CHAPTER III

Differential Equations of Higher Order

Partial differential equations of higher than first order present so many diverse aspects that a unified general theory (as in Chapter II) is not possible. There is a decisive distinction between several types of differential equations, called "*elliptic*", "*hyperbolic*", and "*parabolic*", each of which shows an entirely different behavior regarding properties and construction of solutions.

In this chapter we shall introduce this classification, guided by examples of physical interest. Moreover we shall discuss in a preliminary way methods of approach toward the solution of relevant problems. The subsequent chapters will be primarily concerned with a systematic theory of elliptic and hyperbolic problems.

Some classical differential equations of second order for a function $u(x, y, z)$ are representative examples:

The *Laplace equation* (elliptic type): $u_{xx} + u_{yy} + u_{zz} = 0$,

The *wave equation* (hyperbolic type): $u_{xx} + u_{yy} - u_{zz} = 0$,

The *heat equation* (parabolic type): $u_z = u_{xx} + u_{yy}$.

§1. *Normal Forms for Linear and Quasi-Linear Differential Operators of Second Order in Two Independent Variables*

For linear and also for quasi-linear differential equations of second order (or corresponding systems of two first order equations) in two independent variables, the classification can be carried out by explicit elementary steps without reference to general theory. It originates from the attempt to find simple normal forms.

1. Elliptic, Hyperbolic, and Parabolic Normal Forms. Mixed Types. A linear differential operator of second order for the function $u(x,y)$ is given by

(1) $$L[u] = au_{xx} + 2bu_{xy} + cu_{yy};$$

the coefficients a, b, c are assumed to be continuously differentiable and not simultaneously vanishing functions of x and y in a domain G. We consider

(2) $$L[u] + g(x, y, u, u_x, u_y) = L[u] + \cdots,$$

where the differential expression $g(x, y, u, u_x, u_y)$ is not necessarily linear and contains no second derivatives. Our object is to transform the differential operator (2) or the corresponding differential equation

(3) $$L[u] + \cdots = 0$$

into a simple normal form by introducing new independent variables

(4) $$\xi = \phi(x,y)\,,\quad \eta = \psi(x,y)\,.$$

Denoting by $u(\xi,\eta)$ the function into which $u(x, y)$ is transformed, we have the relations

$$u_x = u_\xi \phi_x + u_\eta \psi_x\,,$$
$$u_y = u_\xi \phi_y + u_\eta \psi_y\,,$$

$$u_{xx} = u_{\xi\xi}\phi_x{}^2 + 2u_{\xi\eta}\phi_x\psi_x + u_{\eta\eta}\psi_x{}^2 + \cdots,$$

$$u_{xy} = u_{\xi\xi}\phi_x\phi_y + u_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + u_{\eta\eta}\psi_x\psi_y + \cdots,$$

$$u_{yy} = u_{\xi\xi}\phi_y{}^2 + 2u_{\xi\eta}\phi_y\psi_y + u_{\eta\eta}\psi_y{}^2 + \cdots$$

(Here again the dots mean terms in which no second order derivatives of u appear.) Thus the differential operator (1) assumes the form

(5) $$\Lambda[u] = \alpha u_{\xi\xi} + 2\beta i_{\xi\eta} + \gamma u_{\eta\eta}$$

with

(6) $$\begin{cases} \alpha = a\phi_x{}^2 + 2b\phi_x\phi_y + c\phi_y{}^2 \\ \beta = a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y\,. \\ \gamma = a\psi_x{}^2 + 2b\psi_x\psi_y + c\psi_y{}^2 \end{cases}$$

Moreover, a, b, c and α, β, γ are related by

(7) $$\alpha\gamma - \beta^2 = (ac - b^2)(\phi_x\psi_y - \phi_y\psi_x)^2\,,$$

$$\begin{aligned} &\alpha\gamma - \beta^2 \\ &= (a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2)(a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2) + \cdots \\ &= (ac - b^2)(\phi_x\psi_y - \phi_y\psi_x)^2 \end{aligned}$$

and by the identity for the "*characteristic quadratic form*"

$$Q(l,m) = al^2 + 2blm + cm^2 = \alpha\lambda^2 + 2\beta\lambda\mu + \gamma\mu^2\,,$$

where the variables l, m and λ, μ are connected at a fixed point x,y by the linear transformation (corrected the misprinting)

$$\begin{cases} l = \lambda\phi_x + \mu\psi_x \\ m = \lambda\phi_y + \mu\psi_y \end{cases}.$$

$$\begin{aligned} &Q(l,m) \\ &= a(\lambda\phi_x + \mu\psi_x)^2 + 2b(\lambda\phi_x + \mu\psi_x)(\lambda\phi_y + \mu\psi_y) \\ &+ c(\lambda\phi_y + \mu\psi_y)^2 \\ &= (a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2)\lambda^2 \\ &+ 2(a\phi_x\psi_x + b\phi_x\psi_y + b\phi_y\psi_x + c\phi_y\psi_y)\lambda\mu \\ &+ (a\psi_x^2 + 2b\psi_x\psi_y + \psi_y^2)\mu^2 \\ &= \alpha\lambda^2 + 2\beta\lambda\mu + \gamma\mu^2 \end{aligned}$$

In the transformation (4) two functions ϕ, ψ are at our disposal so that we may impose two conditions on the transformed coefficients α, β, γ aiming at simple normal forms of the transformed differential equation (5).

We consider the following sets of conditions:

I. $\alpha = \gamma\,,\ \ \beta = 0\,,$

II. $\alpha = -\gamma\,,\ \ \beta = 0$ (or $\alpha = \gamma = 0$),

III. $\beta = \gamma = 0\,.$

Which of these conditions can be satisfied by the transformations – of course always real transformations are assumed -- depends on the algebraic character of the form $Q(l, m)$, or geometrically speaking on the character of the quadratic curve in the l, m-plane, for fixed

$x, y: Q(l, m) = 1$; this curve may be an ellipse, a hyperbola, or a parabola. Accordingly at a point *x, y* we call the operator $L[u]$

I. elliptic if $ac - b^2 > 0$,

II. hyperbolic if $ac - b^2 < 0$,

III. parabolic if $ac - b^2 = 0$.

The corresponding normal forms of the differential operator are

I. $\Lambda[u] = \alpha(u_{\xi\xi} + u_{\eta\eta}) + \cdots$,

$$\text{II.} \quad \begin{cases} \Lambda[u] = \alpha(u_{\xi\xi} - u_{\eta\eta}) + \cdots \\ \qquad or \\ \Lambda[u] = 2\beta u_{\xi\eta} + \cdots \end{cases}$$

III. $\Lambda[u] = \alpha u_{\xi\xi} + \cdots$

and the normal forms of the differential equation are

I. $u_{\xi\xi} + u_{\eta\eta} + \cdots = 0$,

$$\text{II.} \quad \begin{cases} u_{\xi\xi} - u_{\eta\eta} + \cdots = 0 \\ \qquad or \\ u_{\xi\eta} + \cdots = 0 \end{cases},$$

III. $u_{\xi\xi} + \cdots = 0$.

For fixed *x, y*, such a normal form can always be obtained simply by the linear transformation which takes Q into the corresponding normal form However, assuming that the operator L is of the same type in every point of a domain G we want to find functions ϕ and ψ which will transform $L[u]$ into a *normal form at every point of* G. Success depends on whether certain first order systems of linear partial differential equations can be solved.

Without loss of generality we may assume $a \neq 0$ everywhere in the domain G; otherwise, either the equivalent assumption $c \neq 0$ would hold or our expression would already be in the normal form II.

To determine transformation functions ϕ and ψ for the whole domain G, we first assume that $L[u]$ is hyperbolic in G, and that the new coefficients must satisfy the condition $\alpha = \gamma = 0$. Equations (6) then lead to the quadratic equation

$$(8) \qquad Q = a\lambda^2 + 2b\lambda\mu + c\mu^2 = 0$$

for the ratio λ / μ of the derivatives ϕ_x / ϕ_y and ψ_x / ψ_y.

1) Hyperbolic

If $L[u]$ is *hyperbolic* in G, then $ac - b^2 < 0$, and thus equation (8) has two distinct real solutions λ_1 / μ_1 and λ_2 / μ_2. Since $a \neq 0$, we may assume that

$$\mu_1 = \mu_2 = 1\,;$$

then (8) defines the quantities λ_1 and λ_2 in G as continuously differentiable functions of x and y. Thus in the hyperbolic case we obtain the normal form

(9) $$\beta u_{\xi\eta} + \cdots = 0$$

by determining the transformation functions ϕ and ψ from the differential equations

(10) $$\phi_x - \lambda_1 \phi_y = 0\,, \quad \psi_x - \lambda_2 \psi_y = 0\,.$$

These two first order linear homogeneous partial differential equations yield, in fact, two families of curves $\phi = const.$ and $\psi = const.$ which also may be defined as the families of solutions of the ordinary differential equations

$$y' + \lambda_1 = 0\,, \quad y' + \lambda_2 = 0$$

or

$$ay'^2 - 2by' + c = 0\,,$$

where y is considered as a function of x along the curves of the family.

The relation

$$\lambda_1 - \lambda_2 = \frac{2}{a}\sqrt{b^2 - ac}$$

shows that curves of the two families cannot be tangent at any point of G, and that $\phi_x \psi_y - \phi_y \psi_x \neq 0$. If $\alpha = \gamma = 0$, equation (7) implies $\beta \neq 0$.

The curves $\xi = \phi(x, y) = const.$ and $\eta = \psi(x, y) = const.$ are called the *characteristic curves of the linear hyperbolic differential operator* $L[u]$.

Since we may divide (9) by β, we can state: *If* $L[u]$ *is hyperbolic, i.e.,* $ac - b^2 < 0$, *the second order differential equation* (3) *may be transformed into the normal form*

(11) $$u_{\xi\eta} + \cdots = 0$$

by introducing the two families of characteristic curves $\xi = const.$ *and* $\eta = const.$ *as coordinate curves.*

2) Elliptic

If $ac - b^2 > 0$ holds, then the operator (2) is *elliptic* in G. In this case the quadratic equation (8) has no real solutions, but it has two conjugate complex solutions λ_1 and λ_2 which are continuous complex-valued functions of the real variables x, y. The equations $\alpha = \gamma = 0$ are satisfied by no family of real curves $\phi = const.$; i.e.,

there are no characteristic curves. If, however, *a, b, c* are analytic functions of x, y and if we assume that $\phi(x,y)$ and $\psi(x,y)$ are analytic, then we may consider the differential equations (10) for complex *x* and *y*, and, as before, transform them into the new variables ξ and η which become complex conjugate. Introducing real independent variables ρ and σ by the equations

(12) $$\frac{\xi+\eta}{2}=\rho\,,\quad \frac{\xi-\eta}{2i}=\sigma$$

$\xi=\phi=\rho+i\sigma,$
$\eta=\psi=\rho-i\sigma.$

we obtain $4u_{\xi\eta}=u_{\sigma\sigma}+u_{\rho\rho}$. Thus we arrive at the *normal form*

(13) $$\Delta u+\cdots=u_{\sigma\sigma}+u_{\rho\rho}+\cdots=0$$

in the elliptic case.

For the preceding transformation involving complex quantities we had to require a condition of analyticity for these coefficients, a condition incisive but essentially alien to the problem. To avoid this restriction, we may use the following procedure (not involving complex quantities) of transforming an elliptic expression to normal form: Writing ρ and σ instead of ξ and η in equations (3), (4), we stipulate the conditions

$$\alpha=\gamma\,,\quad \beta=0$$

or, explicitly,

$$\begin{cases} a\rho_x{}^2+2b\rho_x\rho_y+c\rho_y{}^2=a\sigma_x{}^2+2b\sigma_x\sigma_y+c\sigma_y{}^2 \\ a\rho_x\sigma_x+b(\rho_x\sigma_y+\rho_y\sigma_x)+c\rho_y\sigma_y=0 \end{cases}.$$

$\alpha=a\phi_x^2+2b\phi_x\phi_y+c\phi_y^2$
$=[a(\rho_x^2-\sigma_x^2)+2b(\rho_x\rho_y-\sigma_x\sigma_y)+c(\rho_y^2-\sigma_y^2)]$
$+2i[a\rho_x\sigma_x+b(\rho_y\sigma_x+\rho_x\sigma_y)+c\rho_y\sigma_y]$

$\gamma=a\psi_x^2+2b\psi_x\psi_y+c\psi_y^2$
$=[a(\rho_x^2-\sigma_x^2)+2b(\rho_x\rho_y-\sigma_x\sigma_y)+c(\rho_y^2-\sigma_y^2)]$
$-2i[a\rho_x\sigma_x+b(\rho_y\sigma_x+\rho_x\sigma_y)+c\rho_y\sigma_y]$

$\beta=a\phi_x\psi_x+b(\phi_x\psi_y+\phi_y\psi_x)+c\phi_y\psi_y$
$=a(\rho_x^2+\sigma_x^2)+2b(\rho_x\rho_y+\sigma_x\sigma_y)+c(\rho_y^2+\sigma_y^2)$

These differential equations can be reduced by ail elementary algebraic manipulation to the following first order system of linear partial differential equations:

(14) $$\sigma_x=\frac{b\rho_x+c\rho_y}{W}\,,\quad \sigma_y=-\frac{a\rho_x+b\rho_y}{W}\,,$$

where

$$W^2=ac-b^2$$

and either sign is permissible for W. From these so-called *Beltrami differential equations,* we obtain immediately, by eliminating one of the unknowns (e.g., σ), the following second order differential equation for the other quantity:

(15) $$\frac{\partial}{\partial x}\frac{a\rho_x+b\rho_y}{W}+\frac{\partial}{\partial y}\frac{b\rho_x+c\rho_y}{W}=0\,.$$

The transformation of the differential equation to normal form (13) in a neighborhood of a point is given by any pair of functions ρ,σ

satisfying (14) and having a nonvanishing Jacobian

$$\begin{vmatrix} \frac{\partial \sigma}{\partial x} & \frac{\partial \sigma}{\partial y} \\ \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} \end{vmatrix} = \sigma_x \rho_y - \sigma_y \rho_x = \frac{1}{W}(a\rho_x{}^2 + 2b\rho_x\rho_y + c\rho_y{}^2).$$

Such functions are determined once we have a solution of (15) with nonvanishing gradient. We shall see in Ch. IV, §7 that under certain smoothness assumptions on the coefficients, (e.g., existence of continuous derivatives up to second order of a, *b, e*) such a solution always exists -- at least locally -- and hence a normalizing parameter system ρ, σ may be introduced in a neighborhood of any point.

3) Parabolic

The third case is the *parabolic* case: $ac - b^2 = 0$. The quadratic equation (8) then has one real root, and we can accordingly introduce one family of curves $\xi = \phi(x, y)$ in such a way that $\alpha = 0$ holds; then, on account of relation (7), we must also have $\beta = 0$, while, for instance, for $\psi = x$ in *G*, $\gamma = a \neq 0$. In the parabolic case we obtain the *normal form*

$$u_{\eta\eta} + \cdots = 0.$$

The theorem stated at the beginning has thus been proved.

Note that the transformation to the normal form is by no means uniquely determined. For example, in the elliptic case the normal form remains unchanged if we subject the ρ, σ -domain to any conformal mapping.

2. Examples. Several examples of the different types of differential equations have already been discussed in Ch. I, §1. The simplest *hyperbolic* equation (that of the vibrating string) $u_{xx} - u_{tt} = 0$ was completely solved. The prototype of the *elliptic* differential equation is the potential equation $\Delta u = u_{xx} + u_{yy} = 0$ (see, e.g., Ch. I, §1). The *parabolic* equation of heat conduction $u_t - u_{xx} = 0$ was discussed in Ch. I, §3.

From the "type" of the equation we shall deduce important properties which suggest not only methods of solution but also criteria to determine whether or not problems are reasonably posed.

1) $u_{xx} + yu_{yy} = 0$

Incidentally, a given differential equation may be of different type in different domains (mixed types); for example, the equation

(16) $$u_{xx} + yu_{yy} = 0$$

is elliptic for $y > 0$ and hyperbolic for $y < 0$, since $ac - b^2 = y$.

In the domain $y < 0$ equation (8), i.e., the equation

$$\lambda^2 + y\mu^2 = 0$$

possesses the two real roots $\lambda / \mu = \pm\sqrt{-y}$; thus the two differential equations

(17) $$\phi_x + \sqrt{-y}\phi_y = 0, \quad \psi_x - \sqrt{-y}\psi_y = 0$$

hold for ϕ and ψ. They have the solutions

$$\begin{cases} \phi = x + 2\sqrt{-y} \\ \psi = x - 2\sqrt{-y} \end{cases}.$$

By the transformation

(18) $$\begin{cases} \xi = x + 2\sqrt{-y} \\ \eta = x - 2\sqrt{-y} \end{cases},$$

(16) assumes the hyperbolic normal form

(19) $$u_{xx} + yu_{yy} = 4u_{\xi\eta} + \frac{2}{\xi - \eta}(u_\xi - u_\eta) = 0$$

for $y < 0$. The characteristic curves are given by the parabolas

$$y = -\frac{1}{4}(x - c)^2;$$

in particular, the curves $\phi = const.$ are the branches of the parabolas having positive slope, the $\psi = const.$ those having negative slope (cf. Figure 3).

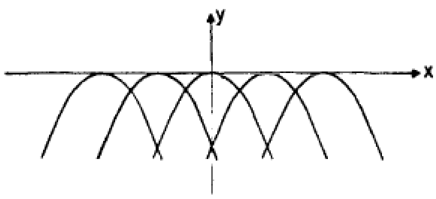

FIGURE 3

For y > 0 we write

(20) $$\begin{cases} \xi = x \\ \eta = 2\sqrt{y} \end{cases};$$

by this transformation, equation (16) assumes the elliptic normal form

(21) $$u_{xx} + yu_{yy} = u_{\xi\xi} + u_{\eta\eta} - \frac{1}{\eta}u_\eta = 0$$

2) $u_{xx} + xu_{yy} = 0$

Similarly, the differential equation

(22) $$u_{xx} + xu_{yy} = 0$$

known as "*Tricomi's equation*" is elliptic for $x > 0$ and hyperbolic for

$x < 0$ because $ac - b^2 = x$.

In the: half-plane $x < 0$, the equations

(23) $$\begin{cases} \xi = \phi(x,y) = \frac{3}{2}y + (\sqrt{-x})^3 \\ \eta = \psi(x,y) = \frac{3}{2}y - (\sqrt{-x})^3 \end{cases}$$

transform (22) into the normal form

(24) $$u_{xx} + xu_{yy} = 9x\left[u_{\xi\eta} - \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta)\right] \quad (\xi > \eta)$$

The characteristic curves are the cubic parabolas

$$y - c = \pm\frac{2}{3}(\sqrt{-x})^3 ;$$

the branches with a downward direction yield the curves $\phi = const.$. those directed upward yield the curves $\psi = const.$ (cf. Figure 4).

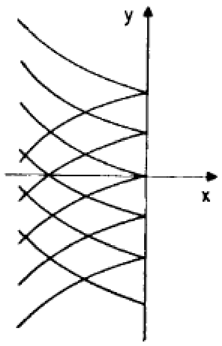

FIGURE 4

For x > 0 we write

$$\begin{cases} \xi = \frac{3}{2}y - i\sqrt{x^3} \\ \eta = \frac{3}{2}y + i\sqrt{x^3} \end{cases},$$

and set

(25) $$\begin{cases} \rho = \frac{\xi + \eta}{2} = \frac{3}{2}y \\ \sigma = \frac{\xi - \eta}{2i} = -\sqrt{x^3} \end{cases};$$

we obtain by this transformation the normal form

(26) $$u_{xx} + xu_{yy} = \frac{9}{4}x\left[(u_{\rho\rho} + u_{\sigma\sigma}) + \frac{1}{3\sigma}u_\sigma\right].$$

The functions (25) satisfy the *Beltrami differential equations*

(27) $$\begin{cases} \sigma_x = -\sqrt{x}\rho_y \\ \sigma_y = \frac{1}{\sqrt{x}}\rho_x \end{cases}.$$

3. Normal Forms for Quasi-Linear Second Order Differential Equations in Two Variables.

4. Example. Minimal Surfaces. Let us consider the differential equation of minimal surfaces

(35) $$(1 + q^2)r - 2pqs + (1 + p^2)t = 0 ;$$

since $ac - b^2 = 1 + p^2 + q^2 > 0$ this differential equation is everywhere elliptic and can be transformed into a normal system of

the form (34). In fact, a short calculation yields the following equations:

$$(36) \qquad \begin{cases} x_\rho^2 + y_\rho^2 + u_\rho^2 = x_\sigma^2 + y_\sigma^2 + u_\sigma^2 \\ x_\rho x_\sigma + y_\rho y_\sigma + u_\rho u_\sigma = 0 \end{cases}, \text{ or } \begin{cases} \boldsymbol{x}_\rho^2 = \boldsymbol{x}_\sigma^2 \\ \boldsymbol{x}_\rho \boldsymbol{x}_\sigma = 0 \end{cases},$$

$$(37) \qquad \Delta\boldsymbol{x}(\boldsymbol{x}_\rho \times \boldsymbol{x}_\sigma) = 0 \quad \text{with} \quad \Delta\boldsymbol{x} = \boldsymbol{x}_{\rho\rho} + \boldsymbol{x}_{\sigma\sigma}) = 0 .$$

This system can be put into a simpler form: Differentiating equations (36) we have

$$\boldsymbol{x}_{\rho\rho}\boldsymbol{x}_\rho = \boldsymbol{x}_{\rho\sigma}\boldsymbol{x}_\sigma \quad \text{and} \quad \boldsymbol{x}_{\sigma\sigma}\boldsymbol{x}_\rho = -\boldsymbol{x}_{\rho\sigma}\boldsymbol{x}_\sigma ;$$

hence,

$$\boldsymbol{x}_\rho \Delta\boldsymbol{x} = 0 \quad \text{and} \quad \boldsymbol{x}_\sigma \Delta\boldsymbol{x} = 0 .$$

On the other hand, equation (37) implies that $\Delta\boldsymbol{x}$ is a linear combination $\Delta\boldsymbol{x} = \alpha\boldsymbol{x}_\rho + \beta\boldsymbol{x}_\sigma$ of the vectors $\boldsymbol{x}_\rho$ and $\boldsymbol{x}_\sigma$. Hence $\alpha = \beta = 0$ and thus $\Delta\boldsymbol{x} = 0$. Thus, a minimal surface in parametric representation with suitable parameters ρ and σ may be characterized by the following conditions: Each of the three coordinates *u, x, y* satisfies the potential equation, i.e.,

$$(38) \qquad \Delta x = 0 , \quad \Delta y = 0 , \quad \Delta u = 0 .$$

Moreover, they satisfy the conditions

$$(39) \qquad \begin{cases} A = \boldsymbol{x}_\sigma^2 - \boldsymbol{x}_\rho^2 = 0 \\ B = 2\boldsymbol{x}_\rho \boldsymbol{x}_\sigma = 0 \end{cases}.$$

With the usual notation of differential geometry

$$E = \boldsymbol{x}_\rho^2 , \quad F = \boldsymbol{x}_\rho \boldsymbol{x}_\sigma , \quad G = \boldsymbol{x}_\sigma^2 ,$$

(39) implies the conditions

$$E - G = 0 , \quad F = 0$$

for the *first fundamental form of the surface*.

These additional conditions apparently add two new differential equations to the three equations (38); however, they merely represent a boundary condition. We need not impose the additional restrictions (39) in a whole two-dimensional ρ,σ-domain, but only along some closed curve in the ρ,σ-domain. From equations (38), the two relations

$$A_\rho = B_\sigma , \quad A_\sigma = -B_\rho$$

follow immediately. They characterize *A* + *iB* as an analytic function of the complex variables $\rho + i\sigma$; therefore, *A* + *iB* vanishes identically if the real part *A* vanishes on any closed curve (e.g., the boundary) and if *B* is zero at some point.

Two immediate conclusions are significant for the theory of

minimal surfaces:

(1) *The mapping of the* ρ,σ *-plane on the minimal surface is conformal.*

(2) *The representation of the minimal surface by harmonic functions is equivalent to the classical Weierstrass representation by means of analytic functions of the complex variable*

$$\rho + i\sigma = \omega .$$

To obtain the formulas of *Weierstrass,* we consider the potential functions x, y, u of ρ,σ as the real parts of analytic functions $f_1(\omega)$, $f_2(\omega)$, $f_3(\omega)$. If $\tilde{x},\tilde{y},\tilde{u}$ are the conjugate potential functions, we have

$$x + i\tilde{x} = f_1(\omega), \quad y + i\tilde{y} = f_2(\omega), \quad u + i\tilde{u} = f_3(\omega).$$

Since, by the Cauchy-Riemann differential equations,

$$x_\sigma = -\tilde{x}_\rho, \quad y_\sigma = -\tilde{y}_\rho, \quad u_\sigma = -\tilde{u}_\rho,$$

we have

$$x_\rho - ix_\sigma = f'_1(\omega), \quad y_\rho - iy_\sigma = f'_2(\omega), \quad u_\rho - iu_\sigma = f'_3(\omega),$$

so that conditions (39) become

$$\phi(\omega) = E - G - 2iF = \sum_{\nu=1}^{3} f'_\nu(\omega)^2 = 0 .$$

Thus, all minimal surfaces may be represented by

$$x = \mathrm{Re}(f_1(\omega)), \quad y = \mathrm{Re}(f_2(\omega)), \quad u = \mathrm{Re}(f_3(\omega)),$$

where the otherwise arbitrary analytic functions $f_\nu(\omega)$ are subject to the condition

$$\sum_{\nu=1}^{3} f'_\nu(\omega)^2 = 0 .$$

Instead of ω we may introduce one of the functions f_ν, e.g., $f_3(\omega)$, as the independent variable. Therefore, the totality of minimal surfaces depends essentially on only one arbitrary analytic function of a complex variable.

5. Systems of Two Differential Equations of First Order.

§2. ***Classification in General and Characteristics***

6. Examples. Maxwell's and Dirac's Equations. It will be easy for the reader to identify the wave equation as hyperbolic, Laplace's equation as elliptic, Cauchy-Riemann's equations $u_x - v_y = 0$, $u_y + v_x = 0$ as an elliptic system, $u_x - v_y = 0$, $u_y - v_x = 0$ as a hyperbolic system, and $u_x = v$, $u_y = v_x$ as a parabolic system.

For the elliptic case we give the following additional examples:

First

$$\Delta\Delta u = 0 \quad \text{or} \quad \sum_{i,k=1}^{n} \frac{\partial^4 u}{\partial x_i^2 \partial x_k^2} = 0 ,$$

with the characteristic form

$$Q = \left(\sum_{i=1}^{m} \phi_i^2 \right)^2 ,$$

and second, the differential equation

$$\sum_{i=1}^{n} \frac{\partial^4 u}{\partial x_i^4} = 0$$

with the characteristic form

$$Q = \sum_{i=1}^{n} \phi_i^4 .$$

An example of a "parabolic" differential equation is

$$u_{tt} = \Delta\Delta u$$

for *n + 1* independent variables with the time variable $x_0 = t$ singled out. Here the characteristic form $\left(\sum_{i=1}^{n} \phi_i^2 \right)^2$ is degenerate since it does not contain the variable ϕ_0.

The operator

$$\left(\Delta - \frac{\partial^2}{\partial t^2} \right) \left(\Delta - 2 \frac{\partial^2}{\partial t^2} \right) u = \Delta\Delta u - 3\Delta u_{tt} + 2u_{tttt}$$

is hyperbolic because its characteristic form in the variables $\phi_1, \cdots, \phi_n$, $\phi_0 = \tau$,

$$Q = \left(\sum_{i=1}^{n} \phi_i^2 - \tau^2 \right) \left(\sum_{i=1}^{n} \phi_i^2 - 2\tau^2 \right),$$

clearly has the required property.

On the other hand, the operator

$$\left(\Delta - \frac{\partial^2}{\partial t^2} \right) \left(\Delta + \frac{\partial^2}{\partial t^2} \right) u = \Delta\Delta u - \frac{\partial^4 u}{\partial t^4}$$

represents an intermediate type; it is neither elliptic nor parabolic nor hyperbolic since the form

$$Q = \left(\sum_{i=1}^{n} \phi_i^2 \right)^2 - \tau^4$$

has only two, not four, real roots τ if the values of the variables $\phi_1, \cdots, \phi_n$ are fixed.

Beltrami equation

Further examples of systems of first order are the Beltrami differential equations.

$$\begin{cases} Wu_x - bv_x - cv_y = 0 \\ Wu_y + av_x + bv_y = 0 \end{cases},$$

where the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is assumed to be positive definite. Here the corresponding characteristic form is

$$Q(\phi) = \begin{vmatrix} -W\phi_1 & b\phi_1 + c\phi_2 \\ W\phi_2 & a\phi_1 + b\phi_2 \end{vmatrix} = -W(a\phi_1^2 + 2b\phi_1\phi_2 + c\phi_2^2) .$$

In the special case where W=1, a=c=1, b=0 (Cauchy-Riemann) it is

$$Q(\phi) = -(\phi_1^2 + \phi_2^2) .$$

Maxwell equation

Dirac equation

The characteristic equation belonging to *Dirac's differential equations* is similar to that of the Maxwell equations. The Dirac equations involve a system of four complex-valued functions

$$u = (u_1, u_2, u_3, u_4)$$

for four variables x_1, x_2, x_3 and x_4 (where $x_4 = t$). To formulate them simply, we introduce the following matrices:

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The equations then are

$$\sum_{k=1}^{4} \alpha_k \left(\frac{\partial}{\partial x_k} - a_k \right) u - \beta b u = 0$$

Here the vector (a_1, a_2, a_3) is proportional to the magnetic potential, $-a_4$ to the electric potential, and b to the rest-mass. According to our rules, the characteristic determinant becomes

$$Q(\phi) = \left| \sum_{k=1}^{4} \alpha_k \phi_k \right| = (\phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2)^2 ,$$

a form of the fourth degree in the variables $\phi_1, \phi_2, \phi_3, \phi_4$. Thus the characteristic manifolds are again the same as those for the wave equation.

§3. *Linear Differential Equations with Constant Coefficients*

2. Fundamental Solutions for Equations of Second Order. Irrespective of order or variability of coefficients, for all linear differential equations, elliptic or hyperbolic, "*fundamental solutions*" defined by certain singularities play an important role, as will become apparent in the next chapters.' Here we shall merely insert a brief preliminary discussion for elliptic second order equations with constant coefficients. We consider the equation

$$L[u] = \Delta u + \rho u = 0$$

and start by asking for "*fundamental*" solutions which depend only on the distance $r = \sqrt{\Sigma (x_i - \xi_i)^2}$ from the point x to a parameter point ξ. By transforming the Laplacian operator to polar coordinates, we obtain (see Vol. I, p. 225)

$$u_{rr} + \frac{n-1}{r} u_r + \rho u = 0 . \tag{10}$$

As is easily verified, the function

$$w(r) = \frac{u_r}{r}$$

satisfies the same equation as u with $n - 1$ replaced by $n + 1$:

$$w_{rr} + \frac{n+1}{r} w_r + \rho w = 0 . \tag{11}$$

Thus, again writing u instead of w, we obtain the "*fundamental solutions*" u for any n by recursion as soon as we know u for $n = 2$ and $n = 3$ from the ordinary differential equations

$$u'' + \frac{1}{r} u' + \rho u = 0$$

$$\begin{aligned} w_r &= \frac{u_{rr}}{r} - \frac{u_r}{r^2} \\ &= \frac{1}{r}\left(-\frac{n-1}{r} u_r - \rho u \right) - \frac{u_r}{r^2} \\ &= -\frac{n u_r}{r^2} - \frac{\rho u}{r} \end{aligned}$$

$$w_{rr} = -\frac{n u_{rr}}{r^2} - n u_r \frac{-2}{r^3} - \frac{\rho u_r}{r} + \frac{\rho u}{r^2}$$

$$\begin{aligned} w_{rr} + \frac{n+1}{r} w_r &= -\frac{n u_{rr}}{r^2} - n u_r \frac{-2}{r^3} - \frac{\rho u_r}{r} + \frac{\rho u}{r^2} + \frac{n+1}{r}\left(-\frac{n u_r}{r^2} - \frac{\rho u}{r} \right) \\ &= -\frac{n u_{rr}}{r^2} + \frac{1}{r^3}\{2n u_r - n(n+1) u_r\} - \frac{\rho u_r}{r} + \frac{\rho u}{r^2}\{1 - (n+1)\} \\ &= -\frac{n u_{rr}}{r^2} + \frac{u_r}{r^3}(-n^2 + n) - \frac{\rho u_r}{r} + \frac{\rho u}{r^2} \\ &= -\frac{n}{r^2}\left\{ u_{rr} + \frac{n-1}{r} u_r + \rho u \right\} - \frac{\rho u_r}{r} \\ &= -\frac{\rho u_r}{r} \\ &= -\rho w \end{aligned}$$

and

$$u'' + \frac{2}{r}u' + \rho u = 0 ,$$

respectively. For $\rho = 0$, i.e., for the Laplace equation, the solutions, except for a constant factor at our disposal, are

$$u = \log\frac{1}{r} , \text{ and } u = \frac{1}{r} .$$

Thus one obtains for any $n \geq 3$ the fundamental solutions

$$u = const.r^{2-n} .$$

With $\rho \neq 0$, say $\rho = \omega^2$, we find in complex notation for n = 1,

$$u = e^{i\omega r} .$$

Hence for n = 3,

$$u = i\omega\frac{e^{i\omega r}}{r} ,$$

and for n = 5,

$$u = -\omega^2\left(\frac{1}{r^2} - \frac{1}{i\omega}\frac{1}{r^3}\right)e^{i\omega r} ,$$

etc. Thus all solutions for odd n can be expressed in terms of trigonometric functions (or hyperbolic functions if $\omega^2 < 0$).

With even n we have, for n =2,

$$u = \alpha J_0(\omega r) + \beta N_0(\omega r) + regular\_function ,$$

where J_0 and $N_0 = (2/\pi)J_0(\omega r)\log r + \cdots$ are the Bessel function and the Neumann function of order zero, respectively, and α, β are constants. If α is chosen as zero we find for n = 4 the singular solution

$$u = \frac{J_0(\omega r)}{r^2} + \frac{\omega}{r}J'_0(\omega r)\log r + \cdots .$$

($J'_0(\omega r)/r$ is regular for r = 0.) This solution we call the *fundamental solution*. One easily ascertains in general: For odd n > 1 we have the singular ("*fundamental*") solutions

$$u = \frac{U}{r^{n-2}} + \cdots ,$$

and for even n

$$u = \frac{U}{r^{n-2}} + W\log r + \cdots ,$$

where the dots denote regular terms and where U and W are regular solutions of $L[U] = L[W] = 0$.

For $\rho < 0$, i.e., for imaginary ω, corresponding relations also

prevail.

In the case of the hyperbolic differential equation

(12) $$L[u] = u_{tt} - \Delta u - \rho = 0 \quad (x = x_1, \cdots, x_n)$$

a quite parallel reasoning leads to the following result:

We seek singular "fundamental solution" of (12) which depend only on the "hyperbolic distance"

$$r = \sqrt{(t-\tau)^2 - \sum_{\nu=1}^{n} (x_\nu - \xi_\nu)^2}$$

from the point *t, x* to the parameter point τ, ξ in the space of $n = m+1$ dimensions. For $u(r)$ we obtain the ordinary differential equation

$$u'' + \frac{n-1}{r} u' - \rho u = 0 \, .$$

As before, fundamental solutions which are singular on the cone $r = 0$ are found to be of the form described above for the elliptic case. The main difference is that now the singularity is spread over a whole cone and that outside the cone the function u is not defined or may be defined as identically zero, while in the elliptic case only the point $x = \xi$ is singular. The significance of such fundamental solutions (which may be modified by multiplication with a constant and addition of any regular solution of L[u] = 0) will become clear in Chapter VI. In Volume I, we have already met such solutions in the form of *Green's function* (see Vol. I, Ch. V, §14).

It may be stated here that these fundamental solutions $u(x;\xi)$ as functions of the point x and the parameter point ξ have the following basic property:

In the elliptic case the integral

$$v(x) = \iint_G f(\xi_1, \cdots, \xi_n) u(x, \xi) d\xi_1 \cdots d\xi_n$$

extended over a domain G including the point x satisfies with a suitable constant c the Poisson equation

$$L[v] = cf(x) \, .$$

In particular, for $n = 3$ the integral

$$cv = \iiint_G f(\xi) \frac{e^{i\omega r}}{i\omega r} dx_1 \cdots dx_n$$

satisfies the "reduced wave equation"

$$\Delta v + \omega^2 v = 0$$

In the hyperbolic case it can be proved that v(x) also satisfies the

differential equation if the domain of integration G fills the characteristic cone issuing from the point x into the ξ-space. (See Ch. VI, §15.)

3. Plane Waves. Turning to equations of arbitrary order k we write the differential equation in n independent variables $x_1,\cdots,x_n$ again in the symbolic form

(13) $$(P_k D_i + P_{k-1} D_i + \cdots + P_0)u + f = 0,$$

where P_k is a homogeneous polynomial with constant coefficients of degree k in the symbols $D_i = \partial / \partial x_i$ $(i = 1, \ldots, n)$ and f denotes a given function of the independent variables. We need consider only the homogeneous equation; i.e., we assume $f = 0$. The nonhomogeneous equation can then be easily treated (see, e.g., §4).

The basic fact is: For any number of independent variables the homogeneous equation (13) possesses solutions in the form of exponential functions $e^{(ax)}$, where

$$(ax) = a_1 x_1 + a_2 x_2 + \cdots$$

with constants a_ν. (On occasion we shall also write (a,x) or $a \cdot x$) The necessary and sufficient condition for u to be a solution is that $a = (a_1,\cdots)$ satisfy the algebraic equation of degree k

(14) $$Q^*(a) = P_k(a) + P_{k-1}(a) + \cdots + P_0 = 0,$$

which defines an algebraic surface $Q^*(a) = 0$ of degree k in the space of the coordinates $a_1, a_2, \cdots$. The classification into types, however, refers more simply to the homogeneous equation

$$Q(a) = P_k(a) = 0;$$

this "*characteristic equation*" depends only on the principal part of the differential equation; it determines the normals of characteristic surface elements in agreement with the definitions of §2.

For example, in three dimensions for the Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = \Delta u = 0$$

we obtain the relation

$$a_1^2 + a_2^2 + a_3^2 = 0.$$

Hence at least one of the exponents a_ν, is imaginary; the corresponding solutions might be written, e.g., in the form

$$e^{xa_1 + ya_2} e^{iz\sqrt{a_1^2 + a_2^2}}.$$

For the wave equation we have solutions $e^{i(a_1x_1 + a_2x_2 + a_3x_3 - a_4 t)}$ with $a_1^2 + a_2^2 + a_3^2 - a_4^2 = 0$, and for the "reduced" wave equation $\Delta u + \omega^2 u = 0$ we have the relation $a_1^2 + a_2^2 + a_3^2 = \omega^2$. For the heat

equation $u_t = \Delta u$, we have the relation $a_1^2 + a_2^2 + a_3^2 - a_4 = 0$.

If the equation $Q(a) = 0$ cannot be satisfied by real values of $a_1, \cdots, a_n$, then the differential equation is called *elliptic*.

4. Plane Waves Continued. Progressing Waves. Dispersion.

In the following sections we shall be primarily concerned with solutions which represent *propagation* phenomena, in particular with *plane waves* arising in the *hyperbolic* case. In addition to the n independent space-variables x we shall consider a further variable $x_0 = t$; we form the inner product $(ax) = a_1 x_1 + \cdots + a_n x_n = A$ with the n-dimensional vector $a : (a_1, \cdots, a_{n)}$ and define the *phase*

$$B = (ax) - bt = A - bt$$

with a constant $a_0 = -b$.

Let us assume first that the differential equation contains only the principal part, i.e., the terms of order *k*, or in other words that $P_\kappa = 0$ for $\kappa < k$. Then the important fact holds: Not only the exponential functions as above are solutions but quite generally all functions of the form

(15) $$u = f(B)$$

are solutions, where the *wave form* $f(B)$ is an *arbitrary function* of the phase $B = A - bt$ and the coefficients a_ν, b are subject to the characteristic equation $Q(-b, a) = 0$. (Compare §2, 4.)

Provided that this equation can be satisfied by real values of $a_1, \cdots, a_n$ and b, the functions $f(B)$ represent *undistorted progressing waves*.

By the term *progressing plane wave* for a homogeneous linear differential equation $L[u]=0$ we mean a solution of the form (15).

Plane waves of this type have constant values on every phase plane of the family

$$B = (ax) - bt = const.$$

in the $(n + 1)$-dimensional x, t-space.

To motivate the term "progressing wave" we consider the n-dimensional space R_n of the space variables $x_1, x_2, \cdots, x_n$ in which the "field" u varies with the time t. A solution u of the form (15) is constant on a whole plane of *constant phase* B of a family of parallel planes. A plane with constant phase moves with constant speed parallel to itself through the space R_n.

If we set

$$a_l = \rho\alpha_l, \ \sum_{l=1}^{n}\alpha_l^2 = 1, \ \rho^2 = \sum_{l=1}^{n} a_l^2, \ b = \rho\gamma,$$

$$B = A - bt = \rho\left(\sum_{i=1}^{n}\alpha_i x_i - \gamma t\right) = \rho((\alpha x) - \gamma t) = \rho E,$$

and write

$$u = f(B) = \phi(E),$$

we obtain a representation in which the numbers α_l are the "direction cosines" of the normals to the plane waves and γ denotes the *speed of propagation of the waves*. E is again called the *phase of the wave,* and the function ϕ**,** or f, is called the *wave form.*

For example, the ordinary wave equation in n space variables, $\Delta u - u_{tt} = 0$, admits plane waves of the form

$$u = \phi((\alpha x) - t);$$

here the coefficients α_l may form an arbitrary unit vector α with $\alpha^2 = 1$ and the wave form ϕ may be an arbitrary function.

In other words, *the wave equation* $\Delta u - u_{tt} = 0$ *is solved by plane waves of arbitrary direction and arbitrarily given form*; *all these waves progress with the speed* $\gamma = 1$.

The waves $f(B)$ are called *undistorted* or *free of dispersion* because they represent for an *arbitrary* form of the wave or signal $f(B)$ an undistorted translation with speed γ (in the direction α of the normal of the phase planes).

If, for an arbitrary direction α, the characteristic equation $Q = 0$ possesses k real and different roots γ, i.e., if there are k different speeds possible for undistorted waves in every direction, the speeds depending in general on the direction α, then the differential equation (13) is called *hyperbolic*. (We shall later generalize this definition by admitting multiple roots in certain cases.) This definition of hyperbolicity referring to the characteristic equation $Q = 0$ is also retained if the differential equation contains terms of lower order.

…

5. Examples. Telegraph Equation. Undistorted Waves in Cables.

For the wave equation $(1/c^2)u_{tt} = \Delta u$, progressing undistorted plane waves with the speed c and the arbitrary form

$$\phi\left(\sum_{l=1}^{n}\alpha_l x_l - ct\right), \ \sum_{l=1}^{n}\alpha_l^2 = 1$$

are possible in every direction. A more general example is given by the telegraph equation

(19) $$u_{tt} - c^2 u_{xx} + (\alpha + \beta) u_t + \alpha\beta u = 0,$$

satisfied by the voltage or the current u as a function of the time t and the position x along a cable; here x measures the length of the cable from an initial point. Unless $\alpha = \beta = 0$, this equation represents dispersion. If we introduce $v = e^{(1/2)(\alpha+\beta)t}$, we obtain the simpler equation

$$v_{tt} - c^2 v_{xx} - \left(\frac{\alpha - \beta}{2}\right)^2 v = 0$$

for the function v. This new equation represents the dispersionless case if and only if

(20) $$\alpha = \beta.$$

In this case the original telegraph equation, of course, possesses no absolutely undistorted wave solutions of arbitrarily prescribed form. However, our result may be stated in the following way: *If condition* (20) *holds, the telegraph equation possesses damped, yet "relatively" undistorted, progressing wave solutions of the form*

(21) $$u = e^{-(1/2)(\alpha+\beta)t} f(x \pm ct),$$

with arbitrary f, progressing in both directions of the cable.

This result is important for telegraphy; it shows that, given appropriate values for the capacity and inductance of a cable, signals can be transmitted - damped in time - in a relatively undistorted form (cf. Ch. V, App. 2).

6. Cylindrical and Spherical Waves. The principle of superposition leads to other important forms of solutions for our differential equations, in particular to *cylindrical* and *spherical waves.*

(a) Cylindrical Waves. The wave equation in two dimensions

(22) $$u_{xx} + u_{yy} - u_{tt} = 0$$

is solved for arbitrary θ by $\exp\{i\rho(x\cos\theta + y\sin\theta)\}\exp\{i\rho t\}$, where ρ is a number which can be arbitrarily chosen. Integration of this "*plane wave*" with respect to the direction angle θ yields the new solution

$$u(x,y,t) = e^{i\rho t}\int_0^{2\pi} \exp\{i\rho r\cos(\theta - \phi)\}d\theta = 2\pi e^{i\rho t} J_0(\rho r),$$

where the polar coordinate r is introduced by $x = r\cos\phi$, $y = r\sin\phi$. This solution represents a *standing wave.*

Thus, a *rotationally symmetric* solution of the *wave equation* (*22*),

a so-called *cylindrical wave*, *is given by the Bessel function Jo* . This solution is regular at the origin $r = 0$.

By the superposition of plane waves we can also construct a solution which is *singular at the origin*, corresponding to a *radiation process* (cf. §4) *with a source at the origin.* For this construction we use improper waves. We consider the complex path L of integration in the θ-plane illustrated in Figure *5* (cf. Vol. I, Ch. VII), and form the complex integral

$$u = e^{i\rho t}\int_L e^{i\rho r\cos\theta}d\theta = \pi e^{i\rho t}H_0^1(\rho r),$$

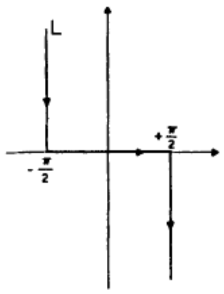

FIGURE 5

where H_0^1 denotes the *Hankel function*. Then u is a solution of the wave equation.

Both cylindrical waves are periodic in *t*, of course, and oscillating but not periodic in the space variable *r*.

(b) Spherical Waves. In three-dimensional space the situation is slightly different. From the solution

$$\exp\{i\rho t\}\exp\{iGr(\alpha x + \beta y + \gamma z)\} = \exp\{i\rho t\}w$$

we obtain by integrating w over the unit sphere in α,β,γ-space the new function

$$v = \iint_\Omega e^{i\rho(\alpha x+\beta y+\gamma z)}d\Omega$$

where $d\Omega$ is the surface element of the unit sphere. Since this function is evidently invariant under rotation of the coordinate system we may, for purposes of calculation, set $x = y = 0$, $z = r$. Introducing polar coordinates θ,ϕ in α,β,γ-space, we obtain

$$v = \int_0^{2\pi}d\phi\int_0^{\pi} e^{i\rho r\cos\theta}\sin\theta d\theta$$

or

$$v = \frac{4\pi}{\rho}\frac{\sin\rho r}{r}.$$

Thus $\exp\{i\rho t\}(\sin\rho r)/r$ is a *standing spherical wave*, rotationally symmetric, *regular at the origin*, and derived by the superposition of regular progressing plane waves.

Waves with a *singularity at the origin* which correspond to *radiation phenomena* must again be formed by means of improper plane waves. The path of integration L of Figure 6 leads to

$$v = \int_L e^{i\rho r\cos\theta} \sin\theta d\theta = 2\pi \frac{e^{i\rho r}}{i\rho r}. \tag{23}$$

In terms of real quantities, we have simultaneously constructed the two spherical wave forms $(\cos\rho r)/r$ and $(\sin\rho r)/r$, the second of which is the regular one just computed.

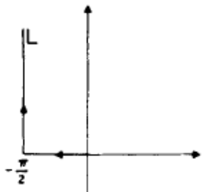

FIGURE 6

We observe: The spherical wave form (23) may be derived by superposition of the plane waves $\exp\{i\rho(\alpha x + \beta y + \gamma z)\}$ for an arbitrary position of the point (x,y,z), with $z > 0$. Independently of the position of this point,

$$2\pi \frac{e^{i\rho r}}{i\rho r} = \int_0^{2\pi} d\phi \int_L e^{i\rho(\alpha x + \beta y \gamma z)} \sin\theta d\theta \tag{24}$$

holds with $x^2 + y^2 + z^2 = r^2$. The easy verification of (24) may be omitted.

Since the wave equation does not contain dispersion terms, we can construct the rotationally symmetric wave

$$u = 2\pi \int_0^{2\pi} \int_0^{\pi} f(t - \alpha x - \beta y - \gamma z) \sin\theta d\theta d\phi$$

with an arbitrary function $f(\lambda)$. This expression is invariant under rotation; thus we may evaluate the integral assuming $x = y = 0$. In polar coordinates we obtain

$$\begin{aligned} u &= 2\pi \int_0^{\pi} f(t - r\cos\theta) \sin\theta d\theta \\ &= \frac{2\pi}{r}[F(t+r) - F(t-r)] \end{aligned},$$

where F, the indefinite integral of f, is arbitrary. Thus, for arbitrary (twice differentiable) F, the function

$$\frac{F(t+r) - F(t-r)}{r}$$

e is a solution. Likewise, each of the functions

$$\frac{F(t+r)}{r} \text{ and } \frac{F(t-r)}{r}$$

in itself is also a solution. This can be easily seen by making appropriate changes in the function f or F as well as by direct verification. These solutions, which obviously are singular at the origin, represent "*progressing spherical waves* attenuated in space".

Moreover, these are the only solutions of the wave equation in three-dimensional space which depend spatially on r alone, because

for a function $u(r, t)$ the expression $\Delta u = u_{xx} + u_{yy} + u_{zz}$ becomes

$$\Delta u = u_{rr} + \frac{2}{r} u_r = \frac{1}{r}(ru)_{rr}$$

(cf. Vol. I, p. 225). Hence the wave equation $\Delta u - u_{tt} = 0$ goes over into the equation

$$\frac{1}{r}[(ru)_{rr} - (ru)_{tt}] = 0 ,$$

whose general solution, according to Ch. I, p. 6, is

$$ru = F(t + r) + G(t - r)$$

with arbitrary F and G.

§6. *Typical Problems in Differential Equations of Mathematical Physics*

1. Introductory Remarks. To find the "general solution", i.e., the totality of all solutions of a partial differential equation, is a problem which hardly ever occurs. The goal usually is to single out specific individual solutions by adding further conditions to the differential equations. For $n + 1$ independent variables these additional restrictions usually refer to n-dimensional manifolds, which sometimes appear as boundaries, sometimes as "initial manifolds", and sometimes as discontinuity surfaces of domains within which the solutions are to be found ('*boundary*", "*initial*", and "*jump conditions*"). In particular, *initial value problems*, or "*Cauchy problems*", were discussed in §4: Along the plane $x0 = t$ values for the solution u and, as the case requires, for its derivatives with respect to t, were prescribed as functions of the coordinates $x_1, x_2, \cdots, x_n$. We seek a solution u for $t \geq 0$ which at $t = 0$ represents the prescribed "initial state". Such solutions of initial value problems may be continued for $t < 0$, so that the manifold $t = 0$ lies within the domain of definition of the solution. In other words, the state at $t = 0$ may be interpreted as the result of a previous state whose continuation into the future is governed by the same laws. In the case of differential equations of first order, for which the solution of the initial value problem was constructed in Chapter II, a continuation of this kind is automatic. For problems of higher order a similar result is contained in Ch. I, §7 where analytic differential equations and analytic initial conditions are treated. However, to assume that either the differential equations or the initial

conditions are analytic is in general an artificial restriction ; moreover, even for analytic differential equations the analytic character of all the solutions is not evident *a priori.* It is therefore reasonable to consider initial or boundary value problems without regard to continuation of solutions beyond these boundaries.

A typical *boundary value problem* ……