CHAPTER I

Introductory Remarks

§1. *General Information about the Variety of Solutions*

1. Examples. For an ordinary differential equation of n-th order, the totality of solutions (except possible "singular" solutions) is a function of the independent variable x which also depends on n arbitrary integration constants $c_1, c_2, \cdots, c_n$. Conversely, for every n-parameter family of functions

$$u = \phi(x; c_1, c_2, \cdots, c_n),$$

there is an n-th order differential equation with the solution $u = \phi$ obtained by eliminating the parameters $c_1, c_2, \cdots, c_n$ from the equation $u = \phi(x; c_1, c_2, \cdots, c_n)$ and from the n equations

$$\begin{cases} u' = \phi'(x; c_1, c_2, \cdots, c_n) \\ \cdots \\ u^{(n)} = \phi^{(n)}(x; c_1, c_2, \cdots, c_n) \end{cases}$$

For partial differential equations the situation is more complicated. Here too, one may seek the totality of solutions or the *"general solution"*; i.e., one may seek a solution which, after certain "arbitrary" elements are fixed, represents every individual solution (again with the possible exception of certain "singular" solutions). In the case of partial differential equations such arbitrary elements can no longer occur in the form of constants of integration, but must involve arbitrary functions; in general, the number of these arbitrary functions is equal to the order of the differential equation. These arbitrary functions depend on one independent variable less than the solution u. A more precise statement of the situation is implied in the existence theorem of §7. In the present section however, we merely collect information by studying a few examples.

1) The differential equation

$$u_y = 0$$

for a function $u(x, y)$ states that u does not depend on y; hence,

$$u = w(x)$$

where $w(x)$ is an arbitrary function of x.

2) For the equation

$$u_{xy} = 0,$$

one immediately obtains the general solution

$$u = w(x) + v(y).$$

3) Similarly, the solution of the nonhomogeneous differential equation

$$u_{xy} = f(x, y)$$

is

$$u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} f(\xi, \eta) d\xi d\eta + w(x) + v(y)$$

with arbitrary functions w and v and fixed values x_0, y_0.

More generally, one may replace the integral by an area integral if one takes, as the region of integration Δ, a "triangle" such as that in Figure 1, whose curved boundary consists of a curve C: $y = g(x)$ or $x = h(y)$ which is not intersected more than once by any of the curves x = const. or y = const. Then,

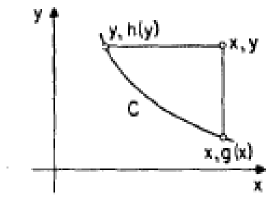

FIGURE 1

(2) $$u(x, y) = \iint_{\Delta} f(\xi, \eta) d\xi d\eta + w(x) + v(y)$$

$$u_x = \int_{g(x)}^{y} f(\xi, \eta) d\eta + w'(x), \quad u_y = \int_{h(x)}^{x} f(\xi, \eta) d\xi + v'(y)$$

The special solution of the differential equation for $w(x) = v(y) = 0$ satisfies the condition $u = u_x = u_y = 0$ for all points (x, y) on the curve C.

4) The partial differential equation

$$u_{x=u_y}$$

is transformed into the equation

$$2\omega_\eta = 0$$

by the transformation of variables

$$x + y = \xi, \quad x - y = \eta, \quad u(x, y) = \omega(\xi, \eta).$$

The "*general solution*" of the transformed equation is $\omega = w(\xi)$; therefore

$$u = w(x + y).$$

Similarly, if α and β are constants, the general solution of the differential equation

$$\alpha u_x + \beta u_y = 0$$

is

$$u = w(\beta x - \alpha y).$$

5) According t o elementary theorems of the differential calculus, the partial differential equation

$$u_x g_y - u_y g_x = 0,$$

where $g(x\ y)$ is any given function of x,y states that the Jacobian

$$\frac{\partial(u,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = u_x g_y - u_y g_x$$

of *u, g* with respect to *x,* vanishes. This means that *u* depends on *g***,** i.e., that

(3) $$u = w[g(x,y)],$$

where *w* is an arbitrary function of the quantity *g*. Since, conversely, every function *u* of the form (3) satisfies the differential equation $u_x g_y - u_y g_x = 0$, we obtain the totality of solutions by means of the arbitrary function *w*.

It is noteworthy that the same result holds for the more general -- quasi-linear -- differential equation

$$u_x g_y(x,y,u) - u_y g_x(x,y,u) = 0,$$

where *g* now depends explicitly not only on x, y but on the unknown function *u*(x, y) as well. For, as one sees, the Jacobian of any solution *u* (x , y) and $\gamma(x,y) = g[x,y,u(x,y)]$ vanishes since

$$\begin{aligned} u_x \gamma_y - u_y \gamma_x &= u_x(g_y + g_u u_y) - u_y(g_x + g_u u_x) \\ &= u_x g_y - u_y g_x \\ &= 0 \end{aligned}$$

Thus, even in this case, the solution is given by the relation

(4) $$u(x,y) = W[g(x,y,u)],$$

which is an implicit definition of *u* by means of the arbitrary function *W*.

For instance, the solution *u*(x, y) of the differential equation

$$\alpha(u)u_x - \beta(u)u_y = 0$$

is implicitly defined by

(5) $$u = W[\alpha(u)y + \beta(u)x],$$

(or by $\alpha(u)y + \beta(u)x = w(u)$), so that *u* depends on the arbitrary function *W* in a rather involved way. (An application will be given in §7,1.)

A special case of the differential equation $\alpha(u)u_x - \beta(u)u_y = 0$ is

$$u_y + uu_x = 0;$$

the solution is given implicitly by

$$u = W(-x + uy),$$

where *W* is arbitrary. If *u* = *u*(*x*(*y*),*y*) is interpreted as the velocity of a particle at a point *x* = *x*(**y**) moving with the time *y*, then the differential equation states that the acceleration of all the particles is zero.

6) The partial differential equation of second order

$$u_{xx} - u_{yy} = 0$$

is transformed into

$$4\omega_{\xi\eta} = 0$$

by the transformation

$$x + y = \xi\,,\quad x - y = \eta\,,\quad u(x,y) = \omega(\xi,\eta)\,.$$

Hence, according to example 2), its solutions are

$$u(x,y) = w(x+y) + v(x-y)\,.$$

7) In a similar way the general solution of the differential equation

$$u_{xx} - \frac{1}{t^2}u_{yy} = 0$$

for any value of the parameter t is

$$u = w(x+ty) + v(x-ty)\,.$$

In particular, the functions

$$u = (x+ty)^n \quad \text{and} \quad u = (x-ty)^n$$

are solutions; i.e.,

$$t^2 u_{xx} - u_{yy}$$

vanishes for all x, y and for all real t.

8) According to elementary algebra, if a polynomial in t vanishes for all real values of t, then it vanishes for all complex values of t as well. Thus, if we substitute $t = i = \sqrt{-1}$, the differential equation of example **7)** is transformed into the *potential equation*

$$\Delta u \equiv u_{xx} + u_{yy} = 0\,;$$

for this equation we obtain solutions of the form

$$(x+iy)^n = P_n(x,y) + iQ_n(x,y)\,,$$

$$(x-iy)^n = P_n(x,y) - iQ_n(x,y)\,,$$

where P_n and Q_n are polynomials with real coefficients which must themselves satisfy the potential equation. Letting n range over the numbers 0, 1, 2,…, then, we have found infinitely many solutions of the potential equation but, in contrast to the previous examples, so far only denumerably many solutions.

In polar coordinates r,θ defined by $x = r\cos\theta$, $y = r\sin\theta$, we have

(6) $$P_n(x,y) = r^n \cos n\theta\,,\quad Q_n(x,y) = r^n \sin n\theta\,.$$

For any arbitrary real α, the functions

$$P_\alpha(x,y) = r^\alpha \cos\alpha\theta\,,\quad Q_\alpha(x,y) = r^\alpha \sin\alpha\theta$$

also satisfy the potential equation in any region of the x, y-plane excluding the origin $x = y = 0$. This is immediately verified after

transforming Δu into polar coordinates (cf. Vol. I, p.226):

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{1}{r^2} u_{\theta\theta} .$$

If we choose two functions $w(\alpha)$ and $v(\alpha)$ in such a way that the first and second order derivatives of the integrals

$$\int_a^b w(\alpha) r^\alpha \cos \alpha\theta d\alpha \quad \text{and} \quad \int_a^b v(\alpha) r^\alpha \sin \alpha\theta d\alpha$$

can be obtained by differentiation under the integral sign, then we can construct a family of solutions, depending on two arbitrary functions w and v**,** of .the form

$$\int_a^b r^\alpha (w(\alpha) cis\, \alpha\theta + v(\alpha) \sin \alpha\theta) d\alpha .$$

9) As an example of a differential equation of higher order, we consider

$$u_{xxyy} = 0$$

and find that

$$u(x, y) = w(y) + x w_1(y) + v(x) + y v_1(x)$$

is its general solution.

10) If the number of independent variables is greater than two, then arbitrary functions depending on two or more variables occur in the general solution. For example, the differential equation

$$u_z = 0$$

for the function $u(x,y,z)$ has the general solution

$$u = w(x, y) .$$

§3. *Methods of Integration for Special Differential Equations*

1. Separation of Variables. For many differential equation problems in mathematical physics families of solutions depending on arbitrary parameters can be obtained by special methods, although these methods do not give the totality of solutions directly.

The most important of these methods is *separation of variables*; it will be illustrated by several examples.

1) Consider the equation

$$u_x^2 + u_y^2 = 1 ;$$

assuming

$$u(x, y) = \phi(x) + \psi(y) ,$$

we obtain

$$(\phi'(x))^2 + (\psi'(y))^2 = 1$$

or

$$(\phi'(x))^2 = 1 - (\psi'(y))^2 .$$

Since the right side is independent of x and the left side independent of y, both sides are independent of x as well as y, hence equal to the same constant α^2; thus we obtain immediately the family of solutions

(1) $$u(x, y) = \alpha x + \sqrt{1-\alpha^2}\, y + \beta$$

with the two arbitrary parameters α and β.

2) Similarly, the differential equation

$$u_x^2 + u_y^2 + u_z^2 = 1$$

for a function u of the three variables x, y, z, leads, if we assume that $u = \phi(x) + \psi(y) + \chi(z)$, to the family of solutions

(2) $$u = \alpha x + \beta y + \sqrt{1-\alpha^2-\beta^2}\, z + \gamma$$

depending on three arbitrary parameters α, β, γ.

3) The tentative hypothesis

$$u = \phi(x) + \psi(y)$$

applied to the differential equation

$$f(x)u_x^2 + g(y)u_y^2 = a(x) + b(y)$$

leads, as in previous examples, to

(3) $$u(x, y) = \int_{x_0}^{x} \sqrt{\frac{a(\xi)+\alpha}{f(\xi)}}\, d\xi + \int_{y_0}^{y} \sqrt{\frac{b(\eta)+\alpha}{g(\eta)}}\, d\eta + \beta$$

where α and β are arbitrary constants.

4) A transformation of the independent variables frequently makes separation of variables successful afterwards. For example, the equation

$$u_x^2 + u_y^2 = \frac{k}{r} - h \quad (r^2 = x^2 + y^2;\ k, h \text{ constants})$$

for $u(x, y)$ occurring in the *two body problem* of celestial mechanics is transformed into the equation

$$u_r + \frac{1}{r^2}u_\theta^2 = \frac{k}{r} - h$$

or

$$r^2 u_r^2 + u_\theta^2 = kr - hr^2$$

for $u(r,\theta)$ in polar coordinate r,θ. Hence, formula (3) yields the family of solutions

(4) $$u = \int_0^r \sqrt{\frac{k}{\rho} - h - \frac{\alpha^2}{\rho^2}}\, d\rho + \alpha\theta + \beta$$

which depends on two arbitrary parameters α, β.

the second order, it is often profitable to set

5) In the case of linear differential equations, particularly those of the second order, it is often profitable to set

$$u(x,y) = \phi(x)\psi(y)$$

(examples are given in Vol. I, Ch. V, §§3-9). For the *heat equation*

(5) $$u_{xx} - u_y = 0$$

we have

$$\phi''(x) : \phi(x) = \psi'(y) : \psi(y)$$

and therefore both the right and the left sides must be constant. One may assume this constant to be either positive or negative and denote it, accordingly, by ν^2 or $-\nu^2$; thus one obtains the two families of solutions

$$u = a \sinh \nu(x - \alpha) e^{\nu^2 y},$$

$$u = a \sin \nu(x - \alpha) e^{-\nu^2 y}.$$

The latter plays a particular role in mathematical physics; if u is the temperature, y the time, and x a space coordinate, it describes a temperature distribution which tends to zero as time progresses.

2. Construction of Further Solutions by superposition. Fundamental Solution of the Heat Equation. Poisson's Integral. From the solutions of linear differential equations containing parameters, further solutions may be obtained by summation, integration, and differentiation processes. Since many such examples are given in Vol. I , Ch. V, only a few more will be discussed here.

In order to obtain another solution of the heat equation, we integrate the solution $e^{-\nu^2 y} \cos \nu x$ with respect to the parameter ν between the limits $-\infty$ and ∞ and find the new solution

$$u = \int_{-\infty}^{\infty} e^{-\nu^2 y} \cos \nu x \, d\nu \qquad (y>0).$$

The integral on the right can easily be computed and yields

(6) $$u = \sqrt{\frac{\pi}{y}} e^{-x^2/4y},$$

the "*fundamental solution*" of the heat equation.

As a second example for the principle of superposition, we give the solution of the *boundary value problem for the potential equation* $\Delta u = 0$ for the circular disk $r^2 = x^2 + y^2 < 1$; for $r = 1$, the boundary values of u are given as a (continuously differentiable) function $g(\theta)$ of the polar angle θ. Let

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\phi) \cos n\phi d\phi, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\phi) \sin n\phi d\phi$$

be the coefficients of the Fourier series for $g(\theta)$; then

$$u(x,y) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) r^\nu$$
$$= \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu P_\nu(x,y) + b_\nu Q_\nu(x,y))$$

converges uniformly for $r \le q < 1$. This series, which may be differentiated twice term by term for $r \le q$, represents a superposition of the potential functions P_n and Q_n considered in example 8) of §1,1. Hence it is a harmonic function, and moreover solves the boundary value problem. In the interior of the circle, we may interchange summation and integration and obtain

$$u(x,y) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\phi) \left[\frac{1}{2} + \sum_{\nu=1}^{\infty} r' \cos \nu(\theta - \phi) \right] d\phi.$$

Writing $2\cos\alpha = e^{i\alpha} + e^{-i\alpha}$ and summing the geometric series thus obtained under the integral sign we arrive, after a trivial manipulation, at the expression

(7) $$u(x,y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} g(\phi) d\phi,$$

which represents the solution of the boundary value problem by means of the *Poisson integral* (cf. Ch. IV, and also Vol. I, p.514).

§4. *Geometric Interpretation of a First Order Partial Differential Equation in Two Independent Variables. The Complete Integral*

1. Geometric Interpretation of a First Order Partial Differential Equation. Geometric intuition is of great help in the theory of integration of first order partial differential equations for a function $u(x, y)$ of two independent variables. Given the differential equation

$$F(x, y, u, p, q) = 0\,,$$

$F_p^2 + F_q^2 \neq 0$, with the abbreviation $p = u_x$, $q = u_y$. Then for every integral surface through the point P with coordinates x, y, u the quantities p and q, which determine the position of the tangent plane at this point, must satisfy condition (1). The tangent plane of an integral surface at the point P is restricted to positions which belong to the manifold characterized by equation (1). For a given point $P:(x, y, u)$, this manifold is in general a one-parameter family (for example, for $p^2 + q^2 = 1$ the family is $p = \cos t$, $q = \sin t$ with the parameter t). If F is linear in p and q then this family of possible tangent planes forms an *axial pencil of planes* through a straight line called the ***"****Monge axis****"***. For the present, we ignore this special case of the first order "quasi-linear" equation which will be discussed in §5; instead we assume that, at every point P considered, our family of planes envelops a genuine cone, the ***"****Monge cone****"***. Thus, in a domain of the ***x****,* *y,* *u*-space the differential equation is represented geometrically by a "*cone field*" just as an ordinary differential equation of the first order is represented by a *direction field*. To find a solution means to find a surface which at each of its points touches the corresponding Monge cone (or "fits" into the cone field).

As in the case of ordinary differential equations, geometric visualization makes the following theorem evident: If *a family of solutions*

(2) $$u = f(x, y, a)$$

of the differential equation $F(x, y, u, p, q) = 0$ *depending on a parameter a possesses an envelope, then this envelope is also a solution.*

Indeed, the envelope of a family of integral surfaces has at each point P a tangent plane which touches the Monge cone there; this tangent plane is the same as that of the integral surface of the family which touches the envelope at *P.*

Analytically, one is led to this statement in the following way: the envelope is obtained by expressing a as a function of x and y from the equation

(3) $$f_a(x, y, a) = 0\,,$$

and then inserting this function $a(x,\ y)$ into f; thus the envelope appears in the form

$$u = f(x, y, a(x, y)) = \psi(x, y).$$

Then, using (3), we have

$$\psi_x = u_x = f_x + f_a a_x = f_x, \quad \psi_y = u_y = f_y + f_a a_y = f_y.$$

Accordingly, the values of $\psi(x_0, y_0)$, $\psi_x(x_0, y_0)$, $\psi_y(x_0, y_0)$ coincide, at a fixed point (x_0, y_0), with the values of $f(x, y, a_0)$, $f_x(x, y, a_0)$, $f_y(x, y, a_0)$, respectively, where $a_0 = a(x_0, y_0)$. Therefore, since the differential equation is satisfied by the function $u = f(x, y, a_0)$ at the point (x_0, y_0), it is also satisfied by $u = f(x, y, a(x, y)) = \psi(x, y)$.

2. The Complete Integral.

3. Singular Integrals.

4. Examples. We consider the two-parameter family of functions

(9) $$(x-a)^2 + (y-b)^2 + u^2 = 1,$$

i.e. the totality of spheres of radius 1 in x, y, u-space whose centers lie in the *x, y*-plane. These functions form a complete integral of the differential equation

(10) $$u^2(1 + p^2 + q^2) = 1.$$

If we set $b = w(a)$, thus singling out from all the spheres that one-parameter family whose centers lie on the curve $y = w(x)$ in the *x,y*-plane, then the envelope of this family, i.e., the surface obtained by eliminating *a* from

(11) $$\begin{cases} (x-a)^2 + (y-w(a))^2 + u^2 = 1 \\ x - a + w'(a)(y - w(a)) = 0 \end{cases},$$

yields another solution. Every such envelope is a *tubular surface* whose axis is $y = w(x)$.

The total two-parameter family (9) also has another envelope consisting of the planes $u = 1$ and $u = -1$; this is intuitively clear and can be confirmed analytically by eliminating *a* and *b* from

(12) $$\begin{cases} (x-a)^2 + (y-b)^2 + u^2 = 1 \\ x - a = 0 \\ y - b = 0 \end{cases}.$$

Since these surfaces satisfy the differential equations (10), they constitute the singular solutions of (10). We are also led to these surfaces if we eliminate the quantities *p* and *q* from the equations

(13) $$\begin{cases} F = u^2(1 + p^2 + q^2) = 1 \\ F_p = 2u^2 p = 0 \\ F_q = 2u^2 q = 0 \end{cases}.$$

Another example is furnished by *Clairaut's differential equation*

(14) $$u = xu_x + yu_y + f(u_x, u_y),$$

which occurs frequently in applications. We start with the two-parameter family of planes

(15) $$u = ax + by + f(a,b),$$

where *f*(*a*, *b*) is a prescribed function of the parameters *a* and *b*. Since $u_x = a$, $u_y = b$, this family satisfies the partial differential equation (14). There *D* = 1 (see formula (5)), hence *u* given by (15) is a complete integral of Clairaut's differential equation.

Again we form envelopes to obtain the general solution of this equation; choosing an arbitrary function *b* = *w*(*a*), we eliminate *a* from the equations

(16) $$\begin{cases} u = ax + yw(a) + f(a, w(a)) \\ 0 = x + yw'(a) + f_a + f_b w'(a) \end{cases}.$$

The singular solution of Clairaut's equation is of importance. We obtain it as the envelope of the two-parameter family (15), i.e., by eliminating *a* and *b* from the equations

(17) $$\begin{cases} u = ax + by + f(a,b) \\ x = -f_a \\ y = -f_b \end{cases}.$$

If we differentiate the differential equation (14) with respect to $u_x = p$, the rule in article 3 leads to the same formulas. (Compare §6, 3 where a different point of view is presented.)

§6. *The Legendre Transformation*

1. The Legendre Transformation for Functions of Two Variables.

The integration of certain classes of differential equations can be considerably simplified by applying the "*Legendre transformation*". This transformation is suggested by the geometric interpretation of the differential equation if we represent the integral surface by its tangent plane coordinates instead of by point coordinates.

For the description of a surface in the *x, y, u*-space, there are two dual possibilities. Either one may give the surface as a point set determined by a function *u*(*x, y*), or one may regard it as the envelope of its tangent planes, i.e., set up the equation which a plane must satisfy in order to be tangent to the surface. If $\bar{x}, \bar{y}, \bar{u}$ are the running coordinates of a plane whose equation is

$$\bar{u} - \xi\bar{x} - \eta\bar{y} + \omega = 0,$$

then we call ξ, η, ω the coordinates of this plane. Since the plane tangent to the surface $u(x,$ y) at the point $(x,$ y, $u)$ has the equation

$$\bar{u} - u - (\bar{x} - x)u_x - (\bar{y} - y)u_y = 0 \,,$$

its plane coordinates are

$$\xi = u_x \,, \quad \eta = u_y \,, \quad \omega = xu_x + yu_y - u \,.$$

Now, the surface considered is determined also if ω is given as a function of ξ and η, by which the two-parameter family of tangent planes is characterized. We can find the dependence $\omega(\xi, \eta)$ from $u(x, y)$ by determining the values x and y as functions of ξ and η from the equations

$$\xi = u_x \,, \quad \eta = u_y$$

and by substituting them into the equation

$$\omega = xu_x + yu_y - u = x\xi + y\eta - u \,.$$

Conversely, in order to determine the point coordinates from the tangent plane coordinates, we form the partial derivatives of the function $\omega(\xi, \eta)$. Since $\xi = u_x$ and $\eta = u_y$, we immediately have

$$\omega_\xi = x + \xi \frac{\partial x}{\partial \xi} + \eta \frac{\partial y}{\partial \xi} - u_x \frac{\partial x}{\partial \xi} - u_y \frac{\partial y}{\partial \xi} = x$$

and, similarly,

$$\omega_x = y \,.$$

Thus we obtain the system of formulas

$$(1) \qquad \begin{cases} \omega(\xi,\eta) + u(x,y) = x\xi + y\eta \\ \xi = u_x \\ \eta = u_y \\ x = \omega_\xi \\ y = \omega_\eta \end{cases} ,$$

which demonstrates the dual character of the relation between point and tangent plane coordinates.

This transformation of a surface from point coordinates to plane coordinates is called the *Legendre transformation* for functions of two variables. It is essentially different in character from a mere *coordinate transformation*. For, rather than assigning to a single point another point, the system (1) assigns to every surface element (x, y, u, u_x, u_y) a surface element $(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$.

The Legendre transformation is always feasible if the two equations $u_x = \xi$, $u_y = \eta$ can be solved for x and y; this is possible whenever the Jacobian

(2) $$\rho = \begin{vmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} \end{vmatrix} = u_{xx}u_{yy} - u_{xy}{}^2$$

does not vanish for the points of the surface considered. The Legendre transformation evidently fails for surfaces which satisfy the differential equation

$$u_{xx}u_{yy} - u_{xy}{}^2 = 0\,,$$

i.e., for *developable surfaces*. This result can be visualized geometrically. **A** developable surface possesses by definition a one-parameter family of tangent planes which are tangent along straight lines, not merely at points; thus it is not possible to establish a one-to-one correspondence between the points and the tangent planes of the surface.

Finally, to apply the Legendre transformation to second order differential equations, we calculate the transformation for the second derivatives of the functions $u(x, y)$ and $\omega(\xi,\eta)$. To this end we think of the variables x and y in the equations $\xi = u_x$, $\eta = u_y$ as expressed in terms of ξ and η by means of the relations $x = \omega_\xi$, $y = \omega_\eta$. By differentiating $\xi = u_x$, $\eta = u_y$ wit respect to ξ and η, we find

$$\begin{cases} 1 = u_{xx}\omega_{\xi\xi} + u_{xy}\omega_{\xi\eta} \\ 0 = u_{xy}\omega_{\xi\xi} + u_{yy}\omega_{\xi\eta} \\ 0 = u_{xx}\omega_{\xi\eta} + u_{xy}\omega_{\eta\eta} \\ 1 = u_{xy}\omega_{\xi\eta} + u_{yy}\omega_{\eta\eta} \end{cases}$$

or, in matrix notation,

$$\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\begin{pmatrix} \omega_{\xi\xi} & \omega_{\xi\eta} \\ \omega_{\xi\eta} & \omega_{\eta\eta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If, for brevity, we set

(3) $$\begin{cases} \omega_{\xi\xi}\omega_{\eta\eta} - \omega_{\xi\eta}{}^2 = \dfrac{1}{\rho} \\ u_{xx}u_{yy} - u_{xy}{}^2 = \rho \end{cases},$$

we obtain

(4) $$\begin{cases} u_{xx} = \rho\omega_{\eta\eta} \\ u_{xy} = -\rho\omega_{\xi\eta} \\ u_{yy} = \rho\omega_{\xi\xi} \end{cases}.$$

2. The Legendre Transformation for Functions of *n* Variables.

3. Application of the Legendre Transformation to Partial Differential Equations. We consider a partial differential equation of at most second order

(7) $$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 .$$

By means of the Legendre transformation we assign the function $\omega(\xi,\eta)$ to an integral surface $u(x,y)$ of this equation. Then the equation $F = 0$ goes over into a differential equation for the function ω, also of a t most second order, namely into

(8) $$G = F(\omega_\xi, \omega_\eta, \xi\omega_\xi + \eta\omega_\eta - \omega, \xi, \eta, \rho\omega_{\eta\eta}, -\rho\omega_{\xi\eta}, \rho\omega_{\xi\xi}) = 0 ,$$

where

$$\rho = \frac{1}{\omega_{\xi\xi}\omega_{\eta\eta} - \omega_{\xi\eta}^{2}} .$$

This differential equation, however, in general yields only the non-developable integral surfaces of the original differential equation, since the Legendre transformation is not applicable to developable surfaces.

Particularly in the case of first order partial differential equations, the Legendre transformation may be profitably applied if the variables x,y and u occur in a simple way while the derivatives u_x, u_y occur in a more complicated manner.

1) As an example, we consider the equation

(9) $$u_x u_y = x$$

which by the Legendre transformation goes over into

(10) $$\xi\eta = \omega_\xi ;$$

its solution can be given immediately by

$$\omega = \frac{1}{2}\xi^2\eta + w(\eta) .$$

On the basis of the transformation formulas, it follows that

(11) $$\begin{cases} x = \xi\eta \\ y = (1/2)\xi^2 + w'(\eta) \\ u = (3/2)\xi^2\eta + \eta w'(\eta) - w(\eta) \end{cases} .$$

If we eliminate ξ and η from these three equations, we obtain the desired solutions of the given differential equation.

2) On the other hand, the differential equation

(12) $$u_x u_y = 1$$

is transformed by Legendre transformation into

$$\xi\eta = 1 .$$

This equation is no longer a differential equation and the

transformation fails here; all solutions of $u_x u_y = 1$ are developable surfaces. This is immediately confirmed by differentiating the equation with respect to x and y:

(13) $$\begin{cases} u_{xx}u_y + u_{xy}u_x = 0 \\ u_{xy}u_y + u_{yy}u_x = 0 \end{cases}.$$

Since the possibility $u_x = u_y = 0$ is excluded because $u_x u_y = 1$, the condition

$$u_{xx}u_{yy} - u_{xy}^2 = 0$$

must be satisfied by every integral surface $u(x, y)$.

The Legendre transformation fails for every differential equation of the form

(14) $$F(u_x, u_y) = 0$$

in the same way.

3) A third example is furnished by the *Clairaut's equation*

(15) $$u = xu_x + yu_y + f(u_x, u_y)$$

already considered in §4, 4. By the Legendre transformation, (15) goes over into the simple equation

(16) $$\omega = -f(\xi, \eta).$$

From this we deduce that the only nondevelopable integral surface of Clairaut's differential equation is represented by equation (16) or, in point coordinates, by

(17) $$\begin{cases} x = -f_\xi(\xi, \eta) \\ y = -f_\eta(\xi, \eta) \\ u = f - \xi f_\xi - \eta f_\eta \end{cases}.$$

The following calculation confirms this conclusion: We differentiate the differential equation (15) and obtain the formulas

$$\begin{cases} (x + f_p)u_{xx} + (y + f_q)u_{xy} = 0 \\ (x + f_p)u_{xy} + (y + f_q)u_{yy} = 0 \end{cases}$$

(where $p = u_x$, $q = u_y$); it follows that, for an integral surface, either

$$D = u_{xx}u_{yy} - u_{xy}^2 = 0$$

or

$$x = -f_p, \quad y = -f_q.$$

But the latter possibility yields precisely the exceptional surface obtained by the Legendre transformation.

4) As another example, we consider the second order differential equation of *minimal surfaces* (see also Vol. I, p. 193).

(18) $$(1+u_y{}^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x{}^2)u_{yy} = 0$$

which is nonlinear in the derivatives of $u(x,y)$. This apparent difficulty can be overcome by transforming (18) by the Legendre transformation into

(19) $$(1+\eta^2)\omega_{\eta\eta} + 2\xi\eta\omega_{\xi\eta} + (1+\xi^2)\omega_{\xi\xi} = 0,$$

i.e., into a linear differential equation. Later (cf. Appendix 1 of this chapter, Ch. III, § 1, 4 and Vol. III) we shall consider other ways of linearizing the differential equation (18), which will yield a simple approach to the theory of minimal surfaces.

5) A similar important application of the Legendre transformation occurs in *fluid dynamics*: Steady flow of a two-dimensional compressible fluid is described by two velocity components *u, v* as functions of the rectangular coordinates *x, y*. Suppose the sound speed *c* is a given function of u^2+v^2. The motion is governed by the first order system of equations

$$\begin{cases} u_y - v_x = 0 \\ (c^2-u^2)u_x - uv(u_y+v_x) + (c^2-v^2)v_y = 0 \end{cases}.$$

Accordingly, there exists a velocity potential $\phi(x,y)$ such that

$$u = \phi_x, \quad v = \phi_y$$

and

$$(c^2-\phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c^2-\phi_y^2)\phi_{yy} = 0.$$

A crucial step in dealing with this nonlinear differential equation of second order is the Legendre transformation

$$\Phi + \phi = ux + vy$$

$$\phi_x = u, \quad \phi_y = v,$$

$$\Phi_u = x, \quad \Phi_v = y.$$

It yields for $\Phi(u,v)$ a linear differential equation of second order

$$(c^2-u^2)\Phi_{vv} + 2uv\Phi_{uv} + (c^2-v^2)\Phi_{uu} = 0$$

which is useful for solving many flow problems.

§7. *The Existence Theorem of Cauchy and Kowalewsky*