V. FLOW IN THREE DIMENSIONS

A. Flow in Nozzles and Jets

B. Conical Flow

86. Qualitative description.

The second type of problem treated in this chapter, that of "***conical flow***", permits a rather far-reaching analysis on the basis of the differential equations. It concerns steady, isentropic, irrotational flow with symmetry about the x-axis and under a further assumption, that the flow is conical, i.e., that the quantities u, ρ, p, q retain constant values on cones (considered infinite) with a common vertex, the origin. Flow satisfying this condition may occur, for instance, at the conical tip of a projectile opposed to a supersonic stream of air.

The flow against a cone is analogous to the flow against a wedge and, as in the case of a wedge, two cases must be distinguished. If the cone angle is not too large, the deflection of the flow is achieved by a shock front which begins at the tip of the cone and, has the shape of a straight cone (Fig. 14a). If, however, the cone angle exceeds a certain extreme value (Fig. 14b), no such conical shock front is possible.

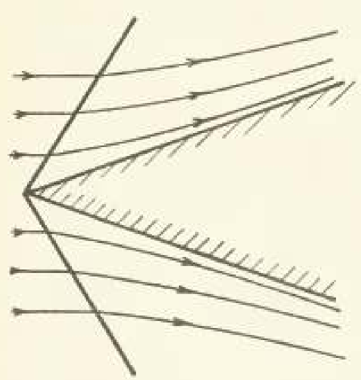

Figure 14a
Conical shock front and conical flow resulting from supersonic flow against a cone with a sufficiently small angle.

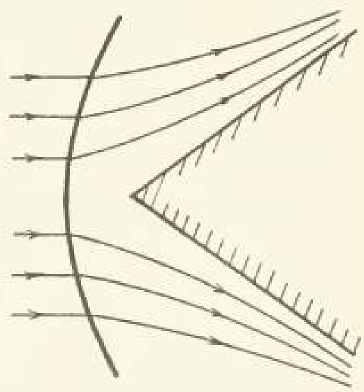

Figure 14b
Curved shock front in supersonic flow against a cone with a large angle.

Instead, a curved shock front stands ahead of the cone. Only the first case can be handled on the basis of the assumption that the flow is conical. We therefore confine ourselves to this case.

In reality projectiles are not represented by infinite cones; they have a conical tip and then taper off, e.g., into a cylindrical shape of finite length. Thus, farther back, the wave from the conical tip interacts with other waves, such as expansion waves coming from the bend of the projectile. It is worth while noting that, in the case of a shock wave standing ahead of the projectile, the distance, under otherwise equal conditions, is the greater the farther the cone extends before tapering off.

Returning now to the idealized case of a strictly conical flow, we may describe the situation qualitatively as follows. Ahead of the shock front the air is in a constant state flowing in the direction of the axis with constant velocity. Since the shock front is a straight cone making everywhere the same angle with the incident flow, the state behind it is also constant and it is therefore clear that the flow is isentropic behind the shock front. Moreover, it can be continued so as to satisfy the basic assumption that the flow is conical. The state of the air beyond the shock cone will, therefore, be constant on co-axial cones. The angle between such a cone and the flow direction approaches zero when this cone approaches the obstacle cone.

87. The differential equations.

For a mathematical treatment, let x be the abscissa along the axis, r be the distance from the axis, u and v be the components of the flow velocity $\boldsymbol{q}$ in the direction of the axis and in the direction perpendicularly away from the axis, respectively. The differential equations for isentropic flow are then

(19) $$v_x = u_r,$$

(20) $$r(\rho u)_x + (r\rho v)_r = 0,$$

where ρ is given by the relation

(21) $$\frac{\rho}{\rho_*} = \left(\frac{c}{c_*}\right)^{2/(\gamma-1)}$$

and Bernoulli's law

(22) $$\left(\frac{c}{c_*}\right)^2 = \frac{1-\mu^2(q/q_*)^2}{1-\mu^2}, \quad q^2 = u^2 + v^2.$$

Inserting (21) and (22) into (20) one has

(23) $$\left(1-\frac{u^2}{c^2}\right)u_x+\left(1-\frac{v^2}{c^2}\right)v_r+\frac{v}{r}-2\frac{uv}{c^2}v_x=0\,.$$

The basic assumption of conical flow now implies that u, v and hence *c* depend only on the ratio

(24) $$t=\frac{x}{r}\,.$$

Equation (19) then becomes

(25) $$v_t+tu_t=0\,,$$

while (23) is reduced to

(26) $$\left(1-\frac{u^2}{c^2}\right)u_t-\left(1-\frac{v^2}{c^2}\right)tv_t+v-2\frac{uv}{c^2}v_t=0\,.$$

Equations (25) and (26) are a pair of differential equations of the first order for the two functions *u* and *v* of *t*. Clearly this pair is equivalent to one equation of the second order for one function. This equation of second order assumes a form, which is particularly amenable to treatment when *v* is introduced as function of *u*. From (25) we have

(27) $$t=-v_u\,.$$

Differentiation of this relation with respect to *t* yields

(28) $$u_t=-\frac{1}{v_{uu}}\,.$$

This relation together with (27) and (25) gives

(29) $$v_t=-\frac{v_u}{v_{uu}}\,.$$

Insertion of equations (27), (28) and (29) into (26) gives

$$\left(1-\frac{u^2}{c^2}\right)+\left(1-\frac{v^2}{c^2}\right)v_u{}^2-vv_{uu}-2\frac{uv}{c^2}v_u=0$$

or

(30) $$vv_{uu}=1+v_u{}^2-\frac{(u+vv_u)^2}{c^2}\,.$$

Every section of a solution of equation (30)[1] gives a flow provided that condition

(32) $$v_{uu}\neq 0$$

is satisfied, because then *x* and *r* can be introduced as independent variables by $v_u=-t=-x/r$. Thus the ray to which values of *u* and *v* are to be attached is determined. The direction of this ray in the *x,y*-plane is evidently normal to the curve $v=v(u)$ at the point (u,v) in

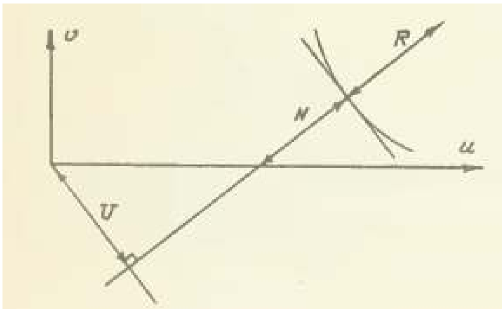

Figure 16

$$R=\frac{N}{1-\frac{U^2}{c^2}}$$

[1] Busemann gives an elegant geometric interpretation of equation (30): See Figure 16 and equation (31). Here R is the radius of curvature.

the hodograph plane.

The flows so obtained are in a certain way analogous to centered simple waves for two-dimensional flows. However, while in the case of two-dimensional steady flow the simple waves are represented in the hodograph plane by two families of fixed characteristics (epicycloids), the images of the special flows considered here in the hodograph plane correspond to a greater variety of curves, namely, a whole family through each point.

88. Conical shocks.

The relations governing the transition through a ***conical shock*** are the same as for the plane oblique shock; the curvature of the shock cone does not enter. When the shock cone is a straight cone, as is assumed, the jumps of u, v, p, and of the entropy are constant along each ray when the assumption of conical flow is satisfied on one side; consequently this assumption remains satisfied on the other side. The flow may continue as a conical flow with constant entropy after crossing the shock. In other words, the assumption of proper conical shocks is compatible with the basic assumption.

Suppose a flow characterized by p_0, ρ_0, u_0, v_0 crosses such a conical shock. (It is to be noted that this can occur only if the speed $q_0 = \sqrt{u_0^2 + v_0^2}$ is supersonic, i.e., if $q_0 > c$). The velocity $q_1 = (u_1, v_1)$ immediately past the shock front is located on the loop of the *strophoid* in the u,v-plane. The inclination of the ray which generates the shock cone is perpendicular to the straight connection between (u_0, v_0) and (u_1, v_1). The positions of the cones corresponding to the cases (a): $v_1 > v_0$ and (b): $v_1 < v_0$ are indicated in Fig. 17.

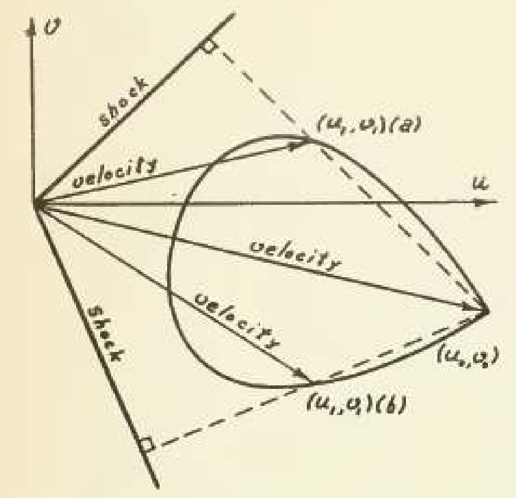

Figure 17
Indicating transition through conical shock front for cases (a): $v_1 > v_0$ and (b): $v_1 < v_0$.

When the flow on either the front or the back side of the shock is to be continued according to differential equation (30), the slope of the u,v-curve is to be so determined that the ray given by (27) coincides with the shock. Since this ray is to be normal to the u,v-curve on the one hand and perpendicular to the straight segment connecting with on the other hand, the u,v-curve should begin or enter in the direction of this segment. The slope of the u,v-curve is thus given by

$$(33) \qquad v_u = \frac{v_1 - v_0}{u_1 - u_0}.$$

The discussion of conical shock fronts by Busemann and by Taylor and Maccoll is restricted to case (a) with $u_0 = q_0 > 0$ and

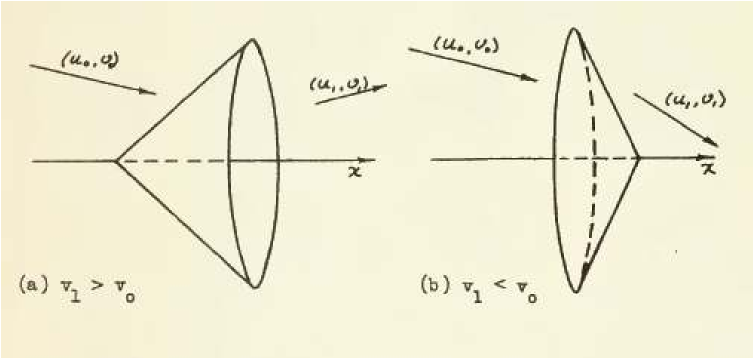

Figure 18
Flow through conical shock fronts.

$v_0 = 0$. This case (Fig. 18a) occurs when a constant axial flow is deflected by a conical projectile. We shall indicate briefly how Busemann treats this problem. Through the shock transition relations the flow velocity (u_1, v_1) past the shock is given (observe that the third transition relation guarantees that the Bernoulli constant $(1/2)\hat{q}^2$ is the same before and after the shock) A solution of equation (30) is to be found whose graph passes through the point (u_1, v_1). The slope v_u of this curve is given by (33). The solution is now to be so continued that $t = x/r$ increases, i.e., in view of (27) v_u decreases up to a point at which the flow and the ray have the same direction, i.e., where $v/u = x/r$, or where the normal passes through the origin; such a point may be called an *end point*. This end point depends on the choice of the point (u_1, v_1) on the strophoid. The manifold of endpoints that can be reached from $(q_0, 0)$ forms a curve which Busemann calls the "*apple curve*" in view of its peculiar shape, see Fig. 19. In this procedure the shock is prescribed and the end direction is found. If the end direction is prescribed one may find the corresponding point on the apple curve by intersecting it with the appropriate ray through the origin. In general, there will be two intersections of which the one corresponding to the weaker shock is likely to occur in reality[2].

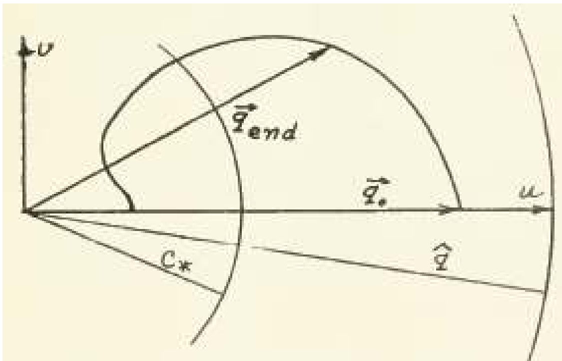

Figure 19
Apple curve showing all possible end velocities that can be reached after crossing a conical shock front from a state with a given incoming velocity.

The values of pressures and angles calculated on the basis of the preceding considerations agree exceedingly well with experimental values (see Taylor and Maccoll [57], [58]).

[2] It may be mentioned that in the procedure of Taylor and Maccoll one begins with the end direction, the shock then being found by following the solution of (26) backwards. This procedure has advantages when single cases are to be investigated.

C. Spherical Waves

89. General remarks.

Spherical wave motion is obviously a subject of basic importance for the study of *explosion waves* in water, air and other media. In spherical motion the velocity is radial and its magnitude as well as that of density, pressure, temperature, and entropy depends only on the distance r from the origin and on the time t. Such motion might be considered in a certain sense as somewhat analogous to **one-dimensional motion in a tube under the influence of a piston**. In the three-dimensional space the piston is replaced by an expanding (or contracting) sphere which impresses a motion on the medium inside or outside.

The simplest model would be that of a "*spherical piston*" pushing at constant velocity into an infinite surrounding medium. Such a model corresponds to the uniform "piston motion" in one dimension as studied in Chapter III, in particular in Art. 41. One should bear in mind, however, that in three-dimensional space an **energy supply at an increasing rate is required** to maintain constant speed of the piston.

In better agreement with actual situations is the assumption that the total energy available for the motion is given. This is the case for spherical *blast waves* caused by the explosion of a given mass of explosive.

While in the first of these two models the shock wave racing ahead of the piston has constant speed so that the shock conditions are compatible with the assumption of isentropic flow on both sides of the discontinuity, this is no longer true of blast waves. In the latter the strength of the shock, and hence the change of entropy, rapidly decreases so that behind the shock front the flow is **no longer isentropic**. Moreover, in blast waves the air or water, after having crossed the shock front and having thereby undergone compression, will rapidly expand again to a pressure in general even below that in front of the shock wave, This *suction phase* is an important feature of motion caused by explosions.

A phenomenon of major importance is that of *reflection of spherical shock fronts*; a contracting spherical wave preceded by a shock front may be "reflected" at the center with the result of

enormous pressure increase behind the reflected shock front.

At the present state of knowledge all that can be done along the lines of mathematical analysis is to find and to discuss some particular solutions of the differential equations of spherical waves which are approximately in agreement with the additional conditions of the problems. One may hope that these solutions display at least qualitatively important features of reality. It is remarkable that such an *unambitious* approach seems to be sufficient to lead to a certain degree of understanding and control of actual phenomena.

90. Analytical formulations.

Assuming that the velocity is radially directed and that the radial component of velocity *u*, the pressure *p* and the density ρ depend only on the distance *r* from the center at the time *t*, the differential equations are (see II (F), Art. 8)

(34) $$u_t = uu_r + \frac{1}{\rho} p_r = 0 ,$$

(35) $$\rho_t + u\rho_r + (u_r + \frac{2u}{r}) = 0 ,$$

(36) $$(p\rho^{-\gamma})_t + u(p\rho^{-\gamma})_r = 0 ,$$

assuming that the medium is polytropic with the adiabatic exponent γ. The third equation expresses the fact that the entropy is constant along the path of a particle. It is not assumed that the entropy is constant throughout since, as stated before, the entropy does not in general remain constant behind a shock front. If the head of the wave is given as a function

(37) $$r = R(t) ,$$

the total energy carried by the wave is expressed as

(38) $$E = 4\pi \int_0^R \left\{ \frac{1}{2}\rho u^2 + \frac{1}{\gamma - 1} p \right\} r^2 dr .$$

E is clearly a function of the time *t*.

Another important quantity, the *impulse* *I* per unit area received by a section of the surface of the sphere at distance *r*, is given by

(39) $$I = \int_T^\infty p dt ,$$

where $T = T(r)$ is the time at which the wave front arrives at the place *r*. $T(r)$ is connected with $R(t)$ through $r = R(T(r))$. Clearly, *I* is a function of *r*.

91. Special solutions.

According to classical procedure, one may obtain particular solutions of the differential equations by assuming a specific form for the solution to reduce the problem to one involving ordinary differential equations. Thus solutions are obtained which have been called *progressing waves.* These are solutions, conveniently assumed in the form

(40) $$u = t^{\beta}\xi U(\xi)\,,\quad \rho = t^{\delta}P(\xi)\,,\quad \frac{p}{\rho} = t^{\varepsilon}\xi^{2}T(\xi)\,,$$

where ξ is the combination

(41) $$\xi = rt^{-\alpha}\,.$$

In other words, a progressing motion is a special motion for which the quantities $ut^{-\beta}$, $\rho t^{-\delta}$, $pt^{-\varepsilon-\delta}$ appear constant for an observer who moves on a path given by $rt^{-\alpha} = const$. The exponents $\alpha, \beta, \delta, \varepsilon$ should be so adjusted that upon insertion of (40) and (41) in (34), (35), (36), equations result which involve only the variable ξ and no longer the variables r and t explicitly. One immediately verifies that to this end one must set

(42) $$\beta = \alpha - 1\,,\quad \varepsilon = 2\beta\,.$$

The equations for U, P, T are then

(34') $$(U-\alpha)(\xi U' + U) + \beta U + \xi T' + 2T + \frac{\xi P'}{P}T = 0\,,$$

(35') $$\alpha)\xi P' + \delta P + (\xi U' + 3U)P = 0\,,$$

(36') $$(U-\alpha)\xi T' - (\gamma-1)(U-\alpha) + \frac{\xi P'}{P}T + \{2\beta - (\gamma-1)\delta\}T = 0\,.$$

It is interesting that after elimination of $\frac{\xi P'}{P}$ by (35'), equations (34') and (36') can be reduced to one equation of first order for T as a function of U.

The equations (34'), (35'), (36') are, of course, amenable to a numerical solution.

When the head of the wave is given by

(37') $$r = R(t) = \Xi t\,,\ \text{or}\ \ \xi = \Xi\,,$$

Ξ being a ***constant*** , we obtain for the energy

(38') $$E = 4\pi t^{5\alpha+\delta-2}\int_0^{\Xi}\left\{\frac{1}{2}U^2 + \frac{1}{\gamma-1}T\right\}P\xi^2\,d\xi\,,$$

and for the impulse per unit area

(39') $$I = \frac{1}{\alpha} r^{(\gamma-1)/\alpha+2} \int_0^{\Xi} \frac{P(\xi)T(\xi)}{\xi^{(\gamma-1)/\alpha+1}} .$$

92. Discussion of Special Cases.

We shall discuss the simplifications resulting from several special assumptions.

(a) If the flow is ***isentropic***, implying constant strength of the shock ahead of it, then relations (40) and (42) require that

(43) $$\delta = \frac{2}{\gamma-1}\beta .$$

If in particular $\delta = \beta = 0$, $\alpha = 1$, the head of the wave $r = \Xi t$ moves with constant velocity Ξ and ρ and p are constant behind it. Such wave motion is therefore compatible with a constant shock front.

As mentioned before, a wave of this type will result if a sphere is suddenly expanded with constant velocity. After crossing the shock front, every air particle acquires the same pressure, density, entropy and velocity. Thereafter, as can be shown, the air particles are further compressed and accelerated and their velocity approaches asymptotically that of the expanding sphere.

(b) Except for the case just mentioned, a shock at the head of the wave is not exactly compatible with a progressing wave. ***Strong shocks***, however, are compatible to a good approximation, if the exponents β and δ are properly related.

Denoting by ρ_0 the density ahead of the wave, and setting the pressure ahead of the wave equal to zero [This is the simplifying approximation corresponding to the assumption of a strong shock.], the shock transition conditions reduce to

(44) $$\rho = \mu^{-2}\rho_0, \quad p = (1-\mu^2)\dot{R}^2\rho_0, \quad u = (1-\mu^2)\dot{R},$$

as can be inferred by setting $p_0 = 0$, $\rho_0 = 0$, $u_0 = 0$, and $\xi = \dot{R}$ in IV (i'),(ii'N), (iii'N). Insertion shows that these relations are compatible with (40), (42), and (37') only if

(45) $$\delta = 0,$$

i.e., if the density remains constant on the paths $r = \xi t^{\alpha}$. For the wave motion behind the shock front one then obtains the boundary values

(46) $$P(\Xi) = \mu^{-2}\rho_0, \quad T(\Xi) = \mu^2(1-\mu^2)(\beta+1)^2,$$
$$U(\Xi) = (1-\mu^2)(\beta+1) .$$

A situation of particular interest arises if the **shock wave contracts** toward the origin and is eventually **reflected** by another progressing wave preceded by a strong shock. Such an occurrence can be expressed in terms of progressing waves of the type considered here only if $\alpha = 0.717$ (or $\alpha = 0.834$ for cylindrical waves). It is very significant (see [64], Fig.4) that the pressure past the reflected shock front is about 26 times the pressure behind the incident shock front (for air, $\gamma = 1.4$) as compared with a 17-fold increase for cylindrical motion and an 8-fold increase for one-dimensional motion.

(c) The condition that the ***energy remains constant*** leads by (38') to the condition

(47) $$\delta = -5\alpha + 2$$

If, in addition, the wave is to possess a strong shock at its head, so that $\delta = 0$, we have

(48) $$\beta = -\frac{3}{5}, \quad \alpha = \frac{2}{5}, \quad \varepsilon = -\frac{6}{5}.$$

The motion of the shock front is then given by

(49) $$r = \Xi t^{2/5}.$$

The pressure behind the shock,

(50) $$p = \frac{4}{25}(1-\mu^2)\rho_0 \Xi^2 t^{-6/5} = \frac{4}{25}(1-\mu^2)\rho_0 \Xi^5 R^{-3},$$

approaches zero as $t \to \infty$. Consequently the assumption that the shock is strong will eventually be violated. As long as this assumption is valid, however, the solution represents a *progressing blast wave*. G. I. Taylor, who first recognized its existence, has carried out the solution numerically and has been able to draw important conclusions from the results (see [63]), although actual blast waves are in general not of this simple, "*progressing*" type.

The difficulties of determining non-progressing spherical waves are very great and for that reason inferences from various approximate treatments have been attempted. The "*incompressible* approximation" arises when one lets γ, and accordingly *c*, become infinite, while ρ remains constant. For water, with $\gamma = 7$, this appears to be acceptable. The "*sonic* approximation" results when the deviation from the state at rest is small so that only linear terms in these deviations need be considered.

Finally it may be mentioned that certain conclusions can be drawn from the differential equations (l), (2), (3) by a purely ***dimensional***

analysis. Any solution, $\tilde{u}(r,t), \tilde{\rho}(r,t), \tilde{p}(r,t)$, leads to a variety of other solutions

(51) $$u = c_0 \tilde{u}\left(\frac{r}{r_0}, \frac{t}{t_0}\right), \quad \rho = \rho_0 \tilde{\rho}\left(\frac{r}{r_0}, \frac{t}{t_0}\right), \quad p = p_0 \tilde{p}\left(\frac{r}{r_0}, \frac{t}{t_0}\right),$$

when $r_0, t_0, c_0, \rho_0, p_0$ are any fixed quantities (of obvious dimensions) satisfying

(52) $$r_0 = c_0 t_0, \quad p_0 = \rho_0 c_0{}^2.$$

One may choose p_0 as the pressure, c_0 as the sound speed ahead of the shock wave. In the case of a blast wave with the energy E_0 one may set

(53) $$r_0 = \left(\frac{E_0}{p_0}\right)^{1/3}.$$

Then one finds for the impulse per unit area (see (39), Art. 91)

(54) $$I = p_0 t_0 \tilde{I}\left(\frac{r}{r_0}\right) = \frac{(E_0 p_0{}^2)^{1/3}}{c_0} \tilde{I}\left(\frac{r}{r_0}\right).$$

In conclusion, it might be emphasized again that the theory of flow in three dimensions is still in a state where one can proceed only by groping for such clues as may come from typical **examples** which can be handled by some **special device**. There is much need for, and some prospect of, progress in this field.