of the source Q. Although some inconvenience is caused because part of this surface is movable (σ), it can easily be circumvented by the consideration that the flow of energy through the surface of the source must simply be equal to the source W_2.

From equation (1.53) the following equation is obtained:

$$\frac{\partial E_2}{\partial t} + \int_S \left[\frac{p_0 \gamma}{\gamma - 1}(\xi_1 + \xi_2) + \frac{\gamma}{\gamma - 1}(\pi_1 \xi_1)_n \right] d\sigma = W_2, \quad (1.61)$$

where n denotes the projection of ξ on the normal to the surface S, $E_2 = \int_V E_2 dV$ is the total energy of the sonic field enclosed within S; and the strength of the source Q is evidently equal to

$$W_2 = \int_\sigma [p_0(\xi_1 + \xi_2)_\nu + (\pi \xi_1)_\nu] d\sigma, \quad (1.62)$$

where ν denotes the projection on the normal to the surface σ. If the control surface is passed outside the sonic field (for example, outside the wave front F, but infinitely near it), from equation (1.61) is obtained

$$\frac{dE_2}{dt} = W_2; \quad E_2 = \int_0^t W_2 dt \quad (1.63)$$

that is, the total radiated energy E_2 is equal to the work of the source Q. On the other hand, if the second form of the conservation law (eq. (1.18)) is treated in the same manner, the following equation results:

$$\frac{dE_1}{dt} = W_2; \quad E_1 = \int_0^t W_2 dt, \quad (1.63')$$

from which it follows that E_1 must be equal to E_2.

From equations (1.54') and (1.58),

$$E_2 - E_1 = \frac{c^2}{\gamma - 1} \int_V (\delta_1 + \delta_2) dV, \quad (1.64)$$

where the integration is over the volume V. The integral $\int_V (\delta_1 + \delta_2) dV$ is the total change of mass of gas in the volume occupied by the sonic field. This change is equal to zero because the substance could not flow out beyond the limits of the wave front; hence $E_1 = E_2$. If the integral over the time period in equations (1.63) or (1.63') is taken over the entire number of periods of vibration of the source and if the fact is taken into account that in this case $d\sigma \cdot p_0 \cdot \int_0^t (\xi_1 + \xi_2)_\nu dt$ is equal to zero (since this integral is equal to the algebraically assumed path of a surface element $d\delta$ of the source Q in the direction along the normal to δ for a complete number of periods), and if the energy obtained over part of a period is neglected,

$$E_2 = E_1 = \int_0^t dt \int_\sigma d\sigma (\pi_1 \xi_1)_\nu = \overline{(\pi_1 \xi_1)_\nu} \sigma t \,, \qquad (1.65)$$

where $\overline{(\pi_1 \xi_1)_\nu}$ is the mean value of the energy flow vector.

Both forms of the conservation law are identical when expressed in integral form. Despite the complete legitimacy and generality of the expressions for E_2 and $\boldsymbol{N}_2$ containing the elements of nonlinear acoustics, in linear acoustics it is entirely possible and more rational under the conditions of a homogeneous and stationary medium to use equations (1.58) for the energy and its flow.

The equivalence of equations (1.54) and (1.58) no longer holds if the medium is nonhomogeneous and in motion. The equations for E_2 and $\boldsymbol{N}_2$ can easily be generalized to the case of a moving medium. Rather complicated expressions are obtained which will not be considered herein.

As will be shown in section 7, it is essential that the relatively simple expressions are obtained for the energy density of sound E and energy flow $\boldsymbol{N}$ resembling expressions (1.58) and containing magnitudes of only linear acoustics in the approximation of geometrical acoustics in a nonhomogeneous and moving medium.

4. Propagation of Sound in a Nonhomogeneous Moving Medium

In the presence of air motion the acoustical phenomena become more complicated. Generally, separation of the acoustical phenomena, in the narrow sense of the word, from the doubly nonlinear processes taking

place In a moving medium is not possible. Thus, for example, the flow, pulsating in velocity if the frequency of these pulsations is sufficiently large, acts on the microphone or ear located in it (not considering phenomena connected with vortex formation on the microphone body itself, see section 28) as a sound of corresponding frequency although the velocity of propagation of these pulsations has nothing in common with the velocity of sound.

The relation between the pressure of these pulsations and their velocity is nonlinear and also differs fundamentally from the relation between the pressure in a sound wave and the velocity of sound vibrations. Finally, the variable nonstationary flow itself can be a source of sound. Phenomena of this kind will be considered later but this section will be concerned exclusively with the problem of the propagation of sound. In order for it to be possible to separate the sound propagated in the medium from the acoustic phenomena arising in the same medium only as a result of its motion, this motion will be assumed to be “soundless”, that is, that the motions I the flow are sufficiently slow so that

$$\tau >> \frac{1}{f}, \qquad (1.66)$$

where τ is the time during which appreciable changes occur in the state of the flow (for example, the period of pulsations of the flow velocity) and f is the frequency of the sound propagated through the medium. This condition requires additional explanations. It depends on the choice of the system of coordinates to which the motion of the flow is referred.

In fact, a general translational motion of the medium has no significance since it simply leads to a transfer of the sound wave. For this reason, it is sufficient that equation (1.66) be satisfied in some one system of the uniformly moving systems of coordinates.

If, for example, a flow is considered in which the propagation of the velocities is stationary (that is, does not depend on the time, but the velocity of the flow periodically changes in space with the period l), then for this flow $\tau = \infty$. If this flow is considered from the point of view of an observer moving with velocity u, the flow will appear to him nonstationary, the period of the velocity pulsations being equal to $\tau' = l / u$.

The phenomenon of the propagation of sound in the two systems of coordinates will differ only in the transport of the sound wave as a whole with velocity u. Since for the present interest is confined to the

propagation of sound, this difference, which can easily be taken into account, is not essential.

When the statement of the problem is broadened and a sound receiver is considered, entirely different results are obtained in these two reference systems. In the first system, in which the flow is stationary, the sound receiver would assume only one frequency f the frequency of sound propagation. In the second system, in addition to this frequency[3] f the receiver would also receive the frequency of pulsations in the flow, that is, $f' = 1/\tau' = u/l$ and the combined frequencies $f_n = f \pm nf'$, n=1,2,3,….

In the following, condition (1.66) is assumed satisfied in any of the possible reference systems. The effect of the flow on the sound propagation will then express itself in two ways: In the first place, the sound will be "carried away" by the flow and, in the second place, it will be dissipated in the nonhomogeneities of this flow.

In the derivation of the fundamental equations of the acoustics of a moving medium, the effect of the **viscosity** and **heat conductivity** of the medium on the sound propagation is ignored. This effect, which can more conveniently be taken into account as a correction, leads to the previously considered absorption of sound. The part played by these factors, which determine irreversible processes in hydrodynamics, may be very appreciable in the formation of the initial state of the medium in which sound is propagated. No less essential in this connection is the effect of the **force of gravity**. Hence the theory of the propagation of sound in a nonhomogeneous and moving medium must have as its basis the general equations of motion of a compressible fluid.

According to equations (1.12), (1.13), and (1.24), these equations are

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0\,, \tag{1.67}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + [\operatorname{rot} \boldsymbol{v}, \boldsymbol{v}] + \nabla \frac{v^2}{2} = -\frac{1}{\rho}\nabla p + \boldsymbol{g} + \nu \Delta \boldsymbol{v} + \frac{\nu}{3}\nabla \operatorname{div} \boldsymbol{v}\,, \tag{1.68}$$

$$\frac{\partial S}{\partial t} + (\boldsymbol{v}, \nabla S) = \frac{\lambda}{\rho} \cdot \frac{\Delta T}{T} + \frac{Q}{\rho T}\,, \tag{1.69}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the medium. Further, equation (1.13) was supplemented by the term g, which represents the effect of the force of gravity. The vector $\boldsymbol{g}$ is the vector of the acceleration of gravity directed always toward the center of the earth. Thus $\rho \cdot \boldsymbol{g}$ is

[3] Actually it changes somewhat because of the Doppler effect; see section 5.

the force of gravity acting on unit volume of the fluid.

Now let sound be propagated in a medium the state of which is described by the magnitudes $\boldsymbol{v}$, p, ρ and S. The initial state of the medium ($\boldsymbol{v}$, p, ρ and S) is considered stable and the sound is considered as a small vibration. All the previously mentioned magnitudes will then receive small increments: ξ, π, δ and σ, respectively, where ξ will be the velocity of the sound vibrations; π, the pressure of the sound; δ, the change in density of the medium; and σ, its change of entropy occurring on passing through a sound wave.

In order to obtain the equations for the elements of the sound wave in equations (1.67), (1.68) and (1.69), $\boldsymbol{v}$ is replaced by $\boldsymbol{v}+\xi$, p by $p+\pi$, ρ by $\rho+\delta$ and S by $S+\sigma$; by restriction to a linear approximation, terms of higher order relative to the small magnitudes ξ, π, δ and σ are rejected. Moreover, as has just been mentioned, the irreversible processes taking place during the sound propagation are ignored, which means that in the linear equations for ξ, π, δ and σ the terms proportional to the viscosity (μ or ν) and the heat conductivity are rejected. On the basis of equations (1.16) and (1.5), the heat Q dissipated in the fluid likewise belong to the number of magnitudes proportional to μ. By the method indicated,

$$\frac{\partial \xi}{\partial t} + [\mathrm{rot}\,\boldsymbol{v}, \xi] + [\mathrm{rot}\,\xi, \boldsymbol{v}] + \nabla(\boldsymbol{v}, \xi) = -\frac{1}{\rho}\nabla\pi + \frac{\nabla p \cdot \delta}{\rho^2}, \tag{1.70}$$

$$\frac{\partial \delta}{\partial t} + (\boldsymbol{v}, \nabla\delta) + (\xi, \nabla\rho) + \rho \cdot \mathrm{div}\,\xi + \delta \cdot \mathrm{div}\,\boldsymbol{v} = 0\,, \tag{1.71}$$

$$\frac{\partial \sigma}{\partial t} + (\boldsymbol{v}, \nabla\sigma) + (\xi, \nabla S) = 0\,. \tag{1.72}$$

The equation of state, which is given in the variables ρ and S, is still to be added to these equations. For small changes of pressure π, and in exactly the same manner as in the preceding section the following is obtained:

$$p = c^2\delta + h\sigma\,;\ c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S,\ \ h = \left(\frac{\partial p}{\partial S}\right)_\rho. \tag{1.73}$$

Equations (1.70), (1.71), (1.72) and (1.73) are the fundamental equations of acoustics for a homogeneous moving medium (eq. (1.74)). Their differences from those known in the literature lie in the fact that they are true in a medium the entropy of which varies from point to point ($\nabla S \neq 0$) and in a flow in which vortices may exist ($\mathrm{rot}\,\boldsymbol{v} \neq 0$).

The approximations made in these equations, in addition to linearity, consist in the fact that no account is taken of the irreversible processes in the sound wave so that the sound wave is considered an **adiabatic process**. This fact is also expressed by equation (1.72). In fact, it follows from this equation that $\frac{d(S+\sigma)}{dt}=0$, that is, the entropy of a given amount of substance remains unchanged with the passage of a sound wave. The entropy of the substance at a given point of space may vary; $\frac{\partial\sigma}{\partial t}\neq 0$.

In this sense the sound wave is not isentropic. The linear character of the equations requires that a small disturbance remain small in the course of time (stability of the initial state). Hence it is not possible with the aid of these equations to describe, for example, such interesting phenomena as the "**sensitive flame**" of a gas burner, the height of which changes sharply under the action of a sound wave.

In other respects the equations are entirely general and it is quite immaterial in what manner the initial state of the medium was formed. In bringing about this state, the force of gravity, the heat conductivity, and the energy flow from the outside (for example, the sun's heat) may be of considerable significance. The effect of all these factors on the sound propagation is taken into account in equations (1.70), (1.71), (1.72), and (1.73) through the magnitudes $\boldsymbol{v}$, p, ρ, and S characterizing the initial medium.

The equation $p=z(\rho,S)$ and equation (1.73) are valid only for a single-component medium. In general, the pressure may depend not only on ρ and S but also on the concentration of the various components. In a complex medium it is necessary to take into account the diffusion of the various components. The corresponding uncomplicated generalization of equations (1.70) to (1.73) will be made in section 13, where the case of sea salt water is considered.

The choice of the thermodynamic variables ρ and S that has been made herein is very convenient for general theoretical considerations. For final numerical computations, however, the variables p and T are more convenient. For this reason, formulas are given expressing the magnitudes $\left(\frac{\partial p}{\partial S}\right)_\rho$ and ∇S entering the equations through the variables p and T.

$$\nabla S = \left(\frac{\partial S}{\partial T}\right)_p \nabla T + \left(\frac{\partial S}{\partial p}\right)_T \nabla p\,,$$

on the basis of the known thermodynamic relations $\left(\frac{\partial S}{\partial T}\right)_p = \frac{c_p}{T}$ (c_p is specific heat at constant pressure), $\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{dT}\right)_p = -\frac{\beta_p}{\rho}$ (β_p is the coefficient of volume expansion and $\beta_p = -\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$).

Hence

$$\nabla S = \frac{c_p}{T}\overline{\nabla} T - \frac{\beta_p}{\rho}\cdot \nabla p\,. \tag{1.74}$$

Further,

$$\left(\frac{\partial p}{\partial T}\right)_S = \left(\frac{\partial p}{\partial S}\right)_\rho \left(\frac{\partial S}{\partial T}\right)_p \quad \text{and} \quad \left(\frac{\partial p}{\partial T}\right)_\rho = -\left(\frac{\partial p}{\partial \rho}\right)_T \left(\frac{\partial \rho}{\partial T}\right)_p .$$

The magnitude

$$\left(\frac{\partial p}{\partial T}\right)_p = -\frac{1}{V^2}\left(\frac{\partial V}{\partial T}\right)_p = -\rho\beta_p \quad \text{and} \quad \left(\frac{\partial p}{\partial \rho}\right)_T = a^2 = \frac{c_v}{c_p}c^2\,,$$

where c^2 is the square of the adiabatic velocity of sound and $\left(\frac{\partial S}{\partial T}\right)_p = \frac{c_p}{T}$. Thus

$$\left(\frac{\partial p}{\partial S}\right)_\rho = \frac{\rho c^2}{c_p}\cdot \beta_p T\,. \tag{1.75}$$

On the basis of equations (1.74) and (1.75) and the medium (c^2, c_p, β_p) and its state (p and T as functions of the coordinates) ∇S and $\left(\frac{\partial p}{\partial S}\right)_\rho$ can easily be found.

The system of fundamental equations (1.70) to (1.73), even if, with the aid of equation (1.73), one variable is eliminated (e.g., δ), contains five unknowns and is therefore very complicated.

Nevertheless, if a complete wave picture of the propagation of sound is to be obtained, these equations cannot be avoided. The main complication lies in the fact that, because the pressure in the medium is a function of two variables (ρ and *T* or, preferably ρ and *S*), then even in a medium at rest where not only vortices of the flow ate absent but where, in general, there is no flow, the right side of equation (1.70) will not be a complete differential of some function and therefore the sound will be vortical ($\mathrm{rot}\,\boldsymbol{\xi} \neq 0$). Considerable simplifications are obtained

when the change in p, ρ and S are small over the length of the sound wave. Geometrical acoustics are considered in greater detail in the next chapter.

For the present, certain special cases of the general system which are not reduced to the approximations of geometrical acoustics are considered.

The most important special case will be the one for which the initial flow is **not vortical** ($\mathrm{rot}\,\boldsymbol{v} = 0$) and the entropy of the medium is **constant** ($\nabla S = 0$).

Under these conditions the pressure in the medium is a function only of the density of the medium so that $\nabla p = c^2 \nabla \rho$. From equation (1.72) it follows that for $\nabla S = 0$, $\sigma = 0$ so that the sound will be propagated isentropically. Then

$$\pi = c^2 \delta .$$

If the potential of the sound pressure is introduced

$$\Pi = \frac{\pi}{\rho}, \tag{1.76}$$

the right side of equation (1.70) will be equal to $-\nabla \Pi$. Therefore the velocity potential of the sound vibrations ϕ can also be introduced

$$\boldsymbol{\xi} = -\nabla \phi . \tag{1.77}$$

The sound will be nonvortical in this case. From equation (1.70)

$$\frac{\pi}{\rho} = \Pi = \frac{\partial \phi}{\partial t} + (\boldsymbol{v}, \nabla) = \frac{d\phi}{dt} . \tag{1.78}$$

Substitution in equation (1.71) of the magnitude Π (for which $\frac{\partial \Pi}{\partial t} = \frac{c^2}{\rho}\frac{\partial \delta}{\partial t}$, $\nabla \Pi = \delta \cdot \overline{\nabla}\left(\frac{c^2}{\rho}\right) + \frac{c^2}{\rho} \cdot \overline{\nabla} \delta$) in place of δ yields the following equation for ϕ:

$$\frac{d^2\phi}{dt^2} = c^2 \cdot \Delta\phi + (\nabla \Pi_0, \nabla \phi) + \frac{d\phi}{dt}(\boldsymbol{v}, \nabla \log c^2), \tag{1.79}$$

where Π_0 is the potential of pressure (heat function) of the initial flow:

$$\Pi_0 = \int \frac{dp}{\rho} . \tag{1.80}$$

Equation (1.79) was derived by N. N. Andreev and I. G. Rusakov (ref. 10) without the last term, which was erroneously omitted. This equation exhaustively describes the propagation of sound in a medium in which the entropy is constant.

A. M. Obukhov (ref. 11) gives an equation which permits an

approximate consideration of the presence of vorticity of the flow but nevertheless makes use of one function, the "quasipotential" ψ. This quasipotential is introduced by the equation

$$\xi = -\nabla\psi + \int^{t}[\text{rot}\,\boldsymbol{v}\times\nabla\psi]dt\,. \tag{1.81}$$

The quasipotential may be introduced only for sufficiently small vorticity of the initial flow, that is, the assumption must be made that

$$\Omega = |\text{rot}\,\boldsymbol{v}| << \omega\,, \tag{1.82}$$

where ω is the cyclical frequency of the sound.

Moreover the assumption is made that $v/c << 1$, so that the initial flow may be taken as incompressible ($\text{div}\,\boldsymbol{v} = 0$). Finally the pressure of the medium is assumed as a function of the density of the medium only. Since $\frac{\partial p}{\partial \rho}$ is considered by A. M. Obukhov as the adiabatic velocity of sound, this implies the assumption that the entropy of the medium is constant. In connection with this assumption, the question arises as to what extent the assumptions of the presence of vorticity ($\text{rot}\,\boldsymbol{v} \neq 0$) and the constancy of the entropy ($\overline{\nabla}S = 0$) generally apply together. The possibility is not excluded, however, that the influence of the vortices on the sound propagation is more effective than the influence of an entropy gradient. These hypotheses are assumed satisfied and ξ is substituted from equation (1.81) into equation (1.70) and, since $\nabla S = 0$, the right side of equation (1.70) will again be $-\overline{\nabla}\Pi$. After simple reductions, the equation, which was found previously, is obtained:

$$\Pi = \frac{\pi}{\rho} = \frac{d\psi}{dt}\,. \tag{1.83}$$

In this case, however, it is true only approximately with an accuracy to $\frac{\Omega^2}{\omega^2}$, $\frac{\Omega}{\omega}\cdot\frac{\boldsymbol{v}}{c}$.

Expressing δ in equation (1.71) in terms of Π and ψ gives the equation of A. M. Obukhov:

$$\frac{d^2\psi}{dt^2} = c^2\Delta\psi + (\nabla\Pi_0, \nabla\psi) + \frac{d\psi}{dt}(\boldsymbol{v}, \nabla\log c^2) + c^2\int^{t}(\nabla\psi, \Delta\boldsymbol{v})dt - \left(\nabla\Pi_0, \int^{t}(\text{rot}\,\boldsymbol{v}, \nabla\psi)dt\right) \tag{1.84}$$

This equation holds with an accuracy up to $\frac{\Omega}{\omega}$, $\frac{\Omega}{\omega}\cdot\frac{\nabla|\Omega|}{k\Omega}$ $\left(k = \frac{\omega}{c}\right)$.

The magnitude $\Delta\boldsymbol{v} = -\,\text{rot}\,\text{rot}\,\boldsymbol{v}$. In this equation, the terms of order $\frac{v^2}{c^2}$

can not be taken into account because in the approximations the assumption was made that $\frac{v}{c} << 1$.[4]

5. Equation for Propagation of Sound in Constant Flow

In many cases the velocity of the flow $\boldsymbol{v}$ may be suitably separated into the **mean velocity** $\boldsymbol{V}$ and the **fluctuating velocity** $\boldsymbol{u}$. The effect of these two components of velocity on the sound propagation may be different. The mean velocity of flow produces the "**drift**" of the sound wave while the second variable part of the flow velocity leads to the **dissipation** of the sound wave. This phenomenon will be considered in more detail later. For the present, attention is concentrated on the effect of the mean flow velocity and the equations are considered for the sound propagation, with the variable part of the flow velocity $\boldsymbol{u}$ ignored. The solution obtained under these conditions is of interest not only as a first step toward the approximate solution of the complete problem with the velocity fluctuations being considered but is of value in itself, especially for the theory of a moving sound source.

In order to obtain an equation for the propagation of sound in a homogeneous forward moving medium, it is sufficient to put $\overline{\nabla}\Pi_0 = 0$ and $\nabla \log c^2 = 0$ in equation (1.79). Expansion of the total derivative with respect to time [$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + (\boldsymbol{v}, \nabla)\left(\frac{\partial\phi}{\partial t} + (\boldsymbol{v}, \nabla\phi)\right)$] yields

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - 2\left(\frac{\boldsymbol{v}}{c^2}, \nabla\frac{\partial\phi}{\partial t}\right) - \frac{(\boldsymbol{v}, \overline{\nabla})(\boldsymbol{v}\nabla\phi)}{c^2} = 0\,. \qquad (1.85)$$

If the X-axis is taken in the direction of the mean velocity and β is set equal to V/c,

$$(1-\beta^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2\beta}{c}\frac{\partial^2\phi}{\partial t\partial x} = 0 \qquad (1.85')$$

For the system of coordinates ξ, η and ζ moving together with the stream $\xi = x - Vt$, $\eta = y$ and $\zeta = z$, equation (1.85') is transformed into the usual **wave equation**

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\zeta^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0\,, \qquad (1.86)$$

as expected, since in this system of coordinates the medium is at rest.

[4] The result of A. M. Obukhov is probably more rigorous and could have successively been obtained as the second approximation of geometrical acoustics (see section 7).

Certain important solutions of equation (1.85’) are now available.

A **plane** sound wave is first considered. In the system of coordinates ξ, η and ζ at rest relative to the air (hence for an observer moving with the stream), this wave has the potential

$$\phi(\xi,\eta,\zeta) = Ae^{i\omega\left(t-(\alpha_1\xi+\alpha_2\eta+\alpha_3\zeta)/c\right)};\ \ \alpha_1^2+\alpha_2^2+\alpha_3^2=1, \tag{1.87}$$

where α_1, α_2 and α_3 are the direction cosines of the normal to the surface of the wave; ω the frequency of the oscillation; and c the velocity of sound. Equation (1.87) is a solution of equation (1.86). According to the previously mentioned transformation, the solution of equation (1.85’) is immediately obtained if ξ is replaced in equation (1.87) by $x - vt$, η by *y*, and ζ by *z*:

$$\phi(x,y,z) = Ae^{i[\omega' t-\omega(\alpha_1 x+\alpha_2 y+\alpha_3 z)/c]}, \tag{1.88}$$

where

$$\omega' = \omega\left(1+\frac{V}{c}\alpha_1\right). \tag{1.89}$$

Thus the sound frequency in a stationary system of coordinates will not be ω but ω'.

This change of the frequency of the sound is the acoustical **Doppler effect**. The effect has an exclusively **kinematic** origin; it depends only on the choice of the system of coordinates. The entire difference in the propagation of a plane wave in a moving medium as compared with a stationary one reduces to this kinematic effect.

Later the Doppler effect will be considered more fully; not only the motion of the observer of the sound will be taken into account but also the motion of the sound source itself, which at present does not enter explicitly in the computation.

A second important form of the solutions of equation (1.85) is presented by sound waves diverging from a certain small **point source** of sound (or, on the contrary, converging to it; In the latter case a sound "sink" is being dealt with, which is a very artificial but mathematically useful concept).

The mathematical expression for the potential of such waves is a generalization of the potential of **spherical waves** for a medium at rest. This potential of spherical waves is a solution of equation (1.86), having the form

$$\chi_0 = \frac{F(t\pm r/c)}{r};\ \ r=\sqrt{\xi^2+\eta^2+\zeta^2}, \tag{1.90}$$

where F is an arbitrary function. The solution with the minus sign is given by waves diverging from a sound source located at the origin of coordinates ($\xi = \eta = \zeta = 0$) and the solution with the plus sign represents the same waves converging to a sound sink at the origin of coordinates. If F is a harmonic function, the following is obtained from equation (1.90)

$$\chi_0 = \frac{e^{i\omega(t \pm r/c)}}{r}, \qquad (1.90')$$

that is, a spherical harmonic wave with frequency ω . In a moving medium in which the propagation of sound is described by equation (1.85') instead of solutions of the form of equation (1.90), the more general expression is obtained.[5]

$$\chi = \frac{F(t + R/c)}{R^*}, \qquad (1.91)$$

where

$$R = \frac{\beta x^* \pm R^*}{\sqrt{1-\beta^2}}, \quad R^* = \sqrt{x^{*2} + y^2 + z^2}, \quad x^* = \frac{x}{\sqrt{1-\beta^2}}. \qquad (1.92)$$

With the substitution of χ from equation (1.91) into equation (1.85), it is not difficult to show that equation (1.91) is in fact the solution of equation (1.85), which moreover transforms into a solution of the form of equation (1.90) for $V = 0$ ($\beta = 0$).

The solution (eq. (1.91)) for a moving medium thus has the same value which equation (1.90) has for a stationary medium; it represents waves diverging from a point source or waves converging to a sink.

6. Generalized Theorem of Kirchhoff

In the theory of the propagation of waves, an important part is played by the theorem of Kirchhoff, which permits expression of the oscillations at any point of space in terms of the oscillations at the surface bounding the space considered (including also the surface at infinity). This theorem is derived for a moving medium, starting from equation (1.85') (ref. 12). This equation, if the coordinate system x^*, y, z contracted in the x-direction is introduced

$$x^* = \frac{x}{\sqrt{1-\beta^2}}; \; y = y; \; z = z \qquad (1.93)$$

assumes the form

[5] The origin of this solution is clarified in detail in section 15.

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\frac{\partial^2\phi}{\partial t\partial x^*} = 0, \tag{1.94}$$

where

$$\Delta = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The singular solution χ (eq. (1.91)) likewise satisfies equation (1.94)

$$\Delta\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} - \frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\frac{\partial^2\chi}{\partial t\partial x^*} = 0. \tag{1.95}$$

The solution χ contains the arbitrary function *F* which, because of later utilization of the solution for the proof of the theorem of interest, is specialized.

$$\chi = \frac{\delta(t+R/c)}{R^*}; \quad R = \frac{\beta x^* + R^*}{\sqrt{1-\beta^2}}, \tag{1.96}$$

where *R* is the distance $\sqrt{x^{*2} + y^2 + z^2}$ from the point *P*, with the coordinates $x^*{}_P, y_P, z_P$ at which the potential ϕ is to be determined to an arbitrary point of the space *Q*, with the coordinates $x^*{}_Q, y_Q, z_Q$, so that $x^* = x_Q - x^*{}_P$, $y = y_Q - y_P$ and $z = z_Q - z_P$.

The function $\delta(\xi)$ is determined such that

$$\begin{cases} \int_a^b f(\xi)\delta(\xi)d\xi = f(0)\_\text{if}\_b > 0, a < 0, \\ \int_a^b f(\xi)\delta(\xi)d\xi = 0\_\text{if}\_b/a > 0. \end{cases} \tag{1.97}$$

Equation (1.97) is assumed valid for any function $f(\xi)$ so that $\delta(\xi)$ is everywhere equal to zero except at the point $\xi = 0$, where $\delta(\xi) = \infty$. Hence $\delta(t+R/c)/R^*$ represents a converging spherical impulse (shock) concentrated about *R* = - *ct*.

A certain surface *S* enclosing the volume Ω in the space *x**, *y*, *z* is considered (see fig. 2 where the surface *S* is formed by two surfaces S_1 and S_2; the volume Ω is crosshatched).

After equation (1.95) is multiplied by ϕ and equation (1.94) by χ, one equation is subtracted from the other and the result is integrated over the volume Ω and over the time t_1 to t_2. Integration over the four-dimensional volume $\Omega(t_2 - t_1)$ yields

$$\int_{t_1}^{t_2} dt \int d\Omega(\phi\Delta\chi - \chi\Delta\phi) + \frac{1}{c^2}\int_{t_1}^{t_2} dt \int d\Omega\left(\chi\frac{\partial^2\phi}{\partial t^2} - \phi\frac{\partial^2\chi}{\partial t^2}\right)$$
$$-\frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c^2}\int_{t_1}^{t_2} dt \int d\Omega\left(\phi\frac{\partial^2\chi}{\partial t\partial x^*} - \chi\frac{\partial^2\phi}{\partial t\partial x^*}\right) = 0 \qquad (1.98)$$

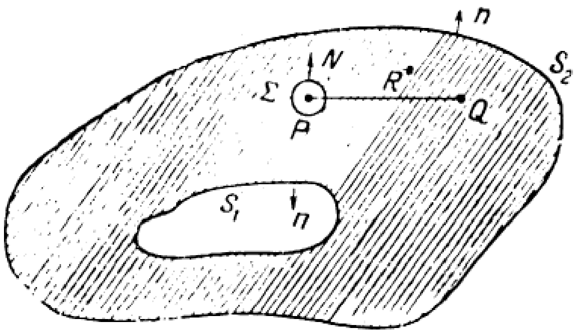

Figure 2.

Application of Green's transformation results in

$$\int_\Omega d\Omega(\phi\Delta\chi - \chi\Delta\phi) = \int_S dS(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}), \qquad (1.99)$$

where $\frac{\partial}{\partial n}$ denotes the derivative along the external normal to the surface S enclosing the volume Ω. At the point P the transformation (eq. (1.99)) will fail because at this point χ becomes infinite. The point P is surrounded by a small surface Σ and the volume $\Delta\Omega$ enclosed by it is excluded from the volume of integration Ω in equation (1.98). The surface Σ (see fig. 2) is considered as part of the surface S. The normal to the small sphere Σ is denoted by N and directed toward the interior of the volume. If Green's transformation, equation (1.99) to equation (1.98), is applied, the following results:

$$\int_{t_1}^{t_2} dt \int_\Sigma dS\left(\phi\frac{\partial\chi}{\partial N} - \chi\frac{\partial\phi}{\partial N}\right) = \int_{t_1}^{t_2} dt \int_\Sigma dS\left(\phi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\phi}{\partial n}\right)$$
$$+\frac{1}{c^2}\int_{t_1}^{t_2} dt \int_{\Omega'} d\Omega\left(\chi\frac{\partial\phi}{\partial t} - \phi\frac{\partial\chi}{\partial t}\right)$$
$$-\frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\int_{t_1}^{t_2} dt \int_{\Omega'} d\Omega\left(\phi\frac{\partial^2\chi}{\partial t\partial x^*} - \chi\frac{\partial^2\phi}{\partial t\partial x^*}\right) \qquad (1.100)$$

The second integral on the right permits carrying out the integration with respect to time

$$
\begin{aligned}
I_2 &= \frac{1}{c^2}\int_{t_1}^{t_2} dt \int_{\Omega} d\Omega \frac{d}{dt}\left(\chi \frac{\partial \phi}{\partial t} - \phi \frac{\partial \chi}{\partial t}\right) \\
&= \int_{\Omega'} d\Omega \left(\chi \frac{\partial \phi}{\partial t} - \phi \frac{\partial \chi}{\partial t}\right)\Bigg|_{t_1}^{t_2}
\end{aligned}
\qquad . \qquad (1.101)
$$

But if t_1 tends to $-\infty$ and t_2 to $+\infty$ so that $t_1 + R/c < 0$ and $t_2 + R/c > 0$, then both χ and $\frac{\partial \chi}{\partial t}$ at t_1 and t_2 are equal to zero on account of the form chosen for χ; hence $I_2 = 0$. The first integral on the right is considered

$$
I_1 = \frac{1}{c^2}\int_{t_1}^{t_2} dt \int_{\Sigma} d\Omega \left(\phi \frac{\partial}{\partial n}\left(\frac{1}{R^*}\right)\delta + \phi \frac{1}{R^*}\frac{\partial R}{\partial n}\cdot\frac{1}{c}\frac{\partial \delta}{\partial t} - \frac{\delta}{R^*}\frac{\partial \phi}{\partial n}\right).
\qquad (1.102)
$$

Integration by parts of the second term with respect to time and use of the property of δ (eq. (1.97)) yield

$$
I_1 = \int_S dS \left(\frac{\partial}{\partial n}\left(\frac{1}{R^*}\right)\cdot \phi_{t=-R/c} - \frac{1}{R^*}\left(\frac{\partial \phi}{\partial n}\right)_{t=-R/c} - \frac{1}{c}\frac{1}{R^*}\frac{\partial R}{\partial n}\left(\frac{\partial \phi}{\partial t}\right)_{t=-R/c}\right),
\qquad (1.103)
$$

where ϕ, $\frac{\partial \phi}{\partial n}$ and $\frac{\partial \phi}{\partial t}$ are taken at the instant $t = -\frac{R}{c}$.

In a similar manner the third integral on the right in equation (1.100) gives

$$
\begin{aligned}
I_3 &= \frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\int_{t_1}^{t_2} dt \int_{\Omega} d\Omega \left(\phi \frac{\partial^2 \chi}{\partial t \partial x^*} - \chi \frac{\partial^2 \phi}{\partial t \partial x^*}\right) \\
&= \frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\left\{\int_{t_1}^{t_2} dt \int_{\Omega} d\Omega \frac{\partial}{\partial x^*}\left(\phi \frac{\partial \chi}{\partial t}\right) - \int_{t_1}^{t_2} dt \int_{\Omega} d\Omega \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x^*}\chi\right)\right\} \\
&= \frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\left\{\int_S \phi \frac{\partial \chi}{\partial t} dS_X - \int_{\Omega} d\Omega \frac{\partial \phi}{\partial x^*}\cdot\frac{\delta(t+R/c)}{R^*}\Bigg|_{t_1}^{t_2}\right\} \\
&= \frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\int_S \left(\frac{\partial \phi}{\partial t}\right)_{t=-R/c}\cdot\frac{1}{R^*} dS
\end{aligned}
\qquad (1.104)
$$

where dS_X is the projection of the area ***n****dS* on the flow velocity (on the *x*-axis). The integral in equations (1.100) on the left is transformed exactly as the first and, since in this case $\frac{\partial}{\partial N}$ is identical with $\frac{\partial}{\partial R^*}$,

$$I_0 = \int_{t_1}^{t_2} dt \int_{\Sigma} d\Sigma \left(\phi \frac{\partial \chi}{\partial N} - \chi \frac{\partial \phi}{\partial N} \right)$$

$$= \int_{\Sigma} d\Sigma \left(\frac{\partial}{\partial R^*} \left(\frac{1}{R^*} \right) \phi_{t=-R/c} - \frac{1}{R^*} \left(\frac{\partial \phi}{\partial N} \right)_{t=-R/c} - \frac{1}{c} \frac{1}{R^*} \frac{\partial R}{\partial R^*} \left(\frac{\partial \phi}{\partial N} \right)_{t=-R/c} \right)$$

(1.105)

and, since $d\Sigma = 4\pi R^{*^2} \cdot dR^*$, as the radius of the sphere R^* approaches zero, the following is obtained:

$$I_0 = -4\pi\phi_{t=0} . \qquad (1.105')$$

Thus on the left the value of the potential at the point *P* at the instant of time *t* = 0 is obtained. Since this instant is arbitrary, if the time origin is everywhere shifted forward by *t* and all the integrals I_1, I_2 and I_3 are collected, the potential at the point *P* at the instant of time *t* will be

$$\phi_P(t) = \frac{1}{4\pi} \int \left\{ \frac{1}{R^*} \left[\frac{\partial \phi}{\partial n} \right] - \frac{\partial}{\partial n} \left(\frac{1}{R^*} \right) [\phi] + \frac{1}{c} \frac{1}{R^*} \left[\frac{\partial \phi}{\partial t} \right] \right\} dS$$
$$- \frac{1}{4\pi} \frac{2\beta}{\sqrt{1-\beta^2}} \frac{1}{c} \int \left\{ \frac{1}{R^*} \left[\frac{\partial \phi}{\partial n} \right] \right\} dS_X ,$$

(1.106)

where the brackets Indicate that the magnitude enclosed by them is taken at the instant of time *t* - *R*/*c*.

For $V_0 = 0$ ($\beta = 0$), *R** = *r* and *R* = *r* and this equation transforms into the usual **equation of Kirchhoff** for a medium at rest.

If the potential depends harmonically on the time so that

$$\phi = \psi e^{i\omega t} \qquad (1.107)$$

then substitution of equation (1.105) in equation (1.104) yields for the amplitude

$$\psi_P = \frac{1}{4\pi} \int \left\{ \frac{\partial \psi}{\partial n} \frac{e^{-ikR}}{R^*} - \psi \frac{\partial}{\partial n} \left(\frac{e^{-ikR}}{R^*} \right) \right\} dS - \frac{2i\beta k}{4\pi R 1 - \beta^2} \int \psi \frac{e^{-ikR}}{R^*} dS_X ,$$

(1.108)

where $k = \omega / c$ is the wave-number vector. If, from the nature of the physical problem, it may be assumed that the disturbances giving rise to the vibrations start within the surface S_1 and not at an infinite time back, they do not have time to be propagated to the surface S_2 at a great distance from S_1. For this reason, if S_2 is shifted to infinity, the values ϕ, $\frac{\partial \phi}{\partial n}$, $\frac{\partial \phi}{\partial t}$ can be assumed equal to zero in it. The volume Ω then takes up the entire space with the exception of S_1 in the interior. If the presence of an infinitely removed surface is "forgotten," it is natural to

call the normal n the interior normal since it is directed inwards from the surface S_1 within which the sources of vibration are concentrated according to the present assumption. Under this condition equations (1.104) and (1.106) may be assumed to give the expression of the potential at any point of space in terms of the values ϕ, $\frac{\partial \phi}{\partial n}$, and $\frac{\partial \phi}{\partial t}$ on the surface S_1 within which (or on it) the sound sources are concentrated.

In conclusion, a certain generalization of this theorem is considered for "volume" sources of sound. It is assumed that equation (1.94) has a right side which is considered as a "volume sound source." The strength of this source is denoted by Q. Equation (1.94) can then be written in the form

$$\varDelta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2\beta}{\sqrt{1-\beta^2}}\cdot\frac{1}{c}\frac{\partial^2\phi}{\partial t\partial x^*} = -4\pi Q\,. \qquad (1.94)$$

Such equations are encountered, for example, in the problem of the dissipation of sound by a turbulent flow (see section 12). If the same operations which were applied to equation (1.94) are applied to this equation, an expression is obtained for ϕ differing from equations (1.i06) and (1.108) by a volume integral. The additional term, on multiplication of equation (1.94) by χ, will be

$$I_4 = -4\pi\int_{t_1}^{t_2} dt \int d\varOmega Q\cdot\chi\,. \qquad (1.109)$$

Integration over t yields (on account of the δ function form of χ)

$$I_4 = -4\pi\int d\varOmega Q_{t=-R/c}\cdot\frac{1}{R^*}\,. \qquad (1.109')$$

Hence, in place of equations (1.106) and (1.108), there are obtained

$$\phi_P(t) = \int \frac{[Q]}{R^*}d\varOmega + \frac{1}{4\pi}\int\left\{\frac{1}{R^*}\left[\frac{\partial\phi}{\partial n}\right] - \frac{\partial}{\partial n}\left(\frac{1}{R^*}\right)[\phi] + \frac{1}{cR^*}\frac{\partial R}{\partial n}\left[\frac{\partial\phi}{\partial t}\right]\right\}dS$$
$$-\frac{1}{4\pi}\frac{2\beta}{\sqrt{1-\beta^2}}\cdot\frac{1}{c}\int\frac{1}{R^*}\left[\frac{\partial\phi}{\partial t}\right]dS_X$$

(1.106')

and

$$\psi_P = \int\frac{Q_0 e^{-ikR}}{R^*}d\varOmega + \frac{1}{4\pi}\int\left\{\frac{\partial\phi}{\partial n}\cdot\frac{e^{-ikR}}{R^*} - \phi\frac{\partial}{\partial n}\left(\frac{e^{-ikR}}{R^*}\right)\right\}dS$$
$$-\frac{1}{4\pi}\frac{2i\beta k}{\sqrt{1-\beta^2}}\int\phi\frac{e^{-ikR}}{R^*}dS_X\,.$$

(1.108')

If the strength of the source depends harmonically on the time

$$Q = Q_0 e^{i\omega t} .$$

The theorems derived herein are used in the theory of wave propagation from a moving source, in particular from an airplane propeller, and in the problem of the occurrence of vortical sound in the motion of bodies in the air.

(End of Chapter 1)