CHAPTER V

Hyperbolic Differential Equations in Two Independent Variables

Introduction

Characteristic curves C have the following properties, each of which can be used as a definition (see also Chapters I, II, III):

1) Along a characteristic curve the differential equation (or, for systems, a linear combination of the equations) represents an *interior differential equation*.

la) Initial data on a characteristic curve cannot be prescribed freely, but must satisfy a *compatibility condition* if these data are to be extended into "*integral strips*".

2) Discontinuities (of a nature to be specified later) of a solution cannot occur except along characteristics.

3) Characteristics are the only possible "branch lines" of solutions, i.e., lines for which the same initial value problem may have several solutions.

For systems of quasi-linear differential equations of first order the initial data or "*Cauchy data*" are simply the values of the unknown functions on the initial curve. The first property is inherent in the basic fact: A direction is characteristic at a point P if there exists a linear combination of the differential equations for which all the unknowns are differentiated at P only in this direction. (A system is *hyperbolic* if it can be replaced by a linearly equivalent one in which each differential equation contains at every point differentiation only in one "*characteristic*" direction.) The second and third properties for hyperbolic systems can also be read off from this special form of the equations.

§ 1. *Characteristics for Differential Equations Mainly of* Second Order

1. Basic Notions. Quasi-Linear Equations. We consider the quasi-linear differential operator of second order

(1) $$L[u] = ar + bs + ct$$

and differential equation

(2) $$L[u] + d = ar + bs + ct + d = 0\,,$$

where

$$r = u_{xx}\,,\quad s = u_{xy}\,,\quad t = u_{yy}$$

and where a, b, c, d are given functions of the quantities x, y, u, $p = u_x$, $q = u_y$ in the region under consideration. (It is always assumed that all functions and derivatives which occur are continuous unless the contrary is expressly stated.)

As in Chapter II, we start with the initial value problem, to extend an *initial strip* into an *integral strip*. We define a strip *C1* of first order as follows: Two functions $x = X(\lambda)$ and $y = Y(\lambda)$ of a parameter λ determine a curve *C0* in the *x,y*-plane; together with a function $u = U(\lambda)$ they determine a curve $\overline{C}_0$ "above *C0*" in *x,y,u*-space. Planes tangent to the curve $\overline{C}_0$ are defined by two additional functions $P(\lambda)$ and $Q(\lambda)$ (with normal given by the direction numbers *P*, *Q*, *-1*) subject to the "*strip relation*"

(3) $$\dot{U} = P\dot{X} + Q\dot{Y}$$

which expresses that the curve $\overline{C}_0$ and the tangent plane arc parallel (here the dot denotes differentiation with respect to the parameter λ). Throughout we assume

$$\dot{X}^2 + \dot{Y}^2 \neq 0 .$$

A given surface *u* (*x,y*) induces such a strip C1 over a base curve *Co* if we identify *U*, *P*, and *Q* of $\overline{C}_0$ with the values of *u, p* , and *q* on *Co*, that is, $U(\lambda) = u(X(\lambda), Y(\lambda))$ and in addition, $P(\lambda) = u_x(X(\lambda), Y(\lambda))$ and $Q(\lambda) = u_y(X(\lambda), Y(\lambda))$. For a strip on a surface *u*(*x*,*y*) we shall write *u* , *p* , *q* (and *x* , y) instead of *U, P*, *Q* (and X, *Y*) whenever the meaning is clear.

Often it is useful to represent the base curve *C0* in the *x,y*-plane by a relation $\phi(x, y) = 0$. We assume that the curve $\phi = 0$ in the *x,y*-plane and also the curve $\overline{C}_0$ on the surface *u* = *u* (*x*, *y*) separate a region $\phi < 0$ from a region $\phi > 0$. We also assume that $\phi = 0$ is a regular curve, i.e., that ϕ_x and ϕ_y do not vanish simultaneously.

Further we define a strip *C2* of second order by considering three additional functions $R(\lambda)$, $S(\lambda)$, $T(\lambda)$ corresponding to the second derivatives *r*, *s*, *t* of a surface *u*(*x*,*y*) through *C1* and satisfying the *strip relations*

$$\dot{P} = R\dot{x} + S\dot{y} , \quad \dot{Q} = S\dot{x} + T\dot{y} .$$

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§2. ***Characteristic Normal Forms for Hyperbolic Systems of First Order***

1. Linear, Semilinear and Quasi-Linear Systems.

2. The Case *k* = 2. Linearization by the Hodograph Transformation. For hyperbolic systems of two equations, just as for

a single differential equation of second order, one may (as in §1) introduce the two families of characteristic curves as coordinate curves. This means one introduces two "characteristic parameters" α and β such that

$$\alpha = \phi(x, y) = const. \quad \text{and} \quad \beta = \psi(x, y) = const.$$

represent the two distinct characteristic families which in the quasi-linear case still depend on the individual solution u^1, u^2 under consideration, but in the linear case are independent of *u* and are a *priori* known. The introduction of characteristic parameters α, β is particularly convenient in the treatment of the nonlinear Monge-Ampère equation (see App. I, §2).

We may consider u^1, u^2, x, y as four functions of the two independent variables α, β and obtain an equivalent system of four equations in characteristic independent parameters

(11a) $$h^{11}u^1_\alpha + h^{12}u^2_\alpha = \gamma^1, \quad h^{21}u^1_\beta + h^{212}u^2_\beta = \gamma^2,$$

(11b) $$x_\alpha = \tau^1 y_\alpha, \quad x_\beta = \tau^2 y_\beta,$$

where the h^{ik}, γ^k, τ^k are known functions of u^1, u^2, x, y, $x_\alpha, \ldots, y_\beta$.

In the linear case the equations (11b) are independent of the equations (1la) ; in the quasi-linear case they are coupled. At any rate, together they replace the original system of two equations by a simpler system of four equations in which, incidentally, the independent variables do not occur explicitly.

In an important special case a quasi-linear system in two unknowns can immediately be reduced to a linear system by simply interchanging the role of dependent and independent variables, i.e., by considering x and y as functions of $u = u^1$ and $u = u^2$. This reduction is possible if the system consists of differential equations which are homogeneous in the derivatives and have the form

$$\begin{cases} a^1 u_x + b^1 u_y + c^1 v_x + d^1 v_y = 0 \\ a^2 u_x + b^2 u_y + c^2 v_x + d^2 v_y = 0 \end{cases},$$

where $a^1, \cdots, d^2$ are functions of *u, v* alone, and if the Jacobian $J = u_x v_y - u_y v_x$ does not vanish. We have

$$u_x = J y_v, \quad u_y = -J x_v,$$
$$v_x = -J y_u, \quad v_y = J x_u$$

and our system is indeed transformed into the linear system

$$\begin{cases} a^1 y_v - b^1 x_v - c^1 y_u + d^1 x_u = 0 \\ a^2 y_v - b^2 x_v + -^2 y_u + d^2 x_u = 0 \end{cases}$$

for $x(u, v)$ and $y(u, v)$.

This transformation plays an important role in fluid dynamics, where u, v denote the velocity components of a steady two-dimensional flow. It is called *hodograph transformation*, because the image in the u,v-plane of a particle path in the x,y-plane is the "*hodograph*" of the particle, that is, the path described by the velocity vector.

§3. *Applications to Dynamics of Compressible Fluids*

The motion of compressible fluids offers instructive and significant illustrations of the concept of characteristics. In Chapter VI we shall consider such flows depending on three space variables and time. Here we restrict ourselves to cases which can be described by only two independent variables.

1. One-Dimensional Isentropic Flow. The differential equations for the velocity $u(x, t)$ and density $\rho(x,t)$ of a one-dimensional isentropic flow are

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho u) = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \end{cases}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} = f'\rho_x = c^2 \rho_x$$

$$L^1[u,\rho] \equiv \rho u_x + u\rho_x + \rho_t = 0\,,$$
$$L^2[u,\rho] \equiv \rho u u_x + \rho u_t + c^2\rho_x = 0\,.$$

The "sound speed" $c(\rho)$ is given by $c = \sqrt{f'(\rho)} > 0$, where $p = f(\rho)$ is a given monotonic function expressing the pressure p in terms of ρ. (The condition $f'(\rho) > 0$ expresses hyperbolicity of the system.) For this first-order quasi-linear system, the characteristic directions $\tau = dx : dt$ are defined by

$$\begin{pmatrix} \rho & u \\ \rho u & c^2 \end{pmatrix}\begin{pmatrix} u \\ \rho \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix}\begin{pmatrix} u \\ \rho \end{pmatrix}_t = 0$$

$$|A - \tau B| = \begin{vmatrix} \rho & u-\tau \\ \rho u - \tau\rho & c^2 \end{vmatrix} = \rho\begin{vmatrix} 1 & u-\tau \\ u-\tau & c^2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & u-\tau \\ u-\tau & c^2 \end{vmatrix} = 0\,,$$

hence $\tau = u \pm c$. The system is hyperbolic since there are two distinct real roots τ. For each of these values of τ, there exists a nontrivial solution (λ^1, λ^2) of the system

$$\begin{cases} \lambda^1 + (u-\tau)\lambda^2 = 0 \\ (u-\tau)\lambda^1 + c^2\lambda = 0 \end{cases}.$$

When $\tau = u + c$, $\lambda^1 : \lambda^2 = c$; when $\tau = u - c$ $\lambda^1 : \lambda^2 = -c$. With each of these values of $\lambda^1 : \lambda^2$, the linear combination $\lambda^1 L^1 + \lambda^2 L^2$ involves differentiation of u and of ρ in the same direction (the corresponding characteristic direction). Accordingly, we have the system of differential equations

$$\Lambda^1[u,\rho] \equiv \rho[u_t + (u+c)u_x] + c[\rho_t + (u+c)\rho_x] = 0\,,$$
$$\Lambda^2[u,\rho] \equiv \rho[u_t + (u-c)u_x] - c[\rho_t + (u-c)\rho_x] = 0\,.$$

We may introduce characteristic parameters $\alpha = \phi(x,t)$, $\beta = \psi(x,t)$ where

$$\psi(x,t) = const.$$

and

$$\phi(x,t) = const.$$

are the characteristic curves defined by

$$-\psi_t : \psi_x = dx : dt = u + c$$

and

$$-\phi_t : \phi_x = dx : dt = u - c$$

Since $-\psi_t : \psi_x = x_\alpha : t_\alpha$ and $-\phi_t : \phi_x = x_\beta : t_\beta$, we obtain the equations

(1a) $$\begin{cases} \rho u_\alpha + c\rho_\alpha = 0 \\ \rho u_\beta - c\rho_\beta = 0 \end{cases},$$

(1b) $$\begin{cases} x_\alpha = (u+c)t_\alpha \\ x_\beta = (u-c)t_\beta \end{cases}$$

for the four functions *u,* ρ*, x, t* of the characteristic parameters α, β (cf. §2,3).

The characteristic curves in this example can be interpreted as representing "sound waves" (small disturbances) whose velocity

$$\frac{dx}{dt} = u \pm c\,,$$

sometimes called "*characteristic speed*", differs from the "*stream velocity*" *u* by $\pm c$.

We point out that equations (la) lead to the so-called "*Riemann invariants*", that is, to functions which are constant along characteristics, in the following way.

We wish to write (la) in the form

$$\frac{dr}{d\alpha} = 0\,,\ \frac{ds}{d\beta} = 0$$

with suitable functions $r(u,\rho)$, $s(u,\rho)$, in order to deduce that *r=const.* along $\beta = const.$ and *s = const.* along $\alpha = const.$ Clearly, we must find functions *g* and *l* such that $r_u = g\rho$, $r_\rho = gc$; $s_u = l\rho$, $s_\rho = -lc$, where the "integrating factors" $g(u,\rho)$ and $l(u,\rho)$ satisfy the compatibility conditions $(g\rho)_\rho = (gc)_u$, $(l\rho)_\rho = (-lc)_u$. One easily finds that

$$g = \frac{1}{2\rho}\,,\ l = -\frac{1}{2\rho}$$

are solutions leading to the *Riemann invariants*

$$2r = u + \int_{\rho_0}^{\rho} \frac{c(\rho')}{\rho'} d\rho' = const. \text{ along } \beta = const.,$$

$$2s = -u + \int_{\rho_0}^{\rho} \frac{c(\rho')}{\rho'} d\rho' = const. \text{ along } \alpha = const.$$

$$\frac{c}{\rho} = \frac{A^{1/2}\gamma^{1/2}\rho^{(\gamma-1)/2}}{\rho} = A^{1/2}\gamma^{1/2}\rho^{(\gamma-3)/2}$$

In the important, case of ideal "polytropic" gases, the pressure is given by

$$\int \rho^{(\gamma-3)/2} d\rho = \frac{1}{(\gamma-1)/2}\rho^{(\gamma-1)/2}$$

$$p = f(\rho) = A\rho^{\gamma},$$

where A and γ are constants, $\gamma > 1$. In this case,

$$\int \frac{c}{\rho} d\rho = \frac{A^{1/2}\gamma^{1/2}}{(\gamma-1)/2}\rho^{(\gamma-1)/2} = \frac{c}{(\gamma-1)/2}$$

$$c^2 = f'(\rho) = A\gamma\rho^{\gamma-1}$$

and if we take $\rho_0 = 0$, the Riemann invariants are

$$r = \frac{u}{2} + \frac{c}{\gamma-1}, \quad s = -\frac{u}{2} + \frac{c}{\gamma-1}.$$

These expressions can be used to solve the initial value problem.

Finally, we note that the nonlinear system $L^1[u,\rho] = 0$, $L^2[u,\rho] = 0$ is of the type which admits linearization by the *hodograph transformation* of §2. Thus the following linearized equations are obtained for x and t as functions of u and ρ:

$$\begin{pmatrix} a^1 \\ b^1 \\ c^1 \\ d^1 \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \\ u \\ 1 \end{pmatrix} \quad \begin{pmatrix} a^2 \\ b^2 \\ c^2 \\ d^2 \end{pmatrix} = \begin{pmatrix} \rho u \\ \rho \\ c^2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} \to \begin{pmatrix} x \\ t \\ u \\ \rho \end{pmatrix}$$

$$\begin{cases} \rho t_p - u t_u + x_u = 0 \\ \rho u t_p - \rho x_p - c^2 t_u = 0 \end{cases}.$$

They are valid as long as the Jacobian $u_x \rho_t - u_t \rho_x$ differs from zero. (For a detailed analysis, see R. Courant and K. O. Friedrichs [1].)

2. Spherically Symmetric Flow. The differential equations

$$L^1[u,\rho] \equiv \rho u_x + u\rho_x + \rho_t = -\frac{2\rho u}{x}$$

$$L^2[u,\rho] \equiv \rho u u_x + \rho u_t + c^2 \rho_x = 0$$

for spherical isentropic flow can be treated in the same way. Since the operators L^1 and L^2 are the same as in the one-dimensional flow (see article 1) , the characteristic directions τ are given by $\tau = u \pm c$. Likewise the values of λ^1/λ^2 are again given, except for a factor of proportionality, by $\lambda^1/\lambda^2 = \pm c$, yielding the system of differential equations

$$\Lambda^1[u,\rho] = -\frac{2c\rho u}{x},$$

$$\Lambda^2[u,\rho] = \frac{2c\rho u}{x}$$

with the same operators Λ^1, Λ^2 as before. The equations analogous to (1a) are

(2a)
$$\begin{cases} \rho u_\alpha + c\rho_\alpha = -\dfrac{2c\rho u}{x} t_\alpha \\ \rho u_\beta - c\rho_\beta = \dfrac{2c\rho u}{x} t_\beta \end{cases}$$

and equations (lb) are unchanged.

3. Steady Irrotational Flow. In a steady two-dimensional irrotational isentropic flow, the velocity components *u, v* satisfy the system

$$L^1[u,\rho] \equiv u_y - v_x = 0$$

$$L^2[u,\rho] \equiv (c^2-u^2)u_x - uv(u_y+v_x) + (c^2-v^2)v_y = 0$$

$$\begin{pmatrix} 0 & -1 \\ c^2-u^2 & -uv \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 & 0 \\ -uv & c^2-v^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_t = 0$$

$$|A-\tau B| = \begin{vmatrix} -\tau & -1 \\ c^2-u^2+\tau uv & -uv-\tau(c^2-v^2) \end{vmatrix}$$
$$= \tau\{uv+\tau(c^2-v^2)\} + \{c^2-u^2+\tau uv\}$$
$$= (c^2-v^2)\tau^2 + 2uv\tau + (c^2-v^2)$$

where the sound speed *c* is a given function of u^2+v^2. In this case, the characteristic directions τ satisfy (corrected)

$$\begin{vmatrix} -\tau & -1 \\ c^2-u^2+uv\tau & -uv-(c^2-v^2)\tau \end{vmatrix} = (c^2-v^2)\tau^2 + 2uv\tau + c^2-u^2 = 0 .$$

At all points where $u^2+v^2>c^2$ (that, is, where the flow is supersonic), this equation has two distinct real roots τ^1,τ^2 and the system is hyperbolic. For each of these values of τ, the corresponding value of $\lambda^1:\lambda^2$ is $-uv-(c^2-v^2)\tau$. By taking the linear combination $\lambda^1L^1+\lambda^2L^2$ in each case, we arrive at the system

$$\Lambda^i[u,v] \equiv (c^2-u^2)(u_y+\tau^i u_x) + (c^2-v^2)\tau^i(v_y+\tau^i v_x) = 0$$

(i=1,2)

The equations for *u, v, x, y* as functions of α,β are

(3a)
$$\begin{cases} (c^2-u^2)u_\alpha + (c^2-v^2)\tau^1 v_\alpha = 0 \\ (c^2-u^2)u_\beta + (c^2-v^2)\tau^2 v_\beta = 0 \end{cases},$$

and

(3b)
$$\begin{cases} x_\alpha = \tau^1 y_\alpha \\ x_\beta = \tau^2 y_\beta \end{cases}$$

Equations (3a) can be written in the equivalent form

(3a')
$$\begin{cases} \tau^2 u_\alpha + v_\alpha = 0 \\ \tau^1 u_\beta + v_\beta = 0 \end{cases}.$$

For the differential equations

$$L^1[u,\rho] \equiv u_y - v_x = 0$$

$$L^2[u,\rho] \equiv (c^2-u^2)u_x - uv(u_y+v_x) + (c^2-v^2)v_y = 0$$

of steady three-dimensional irrotational isentropic flow with cylindrical symmetry, the operators L^1 and L^2 are the same as in

the previous example so that the characteristic directions τ^1, τ^2 again satisfy

$$(c^2 - v^2)\tau^2 + 2uv\tau + c^2 - u^2 = 0 .$$

The equations analogous to (3) in this case are

$$\text{(4a)} \qquad \begin{cases} u_\alpha + \dfrac{1}{\tau^2} v_\alpha = -\dfrac{c^2 v}{(c^2 - u^2)y} x_\alpha \\ u_\beta + \dfrac{1}{\tau^1} v_\beta = -\dfrac{c^2 v}{(c^2 - u^2)y} x_\beta \end{cases}$$

and equations (3b).

Again, the equations of steady irrotational flow allow linearization by the *hodograph transformation* of §2. Provided that

$$u_x v_y - v_x u_y \neq 0 ,$$

we obtain the linear system

$$\begin{cases} x_v - y_u = 0 \\ (c^2 - u^2) y_v + uv(x_v + y_u) + (c^2 - v^2) x_u = 0 \end{cases}$$

for x, y as functions of u and v.

4. Systems of Three Equations for Nonisentropic Flow. In a (nonsteady) one-dimensional nonisentropic flow, the velocity $u(x, t)$, pressure $p(x, t)$ and specific entropy $S(x, t)$ satisfy the system

$$L^1[u, p, S] \equiv \rho c^2 u_x + u p_x + p_t = 0 ,$$
$$L^2[u, p, S] \equiv \rho u u_x + \rho u_t + p_x = 0 ,$$
$$L^3[u, p, S] \equiv u S_x + S_t = 0 ,$$

$$\begin{pmatrix} \rho c^2 & u & 0 \\ \rho u & 1 & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} u \\ p \\ S \end{pmatrix}_x + \begin{pmatrix} 0 & 1 & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ p \\ S \end{pmatrix}_t = 0$$

$$|A - \tau B| = \begin{vmatrix} \rho c^2 & u - \tau & 0 \\ \rho u - \tau\rho & 1 & 0 \\ 0 & 0 & u - \tau \end{vmatrix}$$

$$= \rho \begin{vmatrix} c^2 & u - \tau & 0 \\ u - \tau & 1 & 0 \\ 0 & 0 & u - \tau \end{vmatrix}$$

$$= \rho(u - \tau)\left\{c^2 - (u - \tau)^2\right\}$$

where ρ and $c \neq 0$ are given functions of p and S. The characteristic directions τ are then defined by

$$\begin{vmatrix} c^2 & u - \tau & 0 \\ u - \tau & 1 & 0 \\ 0 & 0 & u - \tau \end{vmatrix} = 0 ;$$

hence, they are

$$\begin{cases} \tau^1 = u + c \\ \tau^2 = u - c , \\ \tau^3 = u \end{cases}$$

and the system is hyperbolic. For each of the characteristic directions τ^1, τ^2, τ^3 there exists a nontrivial solution $\lambda^1, \lambda^2, \lambda^3$ of

$$\begin{cases} c^2 \lambda^1 + (u - \tau)\lambda^2 = 0 \\ (u - \tau)\lambda^1 + \lambda^2 = 0 \\ (u - \tau)\lambda^3 = 0 \end{cases} ,$$

i.e., except for a factor of proportionality,

$$(\lambda^1,\lambda^2,\lambda^2)=\begin{cases}(1,c,0)\_for\_\tau=\tau^1\\(-1,c,0)\_for\_\tau=\tau^2\\(0,0,1)\_for\_\tau=\tau^3\end{cases}$$

By forming the linear combination $\lambda^1L^1+\lambda^2L^2+\lambda^3L^3$, in each case, we obtain the system

$$\Lambda^1[u,p,S]\equiv\rho c[u_t+(u+c)u_x]+p_t+(u+c)p_x=0\,,$$
$$\Lambda^2[u,p,S]\equiv\rho c[u_t+(u-c)u_x]+-[p_t+(u-c)p_x]=0\,,$$
$$\Lambda^1[u,p,S]\equiv S_t+uS_x=0\,.$$

Although we do not introduce characteristic parameters instead of the variables *x, t* as we did when there were two equations for two unknowns, we can, nevertheless, write these equations in a more concise form by letting s1, s 2 , s3 be curve parameters on the three families of characteristic curves. These equations then take the form

$$\begin{cases}\rho c\dfrac{\partial u}{\partial s_1}+\dfrac{\partial p}{\partial s_1}=0\\ \rho c\dfrac{\partial u}{\partial s_2}-\dfrac{\partial p}{\partial s_2}=0\\ \dfrac{\partial S}{\partial s_3}=0\end{cases}.$$

In the present example, the characteristic curves determined by $dx:dt=u+c$ represent sound waves (as in the isentropic case) and the characteristic curves $dx:dt=u$ represent trajectories of particles.

As another example, we consider the differential equations

$$\begin{cases}\rho uu_x+\rho vu_y+c^2\rho_x=0\\ \rho uv_x+\rho vv_y+c^2\rho_y=0\\ \rho(u_x+v_y)+u\rho_x+v\rho_y=0\end{cases}$$

$$\begin{pmatrix}\rho u&0&c^2\\0&\rho u&0\\\rho&0&u\end{pmatrix}\begin{pmatrix}u\\v\\\rho\end{pmatrix}_x+\begin{pmatrix}\rho v&0&0\\0&\rho v&c^2\\0&\rho&v\end{pmatrix}\begin{pmatrix}u\\v\\\rho\end{pmatrix}_y=0$$

$$|A-\tau B|=\begin{vmatrix}\rho u-\tau\rho v&0&c^2\\0&\rho u-\tau\rho v&-\tau c^2\\\rho&-\tau\rho&u-\tau v\end{vmatrix}$$
$$=\rho^2\begin{vmatrix}u-\tau v&0&c^2\\0&u-\tau v&-\tau c^2\\1&-\tau&u-\tau v\end{vmatrix}$$
$$=\rho^2\left\{(u-\tau v)^3-c^2(u-\tau v)-(u-\tau v)\tau c^2\tau\right\}$$
$$=\rho^2\left\{(u-\tau v)^3-c^2(u-\tau v)(1+\tau^2)\right\}$$
$$=\rho^2(u-\tau v)\left\{(u-\tau v)^2-c^2(1+\tau^2)\right\}$$

(where the sound speed $c=c(\rho)\neq 0$ is a known function) for the velocity components *u*(*x,y*), *v*(*x,y*) and the density $\rho(x,y)$ of steady two-dimensional rotational isentropic flow. The characteristic directions τ satisfy (*corrected*)

$$\begin{vmatrix}u-\tau v&0&c^2\\0&u-\tau v&-\tau c^2\\1&-\tau&u-\tau v\end{vmatrix}=(u-\tau v)[(u-\tau v)^2-(1+\tau^2)c^2]=0\,.$$

Two families of characteristic curves *dx:dy* are determined by the roots of the quadratic factor

$$(1+\tau^2)c^2-(u-\tau v)^2=(c^2-v^2)\tau^2+2uv\tau+c^2-u^2=0\,,$$

an equation for τ already considered in our example for the irrotational case. These two families of characteristics will be real and

distinct at points where $u^2 + v^2 > c^2$ (i.e., where the flow is supersonic). Sometimes the characteristic curves of the two families are referred to as *Mach lines*. The third family of characteristic curves is determined by *dx:dy* = *u:v* (i.e., $u - \tau v = 0$) and consists of the *streamlines*, namely, the curves tangent to the velocity vectors. It is easy to see that a streamline cannot have the same direction as a Mach line at any point, so the system is totally hyperbolic for $u^2 + v^2 > c^2$.

For the first two characteristic directions, we obtain, except for a factor of proportionality,

$$(\lambda^1, \lambda^2, \lambda^3) = (-1, \tau^i, u - \tau_v^i) \quad (i = 1, 2)$$

and for the third characteristic direction

$$(\lambda^1, \lambda^2, \lambda^3) = (u, v, 0) .$$

Taking the linear combination $\lambda^1 L^1 + \lambda^2 L^2 + \lambda^3 L^3$ in each case and introducing curve parameters *s1*, *s2*, *s3* on the three families of characteristic curves, we arrive at the system

$$\rho v \frac{\partial u}{ds_i} - \rho u \frac{\partial v}{\partial s_i} - [uv + \tau^i (c^2 - v^2)] \frac{\partial \rho}{\partial s_i} = 0 \quad (i=1, 2)$$

$$\rho u \frac{\partial u}{\partial s_3} + \rho v \frac{\partial v}{\partial s_3} + c^2 \frac{\partial \rho}{\partial s_3} = 0 .$$