ON THE PARTIAL DIFFERENCE EQUATIONS
OP MATHEMATICAL PHYSICS

by

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Introduction

Problems involving the classical linear partial differential equations of mathematical physics can be reduced to algebraic ones of a very much simpler structure by replacing the differentials by **difference quotients** on some (say rectilinear) mesh. This paper will undertake an elementary discussion of these algebraic problems, in particular of the behavior of the solution as the mesh width tends to zero. For present purposes we limit ourselves mainly to simple but typical cases, and treat them in such a way that the applicability of the method to more general difference equations and to those with arbitrarily many independent variables is made clear.

Corresponding to the correctly posed problems for partial differential equations we will treat boundary value and eigenvalue problems for **elliptic difference equations**, and initial value problems for the **hyperbolic or parabolic cases**. We will show by typical examples that the passage to the limit is Indeed possible, i.e., that the solution of the difference equation converges to the solution of the corresponding differential equation; in fact we will find that for elliptic equation in general a difference quotient of arbitrarily high order tends to the corresponding derivative. Nowhere do we assume the existence of the solution to the differential equation problem — on the contrary, we obtain a simple existence proof by using the limiting process.[1] For the case of elliptic equations convergence is obtained independently of the choice of mesh, but we will find that for the case of the initial value problem for hyperbolic equations, convergence is obtained only if the ratio of the mesh widths in different directions satisfies certain inequalities which in turn depend on the position of the characteristics relative to the mesh.

We take as a typical case the boundary value problem of potential theory. Its solution and its relation to the solution of the corresponding difference equation has been extensively treated during the past few years.[2] However in

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[1] Our method of proof may be extended without difficulty to cover boundary value and eigenvalue problems for arbitrary linear elliptic differential equations and initial value problems for arbitrary linear hyperbolic differential equations.

[2] J. leRoux, "Sur le problem de Dirichlet", Journ. de mathem. pur. et appl. (6) 10 (1914), p. 189. R.G. D. Richardson, "A new method in boundary problems for differential equations", Trans. of the Am. Math. Soc. 18 (1917), p. 489ff. H. B. Philips and N. Wiener, "Nets and the Dirichlet Problem", Publ. of M. I. T. (1925).
Unfortunately these papers were not known by the first of the three authors when he prepared his note "On the theory of partial difference equations", Gött. Nachr. 23, X, 1925, on which the present work depends.
See also L. Lusternik, "On an application of the direct method in variation calculus", Recuell dela Société Mathém. de Moscou, 1926. G. Bouligand, "Sur le problème de Dirichlet," Ann. de la soc. polon. de mathém. 4, Cracow 1926.
On the meaning of the difference expressions and on further applications of them, see R. Courant, "Uber dlrekte Methoden in der Variationsrechnung", Math. Annalen 97, p. 711 > and the references given therein.

contrast to the present paper, the previous work has involved the use of quite special characteristics of the potential equation so that the applicability of the method used there to other problems has not been immediately evident.

In addition to the main part of the paper, we append an elementary algebraic discussion of the connection of the boundary value problem of elliptic equations with the random walk problem arising in statistics.

Part I. The Elliptic Case

§1. Preliminary Remarks

1. Definitions

Consider a rectangular array of points in the (x, y) -plane, such that for mesh width $h > 0$ the points of the lattice are given by

$$\left.\begin{matrix} x = nh \\ y = mh \end{matrix}\right\} \quad m, n = 0, \pm 1, \ \pm 2, \ldots$$

Let G be a region of the plane bounded by a continuous closed curve which has no double points. Then the corresponding mesh region, G_h — which is uniquely determined for sufficiently small mesh width — consists of all those mesh points lying in G which can be connected to any other given point in G by a connected chain of mesh points. By a **connected chain of mesh points** we mean a sequence of points such that each point follows in the sequence one of its four neighboring points. We denote as a **boundary point** of G_h a point whose four neighboring points do not all belong to G_h. All other points of G_h we call **interior points**.

We shall consider functions u, v,… of position on the grid, i.e., functions which are defined only for grid points, but we shall denote them as $u(x,y)$, $v(x,y)$,…. For their forward and backward difference quotients we employ the following notation,

$$u_x = \frac{1}{h}[u(x+h,y) - u(x,y)], \quad u_y = \frac{1}{h}[u(x,y+h) - u(x,y)],$$

$$u_{\bar{x}} = \frac{1}{h}[u(x,y) - u(x-h,y)], \quad u_{\bar{y}} = \frac{1}{h}[u(x,y) - u(x,y-h)].$$

In the same way the difference quotients of higher order are formed, e.g.,

$$(u_x)_{\bar{x}} = u_{x\bar{x}} = u_{\bar{x}x} = \frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)\}, \text{ etc.}$$

2. Difference Expressions and Green's Function

In order to study linear difference expressions of second order, we form (using as a model the theory of partial differential equations), a bilinear expression from the forward difference quotients of two functions, u and v,

$B(u,v) = au_x v_x + bu_x v_y + cu_y v_x + du_y v_y + \alpha u_x v + \beta u_y v + \gamma u v_x + \delta u v_y + guv$

where

$$a = a(x,y), \ldots, \quad \alpha = \alpha(x,y), \ldots, g = g(x,y)$$

are functions on the mesh.

From the bilinear expression of first order we derive a difference expression of second order in the following way: we form the sum

$$h^2 \sum_{G_h}\sum B(u,v)$$

over all points of a region G_h. In the mesh, where in $B(u, v)$ the difference quotients between boundary points and points not belonging to the mesh are to be set equal to zero. We now transform the sum through partial summation, i.e., we arrange the sum according to v, and split it up into a sum over the set of interior points, G'_h and a sum over the set of boundary points, Γ_h. Thus we obtain:

(1) $$h^2 \sum_{G_h}\sum B(u,v) = -h^2 \sum_{G'_h}\sum vL(u) - h\sum_{\Gamma_h} vR(u) .$$

$L(u)$ is a linear difference expression of second order defined for all interior points of C_n:

$$\begin{aligned} L(u) = (au_x)_{\bar{x}} + (bu_x)_{\bar{y}} + (cu_y)_{\bar{x}} + (du_y)_{\bar{y}} \\ - \alpha u_x - \beta u_y + (\gamma u)_{\bar{x}} + (\delta u)_{\bar{y}} - gu \end{aligned}.$$

$R(u)$ is, for every boundary point, a linear difference expression whose exact form will not be given here.

If we arrange $\sum_{G_h}\sum B(u,v)$ according to u, we find

(2) $$h^2 \sum_{G_h}\sum B(u,v) = -h^2 \sum_{G'_h}\sum uM(v) - h\sum_{\Gamma'_h} uS(v) .$$

$M(v)$ is called the **adjoint difference expression** of $L(u)$ and has the form

$$\begin{aligned} M(v) = (av_x)_{\bar{x}} + (bv_y)_{\bar{x}} + (cv_x)_{\bar{y}} + (dv_y)_{\bar{y}} \\ + (\alpha v)_{\bar{x}} + (\beta v)_{\bar{y}} - \gamma v_x - \delta v_y - gv \end{aligned},$$

while $S(v)$ is a difference expression corresponding to $R(u)$ for the boundary.

Formulas (1) and (2) give

(3) $$h^2 \sum_{G'_h}\sum (vL(u) - uM(v)) + h\sum_{\Gamma'_h} (vR(u) - uS(v)) = 0 .$$

Formulas (1), (2) and (3) are called **Green's Formula**.

The simplest and most important case results if the bilinear form is symmetric, i.e., if the relations $b = c$, $\alpha = \gamma$, $\beta = \delta$ hold. In this case $L(u)$ is identical with $M(u)$ — the **self-adjoint** case — and it can be derived from the quadratic expression

$$B(u,u) = a{u_x}^2 + 2bu_x u_y + d{u_y}^2 + 2\alpha u_x u + 2\beta u_y u + gu^2 .$$

In the following we shall limit ourselves mainly to expressions $L(u)$ which are self-adjoint. The character of the difference expression $L(u)$ depends principally on the nature of those terms in the quadratic form $B(u, v)$ which are quadratic in the first difference quotients. We call this part of $B(u,u)$ the **characteristic form**:

$$P(u,u) = a{u_x}^2 + 2bu_x u_y + d{u_y}^2 .$$

We call the corresponding difference expression $L(u)$ **elliptic** or **hyperbolic** depending on whether the function $P(u, u)$ of the difference quotients is (positive) definite or indefinite.

The difference expression $\varDelta u = u_{x\bar{x}} + u_{y\bar{y}}$ with which we shall concern ourselves in the following paragraph is elliptic, i.e., it comes from the quadratic expression

$$B(u,u) = {u_x}^2 + {u_y}^2 \quad \text{or} \quad {u_{\bar{x}}}^2 + {u_{\bar{y}}}^2 .$$

The corresponding Green's formulas are[3]

(4) $$h^2 \sum_{G_h}\sum ({u_x}^2 + {u_y}^2) = -h^2 \sum_{G'_h} u\varDelta v - h\sum_{G'_h} uR(u) ,$$

(5) $$h^2 \sum_{G_h}\sum (v\varDelta u - u\varDelta v) + h\sum_{G'_h} (vR(u) - uR(v)) = 0 .$$

The difference expression, $\varDelta u = u_{x\bar{x}} + u_{y\bar{y}}$, is obviously the analogue of the differential expression $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ for a function $u(x, y)$ of the continuous variables x and y. Written out explicitly the difference expression is

$$\varDelta u = \frac{1}{h^2}\{u(x+h,y) + u(y,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)\} .$$

Therefore $\dfrac{h^2}{4}\varDelta u$ is the excess of the arithmetic mean of the functional values at the four neighborhood points over the functional value at the point in question.

Completely similar considerations lead to linear difference expressions of the **fourth order** and corresponding Green's formula, provided one starts

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[3] The boundary expression $R(u)$ may be written as follows: Let $u_0, u_1, \ldots, u_\nu$ be values of the function at a boundary point and at its ν neighboring points ($\nu \le 3$), then $R(u) = \frac{1}{h}(u_1 + \cdots + u_\nu - \nu u_0)$ ·

from bilinear difference expressions which are formed from the difference quotients of second order. Consider for example the difference expression

$$\Delta u = u_{xx\bar{x}\bar{x}} + +2u_{x\bar{x}y\bar{y}} u_{yy\bar{y}\bar{y}} .$$

This corresponds to the quadratic expression

$$B(u,u) = (u_{x\bar{x}} + u_{y\bar{y}})^2 = (\Delta u)^2 ,$$

provided one orders the sum

$$h^2 \sum_{G'_h} \sum \Delta u \Delta v$$

according to v, or equivalently replaces u by Δu in equation (5). One must notice however that in the expression $\Delta\Delta u$, the functional value at a point is connected with the values at its neighboring points and at their neighboring points, and accordingly is defined only for such points of the region G_h as are also interior points of the region $G'_{h'}$ (See Section 5). The entirety of such points we designate as G''_h. We obtain then Green's formula

$$h^2 \sum_{D'_h} \sum \Delta u \cdot \Delta v = h^2 \sum_{D'_h} \sum v \Delta\Delta u + \sum_{\Gamma_h + \Gamma'_h} v \cdot R(u) , \tag{6}$$

where R is a definable linear difference expression for each point of the boundary strip $\Gamma_h + \Gamma'_h$. Γ'_h indicates the set of boundary points of G'_h.

§2. Boundary Value and Eigenvalue Problems

1. The theory of boundary value problems

The boundary value problem for linear elliptic homogeneous difference equations of second order, which corresponds to the classical boundary value problem for partial differential equations, can be formulated in the following way.

Let there be given a self-adjoint elliptic linear difference expression of second order, $L(u)$, in a mesh region, G_h. $L(u)$ results from a quadratic expression $B(u, v)$ which is positive definite in the sense that it cannot vanish if u_x and u_y do not themselves vanish.

A function, u, is to be determined satisfying in G_h the difference equation

$$L(u) = 0,$$

and assuming prescribed values at the boundary points.

Under these requirements there will be exactly as many linear equations as there are interior points of the mesh at which the function u is to be determined.[4] Some of these equations which involve only mesh points whose

[4] If the matrix of the linear system of equations corresponding to an arbitrary difference

neighbors also lie in the interior of the region are homogeneous; others which involve boundary points of the mesh region are inhomogeneous. If the right-hand side of the inhomogeneous equations is set equal to zero, that is if $u = 0$ on the boundary, then it follows from Green's formula (1), by setting $u = v$ that $B(u,u)$ vanishes, and further from the definiteness of $B(u,u)$ that u_x and u_y vanish, and hence that u itself vanishes. Thus the difference equation for zero boundary value has the solution $u = 0$, or in other words the solution is uniquely determined since the difference of two solutions with the same boundary value must vanish. Further, if a linear system of equations with as many unknowns as equations is such that for vanishing right-hand side the unknowns must vanish, then the **fundamental theorem of the theory of equations** asserts that for an arbitrary right-hand side exactly one solution must exist. In our case there follows at once the **existence of a solution** of the boundary value problem.

Therefore we see that for elliptic difference equations the uniqueness and existence of the solution of the boundary value problem are related to each other through the fundamental theorem of the theory of linear equations, whereas for partial differential equations both facts must be proved by quite different methods. The basis for this difficulty in the latter case is to be found in the fact that the differential equations are no longer equivalent to a finite number of equations, and so one can no longer depend on the equality of the number of unknowns and equations.

Since the difference expression

$$\Delta u = 0$$

can be derived from the positive definite quadratic expression

$$h^2 \sum_{G_h}\sum (u_x{}^2 + u_y{}^2),$$

the boundary value problem for the difference expression is uniquely solvable.

The theory for difference equations of higher order is developed in exactly the same way as that for difference equations of second order; for example one can treat the fourth order difference equation

$$\Delta\Delta u = 0 .$$

In this case the values of the function must be prescribed on the boundary strip $\Gamma_h + \Gamma'_h$. Evidently here also the difference equation yields just as many linear equations as there are unknown functional values at the points of

equation of second order, $L(u) = 0$, is transposed, then the transposed set of equations corresponds to the adjoint difference equation $M(v) = 0$. Thus the above self-adjoint system gives rise to a set of linear equations with symmetric coefficients.

G''_h. In order to demonstrate the **existence and uniqueness** of the solution one needs only to show that a solution which has the value zero in the boundary strip $\Gamma_h + \Gamma'_h$ necessarily vanishes identically. To this end we note that the sum over the corresponding quadratic expression

$$h^2 \sum_{G''_h} \sum (\Delta u)^2 \tag{7}$$

for such a function vanishes, as can be seen by transforming the sum according to Green's formula (6). The vanishing of the sum (7) however implies that Δu vanishes at all points of G'_h, and according to the above proof this can only happen for vanishing boundary values if the function u assumes the value zero throughout the region. Thus our assertion is proved, and both the existence and uniqueness of the solution to the boundary value problem for the difference equations are guaranteed.[5]

2. Relation to the minimum problem

The above boundary value problem is related to the following minimum problem: among all functions $\varphi(x, y)$ defined in the mesh region G_h and assuming given values at the boundary points, that function $\varphi = u(x, y)$ is to be found for which the sum

$$h^2 \sum_{G_h} \sum B(\varphi, \varphi)$$

over the mesh region assumes the least possible value. We assume that the quadratic difference expression of first order, $B(u,u)$ is positive definite in the sense described in Section 1, part 2. One can show that the difference equation $L(\varphi) = 0$ results from this minimum requirement on the solution $\varphi = u(x, y)$, where $L(\varphi)$ is the difference expression of second order derived previously from $B(\varphi, \varphi)$. In fact this can be seen either by applying the rules of differential calculus to the sums $h^2 \sum_{G_h} \sum B(\varphi, \varphi)$ as a function of a finite number of values of φ at the grid points, or by employing the usual methods from the calculus of variations.

By way of example, solving the boundary value problem of finding the solution to the equation $\Delta \varphi = 0$ which assumes given boundary values, is equivalent to minimizing the sum $h^2 \sum_{G_h} \sum (\varphi_x{}^2 + \varphi_y{}^2)$ over all functions

[5] For the actual process of carrying through the solution of the boundary value problem by an iterative method, see among others the treatment: "Uber Randwertanjgahen bei partieller Differenzenglerthunger" by R. Courant, Zeitschr. f. angew, Mathematik u. Mechanik 6 (1925), pp. 322-325. Also there is a report by H. Henky, in Zeitschr. f. angew. Math. u. Mech. 2 (1922), p. 58 ff.rr

assuming given values on the boundary strip Γ'_h. Besides this sum there are yet other quadratic expressions in the second derivatives which give rise to the equation $\Delta\Delta u = 0$ under the process of being minimized. For example this is true in G_h for the sum

$$h^2 \sum_{G'_h} \sum (u_{xx}{}^2 + 2u_{xy}{}^2 + u_{yy}{}^2) .$$

That the minimum problem posed above always has a solution follows from the theorem that a continuous function of a finite number of variables (the functional values of φ at the grid points) always has a minimum if it is bounded from below and if it tends to infinity as soon as any of the independent variables goes to infinity.[6]

3. Green's function

It is possible to treat the boundary value problem for the inhomogeneous equation, $L(u) = -f$, in much the same way as the homogeneous case, $L(u) = 0$. For the inhomogeneous case it is sufficient to consider only the case of zero boundary conditions, since different boundary conditions can be taken care of by adding a suitable solution of the homogeneous equation. To solve the linear system of equations representing the boundary value problem, $L(u) = -f$, we first choose as the function $f(x, y)$ a function which assumes the value $-\frac{1}{h^2}$ at the point $x = \xi$, $y = \eta$ of the mesh. If $K(x, y; \xi, \eta)$ is the solution (vanishing on the boundary) of this difference equation which depends on the parametric point (ξ, η), then the solution for an arbitrary boundary condition can be represented by the sum

$$u(x, y) = h^2 \sum_{(\xi,\eta)\in G_h} K(x, y; \xi, \eta) f(\xi, \eta) .$$

The function $K(x, y; \xi, \eta)$ which depends on the points (x,y) and (ξ, η) is called the **Green's function** of the differential expressions $L(u)$. If we call the Green's function for the adjoint expression $M(v)$, $\overline{K}(x, y; \xi, \eta)$, then the equivalence

$$K(\overline{\xi}, \overline{\eta}; \xi, \eta) = \overline{K}(\xi, \eta; \overline{\xi}, \overline{\eta})$$

holds, as can be seen to follow immediately from Green's formula (3) when $u = K(x, y; \xi, \eta)$ and $v = \overline{K}(x, y; \overline{\xi}, \overline{\eta})$. For a self-adjoint difference expression the above relation gives the symmetric expression:

$$K(\overline{\xi}, \overline{\eta}; \xi, \eta) = K(\xi, \eta; \overline{\xi}, \overline{\eta}) .$$

6 It can easily be verified that the hypotheses for the application of this theorem are satisfied.

4. Eigenvalue problems

Self-adjoint difference expressions, $L(u)$, give rise to eigenvalue problems of the following type: find the value of a parameter λ, the eigenvalue, such that in G_h, a solution, the eigenfunction can be found for the difference equation

$$L(u) + \lambda u = 0 ,$$

where u is to be zero on the boundary, Γ_h.

The eigenvalue problem is equivalent to finding the **principle axes** of the quadratic form $B(u,u)$. Exactly as many eigenvalues and corresponding eigenfunctions exist as there are interior mesh points of the region G_h. The system of eigenfunctions and eigenvalues (and a proof of their existence) is given by the minimum problem:

Among all functions, $\varphi(x, y)$, vanishing on the boundary, and satisfying the orthogonality relation

$$h^2 \sum_{G_h}\sum \varphi u^{(\nu)} = 0 \text{ , } (\nu = 1, \ldots, m-1)$$

and normalized such that

$$h^2 \sum_{G_h}\sum \varphi^2 = 1 ,$$

the function, $\varphi = u$, is to be found for which the sum

$$h^2 \sum_{G_h}\sum B(\varphi, \varphi)$$

assumes its minimum value. The value of this minimum is the m-th eigenvalue, and the function for which it is assumed is the m-th eigenfunction.[7]

§3. Connections with the Problem of the Random Walk[8]

The theme of the following is related to a question from the theory of

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[7] From the orthogonality condition on the eigenfunctions,

$$h^2 \sum_{G_h}\sum u^{(\nu)} u^{(\mu)} = 0 \text{ , } (\nu \neq \mu)$$

it follows that each function, $g(x,y)$, which vanishes on the boundary of the mesh can be expanded in terms of the eigenfunctions in the form of

$$g = \sum_{\nu=1}^{N} c^{(\nu)} u^{(\nu)} ,$$

where the coefficients are determined from the equations

$$c^{(\nu)} = h^2 \sum_{G_h}\sum g u^{(\nu)} .$$

In this way in particular the following representation for the Green's functions may be derived

$$K(x, y; \xi, \eta) = -\sum_{\nu=1}^{N} \frac{u^{(\nu)}(x, y) \cdot u^{(\nu)}(\xi, \eta)}{\lambda^{(\nu)}} .$$

[8] Section 3 is not prerequisite to Section 4.

probability, namely the problem of the random walk in a bounded region.[9] We consider the lines of a mesh region G_h as paths along which a particle can move from one grid point to a neighboring one. In this net of streets the particle can wander aimlessly, and it can choose at random one of the four directions leading from each intersection of paths of the net - all four directions being equally probable. The walk ends as soon as a boundary point of G_h is reached because here the particle must be absorbed.

We ask:

1) What is the **probability** $w(P;R)$ that a random walk starting from a point P reaches a particle point R of the boundary?

2) What is the **mathematical expectation** $v(P;Q)$ that a random walk starting from P reaches a point Q, of G_h without touching the boundary?

This probability or mathematical expectation, respectively will be defined more precisely by the following process. Assume that at the point P there is a unit concentration of matter. Let this matter diffuse into the mesh with constant velocity, traveling say a mesh width in unit time. At each meshpoint let exactly one fourth of the matter at the point diffuse outwards in each of the four possible directions. The matter which reaches a boundary point is to remain at that point. If the point of origin P is itself a boundary point, then the matter never leaves that point.

We define the **probability** $w(P, R)$ that a random walk starting from P reaches the boundary point R (without previously attaining the boundary), as the amount of matter which accumulates at this boundary point over an infinite amount of time.

We define the **probability** $E_n(P;Q)$ that the walk starting from the point P reaches the point Q in exactly n steps without touching the boundary by the amount of matter which accumulate in n units of time provided P and Q are both interior points. If either P or Q are boundary points then E_n is set equal to zero.

The value $E_n(P;Q)$ is exactly equal to $1/4^n$ times the number of paths of n steps leading from P to Q without touching the boundary. Thus $E_n(P;Q) = E_n(Q;P)$.

We define the **mathematical expectation** $V(P;Q)$ that one of the paths considered above leads from P to the point Q, by the infinite sum of all of these possibilities,[10]

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[9] The present treatment is essentially different from the familiar treatments which can be carried through, say for example in the case of Brownian motion for molecules. The difference lies precisely in the way in which the boundary of the region enters.

[10] The convergence will be shown below.

$$V(P;Q) = \sum_{\nu=0}^{\infty} E_\nu(P;Q)\,,$$

i.e., for the interior points *P* and *Q* the sum of all the concentrations which have passed through the point *Q* at different times. This will be assigned the expected value 1 for a concentration originating at *Q*.

If the amount arriving at the boundary point *R* in exactly *n* steps is designated as $F_n(P;R)$, then the probability *w*(*P*;*R*) is given by the series

$$w(P;R) = \sum_{\nu=0}^{\infty} F_\nu(P;R)\,.$$

All the terms of this series are positive and the partial sum is bounded by one (since the concentration reaching the boundary can be made up of only part of the initial concentration), and therefore the convergence of the series is assured.

Now it is easy to see that the probability $E_n(P;Q)$, that is, the concentration reaching the point *Q* in exactly *n* steps tends to zero as *n* increases. For if at any point *Q*, from which the boundary point *R* can be reached in *m* steps, we have $E_n(P;Q) > a > 0$, then at least $a/4^a$ of the concentration at *Q* will reach the point *R* after *m* steps. However, since the sum $\sum_{\nu=0}^{\infty} F_\nu(P;R)$ converges, the concentration reaching *R* goes to zero with increasing time, where the value of $E_n(P;Q)$ must itself vanish as time increases; that is the probability of an infinitely long walk remaining in the interior of the region is zero.

From this it follows that the entire concentration eventually reaches the boundary; or in other words that the sum $\sum_R w(P;R)$ over all the boundary points *R* is equal to one.

The convergence of the infinite series for the mathematical expectation *v*(*P*; *Q*)

$$v(P;Q) = \sum_{\nu=0}^{\infty} E_\nu(P;Q)$$

remains to be shown.

To this end we remark that the quantity $E_n(P;Q)$ satisfies the following relations

$$E_{n+1}(P;Q) = \frac{1}{4}\{E_n(P;Q_1) + E_n(P;Q_2) + E_n(P;Q_3) + E_n(P;Q_4)\},\ (n \geq 1),$$

where Q_1 through Q_4 are the four neighboring points of the interior point *Q*. That is, the concentration at the point *Q* at the (*n*+1)-st step consists of 1/4

times the sum of the concentrations at the four neighboring points at the n-th step. If one of the neighbors of Q, for example $Q_1 = R$, is a boundary point then it follows that no concentration flows from this boundary point to Q since the expression $E_n(P;Q)$ is zero in this case. Furthermore, for an interior point, $E_0(P;P) = 1$ and of course $E_1(P;Q) = 0$.

From these relationships we find for the partial sum

$$v_n(P;Q) = \sum_{\nu=0}^{n} E_\nu(P;Q)$$

the equation

$$v_{n+1}(P;Q) = \frac{1}{4}\{v_n(P;Q_1) + v_n(P;Q_2) + v_n(P;Q_3) + v_n(P;Q_4)\},$$

for $P \neq Q$. For $P = Q$,

$$v_{n+1}(P;P) = 1 + \frac{1}{4}\{v_n(P;P_1) + v_n(P;P_2) + v_n(P;P_3) + v_n(P;P_4)\},$$

that is, the expectation that a point goes back into itself consists of the expectation that a non-vanishing path leads from P back again to itself – namely

$$\frac{1}{4}\{v_n(P;P_1) + v_n(P;P_2) + v_n(P;P_3) + v_n(P;P_4)\},$$

together with the expectation unity that expresses the initial position of the concentration itself at this point.

The quantity $v_n(P;Q)$ thus satisfies the following difference equation.[11]

$$\Delta v_n(P;Q) = \frac{4}{h^2} E_n(P;Q), \text{ for } P \neq Q,$$

$$\Delta v_n(P;Q) = \frac{4}{h^2}(E_n(P;Q) - 1), \text{ for } P = Q.$$

$v_n(P;Q)$ is equal to zero when Q is a boundary point.

The solution of this boundary value problem for arbitrary right-hand side is uniquely determined as we have explained earlier (Section 2, Part 1.), and depends continuously on the right-hand side. Since the variables $E_n(P;Q)$ tend to zero, the solution $v_n(P;Q)$ converges to the solution $v(P;Q)$ of

[11] This defines the Δ-operation for the variable point Q.
This equation can be interpreted as an equation of the heat conduction type. That is if the function $v_n(P;Q)$ is considered, not as a function of the index n as in our presentation above, but rather as a function of time, t, which is proportional to n, so that $t = n\tau$ and $v_n(P;Q) = v(P;Q;t) = v(t)$, then the above equations can be written in the following form:

$$\Delta v(t) = \frac{4t}{h^2} \cdot \frac{v(t+\tau) - v(t)}{\tau}, \text{ for } P \neq Q,$$

$$\Delta v(t) = \frac{4t}{h^2} \cdot \left(\frac{v(t+\tau) - v(t)}{\tau} - \frac{1}{\tau}\right), \text{ for } P = Q.$$

For a similar difference equation which has a parabolic differential equation as its limiting form, see Section 6 of the second half of the paper.

the difference equation

$$\Delta v(P;Q) = 0 \quad \text{for} \quad P \neq Q,$$

$$\Delta v(P;Q) = -\frac{4}{h^2} \quad \text{for } P = Q,$$

with boundary values $v(P;R) = 0$.

Thus we see that the mathematical expectation $v(P;Q)$ exists and is non other than the Green's function for the difference equation $\Delta u = 0$, except for a factor of 4. The symmetry of the Green's function is an immediate consequence of the symmetry of the quantity $E_n(P;Q)$ which was used to define it.

The probability $w(P;R)$ satisfies, with respect to P, the relation

$$w(P;P) = \frac{1}{4}\{w(P_1;R) + w(P_2;R) + w(P_3;R) + w(P_4;R)\},$$

and thus the difference equation

$$\Delta w = 0.$$

That is, if P_1, P_2, P_3, P_4 are the four neighboring points of the interior point P, then each path from P to R must pass through one of these four directions, and each of the four is equally likely. Furthermore, the probability of going from one boundary point R to another R' is $w(R;R') = 0$ unless the two points R and R' coincide, in which case $w(R;R) = 1$. Thus $w(P;R)$ is that solution of the boundary value problem $\Delta w = 0$ which assumes the value 1 at the boundary point R and the value 0 at all other points of the boundary. Therefore the solution of the boundary value problem for an arbitrary boundary value $u(R)$ has the simple form $u(P) = \sum_R w(P;R)u(R)$, where the sum is to be extended over all the boundary points.[12] If the function $u \equiv 1$ is substituted for u in this expression, then we check the relation $1 = \sum_R w(P;R)$.

The interpretation given above for Green's function as an expectation yields immediately further properties. We mention only the fact that the Green's function decreases if one goes from the region G to a subregion $\overline{G}$ lying within G; that is the number of possible paths for steps on the mesh leading from one point P to another Q (without touching the boundary), decreases as the region decreases.

Of course corresponding relationships hold for more than two

[12] Moreover it is easy to show that the probability $w(P;R)$ of reaching the boundary is the boundary expression $\tilde{R}(K(P,Q))$, constructed from the Green's function $K(P;Q)$ in terms of Q, where $u(x,y)$ is to be identified with $w(P,Q)$, and $v(x,y)$ with $v(P,Q)$ in Green's formula (5).

independent variables. We note only that other elliptic difference equations admit a similar probability interpretation.

If the limit for vanishing mesh width is considered by methods given in the following section, then the Green's function on the mesh goes over to the Green's function of the potential equation except for a numerical factor; a similar relationship holds between the expression $\frac{w(P;R)}{h}$ and the normal derivative of the Green's function at the boundary of the region. In this way for example the Green's function for the potential equation could be interpreted as the specific mathematical expectation of going from one point to another,[13] without reaching the boundary.

In going over to the limit of a continuum from the mesh, the influence of the direction in the mesh prescribed for the random walk vanishes. This fact suggests that it would be of some interest to consider limiting cases of more general random walks for which the limitations on the direction of the walk are relaxed. This lies outside of the scope of this presentation however and we can only hope to renew the question at some other opportunity.

§4. The Solution of the Differential Equation as a Limiting Form of the Solution of the Difference Equation

1. The boundary value problem of potential theory

In letting the solution of the difference equation tend to the solution of the corresponding differential equation we shall relinquish the greatest possible degree of generality with regard to the boundary and boundary values in order to demonstrate more clearly the character of our method.[14] Therefore we assume that we are given a simply connected region G with a boundary formed of a finite number of arcs with continuously turning tangents. Let $f(x,y)$ be a given function which is continuous and has continuous partial derivatives of first and second order in a region containing G. If G_h is the mesh region with mesh width h belonging to the region G, then let the boundary value problem for the difference equation $\Delta u = 0$ be solved for the same boundary values which the function $f(x,y)$ assumes on the boundary; let $u_h(x,y)$ be the solution. We shall prove that as $h \to 0$ the function $u_h(x,y)$ converges to a function $u(x,y)$ which satisfies in G the partial

[13] Here the a priori expectation of reaching a certain area element is understood to be equal to the area of the element.

[14] We mention however that carrying through our method for more general boundaries and boundary values in no way causes any fundamental difficulty.

differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ and takes the value of $f(x, y)$ at each of the points of the boundary. We shall show further that for any region lying entirely within G the difference quotients of u_h of arbitrary order tend uniformly towards the corresponding partial derivatives of $u(x, y)$.

In the convergence proof it is convenient to replace the boundary condition $u = f$ by the weaker requirement that

$$\frac{1}{r}\iint_{S_r}(u-f)^2\,dxdy \to 0 \quad \text{as} \quad r \to 0,$$

where S_r is that strip of G whose points are at a distance from the boundary smaller than r.[15]

The convergence proof depends on the fact that for any subregion G^* lying entirely within G, the function $u_h(x,y)$ and each of its difference quotients is bounded and uniformly continuous as $h \to 0$ in the following sense: For each of these functions, say $w_h(x,y)$, there exists a $\delta(\varepsilon)$ depending only on the subregion and not on h such that

$$|\, w_h(P) - w_h(P_1)\,| < \varepsilon$$

provided the mesh points P and P_1 lie in the same subregion of G_h and are separated from each other by a distance leas than $\delta(\varepsilon)$.

Once uniform continuity in this sense (equi-continuity) has been established we can in the usual way select from the functions u_h a subsequence of functions which tend uniformly in any subregion G^* to a limit function $u(x,y)$, while the difference quotients of u_h tend uniformly towards the partial derivatives of u. The limit function then possesses derivatives of arbitrarily high order in any proper subregion G^* of G and satisfies $\nabla^2 u = 0$ in this region. If we can show also that u satisfies the boundary condition we can regard it as the solution of our boundary value problem for the region G. Since this solution is uniquely determined, it is clear that not only a partial sequence of the functions u_h but this sequence of functions itself possesses the required convergence properties.

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[15] The weaker boundary value requirement does in fact provide the unique characterization of the solution, as can be seen from the easily proved theorem: If the boundary condition above is satisfied for $f(x,y) = 0$ for a function satisfying the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

in the interior of G and if $\iint_G\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy$ exists, then $u(x, y)$ is identically zero. (See Courant, "Uber die Losung der Differentialgleichungen der Physik", Math Ann. 85 exp. p. 296 ff .)

In the case of two independent variables the boundary values are actually attained, as follows from the weaker requirement; but in the case of more variables the corresponding result cannot in general be expected since there can exist exceptional points on the boundary at which the boundary value is not taken on even though a solution exists under the weaker requirement.

The uniform continuity (equi-continuity) of our quantities may be established by proving the following lemmas.

1) *As* $h \to 0$ *the sums over the mesh region*

$$h^2 \sum_{G_h}\sum u^2 \quad \text{and} \quad h^2 \sum_{G_h}\sum (u_x{}^2 + u_y{}^2)$$

remain bounded.[16]

2) *If* $w = w_h$ *satisfies the difference equation* $\Delta w = 0$ *at a mesh point of* G_h*, and if, as* $h \to 0$ *the sum* $h^2 \sum_{G^*_h}\sum w^2$ *extended over a mesh region* G^*_h *associated with a subregion G* of G remains bounded, then for any fixed subregion G** lying entirely within G* the sum*

$$h^2 \sum_{G^{**}_h}\sum (w_x{}^2 + w_y{}^2)$$

over the mesh region G^{**}_h *associated with G** likewise remains bounded as* $h \to 0$.

Using (1) there follows from this, since all of the difference quotients w of the function u_h *again satisfy the difference equation* $\Delta w = 0$*, that each of the sums*

$$h^2 \sum_{G^*_h}\sum w^2$$

is bounded.

3) *From the boundedness of these sums there follows finally the boundedness and uniform continuity of all the difference quotients themselves.*

2. Proof of the lemmas

The proof of 1) follows from the fact that the functional values u_h are themselves bounded. For the greatest (or least) value of the function is assumed on the boundary[17] and so is bounded by a prescribed finite value. The boundedness of the sum $h^2 \sum_{G_h}\sum (u_x{}^2 + u_y{}^2)$ is an immediate consequence of the minimum property of our mesh function formulated in Part 2 of Section 2 which gives in particular

$$h^2 \sum_{G_h}\sum (u_x{}^2 + u_y{}^2) \le h^2 \sum_{G_h}\sum (f_x{}^2 + f_y{}^2),$$

but as $h \to 0$ the sum on the right tends to the integral

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[16] Here and in the following we drop the index *h* from the grid functions.

[17] We note however with a view to carrying over the method to other differential equations, that we can relax these conditions. To this end we need only to bring into play the inequality (15) or to use the reasoning of the alternative (see Part 4, Section 4).

$$\iint_G \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] dxdy$$

which, by hypothesis, exists.

To prove 2) we consider the quadratic sum

$$h^2 \sum_{\mathrm{int}.Q_1}\sum ({w_x}^2 + {w_{\bar{x}}}^2 + {w_y}^2 + {w_{\bar{y}}}^2),$$

where the summation extends over all the interior points of a square Q_1, (see Fig. 1). We call the values of the function on the boundary S_1 of the square Q_1, w_1, and those on the boundary S_0, of Q_0, w_0. Then Green's formula gives

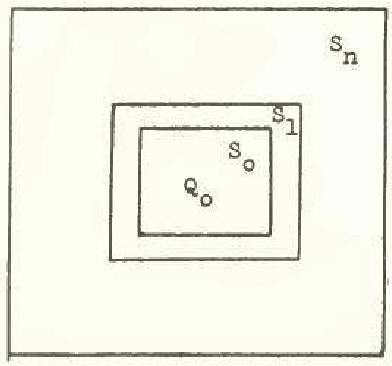

Fig. 1

(8) $$h^2 \sum_{\mathrm{int}.Q_1}\sum ({w_x}^2 + {w_{\bar{x}}}^2 + {w_y}^2 + {w_{\bar{y}}}^2) = \sum_{S_1} w^2 - \sum_{S_0} w^2 - \sum_{C_1} w^2 ,$$

where S_1 and S_0 are respectively the boundary of Q_1 and the square boundary of the lattice points Q_0 lying within S_1, while C_1 consists of the four corner points of the boundary of Q_1.

We now consider a sequence of concentric squares $Q_0, Q_1, \ldots, Q_n$ with boundaries $S_0, S_1, \ldots, S_n$, where each boundary is separated from the next by a mesh width. Applying the formula to each of these squares and observing that we have always

$$2h^2 \sum_{Q_0}\sum ({w_x}^2 + {w_y}^2) \le h^2 \sum_{Q_k}\sum ({w_x}^2 + {w_{\bar{x}}}^2 + {w_{\bar{y}}}^2),\ (k \ge 1)$$

we obtain

$$2h^2 \sum_{Q_0}\sum ({w_x}^2 + {w_y}^2) \le \sum_{S_k} w^2 - \sum_{S_{k-1}} w^2 - \sum_{C_k} w^2 ,\ (1 \le k < n).$$

We strengthen the inequality by neglecting the last non positive term on the right and we then add the *n* inequalities to obtain

$$2nh^2 \sum_{Q_0}\sum ({w_x}^2 + {w_y}^2) \le \sum_{S_n} w^2 - \sum_{S_0} w^2 \le \sum_{S_n} w^2 .$$

Summing this inequality from $n = 1$ to $n = N$ we get

$$N^2 h^2 \sum_{Q_0}\sum ({w_x}^2 + {w_y}^2) \le \sum_{Q_N}\sum w^2 .$$

Diminishing the mesh width *h* we can make the squares Q_0 and Q_n converge towards two fixed squares lying within *G* and having corresponding sides separated by a distance *a*. In this process $Nh \to a$ and we have, independent of the mesh width

(9) $$h^2 \sum_{Q_0}\sum ({w_x}^2 + {w_y}^2) \le \frac{h^2}{a^2} \sum_{Q_N}\sum w^2 .$$

For sufficiently small h this inequality holds of course not only for two squares Q_0 and Q_N but with a change in the constant, a, for any two subregions of G such that one is contained entirely within the other. Thus lemma (2) is proved.[18]

In order to prove the third result, that u_h and all its partial difference quotients w_h remain bounded and uniformly continuous as $h \to 0$, we consider a rectangle R with corners P_0, Q_0, P, Q and with sides P_0Q_0 and PQ of length a parallel to the x-axis.

We start with the relation

$$w(Q_0) - w(P_0) = h \sum_{PQ} w_x - h^2 \sum_R \sum w_{xy} ,$$

and the inequality

$$(11) \qquad | w(Q_0) - w(P_0) | \le h \sum_{PQ} | w_x | + h^2 \sum_R \sum | w_{xy} | ,$$

which is a consequence of it.

We then let the side PQ of the rectangle vary between an initial line P_1Q_1, a distance b from P_0Q_0 and a final line P_2Q_2 a distance $2b$ from P_0Q_0, and we sum the corresponding $\frac{b}{h}+1$ inequalities (11). We obtain the estimate

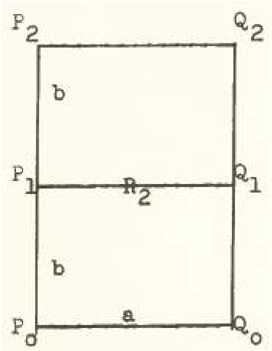

Fig. 2.

$$| w(Q_0) - w(P_0) | \le \frac{1}{b+h} h^2 \sum_{R_2} | w_x | + h^2 \sum_{R_2} \sum | w_{xy} | ,$$

where the summations extend over the entire rectangle, $R_2 = P_0Q_0P_2Q_2$. From Schwarz's inequality it then follows that,

(12)

$$| w(Q_0) - w(P_0) | \le \frac{1}{b} \sqrt{2ab} \sqrt{h^2 \sum_{R_2} \sum {w_x}^2} + \sqrt{2ab} \sqrt{h^2 \sum_{R_2} \sum {w_{xy}}^2} .$$

Since, by hypothesis, the sums which occur here multiplied by h^2 remain bounded, it follows that as $a \to 0$ the difference $| w(P_0) - w(Q_0) | \to 0$ independently of the mesh-width, since for each subregion G^* of G the quantity b can be held fixed. Consequently the uniform continuity (equi-continuity) of $w = w_h$ is proved for the x-direction. Similarly it holds for the y-direction and so also for any subregion G^* of G. The boundedness of the function w_h in G^* finally follows from its uniform

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[18] If we do not assume that $\Delta w = 0$, then in place of the inequality (9) we find

$$(10) \qquad h^2 \sum_{G^{**}} \sum ({w_x}^2 + {w_y}^2) \le c_1 h^2 \sum_{G^*} \sum w^2 + c_2 h^2 \sum_{G^*} \sum (\Delta w)^2$$

for suitable constants c_1 and c_2 independent of h, where G^{**} lies entirely within G^* and G^* in turn is contained in the interior of G.

continuity (equi-continuity) and the boundedness of $h^2 \sum_{G^*}\sum w_h{}^2$.

By this proof we establish the existence of a subsequence of functions u_h which converge towards a limit function $u(x, y)$ and which do so uniformly together with all their difference quotients, in the sense discussed above for every interior subregion of G. This limit function $u(x, y)$ has throughout G continuous partial derivatives of arbitrary order, and satisfies there the potential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

3. The boundary condition

In order to prove that the solution satisfies the boundary condition formulated above we shall first of all establish the inequality

$$(13) \qquad h^2 \sum_{S_{r,h}}\sum v^2 \le A r^2 h^2 \sum_{S_{r,h}}\sum (v_x{}^2 + v_y{}^2) + Brh \sum_{\Gamma_h} v^2 ,$$

where $S_{r,h}$ is that part of the mesh region G_h which lies inside a boundary strip S_r. This boundary strip S_r consists of all points of G whose distance from the boundary is less than r; it is bounded by Γ and another curve Γ_r. The constants A and B depend only on the region and not on the function v nor the mesh width h.

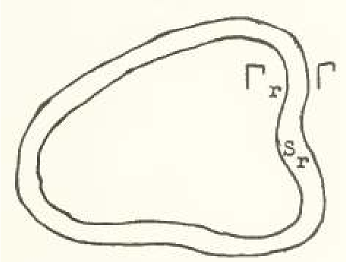

Fig. 3.

In order to prove the above inequality, we divide the boundary, Γ, of G into a finite number of pieces for which the angle of the tangent with one of the x or y-axes is greater than some positive value (say 30°). Let γ, for instance, be a piece of Γ which is sufficiently steep (in the above sense) relative to the x-axis, (See Fig. 4). Lines parallel to the x-axis intersecting γ will cut a section γ_r from the neighboring curve Γ_r, and will define together with γ and γ_r a piece s_r of the boundary strip S_r. We use the symbol $s_{r,h}$ to denote the portion of G_h contained in s_r and denote the associated portion of the boundary Γ_h by γ_h.

We now imagine a parallel to the x-axis to be drawn through a mesh point P_h of $s_{r,h}$. Let it meet the boundary γ_h in a point $\overline{P}_h$. The portion of this line which lies in $s_{r,h}$ we call $p_{r,h}$. Its length is certainly smaller than cr, where the constant c depends only on the smallest angle of inclination of a tangent γ to the x-axis.

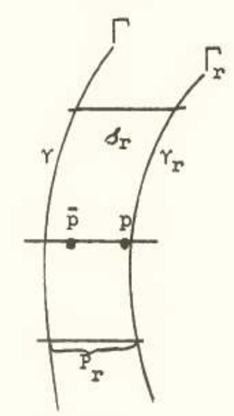

Fig. 4.

Between the values of v at P_h and $\overline{P}_h$ we have the relation

$$v(P_h) = r(\overline{P}_h) \pm h \sum_{P_h \overline{P}_h} v_x .$$

Squaring both sides and applying Schwarz's inequality, we obtain

$$v(P_h)^2 \le 2v(\overline{P}_h)^2 + 2cr \cdot h \sum_{p_{r,h}} v_x{}^2 .$$

Summing with respect to P_h in the x-direction, we get

$$h \sum_{p_r} v^2 \le 2crv(\overline{P}_h)^2 + 2c^2 r^2 h \sum_{p_r} v_x{}^2 .$$

Summing again in the y-direction we obtain the relation

$$h \sum_{S_{r,h}} \sum v^2 \le 2cr \sum_{\Gamma_h} v(\overline{P}_h)^2 + 2c^2 r^2 h \sum_{S_{r,h}} \sum v_x{}^2 . \tag{14}$$

Writing down the inequalities associated with the other portions of Γ and adding all the inequalities together we obtain the desired inequality (13).[19]

We next set $v_h = u_h - f_h$ so that $v_h = 0$ on Γ.

Then since $h^2 \sum_{G_h} \sum (v_x{}^2 + v_y{}^2)$ remains bounded as $h \to 0$, we obtain from (13)

$$\frac{h^2}{r} \sum_{S_{r,h}} \sum v^2 \le \widetilde{H} r , \tag{16}$$

where $\widetilde{H}$ is a constant which does not depend on the function v or the mesh size. Extending the sum on the left to the difference $S_{\nu,h} - S_{\rho,h}$ of two boundary strips, the inequality (16) stills holds with the same constant $\widetilde{H}$ and we can pass to the limit $h \to 0$.

From the inequality (16) we then get

$$\frac{1}{r} \iint_{S_r - S_\rho} v^2 dxdy \le \widetilde{H} r , \ v = u - f.$$

Now letting the narrower boundary strip S_ρ approach the boundary we obtain the inequality

$$\frac{1}{r} \iint_{S_r} v^2 dxdy = \frac{1}{r} \iint_{S_r} (u-f)^2 dxdy \le \widetilde{H} r ,$$

which states that the limit function u satisfies the prescribed boundary condition.

4. Applicability of the Method to Other Problems

Our method is based essentially on the inequalities arising from the lemmas stated previously since the principal points of the proofs follow from

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[19] By similar reasoning we can also establish the inequality

$$h^2 \sum_{G_h} \sum v^2 \le c_1 h \sum_{\Gamma_h} v^2 + c_2 h^2 \sum_{G_h} \sum (v_x{}^2 + v_y{}^2) \tag{15}$$

the two last lemmas in part 1 of Section 4[20]. No use is made of special fundamental solutions or special properties of the difference expression, and therefore the method can be carried over directly to the case of arbitrarily many independent variables as well as to the eigenvalue problem,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\lambda u=0\ .$$

The same sort of convergence relations will obtain in this case as above[21]. Also the method applies to linear partial differential equations of other types, in particular its application to equations with variable coefficients requires only some minor modifications. The essential difference in this case lies only in proving the boundedness of $h^2\sum\sum u_h^2$ which of course does not always hold for an arbitrary linear problem. However in case this sum is not bounded it can be shown that the general boundary value problem for the differential equation in question also possesses effectively no solutions, but that in this case there exist non-vanishing solutions of the corresponding homogeneous problem, i.e., eigenfunctions[22].

5. The boundary value problem $\Delta\Delta u=0$

In order to show that the method can be carried over to the case of differential equations of higher order, we will treat briefly the boundary value problem of the differential equation:

$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=0\,.$$

We seek, in G, a solution of this equation for which the values of u and its first derivative are given on the boundary, being specified there by some function $f(x, y)$.

To this end we assume as previously that $f(x, y)$ together with its first and second derivatives is continuous in that region of the plane containing the region G.

We replace our differential equation problem by the new problem of solving the difference equation $\Delta u=0$ in the mesh region G subject to the condition that at the points of the boundary strip $\Gamma_h+\Gamma_h'$ the function u

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in which the constants c_1 and c_2 depend only on the region G and not on the mesh division.

[20] For an application of corresponding integral inequalities see K. Friedrichs, "Die Rand-und Eigen Wert problems aus der Theorle der elastischen Platten", Math. Ann. 98, p. 222.

[21] Similarly one proves at the same time that every solution of such a differential equation problem has derivatives of every order.

[22] See Courant-Hilbert, Methoden der mathematischen Physik 1, Chap. III, Section 3, where the theory of integral equations is handled with the help of the corresponding principle of the alternative. See also the Dissertation (Gottingen) of W. v. Koppenfels which will appear soon.

takes on the values $f(x, y)$. From Section 2 we know that this boundary value problem has a unique solution. We will show that as the mesh size decreases this solution, in each interior subregion of G, converges to the solution of the differential equation, and that all of its difference quotients converge to the corresponding partial derivatives.

We note first that for the solution $u = u_h$, the sum

$$h^2 \sum_{G'_h} \sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

remains bounded as $h \to 0$. That is, by applying the minimum requirements on the solution (part 2, Section 2) one finds that this sum is not larger than the corresponding sum

$$h^2 \sum_{G'_h} \sum (f_{xx}^2 + f_{xy}^2 + f_{yy}^2)$$

and this converges as $h \to 0$ to

$$\iint_G \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \right) dxdy$$

which exists, by hypothesis.

From the boundedness of the sum

$$h^2 \sum_{G'_h} \sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

follows immediately the boundedness of $h^2 \sum_{G'_h} \sum (\Delta u)^2$ and also that of

$$h^2 \sum_{G'_h} \sum (u_x^2 + u_y^2) \quad \text{and} \quad h^2 \sum_{G'_h} \sum u^2 .$$

That is, for arbitrary w the following inequality holds (see footnote 19),

$$h^2 \sum_{G_h} \sum w^2 \le ch^2 \sum_{G_h} \sum (w_x^2 + w_y^2) + ch \sum_{\Gamma_h} w^2 . \tag{15}$$

Then if one substitutes the first difference quotients of w for w itself in this inequality and applies the expression over the subregion of G_h for which they are defined, then one finds the further inequality,

$$h^2 \sum_{G_h} \sum (w_x^2 + w_y^2) \le ch^2 \sum_{G'_h} \sum (w_{xx}^2 + 2w_{xy}^2 + w_{yy}^2) + ch \sum_{\Gamma_h + \Gamma'_h} (w_x^2 + w_y^2)$$

where again the constant c is independent of the function and of the mesh size. We apply this inequality to $w = u_h$ and thus find the boundedness of the sum over $\Gamma_h + \Gamma'_h$ on the right-hand side; (by definition these boundary sums converge to the corresponding integral containing $f(x, y)$). Thus from the boundedness of

$$h^2 \sum_{G'_h} \sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

follows the boundedness of $h^2 \sum_{G_h} \sum (u_x^2 + u_y^2)$ and thence that of $h^2 \sum_{G_h} \sum u^2$.

For the third step we substitute one after the other the expressions Δu, Δu_x, Δu_y, Δu_{xx}, …, for w in the inequality

$$h^2 \sum_{G^*} \sum (w_x^2 + w_y^2) \leq ch^2 \sum_{G^*} \sum w^2 + ch^2 \sum_{G^*} \sum (\Delta w)^2$$

(see part 2, Section 4) where G^* is a subregion of G containing G^{**} in its interior. The expressions introduced all satisfy the equation $\Delta w = 0$. It follows then that for each expression in turn and for all subregions G^* of G that the sums, $h^2 \sum_{G^*} \sum (w_x^2 + w_y^2)$, that is, $h^2 \sum_{G^*} \sum (\Delta u_x^2 + \Delta u_y^2)$, $h^2 \sum_{G^*} \sum (\Delta u_{xx}^2 + \Delta u_{xy}^2)$, …, are bounded together with the sums:

$$h^2 \sum_{G_h} \sum u^2 \ , \ \ h^2 \sum_{G_h} \sum (u_x^2 + u_y^2), \text{ and } \ h^2 \sum_{G_h} \sum (\Delta u)^2$$

which are already known to be bounded.

Finally we substitute into the inequality (10), in place of w, the sequence of functions u_{xx}, u_{xy}, u_{yy}, u_{xxx},…, for which

$$h^2 \sum_{G_h^*} \sum (\Delta w)^2 \text{ , i.e., } \ h^2 \sum_{G_h^*} \sum (\Delta u_{xx})^2 ,\ldots$$

are bounded as shown above. We then find that for all subregions the sums

$$h^2 \sum_{G_h^*} \sum (u_{xxx}^2 + u_{xxy}^2) \ , \ \ h^2 \sum_{G_h^*} \sum (u_{xyx}^2 + u_{xyy}^2) ,\ldots$$

remain bounded.

From these facts we can now conclude as previously that from our sequence of mesh functions a subsequence can be chosen which in each interior subregion of G converges (together with all its difference quotients) uniformly to a limit function (or respectively its derivatives) which is continuous in the interior of G.

We have yet to show that this limit function which obviously satisfies the differential equation $\Delta\Delta u = 0$ also takes on the prescribed boundary conditions. For this purpose we say here only that, analogous to the foregoing, the expressions

$$\iint_{S_r} (u-f)^2\,dxdy \le cr^2\,,$$

$$\iint_{S_r} \left[\left(\frac{\partial u}{\partial x}-\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}-\frac{\partial f}{\partial y}\right)^2\right]dxdy \le cr^2$$

hold[23].23 That the limit function fulfills these conditions may be seen by carrying over the treatment in Part 3, Section 4 to the function u and its first difference quotient.

From the uniqueness of our boundary value problem we see furthermore that not only a selected subsequence, but also the original sequence of functions u possesses the asserted convergence properties.

(End of Part I)

[23] That the boundary values for the function and its derivatives actually are assumed is not difficult to prove. See for instance the corresponding treatment of K. Friedrichs, *loc. cit.*