III. MOTION IN ONE DIMENSION

A. Integrations of the Differential Equations

17. Introductory remarks

18. Solution of the differential equations.

The differential equations of one-dimensional motion (see Chapter II, Art. 8) can be integrated almost immediately when written in their ***characteristic form***. Replacing the quantity y by t and the quantity v by ρ in the theory of Chapter II, Art. 9, we find, with $c^2 = p'(\rho)$, the expressions $\xi_+ = 1/(u+c)$ and $\xi_- = 1/(u-c)$ (by a straight-forward application of the procedure described there); accordingly, the characteristic form of the differential equations with the ***characteristic parameters*** α, β as independent variables is

(1) $$\begin{cases} I_+ : x_\alpha = (u+c)t_\alpha \\ I_- : x_\beta = (u-c)t_\beta \end{cases}, \quad \begin{cases} II_+ : u_\alpha = -(c/\rho)\rho_\alpha \\ II_- : u_\beta = (c/\rho)\rho_\beta \end{cases}.$$

The system II can be completely solved for u, ρ by the relations

(2) $$\begin{cases} u + G(\rho) = V(\beta) = 2r \\ u - G(\rho) = W(\alpha) = 2s \end{cases},$$

where V and W (and r and s) are arbitrary functions and where $G(\rho)$ is defined by

(3) $$G(\rho) = \int_{\rho'}^{\rho} \frac{c}{\rho} d\rho$$

ρ' being an arbitrary constant. Instead of α and β it is often convenient to introduce the special parameters r and s so that

(4) $$u = r + s, \quad G(\rho) = r - s;$$

then u, ρ and c are known functions of r and s.

Substituting these solutions of II in I, we obtain for x and t as functions of α and β, or of r and s, two linear differential equations of first order.

The structure of the solutions is best described by reference to the characteristics $\beta = \text{constant}$, $\alpha = \text{constant}$ (or r = constant, s = constant), which in the x,t-plane are characteristics C_+ and C_- respectively, and in the u, ρ-plane characteristics Γ_+ and Γ_- respectively. Then we have

(5) $$\begin{cases} \beta = \text{constant\_}on\text{\_}\Gamma_+ : u + G(\rho) = 2r \\ \text{r = constant\_on\_C}_+ : \dfrac{dx}{dt} = u + c \\ \alpha = \text{constant\_}on\text{\_}\Gamma_- : u - G(\rho) = 2s \\ \text{s = constant\_on\_C}_- : \dfrac{dx}{dt} = u - c \end{cases}.$$

Thus for polytropic gases with $\rho' = 0$ we have

$$r = \frac{u}{2} + \frac{c}{\gamma - 1}, \quad s = \frac{u}{2} - \frac{c}{\gamma - 1}$$

and we infer the following basic statement:

$$(6) \qquad \begin{cases} On\_C_+ : \dfrac{dx}{dt} = u + c,\_and\_\dfrac{u}{2} + \dfrac{c}{\gamma - 1} = \text{constant,} \\ On\_C_- : \dfrac{dx}{dt} = u - c,\_and\_\dfrac{u}{2} - \dfrac{c}{\gamma - 1} = \text{constant} \end{cases}.$$

If we consider not u and ρ, but u and the sound speed c as dependent variables, the characteristics Γ become straight lines:

$$(7) \qquad \begin{cases} \Gamma_+ : \dfrac{u}{2} + \dfrac{c}{\gamma - 1} = \text{constant,} \\ \Gamma_- : \dfrac{u}{2} - \dfrac{c}{\gamma - 1} = \text{constant} \end{cases}$$

with $c \geq 0$.

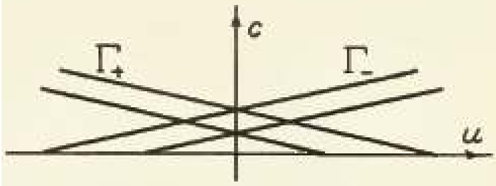


Figure 2
Characteristics
in u,c-plane.

The characteristics Γ_+ and Γ_- are fixed curves in the u, ρ-plane, namely,

$$(8) \qquad \begin{cases} \Gamma_+ : \dfrac{u}{2} + \dfrac{2\sqrt{A\gamma}}{\gamma - 1} \rho^{(\gamma-1)/2} = \text{constant,} \\ \Gamma_- : \dfrac{u}{2} - \dfrac{2\sqrt{A\rho}}{\gamma - 1} \rho^{(\gamma-1)/2} = \text{constant} \end{cases}$$

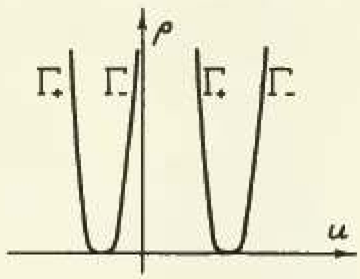


Figure 3
Characteristics
in u,ρ-plane.

as shown in Fig. 3, where the left-hand branches represent the curves Γ_+ and the right-hand branches the curves Γ_-. With u and the enthalpy

$$i = \int_0^\rho \frac{c^2}{\rho} d\rho = \frac{c^2}{\gamma - 1}$$

as dependent variables, the characteristics Γ become ordinary parabolas in the *u,i*-plane:

$$(9) \qquad \begin{cases} \Gamma_+ : u = -2\sqrt{\dfrac{i}{\gamma - 1}} + \text{constant,} \\ \Gamma_- : u = 2\sqrt{\dfrac{i}{\gamma - 1}} = \text{constant.} \end{cases}$$

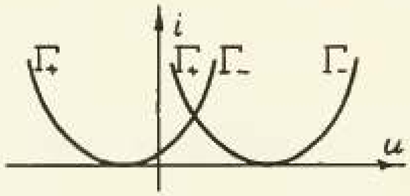


Figure 4
Characteristics
in u,i-plane.

where the left-hand branches of the parabolas represent Γ_+, the right-hand branches Γ_-. For $i \leq 0$, naturally, as for $\rho \leq 0$ or $c \leq 0$, the differential equations lose their physical meaning. No essentially new elements occur in the Lagrangean representation (see Chapter I, Art. 7). With the independent variables

$h = \int_{x_0}^{x} \rho(\xi) d\xi$ (instead of x) and t (instead of y), with the dependent variables u and τ (instead of v), and with $k(\tau) = \rho c = c(\tau)/\tau$, the equations are

$$(10) \qquad \mathrm{I}\begin{cases} C_+ : h_\alpha = k(\tau) t_\alpha \\ C_- : h_\beta = -k(\tau) t_\beta \end{cases}, \qquad \mathrm{II}\begin{cases} \Gamma_+ : u_\alpha = k(\tau) \tau_\alpha \\ \Gamma_- : u_\beta = -k(\tau) \tau_\beta \end{cases}.$$

The characteristics Γ_+ and Γ_- again can be explicitly described:

$$(11) \qquad u \mp \int_0^\tau k(\tau) d\tau = \text{constant}.$$

19. Simple waves.

Our main subject in the present and next few articles will be the motion caused by a **piston** moving in a gas which is initially at rest.

No matter whether the piston recedes from or moves into the gas, not all parts of the gas will be affected instantaneously. There will be a "***wave***" proceeding from the piston into the gas and only the particles which have been reached by the *wave front* will be disturbed from their initial state of rest. (a) If this wave represents a continuous motion, as is always the case if the piston **recedes** from the gas, the wave front progresses with the ***sound speed*** c_0 of the quiet gas. (b) If the piston **moves into** the gas the situation may become more complicated through the emergence of a ***supersonic*** discontinuous ***shock wave*** as we shall see in Section C. At any rate, in Sections A and B we confine ourselves to a consideration of continuous motions satisfying the differential equations (at least near the piston). Such a continuous motion of the gas can be completely determined by the simple wave theory of Chapter II, Art. 16.

In the x,t-representation the undisturbed gas corresponds to a zone of rest (I) adjacent to the x-axis and (as long as the disturbance proceeds at sound speed) bounded by a characteristic $C_+ : x = c_0 t$, which terminates the range of influence of the piston curve L. According to the general theory of Art. 16, the zone (I) of rest is followed beyond the line $x = c_0 t$ by a simple wave (II) generated by a family of straight characteristics C_+.

We shall here give a brief description of the simple waves and then in Sections B and C supply the details of the two cases of receding and impinging piston motions separately.

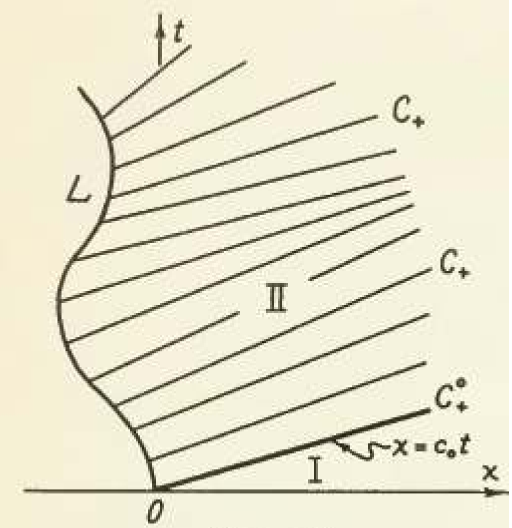


Figure 5
Simple wave (II) adjacent to zone of constant state (I).

Along each conjugate characteristic C_-, which cuts across the

lines C_+, we have, according to equation (5), Art. 18, the relation

$u - G(\rho) = 2s = \text{constant}$,

where the constant is determined by the consideration that all the characteristics intersect an "*initial characteristic*" C_+^0 with values u_0, ρ_0 (e.g., the characteristic terminating the zone of constant state of rest (I)). Thus, throughout the simple wave we have

(12) $$u - G(\rho) = u_0 - G(\rho_0)\,,$$

and, in particular, if the initial characteristic terminates a state of rest,

(13) $$u - G(\rho) = -G(\rho_0)\,.$$

For polytropic gases we have $G(\rho) = 2c(\rho)/(\gamma - 1)$, therefore the basic relation in a forward-facing simple wave (see (7), Art. 18) is

(14) $$u - \frac{2}{\gamma - 1}c = u_0 - \frac{2}{\gamma - 1}c_0\,,$$

or, in particular, if the initial state (0) is a state of rest,

(15) $$u - \frac{2}{\gamma - 1}c = -\frac{2}{\gamma - 1}c_0\,,$$

where c_0 is the sound speed in the quiet gas. With the abbreviation

(16) $$\mu^2 = \frac{\gamma - 1}{\gamma + 1}\,, \quad 1 - \mu^2 = \frac{2}{\gamma + 1}\,,$$

our relation can be written in the form

(17) $$\mu^2(u - u_0) = (1 - \mu^2)(c - c_0)\,.$$

Incidentally, these last equations are nothing but the equation of the single characteristic Γ_- in the u,c-plane which belongs to the simple wave in accordance with the general theory of Chapter II, Art. 16, and which happens to be a straight line also.

B. Continuous Motion. Rarefaction Waves.

20. Rarefaction waves.

We now distinguish between the cases of **expansive** and **compressive** motion and consider first the case of expansive motion caused by a receding piston, assuming from the outset that the medium is a polytropic gas originally at rest with constant density ρ_0 and sound speed c_0. Furthermore, it is assumed that the piston, originally at rest, is withdrawn with increasing speed until ultimately the constant velocity -w is attained. Then the piston curve L will bend backward from 0 to a point B where the slope -w with respect to the t-axis is reached and then continue as a straight line in the same direction, as shown in Fig. 6. We then have a zone (I) of rest:

$0 \le c_0 t \le x$, and an adjacent zone (II) of a simple wave, which we shall call an ***expansion wave*** or ***rarefaction wave*** because, as we shall see, the gas flowing through the zone of this wave steadily decreases in density, and even at a fixed point of the tube the density decreases as long as the point remains in the zone of the simple wave. The particles flow toward the receding piston, starting with zero velocity at the head or front of the rarefaction wave, which is represented by the characteristic $C_+ : x = c_0 t$, and proceed into the zone of constant state with the speed of sound. In this wave the straight lines of the generating family of characteristics C_+ start at the piston curve $L : x = f(t)$.

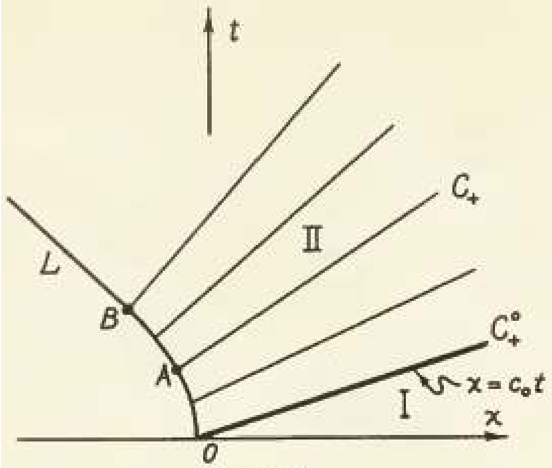


Figure 6
Rarefaction wave (II) resulting from expansive action of piston.

To construct the wave and to prove and amplify the preceding statement we consider a point A on the piston curve L from which a characteristic C_+ is assumed to start into zone (II). At A the velocity u_A of the gas is known, since it is equal to the velocity $dx/dt = f'(t)$ of the piston. We also obtain the value of the sound speed c_A at A by formula (14) of Art. 19:

(18) $$u_A = \frac{2}{\gamma - 1} c_A = u_0 - \frac{2}{\gamma - 1} c_0 .$$

Now the straight line C_+ through A is determined by its slope

$$\frac{dx}{dt} = u_A + c_A ;$$

and on C_+ the values of u, c, ρ, p are now fixed as u_A, c_A, ρ_A, p_A where ρ_A, p_A are given by

(19) $$\frac{p_A}{p_0} = \left(\frac{\rho_A}{\rho_0} \right)^{\gamma}, \quad c_A{}^2 = \gamma \frac{p_A}{\rho_A},$$

which leads immediately to the important relations

(20) $$p = p_0 \left[1 + \frac{\gamma - 1}{2} \frac{u - u_0}{c_0} \right]^{2\gamma/(\gamma - 1)},$$

(21) $$\rho = \rho_0 \left[1 + \frac{\gamma - 1}{2} \frac{u - u_0}{c_0} \right]^{2/(\gamma - 1)},$$

where the subscript A is omitted. With reference to the state on any initial characteristic C_+^0, these express density and pressure in the simple wave in terms of the velocity. If we choose as this initial characteristics the one terminating the state (I) of rest, we have $u_0 = 0$ and obtain

(22) $$p = p_0 \left[1 + \frac{\gamma - 1}{2} \frac{u}{c_0} \right]^{2\gamma/(\gamma - 1)},$$

(23) $$\rho = \rho_0 \left[1 + \frac{\gamma - 1}{2} \frac{u}{c_0} \right]^{2/(\gamma-1)} ,$$

(24) $$c = c_0 + \frac{\gamma - 1}{2} u .$$

Since u is negative in the simple wave, these formulas exhibit the fact that density and pressure decrease as we follow the path of a particle represented in the x,t-plane by a trajectory of the family of characteristics C_+. For increasing t a fixed point in the gas will belong to "later" characteristics C_+, i.e., larger values of $-u$, and hence, as stated before, to smaller values of p and ρ.

The law of rarefaction expressed by (22) and (23) becomes meaningless as soon as $-u > q_e$ where

(25) $$q_e = \frac{2}{\gamma - 1} c_0$$

is the ***escape speed*** of the gas originally at rest. If $-u$ reaches the escape speed, the rarefaction has thinned the gas down to zero density and pressure and the sound speed has likewise decreased to zero. If a rarefaction wave extends to this stage it is called a ***complete rarefaction wave*** as it then ends in a ***vacuum***.

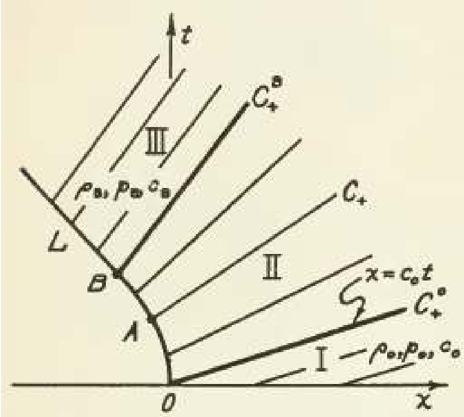


Figure 7
Simple wave (II) connecting two regions (I) and (III) of constant state ($w < q_e$).

For the ***end*** or ***tail*** of the expansion wave there result two possibilities, according to whether or not the terminal speed w of the piston is below the escape speed q_e.

(a) If $w < q_e$, the preceding construction of the simple wave will yield characteristics C_+ for every point A on the piston curve from 0 to B. The rarefaction wave is incomplete and ends at the characteristic C_+^B through B with $u = u_B = -w$ and

(26) $$\begin{cases} c_B = c_0 - \dfrac{\gamma - 1}{2} w \\ p_B = p_0 \left(1 - \dfrac{w}{q_e} \right)^{2\gamma/(\gamma-1)} \\ \rho_B = \rho_0 \left(1 - \dfrac{w}{q_e} \right)^{2/(\gamma-1)} \end{cases} .$$

It is followed by a zone (III) of constant state u_B, ρ_B, p_B, c_B between the tail of the incomplete rarefaction wave and the piston in which the characteristics C_+ are all parallel (as they are in the zone (I) of constant state in front of the simple wave).

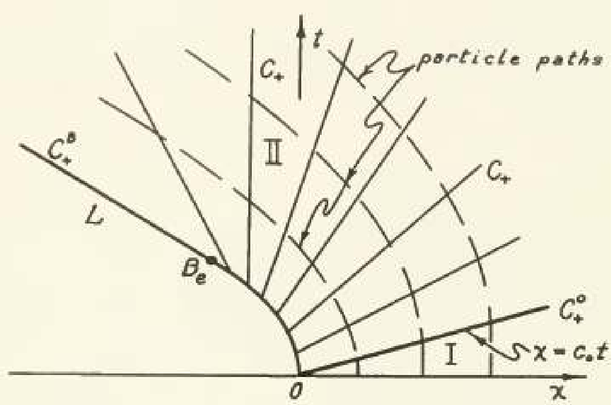


Figure 8
Complete rarefaction ($w = q_e$).

(b) If $w = q_e$ the characteristic C_+^B through $B = B_e$ is tangent to the piston curve, for at B the piston curve has the slope $f'(t) = -w$,

while that of the characteristic C_+^B is $dx/dt = u_B + c_B = -w = -q_e$ since $c_B = 0$. In other words, the wave is just completed at the piston.

(c) If $w > q_e$ the completion of the wave is already achieved before the piston reaches the terminal speed. There will be a point B_e on the piston curve L between 0 and B for which the characteristic $C_+^{B_e}$ is tangent to the piston curve and carries the value zero for density, pressure and sound speed. In this case the rarefaction is completed with this line $C_+^{B_e}$ and beyond it we have a zone (III') of ***cavitation*** , equivalent to a vacuum between the receding piston and the tail of the wave in the gas.

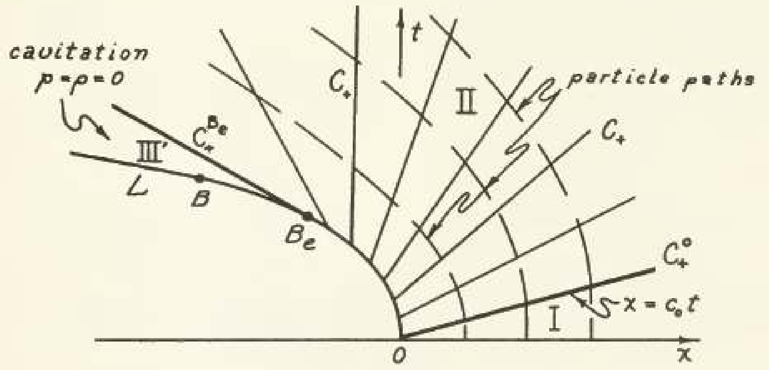


Figure 9
Cavitation ($w > q_e$).

Physically speaking, the escape speed q_e is the speed beyond which a piston cannot recede without separating from the thinned-out gas. If the speed of the piston exceeds q_e, then, as far as the motion of the gas is concerned, it does not matter what the actual value of w is. We might just as well consider w as infinite or imagine the piston or a wall **suddenly removed**, allowing the gas to escape into a vacuum, an interpretation to which the name "escape speed" alludes.

We can summarize our results qualitatively as follows. A piston receding from a gas at rest with speed which never decreases causes an expansion wave of particles moving toward the piston. At the head or front of the wave, which moves into the gas at sound speed, the velocity of the gas is zero. Through the wave the gas is accelerated. (a) If the piston speed w is below the ***escape speed*** q_e, the gas will expand until it has reached the speed w of the piston and then continue with constant velocity, density and pressure. (c) If, however, the piston speed exceeds the escape speed, the expansion is complete and the wave ends in a zone of ***cavitation*** between the ***tail*** and the piston. In any case the wave moves into the quiet gas, while the gas particles move at increasing speed from the wave front to the tail, i.e., from zones of higher pressure and density to zones of lower pressure and density.

A further remark of a general character might be added. In our diagrams it was assumed that the characteristics C_+ of the rarefaction wave diverge from the piston curve L, i.e., that

$dx/dt = u_A + c_A$ decreases, as A moves from to B.

Since along the piston curve ρ decreases and since u and c are functions of ρ in our wave, our statement amounts to

$$\frac{du}{d\rho} + \frac{dc}{d\rho} > 0 .$$

At the piston we have, from (12), $du/d\rho = c/\rho$; hence the preceding relation is equivalent to

$$\frac{1}{\rho} + \frac{1}{c}\frac{dc}{d\rho} > 0$$

or

$$\frac{d}{d\rho}(\log \rho c) > 0 .$$

In other words, a sufficient condition for the desired divergence of the lines C_+ (admitting a general equation of state) is that the acoustic ***impedance*** ρc increases with ρ, a condition certainly satisfied for polytropic gases.

Finally, it should be stated that in our simple waves the conjugate characteristics C_- and the paths of the gas particles can be found by integrating the ordinary differential equations

$$\frac{dx}{dt} = u - c \quad \text{and} \quad \frac{dx}{dt} = u ,$$

respectively, after the functions $u(x,t)$ and $c(x,t)$ have been found by the previous construction.

In a complete rarefaction wave the characteristics C_- as well as the particle paths acquire asymptotically the direction of the last characteristic $C_+^{B_e}$, i.e., the direction $dx/dt = -q_e$.

21. Centered rarefaction waves.

Of particular interest is the case of a ***centered rarefaction wave***, which corresponds to an **idealized piston motion** where the acceleration from rest to a constant terminal velocity -w takes place in an infinitely small time interval, i.e., instantaneously. Then the family of characteristics C_+ forming the simple wave will degenerate into a pencil of lines through the origin O: x =0, t =0 (see Fig. 10).

In the center O the quantities u, ρ, p as functions of x and t are discontinuous, but this discontinuity is **immediately smoothed out** in the subsequent motion. Here we have the first and typical example of an initial discontinuity which immediately resolves into continuous

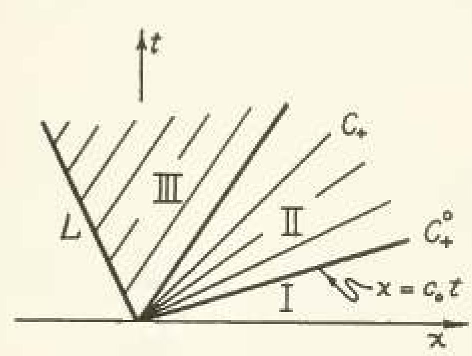


Figure 10
Centered rarefaction
wave ($w < q_e$).

motion.

22. Explicit formulas for centered rarefaction waves.

For ***centered rarefaction waves*** all quantities can be expressed explicitly as simple functions of x and t. With the center at the origin O, we have for each line C_+ through O with flow velocity u

$$\frac{dx}{dt} = \frac{x}{t}.$$

Hence, by virtue of $\frac{dx}{dt} = u + c = u + c_0 + \frac{\gamma - 1}{2}u$ (see equations (6) and (24)), we have in the wave zone (II),

$$\frac{x}{t} = u + (c_0 + \frac{\gamma - 1}{2}u)$$

or

$$u = \frac{2}{\gamma + 1}\left(\frac{x}{t} - c_0\right) = (1 - \mu^2)\left(\frac{x}{t} - c_0\right), \tag{27}$$

where

$$\mu^2 = \frac{\gamma - 1}{\gamma + 1};$$

and from $c = c_0 + \frac{\gamma - 1}{2}u$ we obtain

$$c = \mu^2 \frac{x}{t} + (1 - \mu^2)c_0. \tag{28}$$

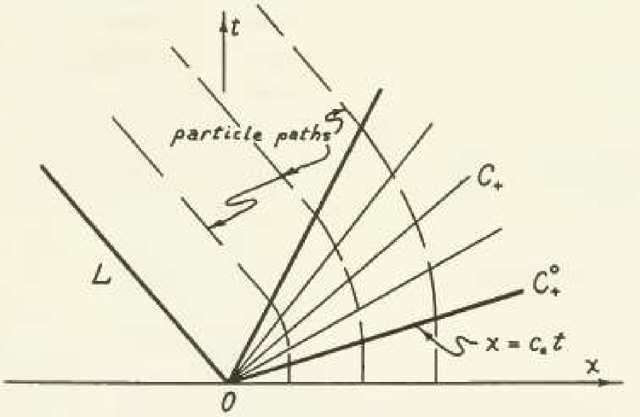


Figure 11
Particle paths in flow
involving centered
rarefaction wave
$(w < q_e)$.

Thus u and c are known explicitly in the zone (II) of the rarefaction wave, and p and ρ can be found (see (22), (23), (25)) by using

$$p = p_0\left(1 + \frac{u}{q_e}\right)^{2\gamma/(\gamma - 1)}, \quad \rho = \rho_0\left(1 + \frac{u}{q_e}\right)^{2/(\gamma - 1)}. \tag{29}$$

We can find the particle paths in (II) by integrating

$$\frac{dx}{dt} = u = (1 - \mu^2)\left(\frac{x}{t} - c_0\right)$$

(see equation (27)). Upon making the substitution

$$x = \xi(t) - \frac{2c_0}{\gamma - 1}t = \xi(t) - \frac{1 - \mu^2}{\mu^2}c_0 t,$$

we obtain

$$\frac{d\xi}{dt} = (1 - \mu^2)\frac{\xi}{t},$$

and by integrating we find that

$$\xi = At^{1 - \mu^2},$$

where A is an arbitrary constant, or

(30) $$x = t\left\{At^{-\mu^2} - \frac{2c_0}{\gamma-1}\right\}.$$

This formula is valid as long as the particles remain in the rarefaction zone (II).

(c) In the case of a **complete rarefaction** ending with zero density for $w \geq q_e$, formula (30) holds for arbitrarily large values of *t*, and we have, for large *t*, the asymptotic expression

(31) $$x \sim -\frac{2c_0}{\gamma-1}t\,.$$

As remarked previously, the fluid remains in the zone (II) of rarefaction, and in the *x,t*-diagram the particle paths acquire asymptotically the direction of the characteristic C_+^e on which the escape speed q_e is attained (see Fig. 12).

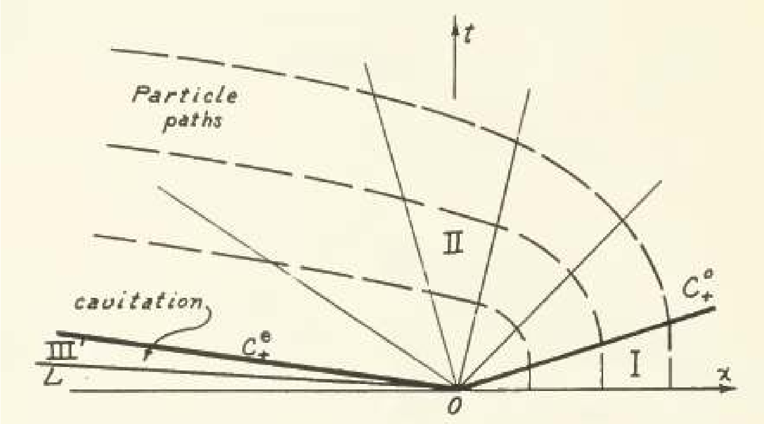


Figure 12
Cavitation behind centered rarefaction wave ($w > q_e$).

(a) For $w < q_e$ the rarefaction wave terminates at the characteristic C_+ on which the velocity has the value u = -*w*, and all the particle paths emerge from (II) parallel to the terminal direction of the piston curve *L* and remain parallel in zone (III).

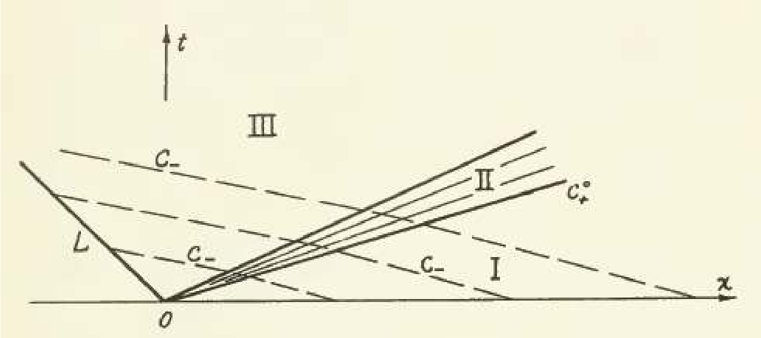


Figure 13
Non-straight characteristics C_- in flow involving centered rarefaction wave ($w < q_e$).

The conjugate characteristics $C\_$ are given by the differential equation

$$\frac{dx}{dt} = u - c = (1-2\mu^2)\frac{x}{t} - 2(1-\mu^2)c_0$$

(see equations (27) and (28)) which leads, with a constant $A\_$ of integration, to

(32) $$x = t\left\{A\_t^{-2\mu^2} - \frac{2c_0}{\gamma-1}\right\}.$$

For $w < q_e$ the characteristics $C\_$ emerge from the (incomplete) rarefaction zone (II) with the slope

(33) $$\frac{dx}{dt} = -w - \left(c_0 - \frac{\gamma-1}{2}w\right) = \frac{\gamma-3}{2}w - c_0\,,$$

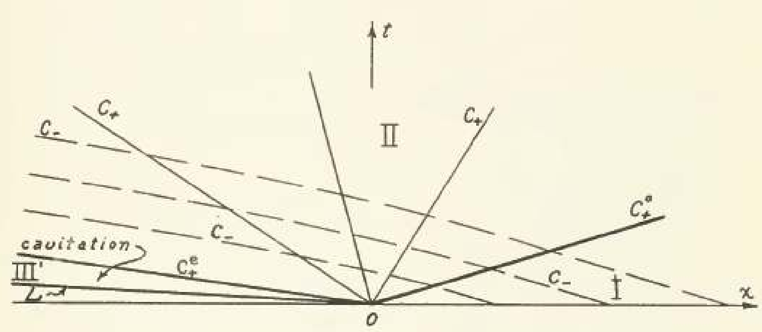


Figure 14
Characteristics C_- in flow involving centered rarefaction wave that ends in cavitation ($w > q_e$).

and then continue as straight lines meeting the piston line *L*: *x* = -*wt*. For $w \geq q_e = 2c_0/(\gamma-1)$ the characteristics $C\_$ again remain within the "*complete rarefaction zone*", and, since $x \sim -\dfrac{2c_0}{\gamma-1}t$,

they approach the particle paths asymptotically.

Obviously these considerations can be generalized to other equations of state.

23. Remark on centered simple waves in Lagrangean coordinates.

We could just as well have developed the theory of simple waves in Lagrange's coordinates, using the equations (38) or (39) of Chapter I, Art. 7. Only a brief remark is made here concerning the characteristics in the Lagrangean form. The characteristics C_+ are given by

(34) $$\frac{dh}{dt} = \rho c = k \,,$$

and the characteristics C_- by

(35) $$\frac{dh}{dt} = -\rho c = -k \,.$$

For centered simple waves the lines C_+ are straight lines in the h,t-plane on each of which $\frac{h}{t} = \text{constant}$. Since $\frac{dh}{dt} = \rho c$ along a line C_+ we find $\rho c = h/t$. In other words, the ***impedance*** is always simply $k = h/t$, no matter what the equation of state is. Consequently for C_- we have $\frac{dh}{dt} = -\frac{h}{t}$, which can be immediately integrated as

(36) $$ht = \text{constant}\,.$$

Thus, for centered rarefaction waves the **non-straight characteristics** are always ***equilateral hyperbolas*** in Lagrangean coordinates.

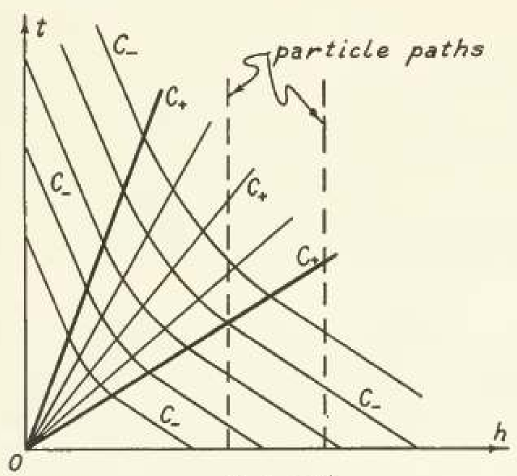


Figure 15
Characteristics in Lagrangean coordinates for centered simple wave.

24. Compression waves.

If a piston is not withdrawn, but is moved **into** the gas-filled tube with a speed which never decreases, or if a receding piston is slowed down or stopped, then a ***contraction wave*** will originate at the piston. The qualitative statements and formulas pertaining to rarefaction waves also apply to contraction waves, except that the forward characteristics C_+ no longer diverge from the piston. Density, pressure and sound speed at the piston increase, and the characteristics C_+, covering zone (II), converge and therefore have an envelope if extended sufficiently far. Certainly the solution constructed above as a simple contraction wave cannot extend beyond the envelope. For if it

did, a forward characteristic C_+, with values u_A, ρ_A corresponding to a position A of the piston, and another with values u_B, ρ_B corresponding to a subsequent position B, would intersect beyond the envelope, and at such a point of intersection unique values of u and ρ would no longer be determined. The analytical extension of our mathematical solution beyond the envelope would therefore be multivalued and hence could no longer represent the state which occurs in reality. Physically speaking, the values of u and ρ are propagated along the characteristics C_+, and the values corresponding to a later position of the piston are propagated with a greater velocity so that they would overtake the values propagated from an earlier position of the piston. An absurdity is inescapable unless we abandon the assumption that the motion remains continuous. Consequently all compressive motion inevitably leads to discontinuities and such discontinuities must occur before or on the envelope.

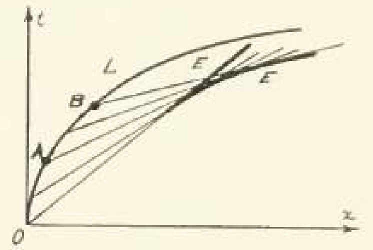


Figure 16
Contraction wave and envelope of characteristics C_+ which bounds possible location of shock line.

25. Position of envelopes for compression waves.

In Section C of this chapter we shall be concerned with these discontinuities. Here a few remarks are added regarding the envelopes formed by the straight characteristics C_+ of a compression wave. Let us consider an ideal gas with the adiabatic exponent γ. The piston curve L corresponding to a compression can be described by

$$L : x = f(t), \quad f'(t) \geq 0 .$$

Then the straight characteristics C_+ are given in terms of a parameter τ by

$$(37) \qquad x = f(\tau) = (t-\tau)\left[\frac{\gamma+1}{2} f'(\tau) + c_0\right],$$

where $f'(\tau) = u_A$ is the velocity of the piston at the position A corresponding to the parameter τ. The envelope is obtained by combining the last equation with

$$(38) \qquad -f'(\tau) = -\frac{\gamma+1}{2} f'(\tau) - c_0 + \frac{\gamma+1}{2}(t-\tau) f''(\tau),$$

which yields

$$(39) \qquad \begin{cases} x = f(\tau) + \left\{c_0 + \dfrac{\gamma+1}{2} f'(\tau)\right\} \dfrac{2c_0 + (\gamma-1) f'(\tau)}{(\gamma+1) f''(\tau)} \\ t = \tau + \dfrac{2c_0 + (\gamma-1) f'(\tau)}{(\gamma+1) f''(\tau)} \end{cases},$$

where τ ranges from zero to the value of the time for which a constant terminal speed is attained or $f''(\tau) = 0$.

As long as $f''(\tau) \geq 0$ we have compressive motion and the envelope exists. In the special case of a piston accelerated from rest with a constant acceleration a according to

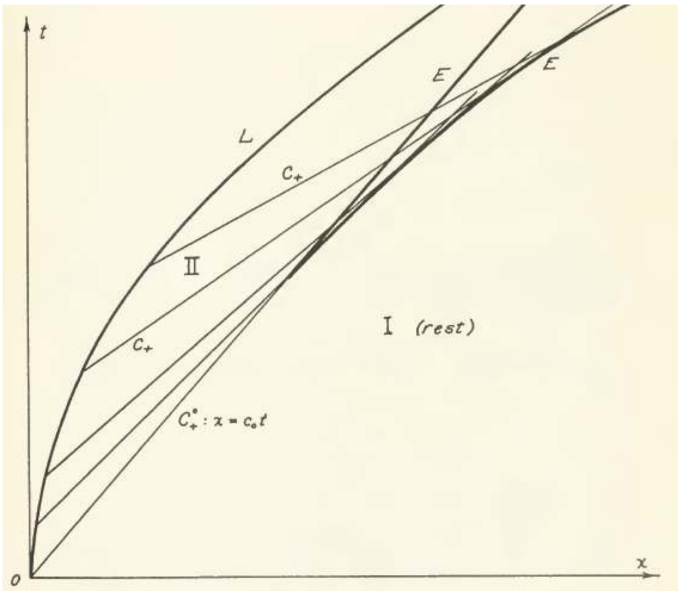


Figure 17
Envelope formed by the characteristics resulting from the piston motion $x = \frac{a}{2} t^2$, indicating the inevitable occurrence of a discontinuity.

$$L : x = \frac{a}{2} t^2, \quad a > 0,$$

the envelope is

$$(40) \qquad \begin{cases} x = \dfrac{\gamma}{2} ar^2 + \dfrac{2\gamma}{\gamma+1} c_0 \tau + \dfrac{2}{\gamma+1} \dfrac{c_0^2}{a} \\ t = \dfrac{2\gamma}{\gamma+1} \tau + \dfrac{2}{\gamma+1} \dfrac{c_0}{a} \end{cases}, \quad \tau > 0,$$

an arc of a parabola beginning at the point

$$(41) \qquad P : x' = \frac{2}{\gamma+1} \frac{c_0^2}{a}, \quad t' = \frac{2}{\gamma+1} \frac{c_0}{a},$$

and tangent at this point to the characteristic line $x = c_0 t$.

We may consider the parabolic arc together with this characteristic line for $t > t'$ as forming a ***cusp***, and it is easily seen that a similar situation prevails for a piston motion given by $x = (a/2)t^2 + \cdots$, where the dots denote terms in t of order higher than two.

We note that the coordinate x' and the time t' of the inevitable beginning of a discontinuity (later described as a ***shock***) will be very close to the origin for large accelerations or for small sound speed c_0.

The motion just considered starts from rest with a sudden acceleration. If there is no initial discontinuity of $f''(t)$ as for $f(t) = (a/3)t^3 + \cdots$, where the dots indicate terms in t of order higher than three, then the envelope has a genuine ***cusp*** with both branches monotonically increasing.

In the typical case $f(t) = (a/3)t^3$ the envelope is given by

$$(42 \qquad \begin{cases} x = \dfrac{a}{3} \tau^3 + (c_0 + a\dfrac{\gamma+1}{2}\tau^2)(c_0 + a\dfrac{\gamma-1}{2}\tau^2)\dfrac{1}{a(\gamma+1)\tau} \\ t = \tau + \dfrac{c_0 + a\dfrac{\gamma-1}{2}\tau^2}{(\gamma+1)a\tau} \end{cases},$$

which shows that to $\tau = 0$ there corresponds a point at infinity on the envelope, as is always the case for $f''(0) = 0$. There will be a minimum value of t along the envelope at the point where $dt/d\tau = 0$, and since

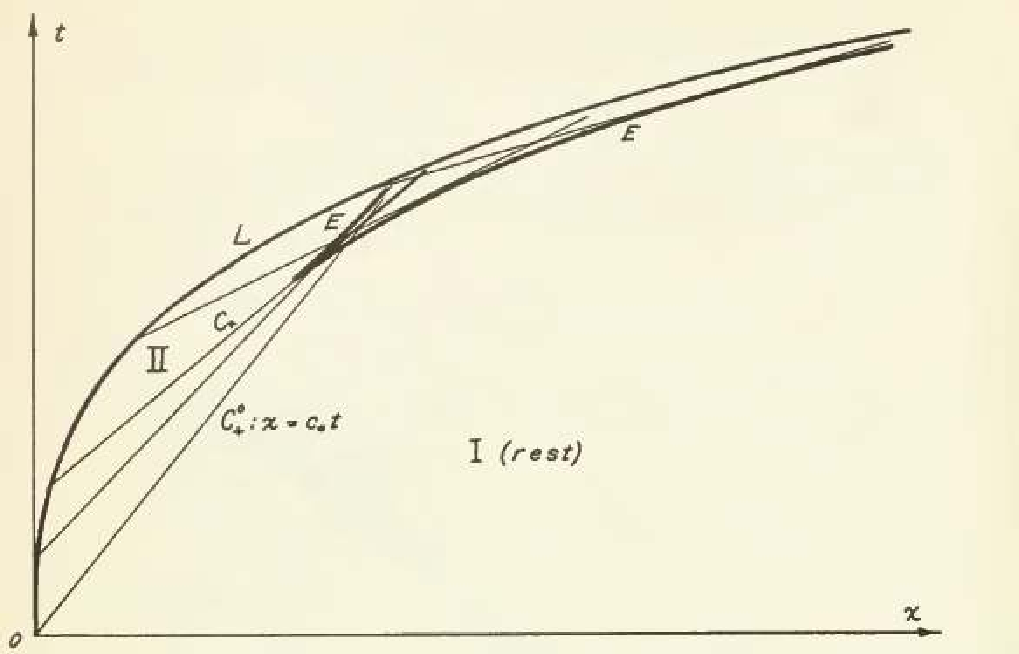


Figure 18
Envelope formed by the characteristics resulting from the piston motion $x = \frac{a}{3}t^3$. (Note that in this case the discontinuity does not begin at the boundary of the region of quiet and that the position of the envelope indicates that the position of the shock front moves closer and closer to the piston.)

$$\frac{dx}{d\tau} \Big/ \frac{dt}{d\tau} = \frac{dx}{dt} = c_0 + \frac{\gamma+1}{2} f'(\tau)$$

is finite, we see that $dx/d\tau = 0$ or the same point on the envelope. This together with the fact that $f'(\tau) > 0$ shows that the point under consideration is really a ***cusp*** as described.

Incidentally, the situation previously discussed for the piston curve $x = (a/2)t^2$ may be considered a ***degenerate*** case where the second branch of the cusp has degenerated into a straight characteristic.

The shape of the envelope may be rather complicated if the piston motion is not simple. Since the fine features of the geometry of the envelope depend on the local behavior of the second and higher derivatives of $f(t)$, however, we must expect that the actual behavior of the flow will not be strongly affected by geometrical complexities of the envelope. As a matter of fact, we shall see that **shock discontinuities always begin infinitely weak** at the cusp of the envelope. Whether they develop afterwards to strong shocks no longer depends on the local factors producing the cusp, but on the **piston motion as a whole**.

C. SHOCK FRONTS

26. Introduction.

As we have seen, certain initial discontinuities are smoothed by centered rarefaction waves, while other motions starting as perfectly continuous contraction waves cannot be maintained without a discontinuity.* The fact is that any compression of the gas by the piston, however slow, will ultimately lead to ***discontinuities*** of velocity, pressure and density.

Hence, for a mathematical description of motions caused by impinging pistons and of many other motions as well, we must abandon, or rather supplement, the mathematical framework employed so far.

One possibility suggests itself immediately. We might try to obtain the necessary generalization from the differential equations of motion directly. In Chapter II, Art. 11, we saw that these differential equations allow discontinuities of the first and higher derivatives of u and ρ across characteristics in the u,t-plane. Such "***sonic discontinuities***" are associated with the differential equations; for example, they arise in initial value problems by passage to the limit from initial values with continuous derivatives to values with local discontinuities and in this limiting process the differential equations remain unchanged. In the case of linear differential equations the same type of limiting process leads to a "***sonic propagation***" even of discontinuities of the dependent functions themselves. For our nonlinear differential equations, however, no such sonic transmission of discontinuities of ρ and u is deducible by a passage to a limit from continuous solutions.

Hence to arrive at an adequate theory we must give up as oversimplified our original description of reality and seek a **closer approximation** to the actual situation by accounting for physical facts neglected in the original differential equations. Accordingly, we should introduce ***viscosity*** and ***heat conduction***, represented by additional (linear) terms of the second order in the differential equations. To the smoothing effect of the force of viscous friction and heat conduction there corresponds the fact that, if the terms of second order are included, the differential equations have **continuous solutions** no matter how small the coefficients of heat conduction and viscosity are.

Observed physical reality now points the way to a remarkable mathematical simplification. For very small values of these

coefficients, the influence of heat conduction and viscosity is negligible except in the **immediate vicinity of sharply defined surfaces** (which may move in time) where velocity, pressure, density and temperature undergo rather **sudden and large changes**. Mathematically speaking, if we let the coefficients of viscosity and heat conduction in the completed differential equations tend to zero, their continuous solution may be expected to converge to solutions of the original differential equations of first order except that certain surfaces emerge across which these solutions have discontinuities in u, ρ , p, c, T.

The values of these quantities at both sides of such discontinuity surfaces are restricted by ***jump conditions*** discovered by Earnshaw, Riemann, Rankine and Hugoniot, and the decisive fact is that the effect of viscosity and heat conduction can be mathematically represented simply by these jump conditions, while otherwise the original differential equations are retained. Instead of attempting to carry out the cumbersome passage to the limit we face the simpler, though still in most cases difficult, problem of determining the **surfaces of discontinuity** in addition to satisfying the jump conditions and, in the regions of continuity, the original differential equations. We shall distinguish two types of such discontinuity surfaces, ***contact surfaces*** and ***shock fronts***. The former are surfaces separating two parts of the medium without flow of substance through the surface; the **shock fronts** are discontinuity surfaces which are **crossed by the flow of gas**. If the shock front moves in time it is called a ***shock wave***. The side of the shock front against which the flow is directed will be called the ***front side*** of the shock, the other the ***back side***. As we shall see, the shock front, observed from the front side, always moves with supersonic speed. In this chapter we are concerned with one-dimensional motion. Hence the shock fronts and contact surfaces are assumed to be planes perpendicular to the x-axis and are represented on the x-axis by points or in the x,t-plane by lines S, henceforth called ***shock lines***, or ***contact lines***, respectively.

27. Shock wave in a tube.

Let us first describe the simplest case of a motion involving a shock wave. The **centered expansion wave** caused by a piston receding at constant speed was studied as a basic type of motion. Just

as basic and typical is the motion caused by a piston starting from rest and suddenly moving with constant speed w into the quiet gas. No matter how small w is, the resulting motion cannot be continuous.

What, then, will happen? Immediately there will appear a **shock front** moving away from the piston with a constant and, as we shall prove, supersonic speed $\dot{\xi}$, uniquely determined by the density and sound speed in the quiet gas and by the piston speed w. In front of the shock the gas is at rest, while behind the shock it moves with the constant velocity w. In the x,t-plane this very simple motion is represented in Fig. 19. The **shock line** S always lies in that region which would be the zone of quiet if the motion were continuous. For a sequence of decreasing values of w the shock line approaches the characteristic $x = c_0 t$ and the **jump** of velocity, pressure and density across the shock approaches zero. The shock becomes ***weak*** and approaches a '***sonic disturbance***."

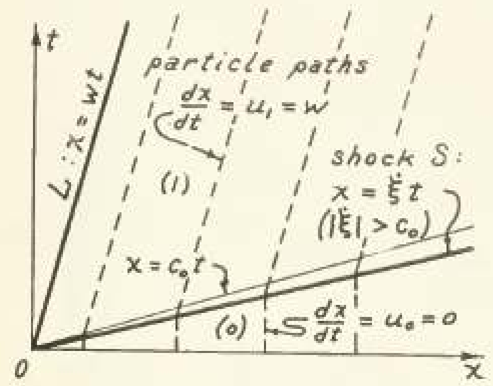


Figure 19
Shock resulting from piston moving into gas with constant velocity.

Before we can substantiate this qualitative description by proof, we must derive and discuss the jump conditions across the shock.

28. Shock conditions

We start from the following basic laws of physics:

(1) Conservation of mass,

(2) Conservation of momentum,

(3) Conservation of energy.

Under the further assumption of continuous velocity, density and pressure, the first two laws would lead to Euler's (or Lagrange's) equations of gas dynamics. Application of these principles to discontinuous motions leads to the corresponding first two jump conditions for shocks. The energy law (3) takes care of a more delicate point. Our original system of differential equations I(14), Art. 3, was supplemented by the equation of state in which we assumed constant entropy in keeping with the supposed adiabatic character of our processes. At first thought, it might seem plausible to suppose that even a shock discontinuity does not entail a change in entropy; in other words, that not only the continuous motion but also the shock involves merely adiabatic changes. Making this assumption, Earnshaw (1855) and Riemann (1860) considered only the conditions (1) and (2). However, as Rankine (1870), Rayleigh (1878) and Hugoniot (1887) observed, this procedure violates the principle of conservation of

energy and thus fails to represent physical reality adequately. One must admit (discontinuous) **changes of entropy** across a shock and stipulate a third shock condition (3) expressing the energy principle. This "***Rankine-Hugoniot***" **discontinuity condition** replaces the assumption of adiabatic changes made for continuous motions.

We shall now derive the conditions that hold **across a discontinuity surface** by applying the three general principles to a column of gas in the tube, the column covering at the time t the interval $a_0(t) < x < a_1(t)$, where $a_0(t)$ and $a_1(t)$ denote the positions of the moving particles that form the ends of the column. By e we denote the internal energy of the gas per unit mass, so that the total energy per unit mass is $e + (1/2)u^2$. Then, for the column, the three basic principles are expressed by the relations

(i) $$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho dx = 0 ,$$

(ii) $$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho u dx = p_{a_0} - p_{a_1} ,$$

(iii) $$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho\left\{\frac{1}{2}u^2 + e\right\} dx = p_{a_0}u_{a_0} - p_{a_1}u_{a_1} ,$$

where $p_{a_0} = p[a_0(t), t]$, etc.

Relation (i) needs no comment, (ii) expresses the fact that the rate of change of momentum of the column equals the total resulting force exerted on the column by the pressure on the two ends, (iii) states that the rate of increase of energy contained in the column is equal to the "power-input", i.e., the work done in unit time by the pressure against the end surfaces of the column (whose velocities are $\dot{a}_0 = u_{a_0}$ and $\dot{a}_1 = u_{a_1}$).

As long as we assume u, ρ, p continuous and differentiable in the whole column, we can easily deduce from the first two of these equations the differential equations of motion I(14), Art. 3. In the present analysis, however, we assume that in the moving column there is a point of discontinuity whose coordinate $x = \xi(t)$ moves with the velocity $\dot{\xi}(t)$.

All of our integrals have the form

$$J = \int_{a_0(t)}^{a_1(t)} \psi(x,t) dx ,$$

the integrand ψ being discontinuous at $x = \xi$. Differentiation leads to

$$
\begin{aligned}
\frac{d}{dt}J &= \frac{d}{dt}\int_{a_0(t)}^{\xi(t)} \psi(x,t)dx + \frac{d}{dt}\int_{\xi(t)}^{a_1(t)} \psi(x,t)dx \\
&= \int_{a_0(t)}^{\xi(t)} \frac{\partial \psi(xt)}{\partial t} dx + \left\{\psi(\xi(t)-0,t)\dot{\xi}(t) - \psi(a_0(t),t)u_0\right\}. \\
&\qquad + \left\{\psi(a_1(t),t)u_1 - \psi(\xi(t)+0,t)\dot{\xi}(t)\right\}
\end{aligned}
$$

The quantities $u_0 = \dot{a}_0(t)$ and $u_1 = \dot{a}_1(t)$ are the velocities at the ends of the column. Our formula holds no matter how short our column is, so long as it contains $x = \xi$ as an interior point. We now perform the limiting process, letting the length of the column approach zero. The first integral on the right-hand side of the last equation then tends to zero. Denoting by ψ_i, u_i the limit values of ψ, u as ξ is approached from the end a_i, we obtain immediately

$$\lim_{a_1 - a_0 \to 0} \frac{d}{dt}J = \psi_1 v_1 - \psi_0 v_0 ,$$

where

$$v_i = u_i - \dot{\xi}$$

is the flow velocity relative to the discontinuity surface. Thus we derive from our three basic equations the following conditions.

Conservation of mass

(i) $\quad \rho_1 v_1 - \rho_0 v_0 = 0$,

or

$$\rho_0 v_0 = \rho_1 v_1 = m ,$$

m being the ***mass flux*** through the surface.

Conservation of momentum

(ii’) $\quad (\rho_1 u_1)v_1 - (\rho_0 u_0)v_0 = p_0 - p_1$,

or

$$\rho_0 u_0 v_0 + p_0 = \rho_1 u_1 v_1 + p_1 .$$

By (i) this relation is equivalent to

(ii) $\quad \rho_0 v_0^2 + p_0 = \rho_1 v_1^2 + p_1 = P$,

which involves only the relative velocities v.

Conservation of energy

(iii’) $\quad \rho_1 \left\{\frac{1}{2}u_1^2 + e_1\right\} v_1 - \rho_1 \left\{\frac{1}{2}u_0^2 + e_0\right\} v_0 = p_0 u_0 - p_1 u_1$,

or

$$\frac{1}{2}\rho_0 v_0 u_0^2 + \rho_0 v_0 e_0 + u_0 p_0 = \frac{1}{2}\rho_1 v_1 u_1^2 + \rho_1 v_1 e_1 + u_1 p_1 .$$

These relations hold across both shock fronts and contact surfaces. These two types of discontinuity surfaces were distinguished by the

property that there is **gas flow across a shock front**, $m \neq 0$, and **no gas flow across a contact surface**, $m = 0$.

In this article we shall consider only **shock discontinuities**, postponing the discussion of contact surfaces to Art. 29.

For shocks ($m \neq 0$) relation (iii') can be simplified. Multiplying equation (ii') by $\dot{\xi}$, subtracting it from (iii'), and using (i), we obtain

$$\text{(iii*)}\qquad \frac{1}{2}v_0^2 + e_0 + \frac{p_0}{\rho_0} = \frac{1}{2}v_1^2 + e_1 + \frac{p_1}{\rho_1} = \frac{1}{2}\hat{q}^2,$$

where $\hat{q}$ is the limit speed introduced in Art. 5, Chapter I. Remembering the definition of the enthalpy $i = e + \frac{p}{\rho}$, we can write

$$\text{(iii)}\qquad \frac{1}{2}v_0^2 + i_0 = \frac{1}{2}v_1^2 + i_1 = \frac{1}{2}\hat{q}^2.$$

As we see, the third shock condition has exactly the form of ***Bernoulli's law***. It differs from it essentially, however, inasmuch as the function which represents the enthalpy i in its dependence on ρ is discontinuous across the shock, since the values i_1 and i_0 correspond to different values η_1 and η_0 of the entropy. [In other words, the change in enthalpy $i_1 - i_0$ across a shock is not equal to $\int_{(0)}^{(1)} \frac{dp}{\rho}$ but equal to $\int_{(0)}^{(1)} \left(\frac{dp}{\rho} + T d\eta \right)$.]

The conditions (i), (ii) and (iii) represent the three shock conditions in a form in which only the relative velocities $v = u - \dot{\xi}$ are involved and not the velocities u and $\dot{\xi}$ separately. It is thus clear that the shock conditions are **invariant under translation** with constant velocity, in accordance with the ***Galilean principle of relativity***.

Elimination of v_0 and v_1 from conditions (i), (ii), (iii*) leads to the following important shock relation:

$$\text{(iii**)}\qquad (\tau_0 - \tau_1)\frac{p_1 + p_0}{2} = e_1 - e_0, \quad \tau = \frac{1}{\rho},$$

which could be interpreted to mean that the **increase in internal energy across the shock front is due to the work done by the mean pressure in performing the compression**. This relation is equivalent to

$$\text{(iii***)}\qquad (\tau_0 + \tau_1)\frac{p_1 - p_0}{2} = i_1 - i_0.$$

Since e, or i, is a known function of ρ and p depending on the

physical properties of the gas, we have **three relations** between **seven quantities** $p_i, \rho_i, u_i, \dot{\xi}$. Hence, if three of these quantities are fixed, there is still a **one-parameter family** of shocks possible.

While the shock relations between the seven quantities are nonlinear and thus do not necessarily define this one-parametric family uniquely, we shall see that under wide conditions, in particular, for polytropic gases, the following **theorems** hold:

> **(A)** *The state (0) on the front side of the shock front and the shock velocity* $\dot{\xi}$ *determine the state (1) on the back side of the shock front.*
>
> **(B)** *The state (0) and the pressure* p_1 *determine the shock front and the complete state (1). The same is true when, instead of* p_1*, the density* ρ_1 *or the velocity* u_1 *is known.*

Generally speaking, in view of the fact that the unknown positions of the shock lines in the *x,t*-plane must be determined, extremely difficult boundary value problems for our differential equations result. Many important special cases, however, are amenable to an analytical treatment as, for instance, when simple piecewise solutions of the differential equations can be fitted together across straight shock lines.

From the mathematical point of view it should be emphasized that for none of the solutions involving shock fronts has a **complete uniqueness proof yet been given**. Therefore, even more than is usually the case in theoretical science, the physical significance of the mathematical solutions must be **verified by experiment**. In gas dynamics mathematical theory is largely a means of finding qualitative and quantitative patterns which may serve to **interpret** experimental data.

29. Contact discontinuities.

The discontinuity conditions (i), (ii'), (iii') admit of a "trivial" or degenerate solution. If the flux *m* through the surface of discontinuity is zero, i.e., if no substance crosses it, then we have $v_0 = v_1 = 0$, hence $u_0 = u_1 = \dot{\xi}$, and from (ii) we infer that $p_0 = p_1$, while (iii') is automatically satisfied (but (iii*) and (iii) can no longer be deduced from (iii')). Such a discontinuity surface is called a ***contact surface***. A contact surface moves with the gas and separates two zones of different density (and temperature); but the pressure and flow velocity are the same on both sides. (It is obvious that in reality such a contact

surface cannot be maintained for an appreciable length of time, for heat conduction between the permanently adjacent particles on either side of the discontinuity would soon make our idealized assumption unrealistic). Henceforth, unless the contrary is stated, we shall always denote as shock only a genuine shock with the flux m through the shock front different from zero.

30. Description of shocks.

We recall the following definitions given in Art. 26. The side of the shock against which the mass flux is directed was called the ***front side***. The other side was called the ***back side***. In other words, the particles cross the shock front from the front toward the back side. This definition is independent of the choice of coordinate system. Pressure and density, as we shall see, are always greater behind the shock than in front of it, and the degree of this increase can be used in various ways to measure the intensity of the shock (see Art. 36). Usually we shall denote the front side with the subscript (0) and the back side of the shock front with (1). We also say that the **shock front faces the front side** or is **directed toward the front side**.

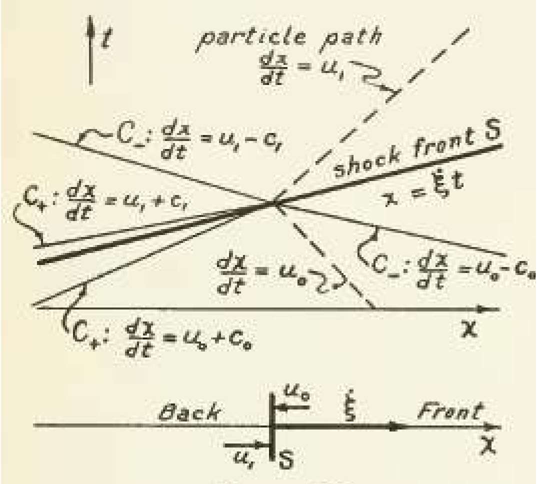


Figure 20
Shock front.

Moving shock fronts are often called ***shock waves*** and it should be clearly understood that the direction in which the shock wave **moves**, given by the sign of $\dot{\xi}$, has nothing to do with the direction toward which it **faces**, i.e., with the distinction of front and back side of the shock, the latter depending only on the relative velocity v.

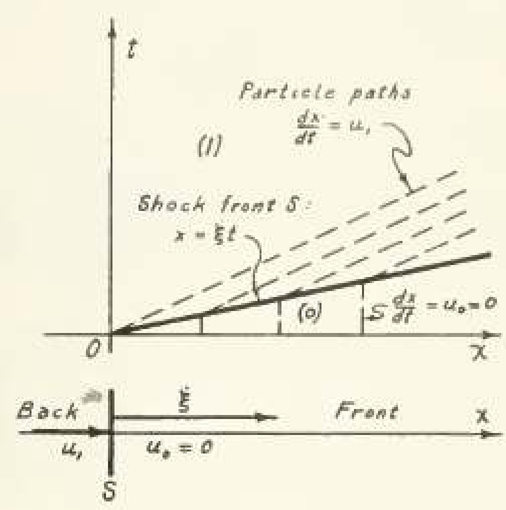


Figure 21
Impinging shock front.

We now discuss **three different interpretations** of a shock front, all of which are equivalent by the ***Galilean principle of relativity***.

(a) First, suppose that the velocity u_0 on the **front side is zero**. Then the shock impinges on a zone (0) of rest with the velocity $\dot{\xi}$ which will be seen to be supersonic when observed from the front side, while the velocity $\dot{\xi} - u_1 = -v_1$ observed from the high-pressure zone on the back side is subsonic. The shock wave moves rapidly into the zone of quiet, enveloping more and more of the gas which, after being overtaken, follows at a speed less than that of the shock front. At the same time the density and pressure are suddenly increased.

(b) Secondly, suppose the velocity u_1 on the **back side is zero**. Then the shock front may be interpreted as receding with the velocity $\dot{\xi}$ leaving behind a high-pressure zone of quiet. Such receding shock waves will be encountered as shock waves reflected from a wall (see

Art. 41).

(c) Finally, suppose that the velocity of the **shock front is zero**, i.e., that the shock front is ***stationary***. (Any shock front will be stationary if observed from a coordinate system moving with the instantaneous shock front velocity $\dot{\xi}$). Such a stationary shock front is simply described by a fixed point $x = \xi$ in the tube into which the gas flows at supersonic speed and behind which it is slowed down(to subsonic speed; while pressure and density are increased. The ***discontinuity conditions*** that hold for stationary shocks ($\dot{\xi} = 0$) can be found immediately by putting $v_i = u_i$ in (i), (ii), (iii), Art. 28:

(i'') $\qquad \rho_0 u_0 = \rho_1 u_1 = m$,

(ii'') $\qquad \rho_0 u_0^2 + p_0 = \rho_1 u_1^2 + p_1 = P$,

(iii'') $\qquad \frac{1}{2}u_0^2 + i_0 = \frac{1}{2}u_1^2 + i_1 = \frac{1}{2}\hat{q}^2$.

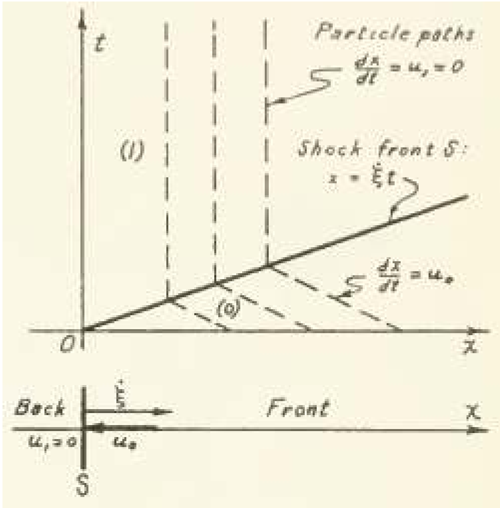


Figure 22
Receding shock front.

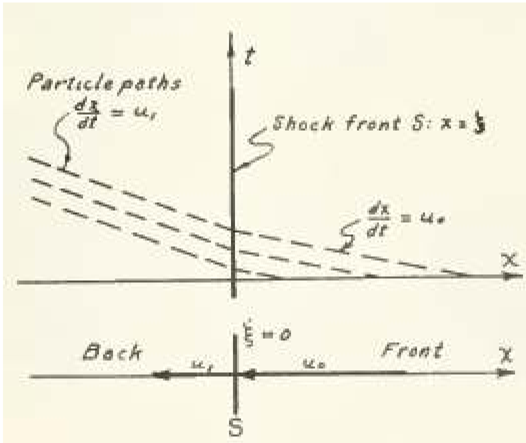


Figure 23
Stationary shock front.

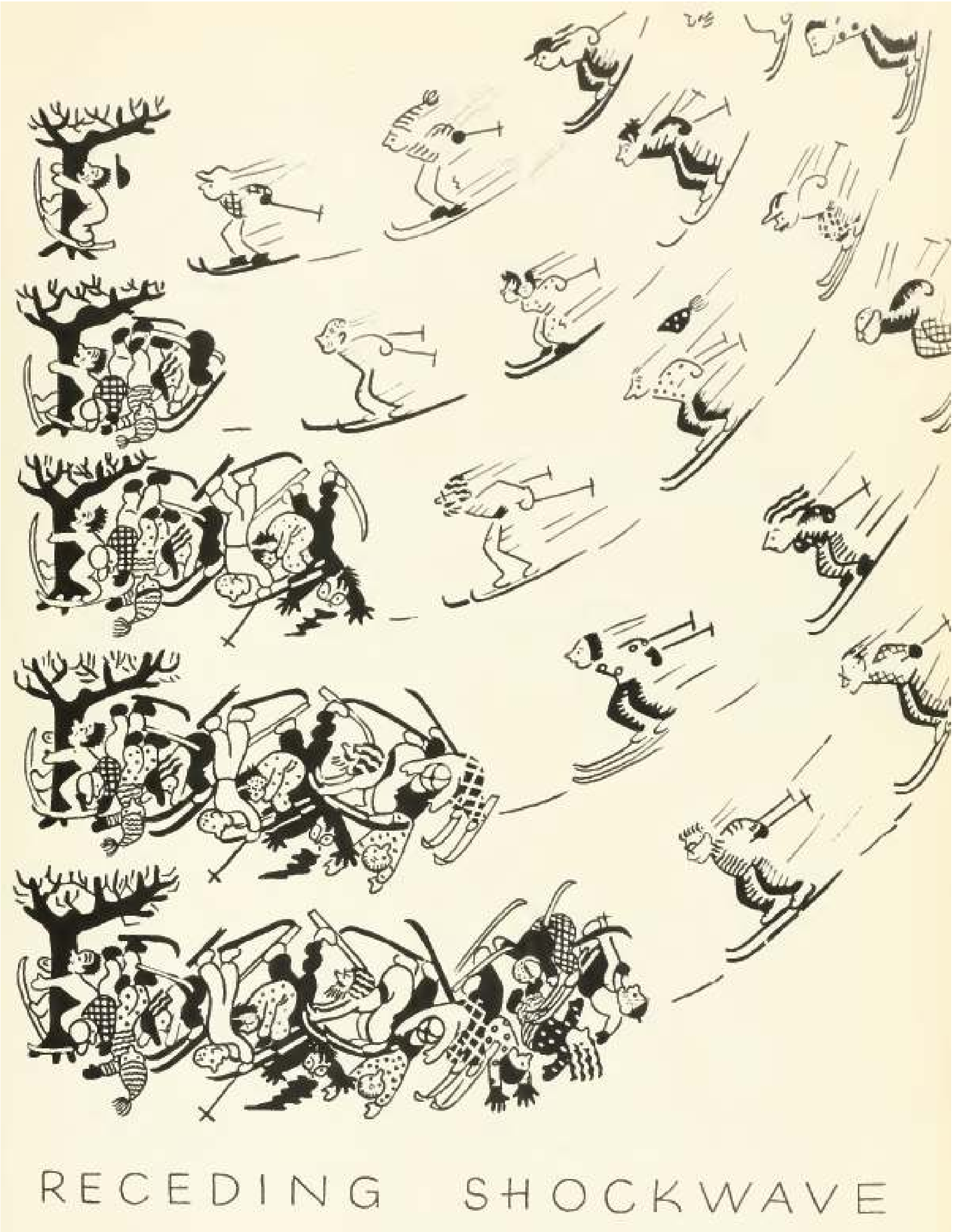
31. Models of shock motion.

Shocks in their different aspects can be visualized by an analogy with a motion of particles such as a **stream of fast automobiles** on a highway. A stationary shock can be produced as follows. We assume a steady flow of traffic at high speed. In such a flow there will be a "**sound speed**", i.e., a speed at which small disturbances occurring in the traffic will spread. If the speed of the travelling cars exceeds this sound speed then a ***steady shock*** will occur at a point where the velocity is suddenly reduced, say by a change from a wide to a narrow road. There the driver in front will suddenly put on his brakes and slow down, being unable to transmit a warning signal to the driver in the rear. The increase in density is obvious and increase in pressure is also immediately represented in our model if we imagine the row of cars separated by springs or buffers with a non-linear law of repulsion.

A ***receding shock wave*** can be pictured in a similar manner. Let us assume, as before, a long column of equally-spaced cars travelling at supersonic speed against an unanticipated obstacle which suddenly brings the first car to a full stop. The second will press close to the first and stop; then the third will be abruptly stopped by the second, and so on. The point separating the stopped cars from the moving cars obviously represents a receding shock front.

A shock front impinging on a zone of rest is represented by the phenomenon of a column of fast moving cars pounding against a row

of widely spaced parked cars.



Models of one-dimensional wave motion by means of individual particles connected by non-linear laws of repulsion are not only suggestive, but may even be used as approximations to actual situations and thus as a basis for numerical computation.

32. Mechanical shock conditions.

We observe that only the third condition explicitly introduces the thermodynamical nature of the substance represented by the energy e or the enthalpy i as a known function of p and ρ. Hence all conclusions drawn solely from the first two shock conditions (the "**mechanical conditions**"),

(i'') $\rho_0 u_0 = \rho_1 u_1 = m$,

(ii'') $\rho_0 u_0^2 + p_0 = \rho_1 u_1^2 + p_1 = P$,

are valid for any medium irrespective of its equation of state. This is true of the relations:

(43) $m(v_1 - v_0) = p_0 - p_1$,

(44) $m^2 = \dfrac{p_0 - p_1}{\tau_0 - \tau_1}$,

(45) $v_1 v_2 = \dfrac{p_0 - p_1}{\rho_0 - \rho_1}$.

Relation (43) follows directly from (ii). Relation (44) follows from (43) by setting $v_1 = m\tau_1$ and $v_0 = m\tau_0$; relation (45) by setting $mv_1 = \rho_0 v_0 v_1$ and $mv_0 = \rho_1 v_1 v_0$.

The velocities v_0, v_1 and the mass flux m obviously have the same sign. Relation (43) then shows that the pressure p changes in the sense opposite to that of the relative speed $|v|$. Relation (44) or (45) shows that the density changes in the same sense as the pressure.

If the shock is compressive, i.e., $\rho_1 > \rho_0$, (which, as we shall see, is always the case for polytropic gases) the pressure increases and the relative speed $|v|$ decreases as the gas crosses the shock front.

Suppose that for fixed entropy η the sound speed c increases monotonically with the density ρ, and that the pressure $p(\rho,\eta)$ also increases monotonically with ρ and η; then (45) yields

$$v_0 v_1 = \frac{p(\rho_1,\eta_1) - p(\rho_0,\eta_0)}{\rho_1 - \rho_0} > \frac{p(\rho_1,\eta_0) - p(\rho_0,\eta_0)}{\rho_1 - \rho_0} = c^2(\bar{\rho},\eta_0),$$

where $\bar{\rho}$ is a properly chosen intermediate value between ρ_0 and ρ_1. Hence

$$v_0 v_1 > c^2(\rho_0,\eta_0) = c_0^2,$$

c_0 being the sound speed in the thinner medium. Since $v_0 > v_1$ (assuming $m > 0$), we have

$$v_0 > c_0,$$

i.e., the ***gas which has not yet reached the shock front flows with supersonic speed*** relative to the front. (For polytropic gases we shall presently see that ***after passing the shock the flow has subsonic speed*** relative to the front, i.e., $v_1 < c_1$).

35. Cases in which the first two shock conditions are sufficient to determine the shock.

Certain further remarks should be made about the role of the first

two shock conditions, the mechanical conditions, in contrast to that of the third, the thermodynamical condition, which is the more trenchant one. There are cases of great practical importance in which the first two conditions alone together with the **pressure-density relation** (***equation of state***) are sufficient to determine the shock phenomenon. In these cases the third shock relation remains valid, of course, but may be considered merely as a means of determining the energy balance after the problem has been solved. These remarks apply to flow in substances in which the ***pressure depends on the density alone*** and not, or not noticeably, on the entropy.

Water, for example, is approximately such a substance, inasmuch as in its equation of state, $p = A\rho^{\gamma} - B$, the coefficients A and B are approximately independent of the entropy. The internal energy for such substances **splits into two parts**, $e = e_1(\rho) + e_2(\eta)$, one depending only on the density, the other only on the entropy. The third shock condition can then be written

$$[e_2]_{(0)}^{(1)} = -\left[\frac{1}{2}v^2 + e_1 + \frac{p}{\rho}\right]_{(0)}^{(1)}.$$

Since the right-hand side is already determined by the first two shock conditions, $[e_2(\eta)]_{(0)}^{(1)}$ can be calculated and the rise in entropy can thus be found.

These remarks also apply to ***weak shocks*** in any substance, i.e., to shocks for which the excess pressure ratio $(p_1 - p_0)/p_0$ is small. For such weak shocks, as we shall see, the entropy rise is very small, in fact of third order in $(p_1 - p_0)/p_0$, and can therefore be safely neglected (see Art. 37).

The theorems (A) and (B) formulated earlier (Art. 28) are valid for cases where the equation of state $p = p(\rho)$ does not depend on the entropy, provided that $p'(\rho) > 0$ and $\dfrac{d^2p}{d\tau^2} > 0$.

To prove Theorem (A) we first show that if ρ_0 and v_0 are prescribed so that v_0 is supersonic, $|v_0| > \sqrt{p'(\rho_0)}$, then the state (1) is uniquely determined; $v_1\ v_0$ is subsonic, $|v_1| < \sqrt{p'(\rho_1)}$. Since

$\frac{d^2 p}{d\tau^2} > 0$, the ratio $\frac{p - p_0}{\tau_0 - \tau}$ increases with ρ. Therefore the equation $\frac{p - p_0}{\tau_0 - \tau} = m^2$, (see (44)), has one solution ρ_1, $\tau_1 = 1/\rho_1$, $p_1 = p(\rho_1)$. From $v_0^2 > p'(\rho_0)$ we have $m^2 > \rho_0^2 p'(\rho_0)$, which is the value that $\frac{p - p_0}{\tau_0 - \tau}$ approaches as $\rho \to \rho_0$. Hence we conclude $\rho_1 > \rho_0$, $|v_1| < |v_0|$. From $v_1^2 < v_1 v_0 = \frac{p_0 - p_1}{\rho_0 - \rho_1} < p'(\rho_1)$, (see (45)) we see that the state (1) is subsonic.

Theorem (B) is equivalent to the following statement. If ρ_0, u_0 and $\rho_1 > \rho_0$ are prescribed, states (0) and (1) are uniquely determined, provided it is in addition stipulated whether (0) should be to the left or to the right of (1). From (44) we find m^2. If (0) is to the left of (1), $m > 0$. Then $v_0 = \tau_0 m$, $\dot{\xi} = u_0 - v_0$, $v_1 = \tau_1 m$ and $u_1 = v_1 + \dot{\xi}$ are determined.

34. Shock relations derived from the differential equations for viscous and heat-conducting fluids.

It seems appropriate to implement the introductory remarks in Art. 26 by a brief and somewhat more subtle analysis of how the shock conditions may be obtained by a passage to the limit of vanishing coefficients μ of ***viscosity*** and λ of ***heat conduction*** from the differential equations involving these factors.

(α) $\qquad \rho_t + (\rho u)_x = 0$, $\qquad$ (Equation of continuity)

(β) $\qquad \rho u_t + \rho u u_x + p_x - \frac{4}{3}\mu u_{xx} = 0$,

(Equation of motion with viscous friction.)

(γ) $\qquad \rho T \eta_t + \rho u T \eta_x = \frac{4}{3}\mu u_x^2 + (\lambda T_x)_x$.

[The left-hand side is the **heat acquired** by a unit volume per unit time. The second term on the right-hand side measures the contribution due to **heat conduction**, while the first term measures the contribution due to **viscous friction**, which is essentially positive in accordance with the second law of thermodynamics.]

We consider a sudden transition in the neighborhood of a point and, with no restriction of generality, we can refer the phenomenon to a coordinate system in which this point is at rest, say at $x = 0$. For

simplicity, assume that in the neighborhood of $x = 0$ the phenomenon can be considered steady, i.e., that we may set $u_t = \rho_t = \eta_t = 0$. Then we rewrite the equations by combining (α) with (β), and (α) and (β) with (γ) in the form of three **conservation laws**

(α') $\quad (\rho u)_x = 0$,

(β') $\quad \left(\rho u^2 + p - \frac{4}{3}\mu u_x\right)_x = 0$,

(γ') $\quad \left[\rho u(\frac{u^2}{2} + i) - \frac{4}{3}\mu u u_x - \lambda T_x\right]_x = 0$,

where the enthalpy i with the differential

$$di = \frac{dp}{\rho} + T d\eta$$

has been introduced. Relation (γ') represents the ***conservation of energy***.

The possibility of writing all three of our conditions as laws of conservation now leads to the ***shock conditions*** in the following way. We integrate the equations (α'), (β'), (γ') between $-\varepsilon$ and ε, where ε is arbitrarily small, with the result

(α'') $\quad [\rho u]_{-\varepsilon}^{\varepsilon} = 0$,

(β'') $\quad \left[\rho u^2 + p - \frac{4}{3}\mu u_x\right]_{-\varepsilon}^{\varepsilon} = 0$,

(γ'') $\quad \left[\rho u(\frac{u^2}{2} + i) - \frac{4}{3}\mu u u_x - \lambda T_x\right]_{-\varepsilon}^{\varepsilon} = 0$,

in which $[f]_{-\varepsilon}^{\varepsilon}$ denotes the difference $f(\varepsilon) - f(-\varepsilon)$. For varying values of λ and μ with the limit $\lambda \to 0$, $\mu \to 0$, we consider a sequence of flows which are assumed to converge to a limit flow except possibly at the point x =0. Relations (α''), (β''), (γ''), not involving quantities at the point x =0, remain valid in the limit. Thus we obtain for the limit flow

$$[\rho u]_{-\varepsilon}^{\varepsilon} = 0,$$

$$[\rho u^2 + p]_{-\varepsilon}^{\varepsilon} = 0,$$

$$\left[\rho u(\frac{u^2}{2} + i)\right]_{-\varepsilon}^{\varepsilon} = 0.$$

When we now let ε approach zero, we obtain the same shock

conditions that we found earlier.

It is clear from the procedure described that the shock conditions depend essentially on the way the differential equations were modified. If this had been done in a different manner, **different shock conditons** might have resulted. (If, for example, the third equation were $(\eta + \lambda\phi)_x = 0$ instead of (γ'). ϕ being a function of the quantities involved, the third shock relation would have been $[\eta]_{(0}^{(1)} = 0$, as was assumed by **Riemann**). This remark shows clearly that the system of shock conditions is ***not merely a mathematical framework*** essentially inherent in the unmodified differential equations but that it depends profoundly on a ***proper accounting for the finer features of physical reality***.

It is interesting to contrast the described **limit process** with a different one which has always been successfully employed for linear differential equations. One considers a set of continuous solutions of the unmodified differential equations

$$(\rho u)_x = 0\,,\ \ (\rho u^2 + p)_x = 0\,,\ \ \eta_x = 0\,,$$

hypothetically assumed to converge to a limit solution which possibly has a discontinuity at x =0. Since for the continuous solutions the relations

$$[\rho u]_{-\varepsilon}^{\varepsilon} = 0\,,\ \ \left[\rho u^2 + p\right]_{-\varepsilon}^{\varepsilon} = 0\,,\ \ [\eta]_{-\varepsilon}^{\varepsilon} = 0$$

hold, the same is true for the limit solution. Of the system of jump conditions obtained on letting $\varepsilon \to 0$, only the third differs from the shock conditions. Thus a **new type of discontinuity** appears to result across which the changes are adiabatic while the two mechanical conditions remain. Such a reasoning would, however, be **fallacious**. Since the relation $(\frac{1}{2}u^2 + i)_x = 0$ is a consequence of the three above differential equations, the relation $\left[\frac{1}{2}u^2 + i\right]_{-\varepsilon}^{\varepsilon} = 0$ also holds; and consequently the third shock condition also holds for the discontinuities of the new type. The new discontinuities would therefore be shocks in the sense discussed above but **without entropy change**. There are no such discontinuities, since our previous formulas imply changes in all quantities u, p, ρ, and η across the

shock front. This argument shows that **continuous solutions of the unmodified differential equations never can approximate discontinuous solutions** (see the remarks in Art. 1, Chapter I).

35. Critical speed and Prandtl's relation for polytropic gases.

The third or thermodynamical shock condition becomes particularly simple in the case of polytropic gases. Then we have for the enthalpy

$$i = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{1-\mu^2}{2\mu^2 c^2}\text{, with }\mu^2 = \frac{\gamma - 1}{\gamma + 1};$$

hence the condition (iii) becomes

(iii γ) $\quad \mu^2 v_0^2 + (1-\mu^2)c_0^2 = \mu^2 v_1^2 + (1-\mu^2)c_1^2 = c_*^2$,

where c_* is the critical speed (Art. 5, Chapter I). Due to this algebraic form of the third condition, the relations between the various quantities on both sides of the shock front and the velocity $\dot{\xi}$ of the shock front are of a purely algebraic character.

The relation between the relative velocities v_0, v_1 on both sides of the shock can be put in a very elegant and useful form, due to ***Prandtl***, i.e.,

(iiip) $\quad v_0 v_1 = c_*^2$.

This fundamental relation involves velocities only and ***does not refer explicitly to thermodynamic quantities*** such as pressure or density.

To prove **Prandtl's relation** we derive from (ii) of Art. 28, (iii γ) and $(1-\mu^2)\gamma = 1 + \mu^2$ the relations

$$\mu^2 p + p_1 = \mu^2 v_1^2 \rho_1 + (1+\mu^2)p_1 = c_*^2 \rho_1,$$
$$\mu^2 p + p_0 = \mu^2 v_0^2 \rho_0 + (1+\mu^2)p_0 = c_*^2 \rho_0.$$

Subtracting, we find

$$p_1 - p_0 = c_*^2(\rho_1 - \rho_0)$$

or

$$\frac{p_1 - p_0}{\rho_1 - \rho_0} = c_*^2$$

and relation (iiip) follows by (45), Art. 32.

Prandtl's relation is evidently equivalent to the transition formula

(iii'p) $\quad \dfrac{c_*}{v_1} + \dfrac{v_1}{c_*} = \dfrac{c_*}{v_0} + \dfrac{v_0}{c_*}$

($v_1 \neq v_0$ being assumed), which in turn can be proved directly as follows. From the first two shock conditions we infer that

$$\frac{p_1}{\rho_1 v_1} + v_1 = \frac{p_0}{\rho_0 v_0} + v_0 .$$

This and the third condition (iiiγ)

$$\frac{2\gamma}{\gamma-1}\frac{p}{\rho} = \frac{\gamma+1}{\gamma-1}c_*^2 - v^2$$

imply that

$$\frac{p}{\rho v} + v = \frac{\gamma+1}{2\gamma}\frac{c_*^2}{v} - \frac{\gamma-1}{2\gamma}v + v = \frac{\gamma+1}{2\gamma}\left(\frac{c_*^2}{v} + v\right)$$

has the same value on both sides, whence (iii'p) results immediately.

Incidentally, Prandtl's relation exhibits the fact that **if a shock is weak**, i.e., if v_0 is approximately equal to v_1, then the shock is approximately a **sonic disturbance**, for it follows from $v_0 = v_1$ that both have the common value c_*; hence the weak discontinuity progresses approximately with sound speed relative to the gas.

As a first, immediate consequence of **Prandtl's relation** we state that the speed of the gas relative to the shock front is **supersonic on the front side**, **subsonic on the back side** of the shock front.

For, formula (iiip) shows that $|v_0|>|v_1|$ implies $|v_0|>c_*$ and $|v_1|<c_*$, and our assertion follows immediately from the definition of the critical speed c_*.

As a further application we derive important relations between the speed $\dot{\xi}$ of the shock front, the flow velocities u_0 and u_1 on both sides and the sound speed on one side of the shock front.

By substituting $v_i = u_i - \dot{\xi}$ in (iiip) and using the definition of c_* we obtain

$$(46)\qquad \begin{aligned}(u_0-\dot{\xi})(u_1-\dot{\xi}) = c_*^2 &= \mu^2(u_0-\dot{\xi})^2 + (1-\mu^2)c_0^2 \\ &= \mu^2(u_1-\dot{\xi})^2 + (1-\mu^2)c_1^2\end{aligned} .$$

This is a quadratic equation for the shock velocity $\dot{\xi}$ and the state on one side if the velocity u_1 on the other side is given. In particular, assuming that (0) is a state of rest, i.e., $u_0 = 0$, and writing $u_1 = w$, we have for $\dot{\xi}$ the equation

$$(47)\qquad (1-\mu^2)\dot{\xi}^2 - w\dot{\xi} = (1-\mu^2)c_0^2 ,$$

ore, referring to the state (1)

$$(48)\qquad (1-\mu^2)(\dot{\xi}-w)^2 + w(\dot{\xi}-w) = (1-\mu^2)c_1^2 ,$$

relations that will soon prove useful.

36. Relations referring to the strength of a shock for polytropic gases.

It is convenient for various considerations to introduce the notion of shock strength. Several parameters offer themselves as a measure for the strength of a shock:

the **excess pressure ratio** $\dfrac{p-p_0}{p_0}$,

the **condensation** $\dfrac{\rho_1-\rho_0}{\rho_0}$,

the parameter M_0^2-1,

$M_0 = v_0 / c_0$ being the Mach number of the incoming flow relative to the shock front. We write down the relations between these quantities for polytropic gases.

As stated before, (iii**), Art. 28, the third shock condition can be expressed in the form

$$(\tau_0-\tau_1)\frac{p_0+p_1}{2} = e_1 - e_0,$$

which, for polytropic eases with $e = \dfrac{1}{\gamma-1}\dfrac{p}{\rho} = \dfrac{1-\mu^2}{2\mu^2}\tau p$, assumes the form $(\tau_0 - \mu h 2\tau_1)p_0 = (\tau_1 - \mu h 2\tau_0)p_1$. This yields the important formula

(49) $$\frac{p_1}{p_0} = \frac{\rho_1-\mu^2\rho_0}{\rho_0-\mu^2\rho_1},$$

which is equivalent to the relation

(59) $$\frac{p_1-p_0}{p_0} = \frac{1+\mu^2}{1-\mu^2\dfrac{\rho_1}{\rho_0}}\frac{\rho_1-\rho_0}{\rho_0} = \frac{\gamma}{\left(\dfrac{\rho_1-\rho_0}{\rho_0}\right)^{-1} - \dfrac{\gamma-1}{2}},$$

connecting the excess pressure ratio with the condensation **without involving velocities**.

Inversely we have

(51) $$\frac{\rho_1}{\rho_0} = \frac{p_1+\mu^2 p_0}{p_0+\mu^2 p_1},$$

or

(52) $$\frac{\rho_1-\rho_0}{\rho_0} = \frac{1-\mu^2}{1+\mu^2\dfrac{p_1}{p_0}}\frac{p_1-p_0}{p_0} = \frac{1}{\gamma\left(\dfrac{p_1-p_0}{p_0}\right)^{-1} + \dfrac{\gamma-1}{2}}.$$

We note that the condensation approaches the finite value $(1-\mu^2)/\mu^2 = 2/(\gamma-1)$ when the excess pressure ratio becomes infinite.

Relation (51) shows that the compression ρ_1/ρ_0 is always

restricted to the range

(53) $$\mu^2 < \frac{\rho_1}{\rho_0} < \frac{1}{\mu^2};$$

so that the compression is never more than μ^{-2}-fold. For $\gamma = 1.4$ the density compression is therefore always less than 6-fold and for $\gamma = 1.2$ the limit of compression is 11-fold. For later purposes we note

(54) $$\frac{\tau_0 - \tau_1}{\tau_0} = \frac{1-\mu^2}{\frac{p_1}{p_0} + \mu^2} \frac{p_1 - p_0}{p_0}$$

which follows from (52).

The relation between the Mach number and the **excess pressure ratio** is particularly simple. To derive it we substitute v_0 / v_1 for ρ_1 / ρ_0 in relation (49) and eliminate v_1 by Prandtl's relation $v_0 v_1 = c_*^2$. Thus we obtain the relation

$$\frac{p_1}{p_0} = \frac{v_0^2 - \mu^2 c_*^2}{c_*^2 - \mu^2 v_0^2}$$

from which, by $c_*^2 = \mu^2 v_0^2 + (1-\mu^2) c_0^2$, we obtain

(55) $$\frac{p_1}{p_0} = (1+\mu^2) M_0^2 - \mu^2,$$

or

(56) $$\frac{p_1 - p_0}{p_0} = (1+\mu^2)(M_0^2 - 1).$$

37. Change in entropy.

For poly tropic gases the change in entropy $\Delta\eta$ across a shock is obtained from the expression for η given in I(5), Art 2. We find, with c_τ denoting the specific heat at constant volume, that

(57) $$\Delta\eta = \eta_1 - \eta_0 = \frac{1}{c_\tau} \log\left(\frac{p_1}{\rho_1^\gamma} \Big/ \frac{p_0}{\rho_0^\gamma} \right),$$

so that, by (49),

(58) $$\Delta\eta = \frac{1}{c_\tau}\left[-\gamma \log\frac{\rho_1}{\rho_0} + \log\left(\frac{\rho_1}{\rho_0} - \mu^2 \right) - \log\left(1 - \mu^2 \frac{\rho_1}{\rho_0} \right) \right].$$

Furthermore, for the **ratio of absolute temperatures** we have

(59) $$\frac{T_1}{T_0} = \frac{p_1 / p_0}{\rho_1 / \rho_0} = \frac{\rho_1/\rho_0 - \mu^2}{\rho_1/\rho_0 (1 - \mu^2 \cdot \rho_1/\rho_0)}.$$

With the aid of these relations we can characterize the

thermodynamic changes which occur across a shock front. The equivalence of $\rho_1 > \rho_0$ and $p_1 > p_0$ follows immediately from (49) and (51). Each of these inequalities is then equivalent to $T_1 > T_0$ by (59). Now

(60)
$$\frac{d(\Delta\eta)}{d\left(\frac{\rho_1}{\rho_0}-1\right)} = \frac{1}{c_\tau}\left[-\frac{\gamma}{\frac{\rho_1}{\rho_0}} + \frac{1}{\frac{\rho_1}{\rho_0}-\mu^2} + \frac{\mu^2}{1-\mu^2\frac{\rho_1}{\rho_0}}\right] = \frac{\gamma\mu^2}{c_\tau}\frac{\left(\frac{\rho_1}{\rho_0}-1\right)^2}{\frac{\rho_1}{\rho_0}(\frac{\rho_1}{\rho_0}-\mu^2)(1-\mu^2\frac{\rho_1}{\rho_0})}.$$

Since $\mu^2 < 1$ and since by (53) $\mu^2\frac{\rho_1}{\rho_0} < 1$, this derivative is positive if $\rho_1/\rho_0 \neq 1$, i.e., $\rho_1 \neq \rho_0$. Furthermore, for $\rho_1/\rho_0 = 1$, $\Delta\eta = 0$. The change in entropy $\Delta\eta = \eta_1 - \eta_0$ is therefore positive for $\rho_1/\rho_0 - 1 > 0$ and negative for $\rho_1/\rho_0 - 1 < 0$. Thus we have established that for polytropic gases any one of the inequalities $\rho_1 > \rho_0$, $p_1 > p_0$, $T_1 > T_0$, $\eta_1 > \eta_0$ **implies all the others**. Thus for any shock in a polytropic gas all the quantities change monotonically with ρ_1/ρ_0. As stated before, shocks in polytropic gases are always **compressive**; upon crossing the shock front the gas acquires higher pressure, temperature, density and entropy. This follows from the ***second law of thermodynamics***, which stipulates that the entropy increases from the front side to the back side of the shock front.

A point of great importance is the following. The **change in entropy** across a shock front is only of the **third order** in the shock strength (i.e., in any of the quantities introduced to measure the shock strength in Art. 36). Hence, for weak shocks the jump in entropy is very small and may be neglected. Accordingly, we may treat a weak shock as an adiabatic change, and need consider only the first two shock conditions, as Riemann did in his incomplete theory (see Art. 33).

That the change in entropy $\Delta\eta$ is of **third order** in small condensation $(\rho_1 - \rho_0)/\rho_0$ can be seen immediately from (60) which shows that the derivative of $\Delta\eta$ is of **second order** in the condensation. Indeed, integration of (60) with respect to $\rho_1/\rho_0 - 1$ leads to

$$(61) \qquad \Delta\eta \cong \frac{1}{c_\tau}\frac{\gamma(\gamma^2-1)}{12}\left(\frac{\rho_1-\rho_0}{\rho_0}\right)^3,$$

where terms of higher order than three in the condensation are neglected. Using (52), this can be written in terms of the excess pressure ratio as

$$(62) \qquad \Delta\eta \cong \frac{1}{c_\tau}\frac{\gamma^2-1}{12\gamma^2}\left(\frac{p_1}{p_0}-1\right)^3,$$

again neglecting terms of order higher than three. For shocks in water the condensation is small enough to make the assumption $\Delta\eta = 0$ reasonably valid, except when the shock is excessively strong. For gases the assumption is still justified when the shock is moderately strong.

38. The state on one side of the shock front determined by the state on the other side.

The variety of relations derived for shock transitions in polytropic gases leads to simple schemes for the calculation of the state (1) when the state (0) and one additional quantity are given, and thus incidentally to a proof of the theorems A and B stated in Art. 28.

(A) Given $p_0, \rho_0, u_0, \dot{\xi}$.

We find first

$$v_0 = u_0 - \dot{\xi}, \quad c_0^1 = \gamma\frac{p_0}{\rho_0};$$

then

$$c_*^2 = \mu^2 v_0^2 + (1-\mu^2)c_0^2,$$

whereupon

$$v_1 = \frac{c_*^2}{v_0}, \quad \rho_1 = \frac{\rho_0 v_0}{v_1}$$

and

$$p_1 = p_0\left\{(1+\mu^2)\frac{v_0^2}{c_0^2} - \mu h2\right\}, \text{ (see (55))},$$

or

$$p_1 = p_0 + \rho_0(v_0^2 - c_*^2).$$

(B) Given $p_0, \rho_0, u_0, p_1 > p_0$.

We first find

$$\rho_1 = \rho_0\frac{\mu^2 p_0 + p_1}{\mu^2 p_1 + p_0}, \text{ (see (51))},$$

then

$$m^2 = -\frac{p_0 - p_1}{\tau_0 - \tau_1}, \text{ (see (44))},$$

and choose $m > 0$ if (0) is t the left of (1); whereupon

$$v_0 = \frac{m}{\rho_0} \quad \text{and} \quad v_1 = \frac{m}{\rho_1},$$

and furthermore,

$$\dot{\xi} = u_0 - v_0, \quad u_1 = v_1 + \dot{\xi}.$$

As a check in computation one can use

$$c_*^2 = \mu^2 v_0^2 + (1-\mu^2)\gamma \frac{p_0}{\rho_0} = \mu h 2 v_1^2 + (1-\mu^2)\gamma \frac{p_1}{\rho_1} = \frac{p_0 - p_1}{\rho_0 - \rho_1}.$$

If the state (0) and the velocity u_1 are given, a similar procedure holds. (See details contained in Art. 33).

39. Geometric representations of shock transitions for polytropic gases.

Prandtl and Busemann have given a suggestive **geometric interpretation** of the conditions for a stationary shock, i.e., of the algebraic relations connecting the state (0) with the state (1) adjacent to the two sides of the shock discontinuity. This representation, briefly described and supplemented here, is not restricted to a polytropic gas.

It is based on diagrams showing the dependence of the pressure p on the flow velocity u, the latter being considered as the independent variable, the quantities η and $\hat{q}$ being kept fixed; this relation is given by a curve $p = p(u)$.

[Explicitly, for a polytropic gas with $p = A(\eta)\rho^\gamma$, we have

(63) $$p = A^{1/(1-\gamma)} \left[\frac{\gamma - 1}{2\gamma} (\hat{q}^2 - u^2) \right]^{\gamma/(\gamma-1)}.]$$

If we change the value of the entropy η, keeping the value $\hat{q}$ of the limit speed fixed, we obtain a one-parametric family of such p,u-curves. These curves do not intersect; in particular, p decreases when the entropy η increases while u is fixed. This is clear from the relation $\frac{\partial}{\partial \eta} p(u,\eta) = -\rho T < 0$, which follows from $\tau dp + T d\eta = di = -u du$.

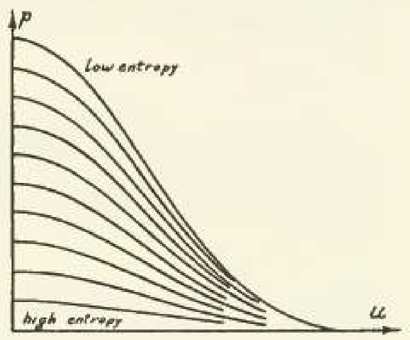


Figure 24
Pressure as a function of flow velocity u for the same limit speed and various constant values of the entropy.

Instead of u we can consider any one of the quantities i, ρ, p as independent variable in our one-parametric family of states with the same value of $\hat{q}$. Thus, from $c^2 = p_\rho$, $i = (1/2)(\hat{q}^2 - u^2)$, we find

$i_u = -u$ and from $i_\rho = \frac{p_\rho}{\rho} = \frac{c^2}{\rho}$ we have $\rho_u = -\frac{\rho u}{c^2}$, hence $p_u = -\rho u = -m$ and $p_{uu} = -\rho\left(1 - \frac{u^2}{c^2}\right)$. The p,u-curves therefore have negative curvature, i.e., are concave toward the u-axis, for subsonic speed, |u| < c, and are convex toward the u-axis for supersonic speed; hence, $u = \pm c$ are points of inflection. As was shown in Art. 5, Chapter I, under the assumption made ($p_{\tau\tau} > 0$) there is just one value c_* (this value for polytropic gases, $c_* = \mu\hat{q}$, being independent of the entropy) such that |u| < c for |u| .< c_* and |u| > c for |u| > c_*. Consequently $p_{uu} < 0$ for |u| < c_* and $p_{uu} > 0$ for |u| > c_*.

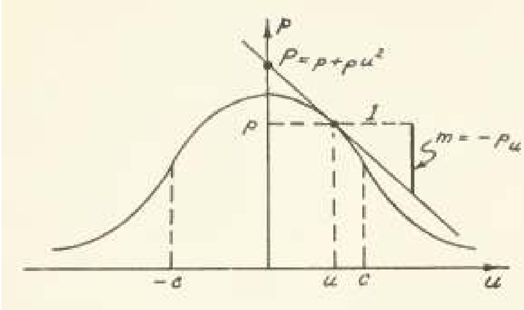


Figure 25
Graphical representation of
$m = \rho u = -p_u$ and $P = p + \rho u^2$.

From $p_u = -\rho u$ the negative slope of our curve is seen to represent the flux crossing a unit section per unit time. The intercept of the tangent on the p-axis is therefore equal to $p + \rho u^2 = P$, a quantity appearing in the second shock condition (ii").

It is now easy to represent a shock transition graphically. Let us suppose that the shock front faces the left side (denoted by the subscript 0); then the flux will come from the left, i.e., we shall have $u_0 > 0$.

The third shock condition (iii") simply states that for the state (0) and the state (1) the limit speed $\hat{q}$ on both sides of the shock is the same. In our graphical representation this condition is accounted for by considering p,u-curves of the family corresponding to the same value of the limit speed $\hat{q}$ with the entropy as parameter. The two states p_0, u_0 and p_1, u_1 on the two sides of the shock front will now be represented by two points on two different curves of the family. Since $\rho u|_{u=u_0} = m = \rho u|_{u=u_1}$, the two tangents at these points are parallel, and by virtue of the shock condition (ii") the two intercepts on the p-axis are equal. Hence the two tangents are identical. Any two states p_0, u_0 and p_1, u_1, which can be connected by a stationary shock are therefore represented by the two points 0 and 1 of contact of a common tangent line to two curves of a family belonging to the same value of the limit speed $\hat{q}$.

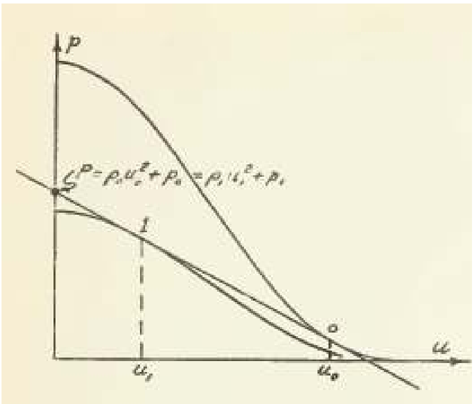


Figure 26
Representation of
shock transition
in p,u-diagram.

From the figure it is clear that of the two velocities u_0, u_1 one must be supersonic, the other subsonic, since at one point $p_{uu} > 0$ and at the other $p_{uu} < 0$. Since $p_\eta < 0$ or, in other words, the

p,u-curve with greater entropy is below the other curve, it is clear that the subsonic state has greater entropy than the supersonic one.

It also follows from a continuity argument that to every supersonic state (0) there is a state (1) which can be connected with (0) by a shock.

The preceding representation of shock transitions is not the only possible or reasonable one. Perhaps the following one, obtained by a Legendre transformation, might prove just as useful. We introduce the flux

$$m = -p_u = \rho u$$

as independent and the expression

$$P = p + mu$$

as dependent variable, and we consider the function $P(m)$. In performing this transformation from p,u to P,m we must realize that by $p_{uu} \neq 0$ for $u \neq c_*$ we are assured of the feasibility of the transformation separately for $0 < |u| < c_*$ and for $c_* < |u| < \hat{q}$. Thus we obtain two branches for $P(m)$, a "***lower***" branch, corresponding to $|u| > c_*$, with $P(0) = 0$, and an "***upper***" branch, corresponding to $|u| < c_*$, with $P(0) = P^0$. For different entropies the functions $P(m)$

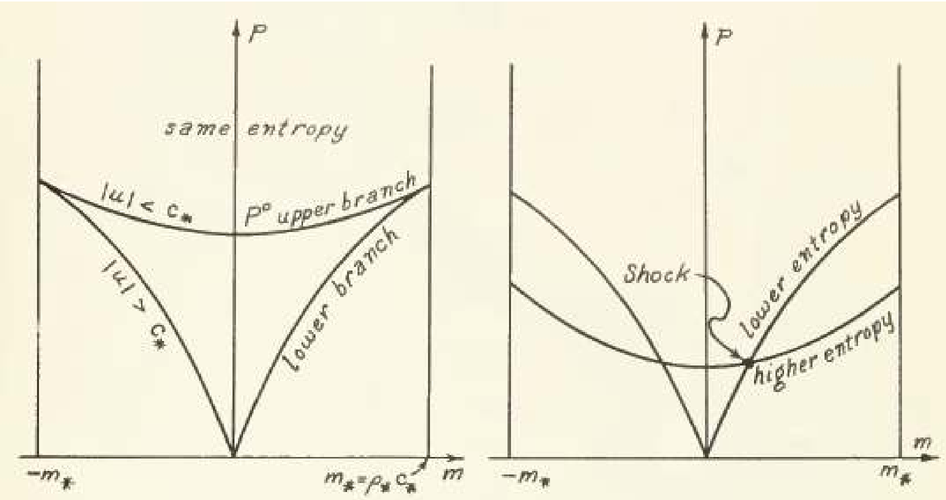


Figure 27

differ only by a constant factor. The lower branches for different entropies do not intersect each other, and the same holds for the upper branches for different entropies. However, it can be seen that through every point on a lower branch one can pass an upper branch connected with a higher entropy. Such an intersection of lower and upper branches for different entropies corresponds to a ***shock transition***, as is obvious from our shock relations.

40. Shock conditions in Lagrangean representation.

A remark concerning the **Lagrangean form of the shock relations** will be useful later (see Art. 7, Chapter I for definitions and notations). If $x(t)$ is the coordinate of a moving particle, $x_0(t)$ referring to a specific "zero"-particle, then any particle is fixed (irrespective of the time) by the Lagrangean coordinate $h=\int_{x_0}^{x}\rho dx$.

With h and t as independent, u and $\tau = 1/\rho$ sa dependent variables, the differential equations are (see I(39), Art. 7)

$$\tau_t = u_h,\ \ u_t = k^2\tau_h, \text{ with } \ k=\frac{c}{\tau}=\rho c,$$

and $x_h=\tau$, $x_t = u$. Now let us consider a shock front S moving relative to the gas, enveloping at the time t a particle with the Lagrangean coordinate $h = h(t)$. Then if $x(h,t)$ is the position of the particle with the coordinates h, t, the position of the shock front is given by

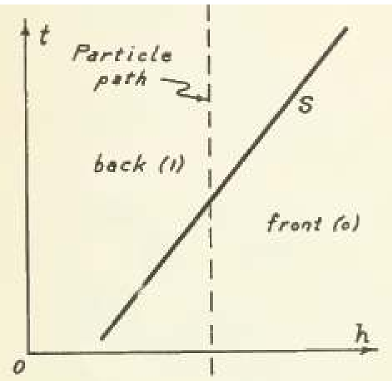


Figure 28
Motion of shock front
in Lagrangean
representation.

$$\xi = x(h(t),t),$$

and the shock velocity is

$$\dot\xi = \tau\dot h + u.$$

With the abbreviations $\tau_1-\tau_0=[\tau]$, $u_1-u_0=[u]$ we immediately obtain the (kinematic) shock condition

(iL) $\qquad \dot h[\tau]+[u]=0,$

which replaces the (automatically satisfied) condition of ***conservation of mass***. We note that $-\dot h$ is the mass crossing the shock front in unit time from the front side to the back side. Consequently, the ***conservation of momentum*** is expressed by

(iiL) $\qquad [p]-\dot h[u]=0,$

which by (iL) in a form invariant under translatory motion is

$$[p]-\dot h^2[\tau]=0,$$

while the ***conservation of energy***, since $v=u-\dot\xi=-\tau\dot h$, is expressed by

$$\left[\frac{1}{2}(u-\dot\xi)^2+i\right]=0,$$

or

(iiiL) $\qquad \frac{1}{2}\dot h^2[\tau^2]+[i]=0,$

the symbol $[f]$ always denoting f_1-f_0.

41. Shock in a uniform compressive motion.

The simplest basic instance of a shock transition is that between two constant states. In Art. 27 we have already given a qualitative description of the typical piston motion producing such a phenomenon. We now supply the quantitative details and proof of the mathematical consistency of our previous description. We are concerned with the problem of a piston moving with the constant velocity w into a zone (0) of rest ($u_0 = 0$) where sound speed is c_0. Then the situation was described as follows. The piston will be preceded by a shock front S moving at supersonic speed $\dot{\xi} > c_0$ into the zone (0) of quiet. Between the impinging piston and the shock we have an ever-widening zone of those particles of the gas which the shock has suddenly accelerated from rest to the piston velocity w and which then continue to move with that velocity.

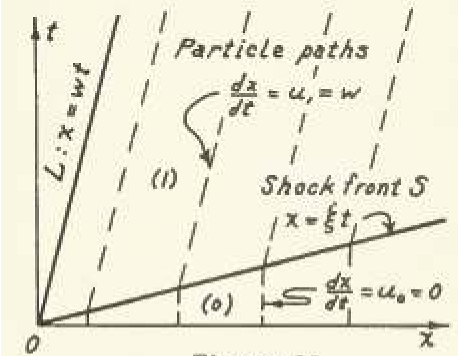


Figure 29
Shock resulting from compressive action of piston moving at a constant velocity.

To substantiate this description, we shall fit it into the framework of the preceding theory by assuming the situation described and then determining the state (1) and the shock velocity $\dot{\xi}$. Since $u_1 = w$, $u = 0$, we find $\dot{\xi}$ from equation (47), obtained in Art. 35,

$$\dot{\xi}^2 - \frac{w}{1-\mu^2}\dot{\xi} - c_0^2 = 0 .$$

The roots of this quadratic equation are

$$(64) \qquad \begin{cases} \dot{\xi}_+ = \dfrac{1}{2}\dfrac{w}{1-\mu^2} + \sqrt{c_0^2 + \dfrac{1}{4}\left(\dfrac{w}{1-\mu^2}\right)^2} \\ \dot{\xi}_- = \dfrac{1}{2}\dfrac{w}{1-\mu^2} - \sqrt{c_0^2 + \dfrac{1}{4}\left(\dfrac{w}{1-\mu^2}\right)^2} \end{cases},$$

of which only the first is positive and corresponds to the situation we are considering. (The physical meaning of the negative root will appear in the next article).

Clearly, the shock velocity $\dot{\xi} = \dot{\xi}_+$ is greater than c_0 and greater than $\dfrac{w}{1-\mu^2}$. The latter observation shows, for example, that for air with $\gamma = 1.4$, $\mu^2 = 1/6$, the shock is at least 20% faster than the piston or the oncoming column of gas.

With the shock velocity thus determined, the description of the basic compressive piston motion is shown to be consistent. Although we have given no proof of **uniqueness**, i.e., we have not

mathematically excluded the possibility of other flow patterns, we accept the preceding reasoning as a **satisfactory theory** for the interpretation of actual phenomena observed under circumstances resembling our idealized model. Having obtained $\dot{\xi}$, we find by the procedure A of Art. 38 the pressure p_1 and the density ρ_1 in the zone adjacent to the piston. For high speed w of the oncoming piston (or gas), i.e., for $w/c_0 >> 1$, we have, by (64) and (55), (51) of Art. 36,

(65) $$\dot{\xi} \sim \frac{w}{1-\mu^2},$$

(66) $$\frac{p_1}{p_0} \sim \frac{1+\mu^2}{(1-\mu^2)^2}\frac{w^2}{c_0^2},$$

(67) $$\frac{\rho_1}{\rho_0} = \frac{1}{\mu^2}.$$

42. Reflection of a shock on a rigid wall.

We shall now discuss a point of great importance, the reflection of a shock. Suppose an oncoming column of gas of constant velocity w behind a shock front impinges on a zone of quiet bounded by a rigid wall. Then the ensuing physical phenomenon can be described as a reflection of the shock wave on the wall, and can be represented mathematically by piecewise constant solutions of the differential equations, satisfying the shock conditions across the **incident** shock wave and the **reflected** shock wave. Under the impact of the incident shock wave the zone (0) of quiet next to the wall will shrink to zero, say at t =0; then a reflected shock will start in the opposite direction and in turn will leave a growing zone of quiet between itself and the wall. The situation can best be grasped from a diagram in the x,t-plane. State (0) is a zone of quiet characterized by the quantities $u_0 = 0$, ρ_0, p_0, c_0. In the state (1) following the incident shock we have $u_1 = w$; in the state (2) adjacent to the wall we again have rest, $u_2 = 0$, but new values ρ_2, p_2, c_2. Our aim is to find the state (2) from the data ρ_0, p_0, w.

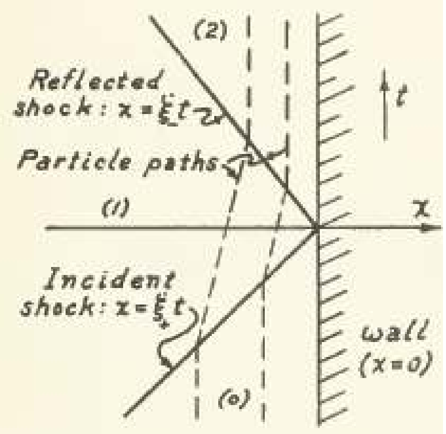


Figure 30
Reflection of
shock wave on
a rigid wall.

To this end we note that the pattern tentatively assumed in Fig. 30 shows a state (1) with flow velocity w and sound speed c_1 connected through a shock with a zone of rest (0) and through another shock with a zone of rest (2). $\dot{\xi}_+$ is the velocity of the incident, $\dot{\xi}_-$ the velocity of the reflected shock; then according to equation (48) of Art.

35, both these velocities must satisfy the same quadratic equation

$$(\dot{\xi}-w)^2+\frac{(\dot{\xi}-w)w}{1-\mu^2}-c_1^2=0 .$$

or the two numbers $M_+=(w-\dot{\xi}_+)/c_1$ and $M_-=(w-\dot{\xi}_-)/c_1$ satisfy the quadratic equation

$$M^2-\frac{w}{(1-\mu^2)c_1}M-1=0 ,$$

so that

$$M_+M_-=-1 .$$

Moreover, the pressure relations following from (55), Art. 36, are

$$\frac{p_0}{p_1}=(1+\mu^2)M_+^2-\mu^2 , \quad \frac{p_2}{p_1}=(1+\mu^2)M_-^2-\mu^2 ,$$

and using $M_+M_-=-1$, we obtain for the **reflected pressure ratio**

$$\frac{p_2}{p_1}=\frac{(2\mu^2+1)\dfrac{p_1}{p_0}-\mu^2}{\mu^2\dfrac{p_1}{p_0}+1} , \tag{68}$$

and for the **excess reflected pressure ratio**

$$\frac{p_2}{p_1}-1=\frac{1+\mu^2}{1+\mu^2\dfrac{p_1}{p_0}}\left(\frac{p_1}{p_0}-1\right). \tag{69}$$

This is the basic relation for the important phenomenon of **reflection**. While in linear wave motion the excess pressure after reflection is simply doubled, we find here a totally different situation. In particular, suppose we have a strong incident shock, i.e., one for which the ratio p_1/p_0 is large. We then find

$$\frac{p_2}{p_1}\sim 2+\frac{1}{\mu^2}=\begin{cases} 8\_(\gamma=1.4) \\ 13\_(\gamma=1.2) \\ 23\_(\gamma=1.1) \end{cases} . \tag{70}$$

Thus, reflection of **strong shocks** results in **considerable increase of pressure** at the wall, a fact obviously of major importance.

For a **weak incident shock** p_1/p_0-1 is small and we find from (69)

$$\frac{p_2}{p_1}\sim\frac{p_1}{p_0} ,$$

in agreement with the well-known facts of ***sonic reflection***.

43. Non-uniform shocks.

In the motion just discussed the situation was greatly simplified by the assumption that the shock establishes the transition from one **constant** state to another, implying a constant speed and strength of the shock wave. In the *x,t*-plane such a shock wave is represented by a straight "**shock line**" whose slope with respect to the *t*-axis is the shock velocity $\dot{\xi}$. Frequently, however, the states on the two sides of the shock front cannot both be considered constant, but are described by more complicated solutions of the differential equations. Moreover, the shock wave will not have a constant velocity, that is, the shock line in the *x,t*-plane will be **curved**. For such shocks the entropy change will in general also vary. Hence, even if the state in front of the shock is of uniform entropy, the gas, after passing the shock front, will no longer have the same entropy throughout. Then we are forced to use the more general differential equations I(14) , (15), (16) , Art. 3, and this is a mathematical complication which has so far precluded any complete theory, though calculations in specific cases are feasible. Fortunately, in many cases of practical importance, the changes in entropy may be neglected with good justification (see Art. 33), and a numerical treatment of the problem becomes more feasible. In such cases we can use the simpler differential equations assuming adiabatic processes, and operate solely with the first two shock conditions disregarding the third, using the latter only for determining thermodynamical quantities after completing the solution.

The motion under the influence of a piston moving at accelerated velocity into quiet gas in a semi-infinite tube exhibits typical features of phenomena involving non-constant shocks. In an earlier discussion (Art. 24) we saw that a simple wave, represented in the *x,t*-plane as in Fig. 31 by a few characteristics, starts from the piston curve L: $x = f(t)$ and moves into the gas. We noticed that if the piston is accelerated, that is, if $d^2 f / dt^2 > 0$, or if $u = \dot{x}$ increases monotonically along L, then the forward characteristics C_+ starting at the piston and sweeping the domain (II) of the simple wave, have monotonically decreasing slopes dx/dt and in general have an envelope (see the examples in Art. 25). Certainly the simple wave (II) cannot extend beyond such an envelope. We can expect the following situation to develop. From the piston curve L a simple wave (II) moves forward into the zone of quiet (I) of the gas. The envelope E of the characteristics may start after a while at a point A (in Fig. 32 this point

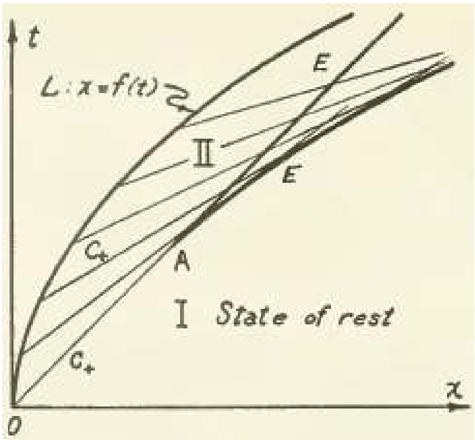


Figure 31
Simple (compression) wave and envelope of characteristics C_+ which sweep it.

is shown on the characteristic C_+, $x = c_0 t$, through the origin, though it is not necessarily situated there). Then a shock line S will begin at A. Through A we draw a characteristic C_- backwards until it meets the piston curve L at B. The curved triangle OAB represents the simple wave (II), which is completely determined by our previous theory. The shock curve S is unknown except that it lies in the region bounded by the lower and upper branches of the envelope formed by the characteristics C_+ issuing from L. This can be inferred from the facts that the converging characteristics C_+ must be cut off before they form an envelope, behind which the state would be ambiguous. Note that the notion of S is subsonic as seen from (III) and supersonic as seen from (I). Below S and OA we have a zone of quiet (I). It is the domain formed by L, AB and S which causes the deeper difficulties in the theory. Since the shock line S is not straight, the shock impresses a different entropy on the different elements of the gas crossing S from the state of rest (I) into the zone (III). In this zone, therefore, the more general differential equations I(14) , (15) , (16) , Art. 3, are to be used. It should again be emphasized that the problem is simplified considerably whenever changes of entropy across S are negligible.

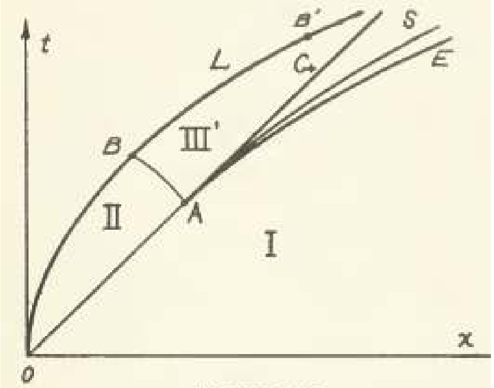


Figure 32
Compressive action of piston and resulting shock.

The shock curve S can be determined according to the following consideration. After passing the shock S from (I), we obtain definite initial values of u, ρ a and the entropy η by means of the shock relations. With these initial values we solve the differential equations I(14), (15), (16), Art. 3, thereby determining u, ρ, η in a zone (III). Now the shock line S is chosen in such a way that on the arc BB' of the piston curve L this solution has the values u prescribed by the velocity of the piston.

This is a very complicated initial-boundary value problem with an unknown boundary, and no general theoretical treatment seems possible. The reverse process, however, can be carried out more easily. Assume that we have a shock line S and determine the initial values on the other side of S according to the shock conditions. Then solve the initial value problem and find the corresponding piston motion as the motion of the flow through B. By carrying out such relatively simple computations for a suitable variety of assumed shock lines S, an assortment of flow patterns can be obtained from which one can choose the one most closely representing a given piston motion.

While determining the shock line S from a given piston motion is a

difficult task, it is at least possible to analyse mathematically the very beginning of the shock, i.e., the line S in the immediate vicinity of A. This problem has been attacked by Hadamard and more recently in an improved way by Calkin.

It should be kept in mind that the shock is weak (sonic) at the beginning, i.e., starts with a pressure ratio $p_1 / p_0 = 1$ at the tip A. Only after the shock line S has bent away from the characteristic direction (which represents sonic disturbances) will the shock become stronger, i.e., p_1 / p_0 will increase.