CHAPTER II

General Theory of Partial Differential Equations of First Order

§1. *General Information about the Variety of Solutions*

The main result of this chapter will be the *equivalence of a first order partial differential equation with a certain system of ordinary differential equations*. The key to the theory is the *concept of characteristics*, which will play a decisive part also in higher order problems.

§1. *Geometric Theory of Quasi-Linear Differential Equations in Two Independent Variables*

1. Characteristic Curves. We shall briefly review quasi-linear differential equations which were treated in Ch. I, §5. First we consider an equation in two independent variables *x, y*:

(1) $$au_x + bu_y = c\,,$$

where *a, b, c* are given functions of *x, y, u* which, in the region considered, are assumed to be continuous together with their first derivatives and to satisfy $a^2 + b^2 \neq 0$.

This partial differential equation may be interpreted geometrically as follows: The integral surface $u(x, y)$ of the differential equation is required to possess at the point, *P*:(*x,y,u*), a tangent plane whose normal has direction numbers $u_x = p$, $u_y = q$ and - 1 connected by the linear equation $ap + bq = c$. According to this equation the tangent planes of all integral surfaces through the point (*x,y,u*) belong to a single *pencil of planes* whose axis is given by the relations

(2) $$dx : dy : du = a : b : c$$

at the point *P*; these pencils and their axes are called *Monge pencils* and *Monge axes*.

The point *P* together with the direction of the Monge axis through *P* constitutes a *characteristic line element*.

The directions of the Monge axes form a direction field in the *x, y, u*-space; the integral curves of this direction field are defined by the system of ordinary differential equations (2) and are called the *characteristic curves* of our partial differential equation. If we introduce a parameter *s* along the characteristic curves the differential equations become

(2’) $$\frac{dx}{ds} = a\,,\ \ \frac{dy}{ds} = b\,,\ \ \frac{du}{ds} = c\,.$$

The projections of the characteristic curves on the x, y-plane are called “*characteristic base curves*”.

To integrate the partial differential equation (1) is the same as to find surfaces which “fit” the Monge field at every point, i.e., surfaces whose tangent plane at every point belongs to the Monge pencil or, in other words, surfaces which are at every point tangent to the Monge axis. Thus we see:

Every surface $u(x,y)$ generated by a one-parameter family of characteristic curves is an integral surface of the partial differential equation.

Conversely, *every integral surface $u(x,y)$ is generated by a one-parameter family of characteristic curves.*

The last statement is easily verified: On every integral surface $u(x,y)$ of the differential equation (l), a one-parameter family of curves $x = x(s)$, $y = y(s)$, $u = u(x(s), y(s))$ can be defined by the differential equations

$$\frac{dx}{ds} = a\,,\ \ \frac{dy}{ds} = b\,,$$

where the function $u(x, y)$ is to be substituted for u in a and b.

Along such a curve, the partial differential equation (1) goes over into the statement $du/ds = c$. Thus our one-parameter family satisfies the relations (2’) and, hence, consists of characteristic curves. Note that the parameter s does not appear explicitly in the differential equations, so that the same integral curves are obtained if s + const. is substituted for s. In this sense, an additive constant in the parameter s is to be considered inessential.

Since the solutions of the system of differential equations (2’) are uniquely determined by the initial values of x, y, u for $s = 0$, we obtain the following theorem:

Every characteristic curve which has one point in common with an integral surface lies entirely on the integral surface.

Moreover, *every integral surface is generated by a one-parameter family of characteristic curves.*

2. Initial Value Problem.

3. Examples

1) To illustrate our results, we consider the differential equation

(4) $$uu_x + u_y = 1$$

(a special case of example 5 in Ch. I, §1). The corresponding characteristic differential equations are

(5) $$\begin{cases} \dfrac{dx}{ds} = u \\ \dfrac{dy}{ds} = 1 \\ \dfrac{du}{ds} = 1 \end{cases};$$

they are solved by

$$\begin{cases} x = x_0 + u_0 s + s^2/2 \\ y = y_0 + s \\ u = u_0 + s \end{cases},$$

where x_0, y_0, u_0 are arbitrary constants of integration. In particular, the family of characteristics intersecting a given initial curve ***C***:

$$x_0 = \phi(t)\,,\quad y_0 = \psi(t)\,,\quad u_0 = \chi(t)$$

is given by

(6) $$\begin{cases} x(s,t) = \phi(x) + s\chi(t) + s^2/2 \\ y(s,t) = \psi(t) + s \\ u(s,t) = \chi(t) + s \end{cases}.$$

The determinant

$$\Delta(s,t) = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{vmatrix} = x_s y_t - x_t y_s = s(\psi_t - \chi_t) + \chi\psi_t - \phi_t$$

has on C the value

(7) $$\Delta = \Delta(0,t) = \chi\psi_t - \phi_t\,.$$

If this determinant Δ is not zero along C, then the parameters s and t can be eliminated from (6); u may then be expressed as a function of x and y. In fact, it now follows that

$$u = y + \chi(t) - \psi(t)\,,$$

$$x = \phi(t) + \chi(t)(y - \psi(t)) + (y - \psi(t))^2/2\,;$$

from the second equation, t is obtained as a function of x and y, provided that

$$D = \phi_t - \chi\psi_t + (y - \psi(t))(\chi_t - \psi_t)$$

is different from zero. As we approach the curve C, $y - \psi(t) \to 0$**;** thus, since $\phi_t - \chi\psi_t \neq 0$, there exists a neighborhood of C in which $D \neq 0$, hence $t = t(x,y)$ and therefore $u = u(x, y)$.

If for C we choose the *characteristic curve*

(8) $$x_0 = t^2/2\,,\quad y_0 = t\,,\quad u_0 = t\,,$$

then (6) goes into

$$x=(s+t)^2/2\,,\quad y=s+t\,,\quad u=st$$

(i,e., again into an expression for the same curve *C*) and does not represent a solution of (4). To solve the initial value problem for such a *C*, we observe that a solution of (4) is given implicitly by the equation

(9) $$x=u^2/2+w(u-y)\,,$$

with an arbitrary function *w*. If we choose *w* in such a way that $w(0)=0$ and that *u* can be determined uniquely from (9), then all the corresponding integral surfaces $u = u(x, y)$ pass through the characteristic curve *C*.

Finally, let *C* be the *noncharacteristic curve*

(10) $$x_0=t^2\,,\quad y_0=2t\,,\quad u_0=t\,.$$

The system (6) goes over into

(11) $$x=s^2/2+st+t^2\,,\quad y=s+2t\,,\quad u=s+t\,.$$

The determinant is

$$\Delta(s,t)=\begin{vmatrix}\dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t}\\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t}\end{vmatrix}=x_s y_t-x_t y_s=(s+t)\cdot 2-(s+2t)\cdot 1=s\,.$$

This vanishes for $s = 0$ although *C* is not characteristic. By eliminating *s* and *t* we have

(12) $$u(x,y)=\frac{y}{2}\pm\sqrt{x-\frac{y^2}{4}}\,.$$

i.e., two surfaces through *C* which, for $x>y^2/4$, satisfy equation (4). The initial value problem is not solved by (12) since the derivatives u_x and u_y do not remain bounded as we approach *C*.

The curve *C* is not an edge of regression of the surface $u = \mathrm{u}(x, y)$; it is; however, singular on the surface in the sense that in the neighborhood of the projection of *C* on the *x,y*-plane *u* is no longer single-valued.

2) In the case of a linear differential equation (1), where the functions *a, b, c* do not depend explicitly on *u*, the vanishing of Δ along a noncharacteristic initial curve means, as we shall see, that the manifold determined by the system

$$x=x(s,t)\,,\quad y=y(s,t)\,,\quad u=u(s,t)$$

is a cylindrical surface perpendicular to the *x, y*-plane. If we assume that

$$\begin{vmatrix} \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \\ \dfrac{\partial u}{\partial s} & \dfrac{\partial u}{\partial t} \end{vmatrix} \equiv y_s u_t - y_t u_s \neq 0$$

on C, then x can be expressed as a function of y and u. We show that $x = f(y,u)$ is independent of u.

First, we note that, for linear differential equations, the relation

$$\Delta_s = (a_x + b_y)\Delta$$

follows from (2') for the determinant Δ, so that Δ vanishes everywhere provided it vanishes on C. Now, from

$$x_s = f_y y_s + f_u u_s\,,$$

$$x_t = f_y y_t + f_u u_t\,,$$

it follows immediately that

$$\Delta = x_s y_t - x_t y_s = f_u(u_s y_t - u_t y_s)$$

hence

$$f_u = 0 \quad \text{or} \quad x = f(y)\,.$$

§8. *Complete Integral and Hamilton-Jacobi Theory*

1. Construction of Envelopes and Characteristic Curves. Consider the partial differential equation

(1) $$F(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0$$

with $p_i = \partial u / \partial x_i$ and $\sum_i F_{p_i}^2 \neq 0$. With a specific solution

(2) $$u = \phi(x_1, x_2, \cdots, x_n, a_1, a_2, \cdots, a_n)$$

depending on n parameters a_i, (a complete integral), assume that the determinant condition

(3) $$D = \left|\phi_{x_i, a_k}\right| \neq 0$$

is satisfied in the part of the x ,u-space considered. Then the envelope of an arbitrary (n - 1)-parameter family of these solutions is also a solution. To prove this we set

$$a_i = \omega_i(t_1, t_2, \cdots, t_{n-1}) \quad (i=1,2,...,n)$$

where the ω_i arc arbitrary functions of the (n–1) parameters t_k . The envelope is obtained by calculating $t_1, t_2, \cdots, t_{n-1}$ from the equations

(4) $$0 = \sum_{i=1}^{n} \phi_{a_i} \frac{\partial \omega_i}{\partial t_\nu} \quad (\nu = 1,2,\cdots,n-1)$$

as functions of $x_1, x_2, \cdots, x_n$ and substituting these t_ν into

$$u = \phi(x_1, \cdots, x_n, \omega_1(t_1, \cdots, t_{n-1}), \cdots, \omega_n(t_1, \cdots, t_{n-1}))$$

The *curves of contact* of a surface given by the complete integral with the envelope will prove to be *characteristic curves*. Such a contact curve corresponds to a fixed system of quantities t_ν, $\partial\omega_i/\partial t_\nu$ and a_i; furthermore relations (4) hold along the curve, which imply, to within a common proportionality factor λ, certain constant values b_i for the ϕ_{a_i}:

(5) $$\phi_{a_i} = \lambda b_i .$$

Using this equation we may assign values λb_i corresponding to given values of the quantities x_i and a_i; then, because of condition (3), we can solve (5) uniquely for the x_i in a neighborhood of the system of values considered and obtain functions

$$x_i(a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, \lambda) .$$

If these functions are substituted into

$$\phi(x_1, x_2, \cdots, x_n, a_1, a_2, \cdots, a_n)$$

we have a curve represented in terms of the parameter λ. Since in the neighborhood considered, any desired values can be given to the quantities a_i and b_i by an appropriate choice of the functions ω_i, we thus obtain a 2n-parameter family of contact curves of our envelope to our complete integral. These curves are characteristic curves of our partial differential equation (1) and, together with

$$p_i = \phi_{x_i}(x_k(a_\nu, b_\nu, \lambda), a_k) ,$$

produce characteristic strips. This follows from the geometric meaning of our strips as strips of contact.

To prove the assertion analytically we differentiate equation (5) with respect to the curve parameter λ:

(6) $$\sum_{k=1}^{n} \phi_{a_i x_k x'_k} = b_i ,$$

where differentiation with respect to λ is denoted by a prime. On the other hand, the differential equation (1) is identically satisfied in the x_i and a_i by $\phi(x_1, x_2, \cdots, x_n, a_1, a_2, \cdots, a_n)$; if we differentiate it with respect to a_i and use (5) we obtain

(7) $$\sum_{k=1}^{n} \phi_{x_k a_i} F_{p_k} + F_u \lambda b_i = 0 .$$

Thus the quantities $-(F_{p_k}/\lambda F_u)$ satisfy the same system of inhomogeneous equations as the x'_k; since the determinant of this system does not vanish, we may conclude that

$$x'_k = -\frac{F_{p_k}}{\lambda F_u} .$$

If we set the expression $-1/\lambda F_u$ (which is different from zero) equal to ρ, then

(8) $$x'_k = \rho F_{p_k}.$$

Furthermore, if we differentiate (1) with respect to x_k, we have

$$F_u p_k + \sum_{i=1}^{n} F_{p_i} \frac{\partial p_i}{\partial x_k} + F_{x_k} = 0$$

or, since, in view of (8),

$$\begin{aligned}\sum_{i=1}^{n} F_{p_i} \frac{\partial p_i}{\partial x_k} &= \frac{1}{\rho}\sum_{i=1}^{n} \frac{\partial p_i}{\partial x_k} x'_i = \frac{1}{\rho}\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_k} x'_i \\ &= \frac{1}{\rho}\sum_{i=1}^{n} \frac{\partial p_k}{\partial x_i} x'_i = \frac{1}{\rho} p'_k\end{aligned},$$

we obtain

$$p'_k = -\rho(F_u p_k + F_{x_k}).$$

Since the curve parameter λ can be so chosen that $\rho = 1$, the characteristic equations (2) of §7 are satisfied by the curves considered.

2. Canonical Form of the Characteristic Differential Equations. A more transparent form can be given to the theory of partial differential equations of the first order and the calculations of article 1 can be simplified if the dependent variable *u* does not appear explicitly in the differential equation. An arbitrary differential equation can always be brought into this special form by increasing artificially the number of independent variables by one.

For this purpose we need only introduce, e.g. (cf. Ch. I, §5), $u = x_{n+1}$ as an independent variable and express a family of solutions

$$u = \psi(x_1, x_2, \cdots, x_n, c)$$

implicitly in the form

$$\phi(x_1, x_2, \cdots, x_n) = c$$

If we replace u_{x_i} by $-\phi_{x_i}/\phi_{x_{n+1}}$ (i = 1, 2,…, n), then we obtain for the new unknown function ϕ a differential equation which does not depend on ϕ explicitly.

For such a differential equation we single out a variable, e.g., $x_{n+1} = x$, and assume the differential equation solved for the derivative of ϕ with respect to this variable. Thus, if instead of ϕ we again write *u,* we may without loss of generality consider differential equations of the form

(9) $$p + H(x_1, x_2, \cdots, x_n, x, p_1, p_2, \cdots, p_n) = 0$$

$$p = u_x, \quad p_i = u_{x_i} \quad (i=1,2,\ldots,n)$$

for function u of the $n+1$ variables $x_1, x_2, \cdots, x_n$.

Then the system of characteristic differential equations, one of which is $dx/ds=1$ (or $x=s$), goes over into the system

$$(10) \qquad \frac{dx_i}{dx} = H_{p_i}, \quad \frac{dp_i}{dx} = -H_{x_i} \quad (i=1,2,\ldots,n);$$

moreover,

$$(11) \qquad \frac{du}{dx} = \sum_{i=1}^{n} p_i H_{p_i} - H, \quad \frac{dp}{dx} = -H_x$$

holds. The equations (10) alone form a system of $2n$ differential equations for $2n$ quantities x_i, p_i. If the functions $x_i(x)$ and $p_i(x)$ are solutions of (10) then $p(x)$ and $u(x)$ are obtained from equations (11) by simple integrations.

In mechanics and in the calculus of variations (cf. Vol. I, Ch. IV, §9 and the present chapter, §9) one is often led to differential equations of the form (10). A system of ordinary differential equations (10)

$$\frac{dx_i}{dx} = H_{p_i}, \quad \frac{dp_i}{dx} = -H_{x_i}$$

associated with a function $H(x_1, x_2, \cdots, x_n, x, p_1, p_2, \cdots, p_n)$ of $2n+1$ variables is called a *canonical system of differential equations.*

The results of this article imply that the integration of the partial differential equation (9) may be reduced to the integration of a canonical system with the same function *H.*

3. Hamilton-Jacobi Theory. Hamilton and Jacobi achieved a major success by recognizing that this relationship may be reversed. To be sure, the integration of a partial differential equation is usually considered as a problem more difficult than that of a system of ordinary differential equations. In mathematical physics one is often led, however, to a system of ordinary differential equations in canonical form. These equations may be difficult to integrate by elementary methods, while the corresponding partial differential equation is manageable; in particular, it may happen that a complete integral is easily obtained, e.g., with the help of the separation of variables (cf. Ch. I, §3). Knowing the complete integral, one can then solve the corresponding system of characteristic ordinary differential equations by processes of differentiation and elimination. This fact, which is contained in the earlier results of §4 and §8, 1 , can be formulated in a particularly simple way for the case of canonical

differential equations and can be verified analytically, independently of the motivation, by envelope construction.

We first formulate anew the concept "*complete integral*" for differential equation (9): We remark that, for every solution u of the differential equation, $u + a$ (with an arbitrary constant a,) is also a solution. If $u = \phi(x_1, x_2, \cdots, x_n, x, a_1, a_2, \cdots, a_n)$ is a solution depending on n parameters a_i such that the determinant

(12) $$\left|\phi_{x_i a_k}\right|$$

is different from zero, then the expression

$$u = \phi + a ,$$

which depends on $n + 1$ parameters, is called a *complete integral*. The principal content of the theory now to be treated is stated in the following theorem analogous to the facts proved in article 1:

If a complete integral $u = \phi(x_1, x_2, \cdots, x_n, x, a_1, a_2, \cdots, a_n) + a$ *is known for the partial differential equation* (9)

$$u_x + H(x_1, x_2, \cdots, x_n, x, u_{x_1}, u_{x_2}, \cdots, u_{x_n}) = 0 ,$$

then, from the equations

(13) $$\phi_{a_i} = b_i , \quad \phi_{x_i} = p_i \quad (i=1,2,\ldots,n)$$

with the 2n arbitrary parameters a_i *and* b_i*, one obtains (implicitly) the 2n-parameter family of solutions of the canonical system of differential equations* (10)

$$\frac{dx_i}{dx} = H_{p_i} , \quad \frac{dp_i}{dx} = -H_{x_i} .$$

Let us assume that from the first n equations (13) the quantities x_i are expressed as functions of x and the $2n$ parameters a_i, b_i -- this is possible because, by assumption, $\left|\phi_{x_i a_k}\right| \neq 0$ -- and let us moreover imagine these values of x_i introduced into the second set of equations (13); we thus obtain functions $x_i(x)$ and $p_i(x)$ which still depend on $2n$ parameters and will be seen to represent the general solution of the system of canonical differential equations. Thereby the solution of the system is reduced to the problem of finding a complete integral of the corresponding partial differential equation.

The shortest proof of this statement is a simple verification similar to that used in article 1. To show that the functions $x_i(x)$ and $p_i(x)$,

so determined, satisfy equations (10) we differentiate the equations $\phi_{a_i} = b_i$ with respect to x and the equation $\phi_x + H(x_i, x, \phi_{x_i}) = 0$ with respect to a_i; we obtain the *2n* equations

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x \partial a_i} + \displaystyle\sum_{k=1}^{n} \frac{\partial^2 \phi}{\partial x_k \partial a_i} \frac{\partial x_k}{\partial x} = 0 \\ \dfrac{\partial^2 \phi}{\partial x \partial a_i} + \displaystyle\sum_{k=1}^{n} H_{p_k} \frac{\partial^2 \phi}{\partial x_k \partial a_i} = 0 \end{cases},$$

from which the first relation of (10) follows, since the determinant $\left|\phi_{a_k x_i}\right|$ does not vanish. To verify the second relation, we differentiate the equations $\phi_{x_i} = p_i$ with respect to x and the equation $\phi_x + H(x_i, x, \phi_{x_i}) = 0$ with respect to x_i and obtain the equations

$$(14) \qquad \begin{cases} \dfrac{dp_i}{d} = \dfrac{\partial^2 \phi}{\partial x \partial x_i} + \displaystyle\sum_{k=1}^{n} \frac{\partial^2 \phi}{\partial x_i \partial x_k} \frac{\partial x_k}{\partial x} \\ 0 = \dfrac{\partial^2 \phi}{\partial x \partial x_i} + \displaystyle\sum_{k=1}^{n} H_{p_k} \frac{\partial^2 \phi}{\partial x_k \partial x_i} + H_{x_i} \end{cases}.$$

Since we have already proved that $dx_i / dx = H_{p_i}$, the second relation of (10) follows immediately.

4. Example. The Two-Body Problem. The motion of two particles P1 and P2 which attract each other is described according to *Newton's law of gravitation* by the differential equations

$$(15) \qquad \begin{matrix} m_1 \ddot{x}_1 = U_{x_1} & m_1 \ddot{y}_1 = U_{y_1} & m_1 \ddot{z}_1 = U_{z_1} \\ m_2 \ddot{x}_2 = U_{x_2} & m_2 \ddot{y}_2 = U_{y_2} & m_1 \ddot{z}_2 = U_{z_2} \end{matrix}$$

where we set

$$U = \frac{\kappa^2 m_1 m_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}.$$

As is easily seen, the motion always remains in a plane; we can therefore choose the plane of motion as the *x, y*-plane of our coordinate system with *P2* placed at the origin. For the position (*x*, *y*) of the particle *P1* we then obtain the equations of motion

$$(16) \qquad m_1 \ddot{x} = U_x, \quad m_1 \ddot{y} = U_y, \quad U = \frac{k^2}{\sqrt{x^2 + y^2}},$$

where $k^2 = \kappa^2 m_1 m_2$.

If we introduce the Hamiltonian function

$$(17) \qquad H = \frac{1}{2}(p^2 + q^2) - \frac{k^2}{\sqrt{x^2 + y^2}},$$

the system (16) finally goes over into the *canonical system* of differential equations

(18) $$\begin{matrix}\dot{x}=H_p \\ \dot{y}=H_q\end{matrix}, \quad \begin{matrix}\dot{p}=-H_x \\ \dot{q}=-H_y\end{matrix},$$

for the quantities x, y, $p=\dot{x}$, $q=\dot{y}$; the integration of these equations is equivalent to the problem of finding a *complete integral* of the partial differential equation

(19) $$\phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2)=\frac{k^2}{\sqrt{x^2+y^2}}.$$

If we introduce polar coordinates r,θ, we obtain from (19)

(20) $$\phi_t+\frac{1}{2}\left(\phi_r^2+\frac{1}{r^2}\phi_\theta^2\right)=\frac{k^2}{r};$$

this equation clearly possesses the family of solutions

(21) $$\phi=-\alpha t-\beta\theta-\int_{r_0}^{r}\sqrt{2\alpha+\frac{2k^2}{\rho}-\frac{\beta^2}{\rho^2}}d\rho$$

which depend on the parameters α,β. By the main theorem of article 3 we then obtain the general solution of (18) in the form

$$\frac{\partial\phi}{\partial\alpha}=-t_0, \quad \frac{\partial\phi}{\partial\beta}=-\theta_0$$

or, explicitly,

(22) $$\begin{cases} t-t_0=-\displaystyle\int_{r_0}^{r}\frac{d\rho}{\sqrt{2\alpha+\frac{2k^2}{\rho}-\frac{\beta^2}{\rho^2}}} \\ \theta-\theta_0=\beta\displaystyle\int_{r_0}^{r}\frac{d\rho}{\rho^2\sqrt{2\alpha+\frac{2k^2}{\rho}-\frac{\beta^2}{\rho^2}}} \end{cases}.$$

The second equation gives us the *trajectory* (or *particle path*); the first determines the motion of the particle on this path as a function of the time t.

If we introduce the variable of integration $\rho'=1/\rho$, the trajectory may be calculated explicitly and is given by

$$\theta-\theta_0=-\arcsin\frac{\frac{\beta^2}{k^2}\frac{1}{r}-1}{\sqrt{1+\frac{2\alpha\beta^2}{k^4}}}$$

or, if we set

$$p=\frac{\beta^2}{k^2}, \quad \varepsilon^2=\sqrt{1+\frac{2\alpha\beta^2}{k^4}},$$

by

$$\theta-\theta_0=-\arcsin\frac{p/r-1}{\varepsilon^2}$$

i.e.,

$$r=\frac{p}{1-\varepsilon^2\sin(\theta-\theta_0)}.$$

The path is an elliptic, a parabola, or a hyperbola, depending on whether $\varepsilon<1$, $\varepsilon=1$, or $\varepsilon>1$.

5. Example. Geodesics on an Ellipsoid. The differential equations of the *geodesics* $u=u(s)$, $v=v(s)$ of a surface

$$x=x(u,v),\quad y=y(u,v),\quad z=z(u,v)$$

may, according to Vol. I, Ch. IV, §9, be written as follows in *canonical form*:

$$\begin{matrix} u_s=H_p \\ v_s=H_q \end{matrix},\quad \begin{matrix} p_s=-H_u \\ q_s=-H_v \end{matrix},\tag{23}$$

where we set

$$\begin{cases} p=Eu_s+Fv_s \\ q=Fu_s+Gv_s \end{cases},$$

and

$$H=\frac{1}{2}\frac{1}{EG-F^2}(Gp^2-2Fpq+Eq^2)$$

with

$$\begin{cases} E=x_u^2+y_u^2+z_u^2 \\ F=x_ux_v+y_uy_v+z_uz_v \\ G=x_v^2+y_v^2+z_v^2 \end{cases}$$

Following article 3, we consider the partial differential equation

$$\phi_s+\frac{1}{2}\frac{1}{EG-F^2}(G\phi_u^2-2F\phi_u\phi_v+E\phi_v^2)=0\tag{24}$$

corresponding to (23); our object is to obtain a *complete integral* of this equation. Now if we set

$$\phi=-\frac{1}{2}s+\psi(u,v),$$

then ψ satisfies the equation

$$G\psi_u^2-2F\psi_u\psi_v+E\psi_v^2=EG-F^2.\tag{25}$$

We are interested in the solution curves, not in the special parametric representation of these curves; it suffices, therefore, to find a one-parameter family of solutions $\psi(u,v,\alpha)$ of (25), from which one obtains the two-parameter family of geodesics in the form

(26) $$\frac{\partial \psi}{\partial \alpha} = C$$

according to the main theorem of article 3.

In the special case of the *ellipsoid*

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \qquad (a,b,c>0),$$

the following parametric representation (cf. Vol. I, p.226) holds, as can easily be verified:

(27) $$\begin{cases} x = \sqrt{\dfrac{a(u-a)(v-a)}{(b-a)(c-a)}} \\ y = \sqrt{\dfrac{b(u-b)(v-b)}{(c-b)(a-b)}} \\ z = \sqrt{\dfrac{c(u-c)(v-c)}{(a-c)(b-c)}} \end{cases}.$$

From this it follows that

(28) $$\begin{cases} E = (u-v)A(u) \\ F = 0 \\ G = (v-u)A(v) \end{cases},$$

where, for brevity, we have set

$$A(u) = \frac{1}{4}\frac{u}{(a-u)(b-u)(c-u)}.$$

For $\psi(u,v)$ we thus obtain the partial differential equation

(29) $$A(v)\psi_u^2 - A(u)\psi_v^2 = (u-v)A(u)A(v),$$

and, if we write $\psi(u,v) = f(u) + g(v)$, we immediately obtain the family of solutions

(30) $$\psi(u,v,\alpha) = \int_{u_0}^{u} \sqrt{A(u')(u'+\alpha)}\,du' + \int_{v_0}^{v} \sqrt{A(v')(v'+\alpha)}\,dv'$$

which depends on the parameter α.

From (26) we find that the *equation of the geodesics on the ellipsoid* is

(31) $$\int_{u_0}^{u} \sqrt{\frac{A(u')}{u'+\alpha}}\,du' + \int_{v_0}^{v} \sqrt{\frac{A(v')}{v'+\alpha}}\,dv' = 2C.$$

Appendix 2 to Chapter I1

Theory of Conservation Laws

In Chapter II we have solved the noncharacteristic initial value problem for a single first order quasi-linear equation. The existence theorem obtained is local, i.e., it was only shown that solutions exist in some neighborhood of the initial curve. Furthermore, in §1 we constructed integral surfaces with an edge of regression; this shows that smooth solutions need not exist in the large.

In this appendix we shall investigate further the occurrence of discontinuities which terminate the regions of existence. Then we shall show how solutions may nevertheless be continued beyond their singularities, provided that the differential equation is interpreted as a "*conservation law*".

Consider quasi-linear equations of the form

(1) $$u_t = au_x \, ,$$

$a = a(u)$ being a function of *u*. Singularities of solutions of such equations may arise from the initial values on the line $t = 0$ in the following manner:

According to the *theory of characteristics*, every solution u remains constant along each characteristic. The slope of a characteristic is $1/a(u)$, and since u is a constant along the characteristic curve, it follows that the slope is a constant and all characteristic curves are straight lines.

From each point x_1 of the initial line $t = 0$ there issues a characteristic line whose slope is determined by the value of u at x_1. Suppose that there is a pair of points x_1 and x_2 on the initial line, say $x_1 < x_2$, where the prescribed values u_1 and u_2 of u are such that

$$a(u_1) < a(u_2) \, .$$

Then the characteristic lines issuing from x_1 and x_2 intersect a t time $t = (x_1 - x_2)/(a(u_1) - a(u_2))$. Since u has different values along the two characteristics, this shows that u cannot be defined as a continuous solution beyond this time.

The presence or absence of *singularities* can also be seen from the following implicit formula for the solution of equation (1) with initial value $u(x,0) = \phi(x)$:

$$u - \phi(x + ta(u)) = 0 \, .$$

According to the *theorem on implicit functions*, u is a regular function

of x and t as long as the derivative of

$$u - \phi(x + ta(u))$$

with respect to u is not zero, i.e.,

$$ta'\phi' \neq 1.$$

This condition is satisfied for t small enough, but will be violated if t becomes larger (if a' arid ϕ' have the same sign, such a value t is positive, otherwise t is negative). At the point where the condition is violated we expect u to become singular.

The foregoing examples show that solutions of quasi-linear equations in general do not exist in the large. But there does exist a theory in the large for solutions of conservation laws.

A *conservation law* for a single function u is an equation of the form

(2) $$\frac{d}{dt}\int_{x_1}^{x_2} u\,dx = f(u(x_2), x_2, t) - f(u(x_1), x_1, t),$$

where f is a given function of u, x, and t. This equation expresses the fact that the total quantity represented by the function u and contained in the interval (x_1, x_2) changes at a rate equal to the "*flux*" f of u through the end points of the interval. This is the form of those laws of physics which ignore dissipative mechanisms and thus express a "phenomenon of conservation".

If u is a differentiable solution of the conservation law (2) then the conservation law (2) is expressed by the quasi-linear differential equation

(2') $$u_t = f_u u_x + f_x = \frac{\partial}{\partial x} f(u, x, t).$$

obtained from (2) by differentiating with respect to x_1 and then setting $x_1 = x_2 = x$. However, as we shall confirm, (2) has discontinuous solutions as well. By admitting discontinuous solutions we shall show that within the class of discontinuous solutions the conservation law *(2) has a solution in the large*, whereas we have seen before that the differential equation *(2') does not*.

Later in Chapters V and VI we shall study "*shock*" discontinuities for systems of conservation laws in any number of dimensions. We shall see there that the qualitative properties of discontinuous solutions of single conservation laws are the same as those of the physically more interesting systems.

Assume that f does not depend explicitly on x and t and abbreviate

f_u as $a = a(u)$. Let $u(x,t)$ be a piecewise differentiable function which is a solution of the integral equation (2). Then u must be a solution of the differential equation

$$u_t = au_x$$

whenever u is differentiable. Letting x_1 and x_2 approach a point of discontinuity from opposite sides, we deduce (see details in Ch. V, §9) the "*jump relation*"

(3) $$[f] = -U[u],$$

where U is the speed of propagation of the discontinuity and the symbol $[g]$ denotes the jump across the discontinuity of the quantity g. In case of small discontinuities we have

$$U = -\frac{[f]}{[u]} \approx -\frac{df}{du} = -a .$$

Since - a is the speed corresponding to the characteristic line direction, we conclude that the small discontinuities propagate with nearly characteristic speed.

Consider the example

(4) $$\frac{d}{dt}\int udx = \frac{1}{2}u^2(x_2) - \frac{1}{2}u^2(x_1) .$$

When the differentiation is carried out, we obtain

$$u_t = uu_x .$$

Dividing by u we have

(5) $$\frac{u_t}{u} = (\log u)_t = u_x .$$

Denoting $\log u$ by v, we ca rewrite (5) as the conservation law

(5') $$\frac{d}{dt}\int vdx = \exp v(x_2) - \exp v(x_1) .$$

The jump relation (3) for the conservation law (4) is

(6) $$\frac{u_1 + u_2}{2} = -U ,$$

where u_1 and u_2 denote values of u on either side of the line of discontinuity. For (5') the jump relation is

$$\frac{e^{v_1} - e^{v_2}}{v_1 - v_2} = -U .$$

From these jump conditions we conclude: If u is a discontinuous solution of (4) then $v = \log u$ is *not* a solution of (5'). One could say that jump relations are not invariant under change of dependent variables; two conservation laws, such as (4) and (5), may correspond

to the same differentia1 equations for smooth solutions, but as conservation laws for discontinuous solutions they need not be equivalent.

Next we show by an example that *solutions of conservation laws are not determined uniquely by their initial data*. We take again the conservation law (4). The function

$$u(x,t)=\begin{cases}1\_for\_2x<-t\\0\_for\_-t<2x\end{cases}$$

is a discontinuous solution of (4), since on either side of the line *2x=-t*, *u* is a constant and thus a smooth solution of equation (4), and across the line of discontinuity *2x=t* the jump condition (6) is satisfied. On the other hand, the function

$$u'(x,t)=\begin{cases}1\_for\_x<-t\\-x/t\_for\_-t<x<0\\0\_for\_0<x\end{cases}$$

is continuous for *t* positive and satisfies the differential equation everywhere except on the lines $x = 0$ and $x = -t$. From this one can easily show by integration that *u'* is a continuous solution of (4). The two solutions *u* and *u'* have the same value at $t = 0$. More generally it is possible to show that for arbitrarily prescribed initial data there exist uncountably many discontinuous solutions with the same prescribed initial data.

Among all these discontinuous solutions with the same initial value there is only one which has physical significance. This solution we shall call the *permissible* one.

What is needed is a mathematical principle characterizing permissible solutions. Such a principle is suggested by the argument used earlier to deduce the occurrence of discontinuity from the crossing of characteristics. A discontinuity is permissible if it prevents the crossing of characteristics. Thus we have the following criterion: A discontinuous solution is *permissible* if every line of discontinuity is crossed by the forward drawn characteristic issuing from either side. Analytically this condition means that for a permissible discontinuity

$$-a(u_L)\geq U\geq -a(u_R),$$

where u_L and u_R denote the value of *u* to the left and right of the line of discontinuity, and *U* denotes the velocity of propagation of the discontinuity.

Germain and Bader have shown that two permissible

discontinuous solutions which agree at $t = 0$ are identical. A more general definition of permissibility and a more general uniqueness theorem were given by O. A. Oleinik.

It is possible to state an analogous condition of permissibility for discontinuities of solutions of systems of conservation laws. When applied to the equations of compressible fluid flow this, condition turns out to be equivalent analytically to the assertion that upon crossing a discontinuity the entropy of the flow increases.

We shall give now an explicit formula for permissible solutions of a conservation law with arbitrarily prescribed initial values. This formula is due to E. Hopf, O. A. Oleinik, and P. D. Lax. We shall assume that $a(u)$ is a monotonic function of u; this implies that $f(u)$ is convex or concave.

Now $g(s)$ is defined as the conjugate of the convex (or concave) function $f(u)$, given by the formula

$$g(s) = \underset{u}{Max}(Min)\{us + f(u)\};$$

we define $b(s)$ as the derivative of g with respect to s. Let $\phi(x)$ be the prescribed initial value

$$u(x,0) = \phi(x) .$$

Define $\Phi(y)$ as the integral of ϕ, i.e.,

$$\frac{d\Phi}{dy} = \phi(y) .$$

Consider the function

$$\Phi(y) + tg\left(\frac{x-y}{t}\right);$$

for fixed x and t this is a continuous function of y. It is easy to show that for t fixed and with the exception of a countable set of values of x, the function has a unique maximum (or minimum) in y, whose position we denote by $y_0(x,t)$. We define

$$u(x,t) = b\left(\frac{x-y_0}{t}\right) \tag{7}$$

and assert:

The function u defined by formula (7) *is a permissible solution of* (1) *with initial value* ϕ.

For a verification of this assertion and further properties of solutions furnished by this formula, see, e.g., P. D. Lax. An existence theorem for conservation laws where f is less restricted has been given

by Kalashnikov.