## II. The Hyperbolic Case

**Section 1. The Equation of the Vibrating String.**

In the second part of this paper we shall consider the initial value problem for linear hyperbolic partial differential equations. We shall prove that under certain hypotheses the solutions of the difference equations converge to the solutions of the differential equations as the mesh size decreases.

We can discuss the situation most easily by considering the example of the approximation to the solution of the wave equation

(1) $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 .$$

We limit ourselves to the particular initial value problem where the value of the solution $u$, and its derivatives are given on the line $t = 0$.

In order to find the corresponding difference equation, we construct in the $(x, t)$ -plane a square grid with lines parallel to the axes and with mesh width $h$. Following the notation of the first part of the paper we replace the differential equation (1) by the difference equation

$$u_{t\bar{t}} - u_{x\bar{x}} = 0 .$$

If we select a grid point, $P_0$, then the corresponding difference equation relates the value of the function at this point to the values at four neighboring points. If we characterize the four neighboring values by the four indices 1, 2, 3, 4 (cf. Fig. 5), then the difference equation assumes the simple form

(2) $$u_1 + u_3 - u_2 - u_4 = 0$$

Note that the value of the function $u$ at the point $P_0$ does not appear itself in the equation.

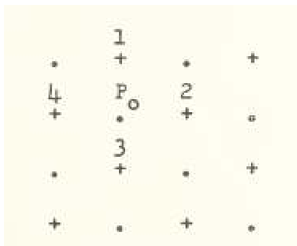

Fig. 5.

We consider the grid split up into two subgrids, indicated in Fig. 5 by dots and crosses, respectively. The difference equation connects the values of the function over each of the subgrids separately, and so we shall consider only one of the two grids. As initial condition the values of the function are prescribed on the two rows of the grid, $t = 0$ and $t = h$. We can give the solution of this initial value problem explicitly; that is, we express the value of the solution at any point $S$ in terms of the values given along the two initial rows. One can see at once that the value at a point of the row $t = 2h$ is uniquely determined by only the three values as the points close to it in the two first rows. The value at a point of the fourth row is uniquely determined by the values of the solution at three particular points in the second and third

rows, and through them it is related to certain values in the first two rows. In general to a point $S$ there will correspond a certain region of dependence in the first two rows; it may be found by drawing the lines $x + t$ = const, and $x - t$ = const., through the point $S$ and extending them until they meet the second row at the points $a$ and $B$, respectively (cf. Fig. 6). The triangle $S\alpha\beta$ is called ***the triangle of determination*** because all the values of $u$ in it remain unchanged provided the values on the first two rows of it are held fixed. The sides of the triangle are called ***lines of determination***.

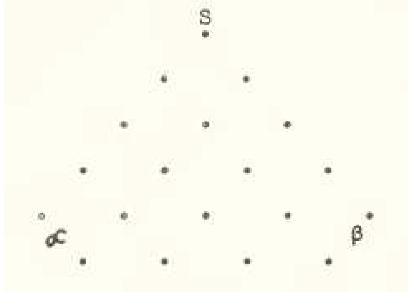

Fig. 6.

If one denotes the differences of $u$ in the direction of the lines of determination by $u'$ and $\grave{u}$, that is,

$$u_1' = u_1 - u_4\,,\quad \grave{u}_1 = u_1 - u_2\,,$$
$$u_2' = u_2 - u_3\,,\quad \grave{u}_4 = u_4 - u_3\,,$$

then the difference equation assumes the form

$u_1' = u_2'$.

I.e., along a line of determination the differences with respect to the other direction of determination are constant, and thus are equal to one of the <u>given</u> differences between the value at two points on the first two rows. Moreover the difference $u_S - u_\alpha$ is a sum of differences $u'$ along the determining line $\overline{\alpha S}$, so that using the remark above, we can obtain the final result (in obvious notation):

$$u_S = u_\alpha + \sum_{\alpha_1}^{\beta_1} u'\,. \tag{3}$$

We now let $h$ go to zero, and let the prescribed values on the second and first rows converge uniformly to a twice continuously differentiable function, $f(x)$, and the difference quotients $\dfrac{u'}{h\sqrt{2}}$ there converge uniformly to a continuously differentiable function $g(x)$. Evidently the right-hand side of (3) goes over uniformly to the expression

$$f(x-t) + \frac{1}{\sqrt{2}}\int_{x-t}^{x+t} g(\xi)\,d\xi \tag{4}$$

if $S$ converges to the point $(x, t)$. This is the well-known expression for the solution of the wave equation (1) with initial values $u(x, 0) = f(x)$ and $\dfrac{\partial u(x,0)}{\partial t} = f'(x) + \sqrt{2}g(x)$. Thus it is shown that as $h \to 0$ the solution of the difference equation converges to the solution of the differential equation provided the initial values converge appropriately

(as above).

**Section 2. On the Influence of the Choice of Mesh. The Domains of Dependence of the Difference and Differential Equations.**

The relationships developed in Section 1 suggest the following considerations.

In the same way that the solution of a linear hyperbolic equation at a point *S* depends only on a certain part of the initial line - namely the "domain of dependence" lying between the two characteristics drawn through *S*, the solution of the difference equation has also at the point *S* a certain domain of dependence defined by the lines of determination drawn through *S*. In Section 1 the directions for the lines of determination of the difference equation coincided with the characteristic directions for the differential equation so that the domains of dependence coincided in the limit. This result however was based essentially on the orientation of the mesh in the $(x, t)$ -plane, and depended furthermore and the fact that a square mesh had been chosen. We shall now consider a more general rectangular mesh with mesh size equal to $h$ (time interval) in the $t$-direction and equal to $kh$ (space interval) in the $x$-direction, where $k$ is a constant. The domain of dependence for the difference equation, $u_{t\bar{t}} - u_{x\bar{x}} = 0$ for this mesh will now either lie entirely within the domain of dependence of the differential equation, $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$, or on the other hand will contain this latter region inside its own domain according as $k < 1$ or $k > 1$, respectively.

From this follows a remarkable fact: if for the case $k < 1$ one lets $h \to 0$, then the solution to the difference equation in general ***cannot*** converge to the solution of the differential equation. In this case a change in the initial values of the solution of the differential equation in the neighborhood of the endpoints $\alpha$ and $\beta$ of the domain of dependence (see Fig. 7.) causes according to formula (4), a change in the solution itself at the point $(x, t)$. For the solution of the difference equation at the point *S* however the changes at the points $\alpha$ and $\beta$ are not relevant since these points lie outside of the domain of dependence of the difference equations. - That convergence does occur for the case $k > 1$ will be proved in Section 3. See for example Fig, 9.

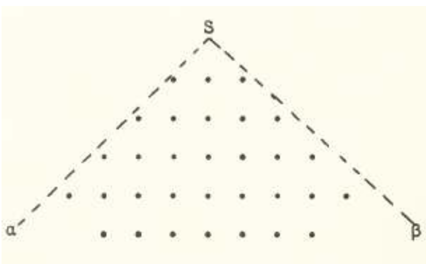

Fig. 7.

If we consider the differential equation

(5) $$2\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$$

in two space variables, $x$ and $y$, and time, $t$, and if we replace it by the corresponding difference equation on a rectilinear grid, then in contrast to the case of only two independent variables it is impossible to choose the mesh division so that the domain of dependence of the difference and differential equations coincide, since the domain of dependence of the difference equation is a quadrilateral while that of the differential equation is a circle. Later (cf. Section 4) we shall choose the mesh division so that the domain of determination of the difference equation contains that of the differential equation in its interior, and shall show that once again convergence occurs.

On the whole an essential result of this section will be that in the case of each linear homogeneous hyperbolic equation of second order one can choose the mesh so that the solution of the difference equation converges to the solution of the differential equation as $h \to 0$, (see for instance Sections 3, 4,7, 8).

**Section 3. Limiting Values for Arbitrary Rectangular Grids.**

Now we consider further the wave equation

(1) $$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0,$$

but impose it now on a rectangular grid with time interval $h$ and space interval $kh$. The corresponding difference equation is

(6) $$L(u)=\frac{1}{h^2}(u_1-2u_0+u_3)-\frac{1}{k^2h^2}(u_2-2u_0+u_4)=0,$$

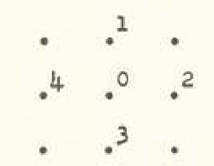

Fig. 8

where the indices represent a "fundamental rhombus" with midpoint $P_0$ and corners $P_1$, $P_2$, $P_3$, $P_4$ (see Fig. 8). According to the equation $L(u)$ = 0 the value of the function $u$ at a point $S$ can be represented by its values on that section of the initial rows $t$ = 0 and $t$ = $h$ cut out by lines of determination through $S$ parallel to the sides of an elementary rhombus. We assume that the initial values are prescribed in such a way that as $t \to 0$ for fixed $k$ the first difference quotients formed from them converge uniformly to given continuous functions on the line $t$ = 0, The initial values can be used to form an explicit representation of the solution of the difference equation (corresponding to (3) in Section 1); however it is too complicated to yield a limiting value easily as $h \to 0$. Thus we will try another approach which will also make it possible for us to treat the general

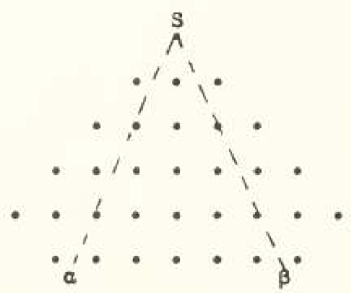

Fig. 9

problem[1].

We multiply the difference expression $L(u)$ by $(u_1 - u_3)$ and form the product using the following identities:

(7) $$(u_1 - u_3)(u_1 - 2u_0 + u_3) = (u_1 - u_0)^2 - (u_0 - u_3)^2,$$

(8) $$\begin{aligned}&(u_1 - u_3)(u_2 - 2u_0 + u_4) = (u_1 - u_0)^2 - (u_0 - u_3)^2\\ &-\frac{1}{2}\left[(u_1 - u_2)^2 + (u_1 - u_4)^2 - (u_2 - u_3)^2 - (u_4 - u_3)^2\right].\end{aligned}$$

Then we obtain

(9) $$\begin{aligned}&2(u_1 - u_3)L(u) = \frac{2}{h^2}(1 - \frac{1}{k^2})\left[(u_1 - u_0)^2 - (u_0 - u_3)^2\right]\\ &+\frac{1}{h^2k^2}\left[(u_1 - u_2)^2 + (u_1 - u_4)^2 - (u_2 - u_3)^2 - (u_4 - u_3)^2\right].\end{aligned}$$

The product (9) is now summed over all elementary rhombuses of the domain of determination. $S\alpha\beta$, The quadratic difference terms on the right-hand side always appear with alternate signs in two neighboring rhombuses so that they cancel out for any two rhombuses belonging to the triangle $S\alpha\beta$. Only the sums of squared differences over the "boundary" of the triangle remain, and we obtain the relation:

(10)

$$\begin{aligned}h^2\sum_{S\alpha\beta} 2\frac{u_1 - u_3}{h}L(u) &= h\sum_{S\alpha}\left[2\left(1 - \frac{1}{k^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{1}{k^2}\left(\frac{u'}{h}\right)^2\right]\\ &+h\sum_{S\beta}\left[2\left(1 - \frac{1}{k^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{1}{k^2}\left(\frac{`u}{h}\right)^2\right]\\ &-h\sum_{I,II}\left[2\left(1 - \frac{1}{k^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{1}{k^2}\left(\frac{u'}{h}\right)^2 + \frac{1}{k^2}\left(\frac{`u}{h}\right)^2\right]\end{aligned}$$

Here $u'$ and $`u$ denote differences in the direction of determination defined in Section 1**,** while $\dot{u}$ designates the difference of the functional values at two neighboring points on a mesh line parallel to the $t$ axis. The range in $\sum_{S\alpha}$ over which $(u')^2$ is taken is the outermost boundary line $S\alpha$ and its nearest parallel neighbor found by shifting the points of $S\alpha$ downward by the amount $h$. There is a similar range for $(`u)^2$ in $\sum_{S\beta}$, and so all of the differences, $u'$, $`u$, and $\dot{u}$ appear once and only once.

For the solution to the problem $L(u)$=0 the right-hand side of (10)

[1] For the following see K. Friedrichs and H. Lewy, "Uber die Eindeutigkeit...etc., Math. Ann. (98, 1928) p. 192 ff, where a similar transformation is used for integrals.

vanishes. The sum over the initial rows I and II which occurs there remains bounded as $h \to 0$ (for fixed *k*); specifically it goes over into an integral of the prescribed function along the initial line. Consequently the sums over $S\alpha$ and $S\beta$ in (10) also remain bounded. If now $k \geq 1$ as we must require (see previous discussion), then $1 - 1/k^2$ is non-negative, and we find that the individual sums

$$h\sum_{S\alpha}\left(\frac{u'}{h}\right)^2, \quad h\sum_{S\beta}\left(\frac{\grave{u}}{h}\right)^2,$$

extended over any line of determination whatever, remain bounded.

From this we can derive the "uniform continuity" (equicontinuity) (cf. Section 4 of the first part of the paper) of the sequence of grid functions in all directions in the plane[2]; since the values of *u* on the initial line are bounded, there must exist a subset which converges uniformly to a limit function *u*(*x, t*).

Both the first and second difference quotients of the function *u* also satisfy the difference equation *L*(*u*) = 0. Their initial values can be expressed through the equation *L*(*u*) = 0 in terms of the first, second and third difference quotients of *u* involving initial values at points on the two initial lines I and II only. We require that they tend to continuous limit functions, that is, that the given initial values *u*(*x,* 0), $u_t(x,0)$ be three times or respectively twice continuously differentiable with respect to *x*.

From this we can apply the convergence considerations set forth above to the first and second difference quotients of *u*, as well as to *u* itself, and we can choose a subsequence such that these difference quotients converge uniformly to certain functions, which must be the first or respectively second derivatives of the limit function *u*(*x, t*). Hence the limit function satisfies the differential equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ which results as the limit of the difference equation *L*(*u*) = 0; it represents indeed the solution of the initial value problem. Since such a solution is uniquely determined, every subsequence of mesh functions converges, and therefore the sequence itself converges to the limit function.

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[2] If $S_1$ and $S_2$ are two points at a distance $\delta$, then one connects them by a path of two segments, $S_1S$ and $SS_2$, where the former is parallel to one line of determination and the latter to the other. Then one finds the appraisal,

$$|u_{S_1} - u_{S_2}| \leq |u_{S_1} - u_S| + |u_S - u_{S_2}| \leq \sqrt{\delta}\sqrt{h\sum_{S_1S}\left(\frac{u'}{h}\right)^2} + \sqrt{\delta}\sqrt{h\sum_{SS_2}\left(\frac{\grave{u}}{h}\right)^2}.$$

**Section 4. The wave equation in three variables**

We treat next the wave equation,

(11) $$2\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$$

and consider its relation to the observations on the domain of dependence discussed in Section 2**.** The domain of dependence of the differential equation (11) is a circular cone with axis parallel to the *t*-direction and with apex angle $\alpha$, where $\tan\alpha=\frac{1}{\sqrt{2}}$. In any rectilinear grid parallel to the axes we introduce the corresponding difference equation

(12) $$2u_{t\bar{t}}-u_{x\bar{x}}-u_{y\bar{y}}=0\,.$$

This equation relates the functional values of $u$ at points of an elementary tetrahedron. In fact it allows the value of the function $u$ at a point $S$ to be expressed uniquely in terms of the values of the function at certain points of the two initial planes $t$ = 0 and $t$ = $h$. At each point $S$ we obtain a pyramid of determination which cuts out from the two base planes two rhombuses as domains of dependence.

If we let the mesh widths tend to zero, keeping their ratios fixed, then we can expect convergence of the sequence of mesh functions to the solution of the differential equation only provided the pyramid of determination contains the cone of determination of the differential equation in its interior. The simplest grid with this property will be one constructed in such a way that the pyramid of determination is tangent to the exterior of the cone of determination. Our differential equation is chosen so that this occurs for a grid of cubes parallel to the axes.

The difference equation (12), in the notation of Fig. 10, assumes for such a grid the form:

(13)

$$L(u)=\frac{2}{h^2}(u'_\alpha-2u_0+u_\alpha)-\frac{1}{h^2}(u_1-2u_0+u_3)-\frac{1}{h^2}(u_2-2u_0+u_4)\,,$$

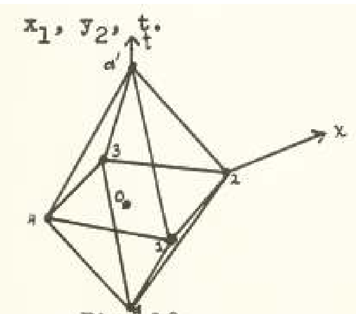

Fig. 10.

in which the functional value, $u_0$, at the midpoint actually cancels out. The values of the solution on the two initial planes must be the values of a function having four continuous derivatives with respect to $x_1 y_2 t$.

For the convergence proof we again use the method developed in Section 3. We construct the triple sum

$$h^2 \sum\sum\sum 2\frac{u'_\alpha - u_\alpha}{h} L(u) = 0$$

for the solution to the difference equation, where the summation is to be extended over all elementary octahedrons of the pyramid of determination emanating from the point $S$. Then almost exactly as before we find that the values of the function $u$ at the interior points of the pyramid of determination cancel out in the summation and that only the values on the two pyramids called $F$, and on the two base surfaces I and II remain.

If we denote by $u'$ the difference of the values of the function at two points connected by a line of an elementary octahedron, then we can write the result as

$$\sum_F \sum (u')^2 - \sum_I \sum_{II} (u')^2 = 0 , \tag{14}$$

where the sum is extended over all surfaces containing differences $u'$; each such difference is to appear only once[3].(26) The double sum over the two initial surfaces stays bounded since it goes over into an integral of the initial values. Therefore the sum over the "surface of determination" $F$ remains bounded.

We now apply these results not to $u$ itself, but to its first, second and third difference quotients, which themselves satisfy the difference equation (13). Their initial values can be expressed using only values on the first two initial planes by means of (13) using first through fourth difference quotients. If $w = w_h$ is one of the difference quotients of any order up to third order, then we know that the sum $h^2 \sum\sum \left(\frac{w'}{h}\right)^2$ extended over a surface of determination remains bounded. From this it follows, through exactly the methods used in Section 4 of the first part of the paper, that the function $u$ and its first and second difference quotients are uniformly continuous (equicontinuous). Thus there exists a sequence of mesh widths decreasing to zero such that these quantities, which are bounded initially, converge to continuous limit functions and, in fact, converge to the solution of the differential equation and to the first and second derivatives of this solution, all exactly as we found earlier (Section 3).

**Appendix. Supplements and generalizations**

[3] The grid ratio has been chosen in such a way that the differences between values of $u$ appearing on the two neighboring surfaces in $F$ do not occur, (as they did in the general case in one dimension treated in Section 3).

## Section 5. Example of a differential equation of first order

We have seen in Section *2* that in the case when the region of dependence of the differential equation covers only a part of the region of dependence of the difference equation, the influence of the rest of the region is not included in the limit. We can demonstrate this phenomenon explicitly by the example of the differential equation of first order, $\frac{\partial u}{\partial t}=0$ if we replace it by the difference equation

(15) $$2u_t - u_x + u_{\bar{x}} = 0 .$$

In the notation of Fig. 5 this becomes

(16) $$u_1 = \frac{u_2 + u_4}{2} .$$

**As** before, the difference equation connects only the points of a submesh with one another. The initial value problem consists of assigning as initial values for $u$ at points $x = 2ih$ on the row $t = 0$ the values, $f_{2i}$, assumed there by a continuous function $f(x)$.

We consider the point $S$ at a distance $2nh$ up along the $t$-axis. It is easy to verify that the solution $u$ at $S$ is represented as

(17) $$u_S = \sum_{i=-n}^{n} \frac{1}{2^{2n}} \binom{2n}{n+i} f_{2i} .$$

As the mesh size decreases, that is as $n \to \infty$, the sum on the right-hand side tends simply to the value $f_0$. This can be demonstrated from the continuity of $f(x)$ and from the behavior of the binomial coefficients as $n$ increases (see the following paragraph).

## Section 6. The equation of heat conduction

The difference equation (16) of Section 5 can also be interpreted as the analogue of an entirely different differential equation, namely the equation of heat conduction,

(18) $$2\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 .$$

In any rectangular mesh with mesh spacing $l$ and $h$ in the time and space directions, respectively, the corresponding difference equation becomes

(19) $$2\left(\frac{u_1 - u_0}{l}\right) = \left(\frac{u_2 + u_4 - 2u_0}{h^2}\right) .$$

In the limit as the mesh size goes to zero the difference equation preserves its form only if $l$ and $h^2$ are decreased proportionately. In particular if we set $l = h^2$, then the value $u_0$ drops out of the

equation and the difference equation becomes

(16) $$u_1 = \frac{u_2 + u_4}{2}.$$

The solution to (16) is given by formula

(17) $$u(0,t) = \sum_{i=-n}^{n} \frac{1}{2^{2n}} \binom{2n}{n+i} f_{2i}.$$

**As** the mesh width decreases, a point $\xi$ on the $x$-axis is always characterized by the index

(20) $$2i = \frac{\xi}{h}.$$

The mesh width $h$ is related to the ordinate $t$ of a particular point by

(21) $$2nh^2 = t.$$

We shall investigate what happens to formula (17) as $h \to 0$, that is $n \to \infty$. Using (21) we write (17) in the form

(22) $$u(0,t) = \sum_{i=-n}^{n} \frac{\sqrt{2n}}{2 \cdot 2^{2n} \sqrt{t}} \binom{2n}{n+i} f_{2i} \cdot 2h_*.$$

For the coefficient of $2hf_{2i} = 2hf(\xi)$ we use the abbreviation

$$\frac{1}{2\sqrt{t}} g_{2n}(\xi) = \frac{\sqrt{2n}}{2 \cdot 2^{2n} \sqrt{t}} \binom{2n}{n + \frac{\xi}{\sqrt{2t}} \sqrt{n}}.$$

The limiting value of the coefficient, which is usually determined by using Stirling's formula, we will calculate here by considering the function $g_{2n}(\xi)$ as the solution of an ordinary difference equation which approaches a differential equation as $h \to 0$. As the difference equation one finds

$$\frac{1}{2h}[g_h(\xi + 2h) - g_h(\xi)] = -\frac{1}{2h} g_h(\xi) \frac{2i+1}{n+i+1}$$

(in which we have written $g_h(\xi)$ instead of $g_{2n}(\xi)$). Or

$$\frac{1}{2h}[g_h(\xi + 2h) - g_h(\xi)] = -g_h(\xi) \frac{\xi + h}{t + h\xi + 2h^2}.$$

$g_h(\xi)$ satisfies the normalization condition

$$\sum_{i=-n}^{n} g_h(\xi) \cdot 2h = 2\sqrt{t}.$$

This sum is over the region of dependence of the difference equation, and as $h \to 0$ this covers the entire $x$-axis.

It can be shown easily that $g_h(\xi)$ converges uniformly to the solution $g(x)$ of the differential equation

$$g'(x) = -\frac{g(x)x}{t}$$

with the auxiliary condition

$$\int_{-\infty}^{\infty} g(x)dx = 2\sqrt{t} \ .$$

From formula (22) after passing to the limit we find

$$u(0,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\xi^2/2t} f(\xi)d\xi$$

which is the known solution of the heat conduction equation.

The results of this section can be carried over directly to the case of the differential equation,

$$4\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

and so on for even more independent variables.

**Section 7. The general homogeneous linear equation of second order in the plane**

We consider the differential equation

(23) $$\frac{\partial^2 u}{\partial t^2} - k^2\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial x} + \gamma u = 0 \ .$$

The coefficients are assumed to be twice continuously differentiable with respect to $x$ and $t$, while the initial values on the line $t = 0$ are assumed three times continuously differentiable with respect to $x$. We replace the differential equation by the difference equation

(24) $$L(u) = u_{t\bar{t}}(x,t) - k^2 u_{x\bar{x}}(x,t) + \alpha u_t + \beta u_x + \gamma u = 0$$

in a grid with time mesh width $h$ and space mesh width $ch$ so that in a neighborhood of the appropriate part of the initial value line the inequality $1 - \frac{k^2}{c^2} > \varepsilon > 0$ holds for the constant, c. The initial values are to be chosen as in Section 3.

For the proof of convergence we again transform the sum,

$$h^2 \sum_{S\alpha\beta}\sum 2\frac{u_1 - u_3}{h} L(u)$$

by using identities (7) and (8). In addition to a sum (see for example (10)) over the doubled boundary of the triangle $S\alpha\beta$ (Fig, 6) one obtains a sum over the entire triangle $S\alpha\beta$ because of the variability of the coefficient $k$ and the presence of lower order derivatives in the differential equation. By using the differentiability of $k$ and the Schwarz inequality one can show that this latter sum is bounded from above by

$$Ch^2 \sum_{S\alpha\beta}\sum \left[ \left(\frac{\grave{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{\dot{u}}{h}\right)^2 + u^2 \right]$$

where the constant $C$ is independent of the function $u$, the mesh width $h$, and, in a certain neighborhood of the initial line, also independent of the point $S$.

Again we can estimate an upper bound for $h^2 \sum_{S\alpha\beta}\sum u^2$ by[4]

$$C_1 h^2 \sum_{S\alpha\beta}\sum \left(\frac{\dot{u}}{h}\right)^2 + C_2 h \sum_{I,II} u^2 ,$$

where the same properties hold for $C_1$ and $C_2$ as are stated above for $C$.

Thus we obtain an inequality of the form

(25)

$$h\sum_{S\alpha}\left[ 2\left(1-\frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{u'}{h}\right)^2 \right] + h\sum_{S\beta}\left[ 2\left(1-\frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{\grave{u}}{h}\right)^2 \right]$$
$$\leq C_3 h^2 \sum_{S\alpha\beta}\sum \left[ \left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{\grave{u}}{h}\right)^2 \right] + D$$

where $D$, for all points $S$ and mesh widths $h$, is a fixed bound for the sums over the initial line.

As vertices of our triangles we choose a sequence of points $S_0$, $S_1$, …, $S_n = S$ lying on a line parallel to the $t$-axis. By summing the corresponding sequence of inequalities (25) after multiplying by $h$ we obtain the following inequality

(26)

$$h^2\sum_{SS\alpha}\left[ 2\left(1-\frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{u'}{h}\right)^2 \right] + h^2\sum_{SS\beta}\left[ 2\left(1-\frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{\grave{u}}{h}\right)^2 \right]$$
$$\leq nh^3 C_3 \sum_{S\alpha\beta}\sum \left[ \left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{\grave{u}}{h}\right)^2 \right] + nhD$$

Now if we notice that one can express a difference $u'$ or $\grave{u}$ in terms of two differences $\dot{u}$ and a difference $\grave{u}$ or respectively $u'$, then we see that the left-hand side of (26) is larger than the simpler form

[4] For proof one makes use of the inequality used in Footnote 25.

$$C_4 h^2 \sum_{S\alpha\beta}\sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{\grave{u}}{h}\right)^2\right],$$

with a suitable constant $C_4$.

Then by confining the discussion to a sufficiently small neighborhood, $0 \le t \le nh = \delta$ of the initial line where $\delta$ is small enough so that

$$C_4 - nhC_3 = C_5 > 0,$$

we find from (26)

$$C_5 h^2 \sum_{S\alpha\beta}\sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{\dot{u}}{h}\right)^2\right] \le \frac{C_4}{C_3 D}. \tag{27}$$

The bound given by (27) when combined with (25) gives a bound on

$$h\sum_{S\alpha}\left(\frac{u'}{h}\right)^2 + h\sum_{S\beta}\left(\frac{\grave{u}}{h}\right)^2,$$

from which, as in Section 3, one can prove the uniform continuity of $u$.

We apply the inequality (25) now, instead of to the function $u$ itself, to its first and second difference quotients, $w$, which also satisfy difference equations whose second order terms are the same as those of (24). In the rest of the terms there will appear lower order differences of $u$ which cannot be expressed in terms $w$, but they will appear in the above argument in the form of quadratic double sums multiplied by $h^2$. This is enough to let us reach the same conclusions for the difference equation for $w$ as we found above for $\boldsymbol{u}$. So we can conclude from this the uniform continuity (equicontinuity) and boundedness of the function $u$ and its first and second derivatives. Consequently a subsequence exists which converges uniformly to the solution of the initial value problem for the differential equation. Again from the uniqueness of the solution we find that the sequence of functions itself converges.

In all of this the assumption must be made that the difference quotients up to third order involving values on the two initial lines converge to continuous limit functions[5].

**Section 8. The initial value problem for an arbitrary linear**

[5] This assumption and also the assumptions on the differentiability of the coefficients of the differential equation, and further on the restriction to a sufficiently small region of the initial line can be weakened in special cases.

**hyperbolic differential equation of second order**

We shall now show that the methods developed so far are adequate for solving the initial value problem for an arbitrary homogeneous linear hyperbolic differential equation of second order. It suffices to limit the discussion to the case of three variables. The development can be extended immediately to the case of more variables. It is easy to see that a transformation of variables can reduce the most general problem of this type to the following: a function $u(x, y, t)$ is to be found which satisfies the differential equation

(28) $$u_{tt} - (au_{xx} 2bu_{xy} + cu_{yy}) + \alpha u_t + \beta u_x + \gamma u_y + \delta u = 0 \text{ ,}$$

and which, together with its first derivative, assumes prescribed values on the surface $t = 0$. The coefficients in Eq. (28) are functions of the variables $x$, $y$, and $t$ and satisfy the condition

$$a > 0 \text{ , } \quad c > 0 \text{ , } \quad ac - b^2 > 0 \text{ .}$$

We assume that the coefficients are three times differentiable with respect to $x$, $y$, and $t$, and that the initial values $u$ and $u_t$ are respectively four and three times continuously differentiable with respect to $x$ and $y$.

The coordinates $x$ and $y$ can be drawn from a given point on the initial plane in such a way that $b = 0$ there. Then of course in a certain neighborhood of this point the conditions

$$a - |b| > 0 \text{ , } \quad c - |b| > 0$$

hold. We restrict our investigation to this neighborhood. We can choose a three times continuously differentiable function $d > 0$ so that

(29) $$\left.\begin{matrix} a - d \\ c - d \\ d - |b| \end{matrix}\right\} > \varepsilon > 0$$

for some constant $\varepsilon$. Then we put the differential equation into the form

(30) $$\begin{aligned} & u_{tt} - (a-d)u_{xx} - (c-d)u_{yy} \\ & -\frac{1}{2}(d+b)(u_{xx} + 2u_{xy} + u_{yy}) \\ & -\frac{1}{2}(d-b)(u_{xx} - 2u_{xy} + u_{yy}) \\ & + \alpha u_t + \beta u_x + \gamma u_y + \delta u = 0 \end{aligned} \text{ .}$$

We now construct in the space a grid of points, $t = lh$, $x + y = mkh$, $x - y = nkl$ ($l,m,n = \ldots - 1, 0, 1, 2 \ldots$) and we replace Eq. (30) by a difference equation $L(u) = 0$ over this mesh. We do this by assigning to each point $P_0$ the following neighborhood: The point $P_{\alpha'}$ or the point

$P_\alpha$ which lies at a distance $h$ or $-h$ respectively along the $t$-axis from $P_0$; also the points $P_1, \cdots, P_8$ which lie in the same ($x$, $y$)-plane with $P_0$ (see Fig. 11**).** These points constitute an "elementary octahedron" with vertices $P_{\alpha'}$, $P_\alpha$, $P_1$, $P_2$, $P_3$, $P_4$. For each grid point lying in the interior of $G$ we replace the derivatives appearing in Eq. (30) by difference quotients over the elementary octahedron about $P_0$.

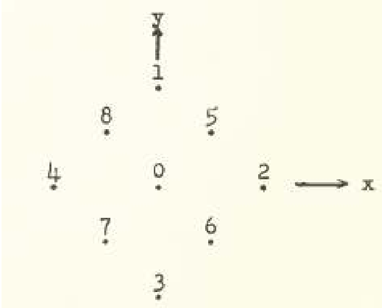

Fig. 11

We replace

$$u_{tt} \text{ by } \frac{1}{h^2}(u_{\alpha'} - 2u_0 + u_\alpha),$$

$$u_{xx} \text{ by } \frac{1}{k^2h^2}(u_2 - 2u_0 + u_4),$$

$$u_{yy} \text{ by } \frac{1}{k^2h^2}(u_1 - 2u_0 + u_3),$$

$$u_{xx} + 2u_{xy} + u_{yy} \text{ by } \frac{4}{k^2h^2}(u_6 - 2u_0 + u_8),$$

$$u_{xx} - 2u_{xy} + u_{yy} \text{ by } \frac{4}{k^2h^2}(u_5 - 2u_0 + u_7).$$

The first derivatives in (30) are replaced by the corresponding difference quotients in the elementary octahedron. The coefficients of the difference equation are given the values assumed by the coefficients of the differential equation at the point $P_0$.

On the initial planes $t = 0$ and $t = h$ we assume that the values of the function are assigned in such a way that as the mesh size approaches zero for fixed $k$, the function approaches the prescribed initial value, and the difference quotients over the two planes up through differences of fourth order converge uniformly to the prescribed derivatives.

The solution of the difference equation $L(u) = 0$ at a point is uniquely determined by the values on the two base surfaces of the pyramid of determination passing through the point.

To prove convergence we construct a sum

$$h^3 \sum\sum\sum 2\frac{u_{\alpha'} - u_\alpha}{h} L(u)$$

over all the elementary octahedrons of the pyramid of determination, and we transform it by using identities **(7)** and (8). This gives one space summation multiplied by $h^3$ and quadratic in the first difference quotients, and also over a double surface a sum which is multiplied by $h^2$ and contains the squares of all the difference quotients of the type $u_\alpha - u_0$, $u_\alpha - u_1$, …, $u_\alpha - u_8$. In this

expression according to (29) the coefficients are larger than some fixed positive constant in all those cases where the ratio $1/k$ between the time and space mesh sizes is taken sufficiently small.

From here on one can proceed exactly as before (Sections 7, 4) and prove that the solution of the difference equation converges to the solution of the differential equation.

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