ACOUSTICAL PROBLEMS
IN
FLUID MECHANICS

A SHORT COURSE

Original text by
F. E. C. Culick
California Institute of Technology

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(to be continued)

INTRODUCTION

Acoustics in general includes the theoretical and experimental studies of the propagation of mechanical disturbances in continua. In simplest form, the wave motions depend only on the macroscopic properties of inertia and compressibility; the influences of electrical and chemical properties arise in special situations. In this course, only wave propagation in gases will be treated, although it will be clear that much of the discussion and results will be applicable to liquids and solids as well.

The formal portions of the subject are for the most part analytical rather than theoretical, and the greater part of the analysis is directed to quite well-defined problems. Appeal to experimental and observational results will often be made, not only to support and justify the analysis, but as a matter of general interest as well.

A major reason that acoustics has been developed so thoroughly is that much can realistically be accomplished within the approximations that the motions have small amplitude and that viscous effects can be ignored. The behavior of acoustic waves is often dominated by conservative processes. The dissipation of mechanical energy to microscopic motions is often a perturbation, particularly if the motions are small. The loss of energy in a few cycles of vibratory motion may be small, but of course the influence is ultimately significant, causing the motions to decay.

Thus, in the first instance, the subject of acoustics or sound comprises compressible, small amplitude (linear), inviscid motions. The properties of the medium are often reasonably assumed to be homogeneous, but many interesting problems arise precisely because of inhomogeneities. If the properties of the medium are homogeneous, then the propagation of small amplitude waves is interesting only if there are boundaries or sources present.

In a very crude way, one may divide the acoustics problems for an electrically neutral, non-reacting medium into four classes: interior problems, exterior problems, problems associated with inhomogeneous properties, and problems dealing with the generation or emission of sound. Interior problems include, for example, the treatment of the normal modes of a chamber, and room acoustics. Exterior problems include the radiation, scattering, and reflection of sound waves. Questions of reflection and refraction arise if the medium is inhomogeneous. Acoustic waves may be generated by moving boundaries, flow past boundaries, heating, or non-uniform motions.

Although most of the effort in this course will be devoted to wave motions, there are many problems for which the approximation of geometrical acoustics is very useful. The basis of that subject will be examined briefly and a few examples given. Geometrical acoustics is analogous to geometrical optics, as wave acoustics is analogous to wave optics. Much of what one learns from studies of acoustics is applicable to optics and other kinds of wave motion as well. Care must be taken in exercising such analogies, however; for example, boundary conditions may differ.

I. SOME FUNDAMENTAL ASPECTS OF ACOUSTICS

1.1 Linearization of the Equations of Motion: the Wave Equation and Velocity Potential

An estimate of the influence of viscous effects is given below in Sect. 1.2. Here the discussion is based on the inviscid equations of motion for a homogeneous medium:

Continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{1.1}$$

Momentum:

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} + \nabla p = 0 \tag{1.2}$$

Energy:

$$\rho \frac{\partial}{\partial t}\left(e + \frac{u^2}{2}\right) + \rho \vec{u} \cdot \nabla \left(e + \frac{u^2}{2}\right) + \nabla \cdot (p\vec{u}) = 0 \tag{1.3}$$

The internal energy is e; the kinetic energy can be eliminated from (1.3) by subtracting the scalar product of (1.2) with $\vec{u}$, thereby giving the equation for e:

$$\begin{array}{c} \left\{ \begin{array}{l} \vec{u} \cdot \rho \dfrac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \rho \vec{u} \cdot \nabla \vec{u} + \vec{u} \cdot \nabla p = 0 \\ \rho \dfrac{\partial}{\partial t}\left(e + \dfrac{u^2}{2}\right) + \rho \vec{u} \cdot \nabla \left(e + \dfrac{u^2}{2}\right) + \nabla \cdot (p\vec{u}) = 0 \end{array} \right. \\ \hline \rho \dfrac{\partial e}{\partial t} + \rho \vec{u} \cdot \nabla e + \nabla \cdot (\rho \vec{u}) - \vec{u} \cdot \nabla p = 0 \end{array}$$

Using the formula

$$\nabla \cdot (p\vec{u}) = p \nabla \cdot \vec{u} + \vec{u} \cdot \nabla p$$

$$\rho \frac{\partial e}{\partial t} + \rho \vec{u} \cdot \nabla e + p \nabla \cdot \vec{u} = 0 \tag{1.4}$$

From the combined First and Second Laws of Thermodynamics, the differential change of entropy is

$$\begin{aligned} Tds &= de + pd\left(\frac{1}{\rho}\right) \\ &= de - \frac{p}{\rho^2} d\rho \end{aligned} \tag{1.5}$$

The change of entropy for a fluid element is therefore given by

$$\frac{Ds}{Dt} = \frac{1}{T}\left[\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}\right]$$

$$\frac{Ds}{Dt} = \frac{1}{T}\left[-\frac{p}{\rho}\nabla\cdot\vec{u}' - \frac{p}{\rho^2}\left(-\rho\nabla\cdot\vec{u}'\right)\right]$$

$$\frac{Ds}{Dt} = 0$$

$$\rho\frac{Ds}{Dt} \equiv \rho\frac{\partial s}{\partial t} + \rho\vec{u}\cdot\nabla s = \rho\frac{De}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} \qquad (1.6)$$

Substitution of (1.l) and (1.4) then gives

$$\rho\frac{Ds}{Dt} = -p\nabla\cdot\vec{u} + \frac{p}{\rho}\left(\rho\nabla\cdot\vec{u}\right) = 0 \qquad (1.7)$$

Because the viscous stresses, heat conduction, and all other irreversible processes have been neglected, the flow is isentropic. Consequently, changes of density may be related to changes of pressure by the following relation

$$d\rho = \left(\frac{\partial\rho}{\partial p}\right)_s dp + \left(\frac{\partial\rho}{\partial s}\right)_p ds$$

$$= \left(\frac{\partial\rho}{\partial p}\right)_s dp \qquad (1.8)$$

$$= \frac{1}{a^2}dp$$

where the symbol a^2 stands for the partial derivative

$$a^2 = \left(\frac{dp}{d\rho}\right)_s \qquad (1.9)$$

and a will be found shortly as the speed of propagation of small disturbances. With (1.8), the continuity equation (1.1) can be written by using $d\rho = \frac{1}{a^2}dp$

$$\frac{1}{a^2}\frac{\partial p}{\partial t} + \rho\nabla\cdot\vec{u} + \frac{1}{a^2}\vec{u}\cdot\nabla p = 0$$

$$\frac{\partial p}{\partial t} + \rho a^2\nabla\cdot\vec{u} + \vec{u}\cdot\nabla p = 0 \qquad (1.10)$$

In all the work in this course, the gas will be assumed to be perfect, so the equation of state is

$$p = \rho RT \qquad (1.11)$$

and for isentropic processes

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{1.12}$$

where p_0 and ρ_0 are reference values. Hence, (1.9) becomes explicitly

$$a^2 = \left(\frac{dp}{d\rho}\right)_s = \frac{\gamma p}{\rho} = \gamma RT \tag{1.13}$$

Moreover, because the internal energy for a perfect gas is independent of the specific volume, v,

$$de = \left(\frac{\partial e}{\partial T}\right)_v dT + \left(\frac{\partial e}{\partial v}\right)_T dv = c_v dT \tag{1.14}$$

Equation (1.10) for the pressure is now

$$\frac{\partial p}{\partial t} + \gamma p \nabla \cdot \vec{u} + \vec{u}\cdot\nabla p = 0 \tag{1.15}$$

With (1.12), the momentum equation (1.2) is

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \frac{1}{\rho}\nabla p = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \frac{1}{\rho_0}\left(\frac{p_0}{p}\right)^{1/\gamma}\nabla p = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \frac{1}{\rho_0}\left(\frac{p_0}{p}\right)^{1/\gamma}\nabla p = 0 \tag{1.16}$$

Equations (1.15) and (1.16) are two equations for the pressure and velocity. An equation for the pressure can be deduced by differentiating (1.15) with respect to time and substituting (1.16) for $\partial\vec{u}/\partial t$:

$$\frac{\partial^2 p}{\partial t^2} + \gamma\frac{\partial p}{\partial t}(\nabla\cdot\vec{u}) + (\gamma p)\nabla\cdot\left(\frac{\partial \vec{u}}{\partial t}\right) + \frac{\partial}{\partial t}(\vec{u}\cdot\nabla p) = 0$$

$$\frac{\partial^2 p}{\partial t^2} + \gamma\frac{\partial p}{\partial t}\nabla\cdot\vec{u} + \gamma p\nabla\cdot\left[-\vec{u}\cdot\nabla\vec{u} - \frac{1}{\rho_0}\left(\frac{p_0}{p}\right)^{1/\gamma}\nabla p\right] + \frac{\partial}{\partial t}\left(\vec{u}\cdot\nabla p\right) = 0$$

Rearrangement leads to

$$\frac{\gamma p}{\rho_0} = \frac{\gamma p_0}{\rho_0}\frac{p}{p_0} = a_0\frac{p}{p_0}$$

$$\frac{\partial^2 p}{\partial t^2} - a_0^2\left(\frac{p}{p_0}\right)\nabla\cdot\left[\left(\frac{p_0}{p}\right)^{1/\gamma}\nabla p\right] = \gamma p\nabla\cdot\left(\vec{u}\cdot\nabla\vec{u}\right) - \gamma\frac{\partial p}{\partial t}\nabla\cdot\vec{u} - \frac{\partial}{\partial t}\left(\vec{u}\cdot\nabla p\right)$$

(1.17)

The boundary condition associated with this equation is set by taking the normal component of (1.16) to give

$$\hat{n}\cdot\frac{\partial\vec{u}}{\partial t}+\hat{n}\cdot\left(\vec{u}\cdot\nabla\vec{u}\right)+\hat{n}\cdot\frac{1}{\rho_0}\left(\frac{p_0}{p}\right)^{1/\gamma}\nabla p=0$$

$$\hat{n}\cdot\nabla p=-\left(\frac{p}{p_0}\right)^{1/\gamma}\rho_0\left[\hat{n}\cdot\frac{\partial\vec{u}}{\partial t}+\hat{n}\cdot\left(\vec{u}\cdot\nabla\vec{u}\right)\right] \qquad (1.18)$$

Equations (1.17) and (1.18) are nonlinear equations, but for most of the problems examined in these notes, the linear forms will be used.

Let ε be a small parameter characterizing the amplitude of time dependent motions superimposed upon a uniform state of a gas at rest:

$$\begin{aligned} p &= p_0+\varepsilon p' \\ \vec{u} &= \varepsilon\vec{u}' \\ \rho &= \rho_0+\varepsilon\rho' \end{aligned} \qquad (1.19)$$

The properties p_0 and ρ_0 are, without loss of generality, taken to be the same as the reference values introduced in (1.12); both p_0 and ρ_0 will here be assumed constant in both space and time. The procedure is now a familiar one: substitute (1.19) and collect terms according to powers of ε. Note that the term $\left(\frac{p}{p_0}\right)^{-1/\gamma}$ is expanded in a Taylor series, giving to second order in ε:

$$\begin{aligned} \left(\frac{p}{p_0}\right)^{-1/\gamma} &= \left(\frac{p_0+\varepsilon p'}{p_0}\right)^{-1/\gamma}=\left(1+\varepsilon\frac{p'}{p_0}\right)^{-1/\gamma} \\ &\approx 1+\left(-\frac{1}{\gamma}\right)\varepsilon\frac{p'}{p_0}+\frac{1}{2}\left(-\frac{1}{\gamma}\right)\left(-\frac{1}{\gamma}-1\right)\varepsilon^2\left(\frac{p'}{p_0}\right)^2-\cdots \\ &= 1-\varepsilon\frac{1}{\gamma}\frac{p'}{p_0}+\varepsilon^2\left(\frac{\gamma+1}{2\gamma^2}\right)\left(\frac{p'}{p_0}\right)^2-\cdots \end{aligned}$$

$$\left(\frac{p}{p_0}\right)^{-1/\gamma}\approx 1-\varepsilon\frac{1}{\gamma}\frac{p'}{p_0}+\varepsilon^2\left(\frac{\gamma+1}{2\gamma^2}\right)\left(\frac{p'}{p_0}\right)^2-\cdots \qquad (1.20)$$

The second term on the LHD of (1.17) is, to second order in ε:

Since

$$\nabla\cdot\left[\left(\frac{p}{p_0}\right)^{-1/\gamma}\nabla p\right]$$

$$= p_0\nabla\cdot\left[\left(1-\frac{\varepsilon}{\gamma}\frac{p'}{p_0}\right)\nabla\left(1+\varepsilon\frac{p'}{p_0}\right)\right]$$

$$= p_0\nabla\cdot\left[\left(1-\frac{\varepsilon}{\gamma}\frac{p'}{p_0}\right)\nabla\left(\varepsilon\frac{p'}{p_0}\right)\right]$$

$$= p_0\nabla\cdot\left[\varepsilon\nabla\left(\frac{p'}{p_0}\right)-\frac{\varepsilon^2}{\gamma}\frac{p'}{p_0}\nabla\left(\frac{p'}{p_0}\right)\right]$$

$$= p_0\nabla\cdot\left[\varepsilon\nabla\left(\frac{p'}{p_0}\right)\right]-p_0\nabla\cdot\left[\frac{\varepsilon^2}{\gamma}\frac{p'}{p_0}\nabla\left(\frac{p'}{p_0}\right)\right]$$

$$= \varepsilon p_0\nabla^2\left(\frac{p'}{p_0}\right)-\varepsilon^2\frac{p_0}{\gamma}\nabla\cdot\left[\frac{p'}{p_0}\nabla\left(\frac{p'}{p_0}\right)\right]$$

$$= \varepsilon p_0\nabla^2\left(\frac{p'}{p_0}\right)-\varepsilon^2\frac{p_0}{\gamma}\left[\frac{p'}{p_0}\nabla^2\left(\frac{p'}{p_0}\right)+\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2\right]$$

Then multiplying p/p_0, we have

$$\left(\frac{p}{p_0}\right)\nabla\cdot\left[\left(\frac{p}{p_0}\right)^{-1/\gamma}\nabla p\right]$$

$$=\left(1+\varepsilon\frac{p'}{p_0}\right)\left\{\varepsilon p_0\nabla^2\left(\frac{p'}{p_0}\right)-\varepsilon^2\frac{p_0}{\gamma}\left[\frac{p'}{p_0}\nabla^2\left(\frac{p'}{p_0}\right)+\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2\right]\right\}$$

$$=\varepsilon p_0\nabla^2\left(\frac{p'}{p_0}\right)-\varepsilon^2\frac{p_0}{\gamma}\left[\frac{p'}{p_0}\nabla^2\left(\frac{p'}{p_0}\right)+\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2\right]+\varepsilon^2\frac{p'}{p_0}p_0\nabla^2\left(\frac{p'}{p_0}\right)$$

So the term of order ε^2 is

$$-\frac{p_0}{\gamma}\left[\frac{p'}{p_0}\nabla^2\left(\frac{p'}{p_0}\right)+\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2\right]+\frac{p'}{p_0}p_0\nabla^2\left(\frac{p'}{p_0}\right)$$

$$=\left(1-\frac{1}{\gamma}\right)p'\nabla^2\left(\frac{p'}{p_0}\right)-\frac{p_0}{\gamma}\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2$$

$$=\frac{\gamma-1}{\gamma}p'\nabla^2\left(\frac{p'}{p_0}\right)-\frac{p_0}{\gamma}\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2$$

Finally we have

$$\left(\frac{p}{p_0}\right)\nabla\cdot\left[\left(\frac{p}{p_0}\right)^{-1/\gamma}\nabla p\right]$$
$$=\varepsilon p_0\nabla^2\left(\frac{p'}{p_0}\right)+\varepsilon^2\left[\frac{\gamma-1}{\gamma}p'\nabla^2\left(\frac{p'}{p_0}\right)-\frac{p_0}{\gamma}\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2\right]$$
$$=\varepsilon p_0\nabla^2\left(\frac{p'}{p_0}\right)+\varepsilon^2 p_0\left[\frac{\gamma-1}{\gamma}\frac{p'}{p_0}\nabla^2\left(\frac{p'}{p_0}\right)-\frac{1}{\gamma}\left\{\nabla\left(\frac{p'}{p_0}\right)\right\}^2\right]$$

Therefore we have

$$a_0^2\left(\frac{p}{p_0}\right)\nabla\cdot\left[\left(\frac{p}{p_0}\right)^{-1/\gamma}\nabla p\right]\approx\varepsilon p_0 a_0^2\left[\nabla^2\left(\frac{p'}{p_0}\right)\right]+\varepsilon^2 p_0 a_0^2\left[\frac{\gamma-1}{\gamma}\left(\frac{p'}{p_0}\right)\nabla^2\left(\frac{p'}{p_0}\right)-\frac{1}{\gamma}\left(\nabla\frac{p'}{p_0}\right)^2\right] \tag{b}$$

Collect terms of first order in ε on the LHS and divide by ε, so (1.17) is

$$\frac{\partial^2 p'}{\partial t^2}-a_0^2\nabla^2 p'$$
$$=\varepsilon\left\{\gamma p_0\nabla\cdot\left(\vec{u}'\cdot\nabla\vec{u}'\right)-\gamma\frac{\partial p'}{\partial t}\nabla\cdot\vec{u}'-\frac{\partial}{\partial t}\left(\vec{u}'\cdot\nabla p'\right)+p_0a_0^2\left[\frac{\gamma-1}{\gamma}\frac{p'}{p_0}\nabla^2\left(\frac{p'}{p_0}\right)-\frac{1}{\gamma}\left(\nabla\frac{p'}{p_0}\right)^2\right]\right\} \tag{1.21}$$

Detail derivation of this equation is as follows:

We now consider the RHS of Equation (1.17)

$$\gamma p\nabla\cdot\left(\vec{u}\cdot\nabla\vec{u}\right)-\gamma\frac{\partial p}{\partial t}\nabla\cdot\vec{u}-\frac{\partial}{\partial t}\left(\vec{u}\cdot\nabla p\right)$$

with Eq. (1.19)

$$p=p_0+\varepsilon p'$$
$$\vec{u}=\varepsilon\vec{u}'$$
$$\rho=\rho_0+\varepsilon\rho'$$

Substituting the perturbation terms we have

$$\gamma p \nabla \cdot (\vec{u}' \cdot \nabla \vec{u}') = \gamma p_0 \left(1 + \varepsilon \frac{p'}{p_0}\right) \nabla \cdot (\varepsilon \vec{u}' \cdot \nabla \varepsilon \vec{u}')$$

$$= \gamma p_0 \nabla \cdot (\varepsilon \vec{u}' \cdot \nabla \varepsilon \vec{u}') + \varepsilon \frac{p'}{p_0} \nabla \cdot (\varepsilon \vec{u}' \cdot \nabla \varepsilon \vec{u}')$$

$$= \varepsilon^2 \gamma p_0 \nabla \cdot (\vec{u}' \cdot \nabla \vec{u}') + \varepsilon^3 \frac{p'}{p_0} \nabla \cdot (\vec{u}' \cdot \nabla \vec{u}')$$

So taking the term of order in ε^2 we have

$$\gamma p \nabla \cdot (\vec{u}' \cdot \nabla \vec{u}') \approx \varepsilon^2 \gamma p_0 \nabla \cdot (\vec{u}' \cdot \nabla \vec{u}')$$

And the second term is

$$-\gamma \frac{\partial p}{\partial t} \nabla \cdot \vec{u} = -\gamma \frac{\partial}{\partial t} \left\{ p_0 \left(1 + \varepsilon \frac{p'}{p_0}\right) \right\} \nabla \cdot \varepsilon \vec{u}$$

$$= -\gamma \left[\frac{\partial p_0}{\partial t} \left(1 + \varepsilon \frac{p'}{p_0}\right) + p_0 \frac{\partial}{\partial t} \left(1 + \varepsilon \frac{p'}{p_0}\right) \right] \nabla \cdot \varepsilon \vec{u}$$

$$= -\gamma p_0 \left[\frac{\partial}{\partial t} \left(1 + \varepsilon \frac{p'}{p_0}\right) \right] \nabla \cdot \varepsilon \vec{u}$$

$$= -\varepsilon^2 \gamma \frac{\partial p'}{\partial t} \nabla \cdot \vec{u}$$

And the third term will be

$$-\frac{\partial}{\partial t} (\vec{u} \cdot \nabla p) = -\frac{\partial}{\partial t} [\varepsilon \vec{u}' \cdot \nabla \{p_0 + \varepsilon p'\}]$$

$$= -\frac{\partial}{\partial t} [\varepsilon \vec{u}' \cdot \{\nabla p_0 + \varepsilon \nabla p'\}]$$

$$= -\frac{\partial \varepsilon \vec{u}'}{\partial t} \cdot \{\nabla p_0 + \varepsilon \nabla p'\} - \varepsilon \vec{u}' \cdot \frac{\partial}{\partial t} \{\nabla p_0 + \varepsilon \nabla p'\}$$

$$= -\varepsilon \frac{\partial \vec{u}'}{\partial t} \cdot \nabla p_0 - \varepsilon^2 \frac{\partial \vec{u}'}{\partial t} \cdot \nabla p' - \varepsilon \vec{u}' \cdot \frac{\partial}{\partial t} \nabla p_0 - \varepsilon^2 \vec{u}' \cdot \frac{\partial}{\partial t} \nabla p'$$

$$= -\varepsilon \frac{\partial \vec{u}'}{\partial t} \cdot \nabla p_0 - \varepsilon^2 \left[\frac{\partial \vec{u}'}{\partial t} \cdot \nabla p' + \vec{u}' \cdot \frac{\partial}{\partial t} \nabla p' \right]$$

So taking the term of order in ε^2 we have

$$-\frac{\partial}{\partial t} (\vec{u} \cdot \nabla p) \approx -\varepsilon^2 \frac{\partial}{\partial t} (\vec{u}' \cdot \nabla p')$$

Finally the RHS of Equation (1.17) is, order in ε^2,

$$\gamma p\nabla\cdot(\vec{u}\cdot\nabla\vec{u})-\gamma\frac{\partial p}{\partial t}\nabla\cdot\vec{u}-\frac{\partial}{\partial t}(\vec{u}\cdot\nabla p)$$
$$=\varepsilon^2\gamma p_0\nabla\cdot(\vec{u}'\cdot\nabla\vec{u}')-\varepsilon^2\gamma\frac{\partial p'}{\partial t}\nabla\cdot\vec{u}-\varepsilon^2\frac{\partial}{\partial t}(\vec{u}'\cdot\nabla p') \qquad \text{(c)}$$

In addition $\dfrac{\partial^2 p'}{\partial t^2}$ must be treated as follows:

$$\frac{\partial^2 p}{\partial t^2}=\frac{\partial^2}{\partial t^2}(p_0+\varepsilon p')$$
$$=\varepsilon\frac{\partial^2 p'}{\partial t^2} \qquad \text{(a)}$$

With (a) and (b) for LHS and (c) for RHS of Eq. (1.17) the dividing by ε, we have Eq. (1.21).

Similarly, the boundary condition (1.18) may be expanded to give

$$\hat{n}\cdot\nabla p'=-\rho_0\frac{\partial\vec{u}'}{\partial t}\cdot\hat{n}-\varepsilon\rho_0\left[\frac{1}{\gamma}\frac{p'}{p_0}\frac{\partial\vec{u}'}{\partial t}\cdot\hat{n}+\hat{n}\cdot(\vec{u}'\cdot\nabla\vec{u}')\right] \qquad (1.22)$$

These nonlinear forms will be used later to examine some problems of wave motions in chambers.

Linear problems are described by the equations obtained in the limit $\varepsilon\to 0$:

$$\frac{\partial^2 p'}{\partial t^2}-a_0^2\nabla^2 p'=0 \qquad (1.23)$$

$$\hat{n}\cdot\nabla p'=-\rho_0\frac{\partial\vec{u}'}{\partial t}\cdot\hat{n} \qquad (1.24)$$

It is often convenient to analyze problems by using the velocity potential ϕ, defined so that the velocity is

$$\vec{u}'=-\nabla\phi \qquad (1.25)$$

The linear forms of (1.15) and (1.16) are directly

$$\frac{\partial p'}{\partial t}+\gamma p_0\nabla\cdot\vec{u}'=0 \qquad (1.26)$$

$$\frac{\partial\vec{u}'}{\partial t}+\frac{1}{\rho_0}\nabla p'=0 \qquad (1.27)$$

Differentiate (1.27) with respect to time, use (1.25), and substitute the velocity potential to find the wave equation

$$\frac{\partial^2 \vec{u}'}{\partial t^2} + \frac{1}{\rho_0} \nabla \frac{\partial}{\partial t} p' = 0$$

$$\frac{\partial^2 \vec{u}'}{\partial t^2} + \frac{1}{\rho_0} \nabla\left(-\gamma p_0 \nabla \cdot \vec{u}'\right) = 0$$

$$\frac{\partial^2 \left(-\nabla \phi\right)}{\partial t^2} + \frac{1}{\rho_0} \nabla\left(-\gamma p_0 \nabla \cdot \left(-\nabla \phi\right)\right) = 0$$

$$\nabla \frac{\partial^2 \phi}{\partial t^2} - \frac{\gamma p_0}{\rho_0} \nabla\left(\nabla^2 \phi\right) = 0$$

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\gamma p_0}{\rho_0} \nabla^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial t^2} - a_0^2 \nabla^2 \phi = 0 \qquad (1.28)$$

The pressure fluctuation is found from (1.26) to be given by

$$\frac{\partial p'}{\partial t} + \gamma p_0 \nabla \cdot \left(-\nabla \phi\right) = 0$$

$$\frac{\partial p'}{\partial t} - \gamma p_0 \nabla^2 \phi = 0$$

$$\frac{\partial p'}{\partial t} - \gamma p_0 \left(\frac{1}{a_0^2} \frac{\partial^2 \phi}{\partial t^2}\right) = 0$$

$$\frac{\partial p'}{\partial t} - \frac{\gamma p_0}{a_0^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

$$p' = \rho_0 \frac{\partial \phi}{\partial t} \qquad (1.29)$$

and the boundary condition is again set by using either (1.27) or (1.29).

1.2 Elementary Solutions to the Wave Equation: Plane, Spherical and Cylindrical Waves

Because the wave equations (1.23) and (1.27) are linear, the important principle of superposition applies. Solutions for more complicated problems can often be constructed by superposing elementary solutions.

Examples of these are covered briefly in this section.

1.2.1 Plane Waves

Let the coordinate x lie in the direction of propagation for plane waves; wavefronts, or planes of constant phase are planes normal to the x-axis. Equation (1.23) is

$$\frac{\partial^2 p'}{\partial t^2} - a_0^2 \frac{\partial^2 p'}{\partial x^2} = 0 \tag{1.30}$$

This can be factored directly in the form

$$\left(\frac{\partial}{\partial t} + a_0 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - a_0 \frac{\partial}{\partial x}\right) p' = 0 \tag{1.31}$$

and a general solution is

$$p' = f(x + a_0 t) + g(x - a_0 t) \tag{1.32}$$

The function $f(x + a_0 t)$ represents a plane wave traveling to the left; and $g(x - a_0 t)$ represents a plane wave traveling to the right both with phase velocity a_0.

The density fluctuations are computed from either (1.9) or (1.12):

$$\rho' = \frac{\rho_0}{\gamma} \frac{p'}{p_0} = \frac{1}{a_0^2} p' = \frac{1}{a_0^2}\left[f(x + a_0 t) + g(x - a_0 t) \right] \tag{1.33}$$

According to the linearized equation (1.26) for the pressure, the velocity fluctuation can be calculated from the pressure by integrating

$$\frac{\partial u'}{\partial x} = -\frac{1}{\gamma p_0} \frac{\partial p'}{\partial t} = \frac{a_0}{\gamma p_0}\left[f'(x + a_0 t) - g'(x - a_0 t) \right]$$

so

$$u' = -\frac{a_0}{\gamma p_0}\left[f(x + a_0 t) - g(x - a_0 t) \right] \tag{1.34}$$

Thus, for a wave traveling to the right,

$$u' = \frac{a_0}{\gamma p_0} g(x - a_0 t) = a_0 \frac{p'}{\gamma p_0} = \frac{p'}{\rho_0 a_0} \tag{1.35}$$

and for a wave traveling to the left,

$$u' = -\frac{p'}{\rho_0 a_0} \tag{1.36}$$

The solution for given initial conditions can be deduced very quickly. Suppose that at t = 0, the pressure and rate of change of pressure are specified functions of x:

$$
\begin{aligned}
&p'(x,0) = P(x) \\
&\frac{dp'}{dt}(x,0) = Q(x)
\end{aligned}
\tag{1.37}
$$

The solution (1.32) gives

$$
\begin{aligned}
&p'(x,0) = f(x) + g(x) = P(x) \\
&\frac{\partial p'}{\partial t}(x,0) = a_0\left[f'(x) - g'(x)\right] = Q(x)
\end{aligned}
\tag{1.38}
$$

Integration of the second gives

$$
\left[f(x) - g(x)\right] = \frac{1}{a_0}\int_{x_0}^{x} Q(\xi)\,d\xi
$$

where x_0 is unspecified at this point. This equation and (1.38) can be solved to give

$$
\begin{aligned}
&f(x) = \frac{1}{2a_0}\int_{x_0}^{x} Q(\xi)\,d\xi + \frac{1}{2}P(x) \\
&g(x) = \frac{-1}{2a_0}\int_{x_0}^{x} Q(\xi)\,d\xi + \frac{1}{2}P(x)
\end{aligned}
\tag{1.40}
$$

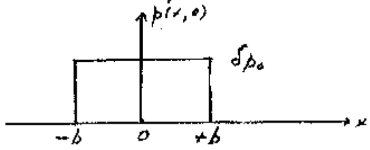
With (1.40), the solution (1.32) is

$$
p' = \frac{1}{2a_0}\int_{x_0}^{x+a_0 t} Q(\xi)\,d\xi + \frac{1}{2}P(x + a_0 t) - \frac{1}{2a_0}\int_{x_0}^{x-a_0 t} Q(\xi)\,d\xi + \frac{1}{2}P(x - a_0 t)
$$

$$
p' = \frac{1}{2}\left[P(x + a_0 t) + P(x - a_0 t)\right] + \frac{1}{2a_0}\int_{x_0 - a_0 t}^{x+a_0 t} Q(\xi)\,d\xi
\tag{1.41}
$$

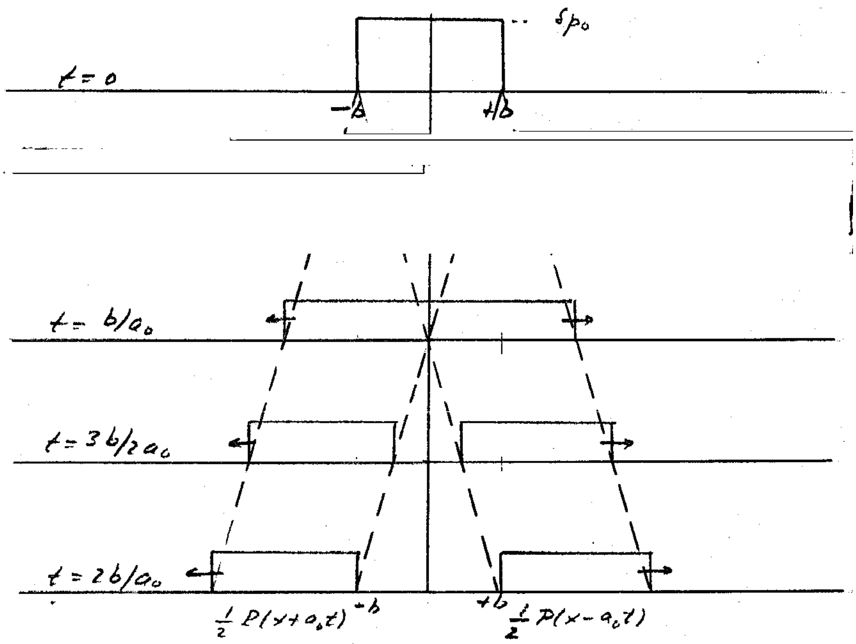
Suppose, for example, that the region $-b \le x \le +b$ is initially pressurized to some value δp_0 and is at rest:

$$
\begin{aligned}
&Q(x) = 0 \\
&P(x) = \begin{cases} \delta p_0 \quad |x| \le b \\ 0 \quad |x| > b \end{cases}
\end{aligned}
\tag{1.42}
$$

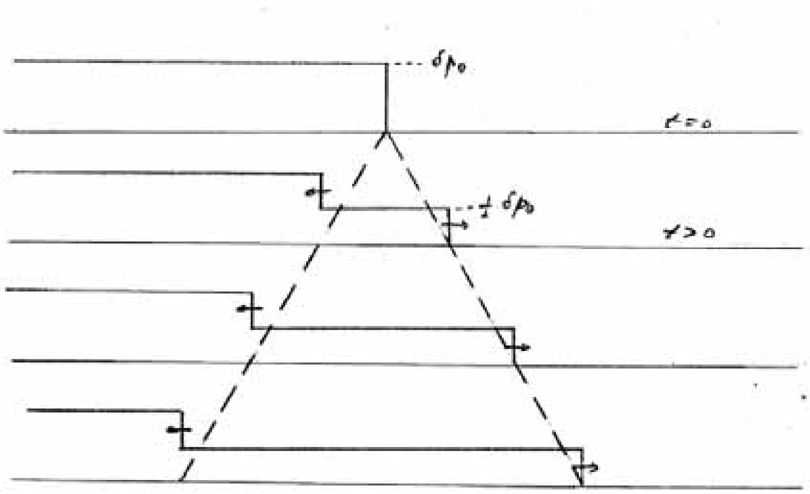


The solution is easily graphed, because with $P(x) = 0$, p' is the sum of a wave traveling to the

left and a wave traveling to the right, each having the shape $\frac{1}{2}P(x)$ and traveling with speed a_0; this is sketched below (sketch A).



Note that if the entire space for $x < 0$ were initially pressurized and at rest, the subsequent wave motion would consist of a compression wave traveling to the right and a rarefaction traveling to the left (sketch B).



The wave traveling to the right is a simple compression wave, an abrupt rise of pressure followed by a region of uniform pressure. The velocity fluctuations are uniform behind the disturbances.

The solution corresponding to the initial condition (1.42) can be written by noting first

that (1.42) is

$$P(x) = \delta p_0 \left[U(x+b) - U(x-b) \right]$$

where $U(x-x_0)$ is the unit step placed at $x = x_0$. Then (1.41) gives

$$p' = \frac{\delta p_0}{2} \left[\left\{ U(x+b+a_0 t) - U(x-b+a_0 t) \right\} + \left\{ U(x+b-a_0 t) - U(x-b-a_0 t) \right\} \right] \tag{1.43}$$

Obviously the terms in curly brackets represent the two rightward and leftward blobs sketched above.

Reflection of a wave can be treated by imposing on the solution (1.41) a boundary condition; for example, suppose $p' = 0$ at $x = 0$ for all t. And again assume the rate of change of pressure to be zero at t =0, so

$Q(x) = 0$, but $P(x) \neq 0$ in $x \geq 0$:

$$\begin{cases} p'(x,0) = P(x) \\ \dfrac{\partial p'}{\partial t}(x,0) = 0 \end{cases} \tag{1.44}$$

$$\begin{cases} p'(0,t) = 0 \\ \dfrac{\partial p'}{\partial t}(0,t) = 0 \end{cases} \tag{1.45}$$

Because $P(x)$ is defined only for positive x, $P(x - a_0 t)$ is meaningless for $t > x/a_0$; hence, the solution (1.41) doesn't work. The difficulty is overcome by considering the motion in <u>all</u> x with the initial condition

$$\begin{cases} p'(x,0) = \eta(x) = \begin{cases} P(x)\_\ for\ \_\ x \geq 0 \\ -P(-x)\_\ for\ \_\ x < 0 \end{cases} \\ \dfrac{\partial p'}{\partial t}(x,0) = 0 \end{cases} \tag{1.46}$$

Then the solution is

$$p'(x,t) = \frac{1}{2} \left[\eta(x + a_0 t) + \eta(x - a_0 t) \right] \tag{1.47}$$

At x=0,

$$p'(0,t) = \frac{1}{2} \left[\eta(a_0 t) + \eta(-a_0 t) \right]$$

$$= \frac{1}{2} \left[P(a_0 t) - P(-a_0 t) \right]$$

$$= 0$$

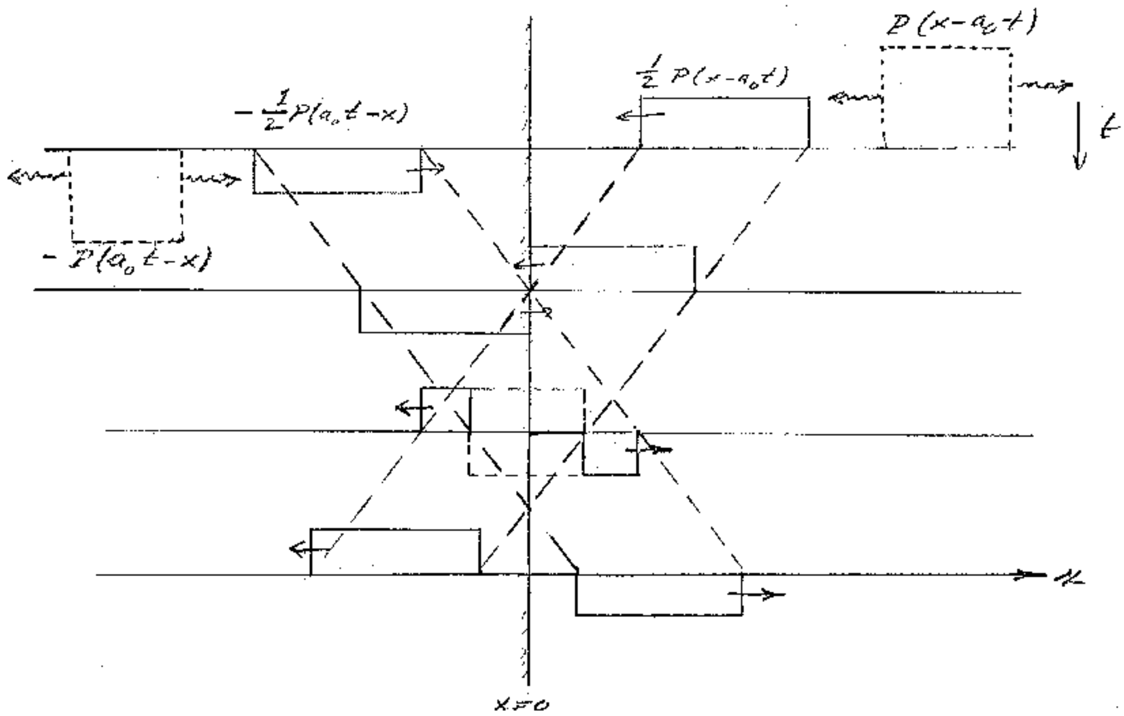
so the boundary condition is satisfied for all t . Now because of the definition (1.46), it follows that

$$\eta(x-a_0t)=\begin{cases}P(x-a_0t)\_\ for\_x\ge a_0t\\ -P(a_0t-x)\_\ for\_x<a_0t\end{cases}$$

Consequently, the solution (1.47) can be written in terms of the function $P(x)$ as:

$$p'(x,t)=\begin{cases}\frac{1}{2}\left[P(x+a_0t)+P(x-a_0t)\right]\_\ for\_x\ge a_0t\\ \frac{1}{2}\left[P(x+a_0t)-P(a_0t-x)\right]\_\ for\_x<a_0t\end{cases}\tag{1.48}$$

An example of such a motion is sketched below:



The waves sketched in x <0 are essentially "image" waves used to satisfy the boundary conditions. They are represented in the formula (1.47) but not in (1.48); the last result gives $p'(x,t)=0$ for all x<0 because $P(\xi)$ has been defined to be zero for $f<0$.

The problem of wave motion in the region $0\le x\le L$ with the boundary condition $p'=0$ at x =0 and x = L is left as a homework problem. With that result, one can describe a pulse barging back and forth in a tube, or a standing wave.

An important special case of plane waves is sinusoidal waves; the dependence on time is $\exp(-i\omega t)$, so f and g necessarily have the forms

$$f(x+a_0t) \to A\exp\left[-\frac{i\omega}{a_0}(x+a_0t)\right]$$

$$g(x-a_0t) \to B\exp\left[-\frac{i\omega}{a_0}(x-a_0t)\right]$$

The ratio $\frac{\omega}{a_0} = k$ is the wavenumber:

$$k = \frac{\omega}{a_0} = \frac{2\pi}{\lambda} \tag{1.49}$$

To see this, consider the function $\cos\left(\frac{\omega}{a_0}x + \omega t\right)$, for example. This has period 2π, and by definition the phase increases by one period if x is changed by one wavelength:

$$2\pi = \frac{\omega}{a_0}\delta x = \frac{\omega\lambda}{a_0}$$

so $\frac{\omega}{a_0} = \frac{2\pi}{\lambda}$.

It is usual to write $\frac{\omega}{a_0}(x+a_0t) = kx + \omega t$. For a plane wave propagating in an arbitrary direction $\vec{k}/|\vec{k}|$, the plane sinusoidal wave is represented by

$$\exp\left[i\left(\vec{k}\cdot\vec{r} \pm a_0t\right)\right]\_or\_\exp\left[-i\left(\omega t \pm \vec{k}\cdot\vec{r}\right)\right]$$

In these notes, the time factor will always be taken as $\exp(-i\omega t)$ and the second form will be used. This is a monochromatic wave; general disturbances can be constructed by superposition.

1.2.2 Spherical Waves

Consider now wave motions propagating in three dimensions, but symmetrically about a point. All properties depend only on the distance r from the point, and the wave equation for pressure is

$$\frac{\partial^2 p'}{\partial t^2} - a_0^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p'}{\partial r}\right) = 0 \tag{1.50}$$

Try a solution of the form $p' = \psi(r,t)/r$, so

$$\frac{\partial^2 p'}{\partial t^2} = \frac{1}{r}\frac{\partial^2 \psi}{\partial t^2}$$

$$\frac{\partial p'}{\partial r} = -\frac{1}{r^2}\psi + \frac{1}{r}\frac{\partial \ \psi}{\partial r}$$
$$\frac{\partial^2 p'}{\partial r^2}\left(r^2 \frac{\partial p'}{\partial r}\right) = r\frac{\partial^2 \psi}{\partial r^2}$$

and ψ satisfies the one-dimensional wave equation,

$$\frac{\partial^2 \psi}{\partial t^2} - a_0^2 \frac{\partial^2 \psi}{\partial r^2} = 0$$

Hence, a general solution has the form

$$p' = \frac{1}{r}\left[F(a_0 t + r) + G(a_0 t - r)\right] \tag{1.51}$$

The first term represents a wave propagating inward, and the second a wave propagating outward.

Consider the initial value problem analogous to that treated for plane waves. Suppose that at t = 0, the pressure and rate of change of pressure are given functions of r :

$$\begin{aligned} &p'(r,0) = V(r) \\ &\frac{\partial p'}{\partial t}(r,0) = W(r) \end{aligned} \tag{1.52}$$

If there are no sources in the field, and no boundaries, the expression (1.51) must remain valid at r =0:

$$\left[p'(r,t)\right]_{r\to 0} = \left[\frac{1}{r}\right]_{r\to 0}\left[F(a_0 t) + G(a_0 t)\right]$$

Thus, $G(\xi) = -F(\xi)$ if p' is to remain finite at the origin, and the general form is

$$p'(r,t) = \frac{1}{r}\left[F(a_0 t + r) - F(a_0 t - r)\right] \tag{1.53}$$

so the rate of change is

$$\frac{\partial p'}{\partial t}(r,t) = \frac{a_0}{r}\left[F'(a_0 t + r) - F'(a_0 t - r)\right] \tag{1.54}$$

To satisfy the initial conditions,

$$F(\xi) - F(-\xi) = \xi V(\xi) \tag{1.55}$$

$$F'(\xi) - F'(-\xi) = \frac{1}{a_0}\xi W(\xi) \tag{1.56}$$

where $F(a_0 t \pm r) \to F(\pm r) = F(\pm\xi)$ and $\xi \le 0$ always because of the definition of r, and the functions V(r), W (r) in (1. 52).

Suppose that the medium is initially at rest, so W = 0 and

$$F'(-\xi) = F'(\xi)$$

The integral of this equation is

$$F(\xi)+F(-\xi)=K \tag{1.57}$$

Where K is a constant. Equations (1.55) and (1.57) can be solved to give

$$F(\xi)=\frac{1}{2}\xi V(\xi)+\frac{K}{2}$$

$$F(-\xi)=-\frac{1}{2}\xi V(\xi)+\frac{K}{2}=\frac{1}{2}(-\xi)V(-(-\xi))+\frac{K}{2}$$

According to the definitions introduced above, $F(\xi)=F(a_0t+r)$ for the inward propagating wave, while for the outward propagating wave,

$$F(-\xi)=\begin{cases} F(a_0t-r)=-\frac{1}{2}(a_0t-r)V(a_0t-r)+\frac{K}{2}\_\ for\_r<a_0t \\ F(r-a_0t)=\frac{1}{2}(r-a_0t)V(r-a_0t)+\frac{K}{2}\_\ for\_r>a_0t \end{cases} \tag{1.58}$$

because the argument ξ of $V(\xi)$ cannot be negative. The solution (1.53) is therefore

$$p'(r,t)=\begin{cases} \frac{1}{2r}\left[(a_0t+r)V(a_0t+r)-(a_0t-r)V(a_0t-r)\right]\_\ for\_r<a_0t \\ \frac{1}{2r}\left[(a_0t+r)V(a_0t+r)+(r-a_0t)V(r-a_0t)\right]\_\ for\_r>a_0t \end{cases} \tag{1.59}$$

because $F(\xi)$ is defined only for $\xi>0$.

Now consider the problem in three dimensions, corresponding to the one-dimensional problem treated above. Let the spherical region $r \le R$ be initially pressurized, $p'(r,0)=\delta p_0$, and at rest, so

$$V(\xi)=\begin{cases} \delta p_0\_\ for\_0<\xi\le R \\ 0\_\ for\_\xi>R \end{cases} \tag{1.60}$$

The motion outside the sphere, r >R , consists of a spherical compression wave followed by a spherical rarefaction wave. Obviously, there is no inward traveling wave, so the solution (1.59) becomes:

$$p'(r,t)=\begin{cases} -\frac{1}{2r}(a_0t-r)\delta p_0\_\ for\_0<a_0t-r\le R \\ 0\_\ for\_a_0t-r>R \end{cases} \qquad r<a_0t \tag{1.61a}$$

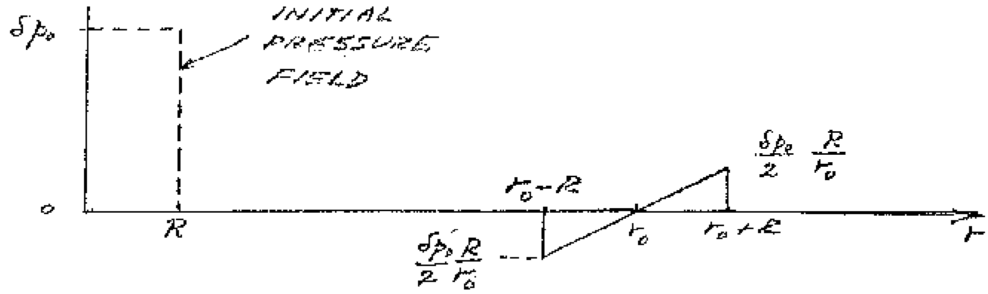
$$p'(r,t)=\begin{cases} +\frac{1}{2r}(r-a_0t)\delta p_0\_\ for\_0<r-a_0t\le R \\ 0\_\ for\_r-a_0t>R \end{cases} \qquad r>a_0t \tag{1.61b}$$

To interpret this result, first note that the lower terms in each set of brackets give the result that the disturbance is confined to the region $a_0t-R<r\le a_0t+R$. The upper terms in (1. 61) can be

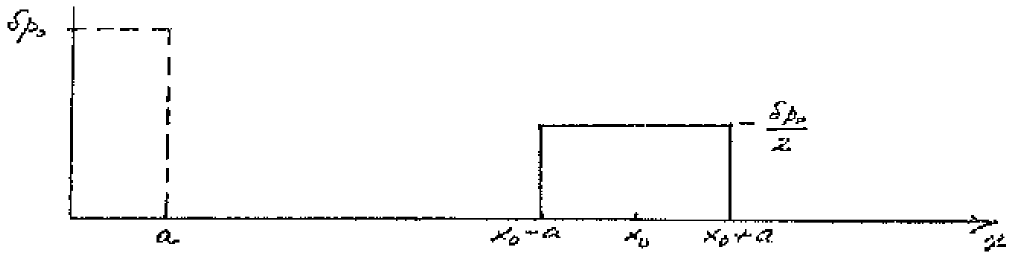
rewritten

$$p'(r,t) = \begin{cases} \left(\dfrac{r - a_0 t}{2r}\right)\delta p_0 \_\ for\ \_\ a_0 t - R \le r < a_0 t \\ \left(\dfrac{r - a_0 t}{2r}\right)\delta p_0 \_\ for\ \_\ a_0 t < r \le a_0 t + R \end{cases} \tag{1.62}$$

For sufficiently large r, and at a fixed time t_0, the pressure varies nearly linearly and is centered at $r_0 = a_0 t_0$ as sketched below.



For comparison, the one-dimensional result is shown.



This example demonstrates an important difference between one-dimensional plane waves and three-dimensional spherical waves. For a plane wave, the gas is only compressed within the disturbance ($p' > 0$), but in the spherical wave, the compression region is followed by a rarefaction. This is due to the reflection of the origin of the wave propagating inward from the edge of the initially pressurized spherical region. To see a bit more clearly what happens, it is helpful to consider the velocity field. For a spherical wave, the linearized momentum equation (1.27) is

$$\frac{\partial u'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial r}$$

With (1.53), the velocity for an outward traveling wave, for example, is

$$\frac{\partial u'}{\partial t} = \frac{1}{\rho_0}\left[\frac{1}{r^2}F(a_0 t - r) + \frac{1}{r}F'(a_0 t - r)\right]$$

The integral from $t = -\infty$ when there is no disturbance to the current time is

$$u' = \frac{1}{\rho_0 a_0}\frac{F(a_0 t - r)}{r} + \frac{1}{\rho_0 r^2}\int_{-\infty}^{t} F(a_0 t' - r)dt',$$
$$-u' = \frac{p'}{\rho_0 a_0} + \frac{1}{\rho_0 r}\int_{-\infty}^{t} p'dt' \qquad \text{(spherical wave)} \qquad (1.63)$$

For plane wave, the corresponding result is (1.34)

$$u' = -\frac{1}{\rho_0 a_0} g(x - a_0 t) = \frac{p'}{\rho_0 a_0} \qquad \text{(plane wave)} \qquad (1.64)$$

Thus, in a plane wave, the velocity vanishes wherever the pressure does, and it is possible to have a disturbance of any shape freely propagating. But in the spherical wave, the velocity is zero outside a pressure disturbance only if the second term in (1. 63) vanishes. The requirement is that $u' \to 0$ for $t \to \infty$, a long time after the pulse has passed a given point r:

$$\int_{-\infty}^{\infty} p'dt = 0 \qquad (1.65)$$

The integral can be converted to an integral over space by the following transformation:

$$\int_{-\infty}^{\infty} p'dt = -\int_{-\infty}^{\infty} \frac{F(a_0 t - r)}{r} dt = \frac{1}{a_0 r}\int_{-\infty}^{\infty} F(\xi)d\xi$$
$$= -\frac{1}{a_0 r}\int_{\infty}^{0} F(a_0 t - r')dr' = \frac{1}{a_0 r}\int_{0}^{\infty} p'r'dr' = 0$$

Thus, for a spherical wave propagating outward, the distribution of pressure must at all times satisfy the condition

$$\int_{0}^{\infty} p'rdr \qquad (1.66)$$

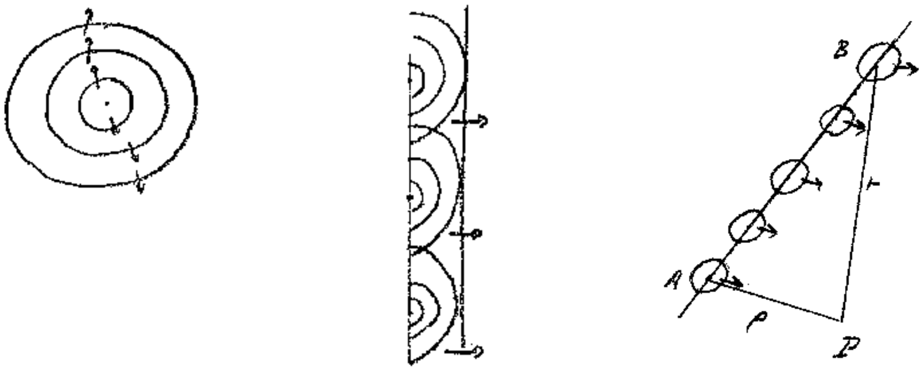
if it is to be confined to a finite region in r.

1.2.3 Cylindrical Waves

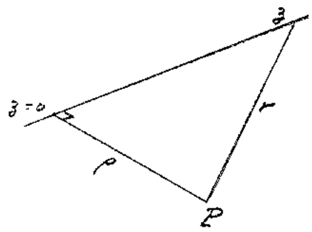
Symmetric waves in two dimensions exhibit a peculiarity not present in either one or two dimensions; there is necessarily a "wake," and discrete pulses simply cannot exist. The reason for this may be seen quite easily by examining the way in which a cylindrically symmetric wave might be generated, at least in principle.

The world is three dimensional, so the elementary source is really a point source emitting spherically symmetric waves. Plane waves may be conceived as the consequence of an infinite planar array of point sources, all emitting waves in phase. Cylindrical waves are generated by an

infinite linear array of point sources emitting waves in phase.



Consider some point P a distance ρ from the line, and consider the waves emitted at the same time from two sources located at points A and B. Obviously, the wave from B arrives at P later than that from the source at A, and also has reduced amplitude because $r > \rho$. But B is an arbitrary point along the line, so even if all the sources on the line simultaneously emit single pulses, an observer at P will be receiving signals forever, although, of course, the amplitude decreases with time.



To make the preceding remarks more quantitative, it is a matter of carrying out the superposition of spherical waves. The elementary outward traveling wave is represented by the second term of (1.51):

$$p'(r,t) = \frac{G(a_0 t - r)}{r} \tag{1.65}$$

Let the origin of coordinates be opposite the observation point P, and let z be measured along the line source:

The distribution of sources may be assumed uniform along the line, so the pressure at P is simply the integral of (1.65) over all z:

$$p'(\rho,t) = \int_{-\infty}^{\infty} \frac{G(a_0 t - r)}{r} dz$$

Because

$$r^2 = z^2 + \rho^2,$$

$$dz = \frac{rdr}{\sqrt{r^2 - \rho^2}},$$

and the integral can be changed to the integral over r :

$$p'(\rho,t) = 2\int_{\rho}^{\infty} \frac{G(a_0 t - r)}{\sqrt{r^2 - \rho^2}} dr$$

Now let $\xi = a_0 t - r$, so $dr = -d\xi$ and

$$p'(\rho,t) = 2\int_{-\infty}^{a_0 t - r} \frac{G(\xi)}{\sqrt{(\xi - a_0 t)^2 - \rho^2}} d\xi \tag{1.66}$$

The pressure at the time t depends on the value of the sources at all earlier times $-\infty < t < t - \rho/a_0$, whereas the pressure at an observation point due to a spherical source depends only on the signal emitted by the source at time $t - r/a_0$.

Suppose that $G(\xi)$ is non-zero only in a small range of ξ, and consider the behavior for large times: $t >> \xi/a_0, \rho/a_0$

$$p'(\rho,t) = 2\int_{\xi_0}^{\xi_0+\delta\xi} \frac{G(\xi)}{\sqrt{(\xi - a_0 t)^2 - \rho^2}} d\xi \to \frac{1}{a_0 t}\int_{\xi_0}^{\xi_0+\delta\xi} G(\xi) d\xi \tag{1.67}$$

Hence, the observed signal for a cylindrical wave can <u>never</u> be discrete; there is necessarily a “wake,” in both time and space, a feature which will be examined again later.

l.3 An Estimate of the Influence of Internal Heat Conduction on Linear Wave Motions

One of the two main assumptions used in many acoustics calculations is that viscous effects may be ignored. It is the purpose of this section to investigate briefly one aspect of this question. To simplify the discussion, only the influence of internal energy transfer by heat conduction will be considered, and only one-dimensional harmonic motions will be treated.

It will be assumed that heat flows according to Fourier's law; in one dimension,

$$q = -\lambda \frac{\partial T}{\partial x} \tag{1.68}$$

where λ is the thermal conductivity. With this included, the one-dimensional energy equation (1.4), written for a perfect gas, becomes

$$\rho \frac{\partial T}{\partial t} + \rho u \frac{\partial T}{\partial x} + \frac{p}{c_v}\frac{\partial u}{\partial x} = \frac{\lambda}{c_v}\frac{\partial^2 T}{\partial x^2} \tag{1.69}$$

The linearized form is

$$\rho \frac{\partial T'}{\partial t} + \frac{p_0}{\rho_0 c_v}\frac{\partial u'}{\partial x} = \frac{\lambda_c}{\rho_0 c_v}\frac{\partial^2 T'}{\partial x^2} \tag{1.70}$$

The linearized continuity equation (1.1) for one-dimensional motions is

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0 \tag{1.71}$$

and if $\partial u'/\partial x$ is eliminated from (l.70), one has

$$\frac{\partial T'}{\partial t} - \frac{p_0}{\rho_0^2 c_v}\frac{\partial \rho'}{\partial t} = \frac{\lambda_c}{\rho_0 c_v}\frac{\partial^2 T'}{\partial x^2} \tag{1.72}$$

Equation (l.27) is the linearized momentum equation, which, for one dimensional motions, is

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0 \tag{1.73}$$

Now for isentropic flow, the pressure is simply a function of the density; more generally, it must be taken as a function of two properties for a simple gas, say, the density and temperature. One may write

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_T d\rho + \left(\frac{\partial p}{\partial T}\right)_\rho dT \tag{1.74}$$

so that (1.73) can be written

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_{T_0}\frac{\partial \rho'}{\partial x} + \frac{1}{\rho_0}\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\frac{\partial T'}{\partial x} = 0 \tag{1.75}$$

in which the partial derivatives are, as noted, evaluated at the undisturbed conditions in the gas. Differentiate this equation with respect to x and again use the continuity equation to find

$$\frac{\partial^2 \rho'}{\partial t^2} - \left(\frac{\partial p}{\partial \rho}\right)_{T_0} \frac{\partial^2 \rho'}{\partial x^2} - \left(\frac{\partial p}{\partial T}\right)_{\rho_0} \frac{\partial^2 T'}{\partial x^2} = 0 \tag{1.76}$$

Equations (1.72) and (1.76) are two equations in the density and temperature which will be used shortly to estimate the influence of heat conduction on the propagation of harmonic waves. Before doing so, a bit of dimensional analysis is illuminating. For a sinusoidal wave, a characteristic length is the wavelength, λ, and a characteristic time is the period, τ. If $\hat{T}$ denotes the amplitude of the motion, then the first and last terms in (1.72) may be estimated as

$$\frac{\partial\, T'}{\partial t} \approx \frac{\hat{T}}{\tau}$$

$$\frac{\lambda_c}{\rho_0 c_v} \frac{\partial^2 T'}{\partial x^2} \approx \frac{\lambda_c}{\rho_0 c_v} \frac{\hat{T}}{\lambda^2}$$

If heat conduction is to be negligible, then one would expect

$$\frac{\lambda_c \hat{T}}{\rho_0 c_v \lambda^2} \bigg/ \frac{\hat{T}}{\tau} = \frac{\lambda_c \tau}{\rho_0 c_v \lambda^2} << 1$$

(for negligible influence of heat conduction).

Because $\tau = 2\pi/\omega$, and $\lambda = 2\pi/k$, this condition becomes

$$\frac{1}{2\pi}\left(\frac{\lambda_c}{\rho_0 c_v}\right)\frac{k^2}{\omega} << 1 \tag{1.77}$$

This combination may be interpreted as the ratio of the characteristic time, $1/\omega$, for changes associated with the wave motion, to the characteristic time, $\rho_0 c_v / \lambda k^2$, for the conduction of heat. If (1.77) is satisfied, the wave motion should be isentropic.

Now assume sinusoidal plane waves, traveling to the right, so

$$\begin{aligned} \rho' &= |\hat{\rho}| e^{i(kx-\omega t)} \\ T' &= |\hat{T}| e^{i(kx-\omega t)} \end{aligned} \tag{1.78}$$

Substitution into (1.72)

$$(-i\omega)|\hat{T}| - \frac{p_0}{\rho_0^2 c_v}(-i\omega)|\hat{\rho}| = \frac{\lambda_c}{\rho_0 c_v} k^2 |\hat{T}|$$

$$\left((i\omega) + \frac{\lambda_c}{\rho_0 c_v} k^2\right)|\hat{T}| - \frac{p_0}{\rho_0^2 c_v}(i\omega)|\hat{\rho}| = 0$$

and into (1.76)

$$(-i\omega)^2|\hat{\rho}| - \left(\frac{\partial p}{\partial \rho}\right)_{T_0}(ik)^2|\hat{\rho}| - \left(\frac{\partial p}{\partial T}\right)_{\rho_0}(ik)^2|\hat{T}| = 0$$

$$\left(\frac{\partial p}{\partial T}\right)_{\rho_0} k^2|\hat{T}| - \left\{\omega^2 - \left(\frac{\partial p}{\partial \rho}\right)_{T_0} k^2\right\}|\hat{\rho}| = 0$$

gives

$$\begin{aligned} &\left(i\omega + \frac{\lambda_c}{\rho_0 c_v}k^2\right)|\hat{T}| - i\left(\omega\frac{p_0}{\rho_0^2 c_v}\right)|\hat{\rho}| = 0 \\ &\left(k^2\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\right)|\hat{T}| - \left(\omega^2 - k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0}\right)|\hat{\rho}| = 0 \end{aligned} \qquad (1.79)$$

This pair of equations has non-trivial solutions only if the determinant of coefficients vanishes, leading to the condition

$$-\left(i\omega + \frac{\lambda_c}{\rho_0 c_v}k^2\right)\left(\omega^2 - k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0}\right) + i\left(\omega\frac{p_0}{\rho_0^2 c_v}\right)\left(k^2\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\right) = 0$$

or

$$\left(i\omega + \frac{\lambda_c}{\rho_0 c_v}k^2\right)\left(k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0} - \omega^2\right) + i\omega k^2\frac{p_0}{\rho_0^2 c_v}\left(\frac{\partial p}{\partial T}\right)_{\rho_0} = 0$$

which can be written

$$(i\omega)\left(k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0} - \omega^2\right) + \frac{\lambda_c}{\rho_0 c_v}k^2\left(k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0} - \omega^2\right) + i\omega k^2\frac{p_0}{\rho_0^2 c_v}\left(\frac{\partial p}{\partial T}\right)_{\rho_0} = 0$$

$$(i\omega)\left(k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0} - \omega^2 + k^2\frac{p_0}{\rho_0^2 c_v}\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\right) + \frac{\lambda_c}{\rho_0 c_v}k^2\left(k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0} - \omega^2\right) = 0$$

$$(i\omega)\left(k^2\left\{\left(\frac{\partial p}{\partial \rho}\right)_{T_0} + \frac{p_0}{\rho_0^2 c_v}\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\right\} - \omega^2\right) + \frac{\lambda_c}{\rho_0 c_v}k^2\left(k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0} - \omega^2\right) = 0$$

or

$$i\left\{k^2\left[\left(\frac{\partial p}{\partial \rho}\right)_{T_0}+\frac{p_0}{\rho_0^2 c_v}\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\right]-\omega^2\right\}+\left(\frac{\lambda_c}{\rho_0 c_v}\right)\left(\frac{k^2}{\omega}\right)\left\{k^2\left(\frac{\partial p}{\partial \rho}\right)_{T_0}-\omega^2\right\}=0$$

(1.80)

If (1. 77) is satisfied, the real part of this equation is negligible; the imaginary part must vanish in any case, giving the dispersion relation

$$\omega^2 = k^2\left[\left(\frac{\partial p}{\partial \rho}\right)_{T_0}+\frac{p_0}{\rho_0^2 c_v}\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\right] \tag{1.81}$$

The speed of propagation (phase speed) is given by

$$a_0^2 = \frac{\omega^2}{k^2} = \left(\frac{\partial p}{\partial \rho}\right)_{T_0}+\frac{p_0}{\rho_0^2 c_v}\left(\frac{\partial p}{\partial T}\right)_{\rho_0} \tag{1.82}$$

when $\left(\lambda_c k^2 / \rho_0 c_v \omega\right) << 1$. If the earlier qualitative remarks are correct, this formula should be the same as (1.9). To show this, a little thermodynamics is required. Suppose that the pressure is regarded as a function of density and temperature, which in turn is treated as a function of density and entropy:

$$p = p(\rho, T(\rho, s))$$

Then the partial derivative in (1.9) is

$$\left(\frac{\partial p}{\partial \rho}\right)_{s_0} = \left(\frac{\partial p}{\partial \rho}\right)_{T_0}+\left(\frac{\partial p}{\partial T}\right)_{\rho_0}\left(\frac{\partial T}{\partial \rho}\right)_{s_0} \tag{1.83}$$

To evaluate the last term, note that for a perfect gas subjected to an isentropic process,

$$p \approx \rho^\gamma \approx \rho^T$$

so

$$T \approx \rho^{\gamma-1}$$

and

$$\frac{\rho_0}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_{s_0} = (\gamma-1)$$

Because $\gamma = c_p / c_v$ and $R = c_p - c_v$, this can be written, with use of the perfect gas law, as

$$\left(\frac{\partial T}{\partial \rho}\right)_{s_0} = (\gamma-1)\frac{T_0}{\rho_0} = \frac{c_p - c_v}{\rho c_v}\frac{p_0}{R\rho_0} = \frac{p_0}{\rho_0^2 c_v}$$

When this is evaluated at the ambient conditions and substituted into (1.83), the result is exactly the right hand side of (1.82), verifying the assertion and the argument offered earlier: the wave propagation is isentropic when the characteristic time for heat conduction is very much less than the period for the wave motion.

If $\left(\lambda_c k^2 / \rho_0 c_v \omega\right) >> 1$, then Eq. (1.80) is satisfied only if

$$a_0^2 = \frac{\omega^2}{k^2} = \left(\frac{\partial p}{\partial \rho}\right)_{T_0} \tag{1.84}$$

For a perfect gas this gives $a_0 = \sqrt{RT_0}$ In this case, the propagation of waves is isothermal. Heat conduction takes place sufficiently rapidly compared with changes associated with the wave motion that the temperature is sensibly constant. Before thermodynamics had been developed, Newton proposed isothermal sound propagation, which, as shown below, is not correct for everyday acoustics; the error was subsequently corrected by Laplace.

Finally, if the ratio $\lambda_c k^2 / \rho_0 c_v$ is neither very large nor very small, eq. (1. 80) is satisfied only if <u>both</u> (1. 81) and (1. 84) are satisfied. These are inconsistent and conflicting requirements. This means that steady sinusoidal waves cannot exist under such conditions. What happens is that the amplitude and shape of the wave must change with time, owing to the conduction of heat from one region of space to another.

To see whether what condition is met in practice, the following representative values for air may be used:

$$\begin{aligned}
\lambda_c &= 0.728 _ cal / cm \cdot K \\
\rho_0 &= 1.2 \times 10^{-3} _ gm / cm^3 \\
c_0 &= 3 \times 10^4 _ cm / \sec \\
c_v &= 0.173 _ cal / gm \cdot K
\end{aligned}$$

Then

$$\frac{\lambda_c}{\rho_0 c_v} \frac{k^2}{\omega} = \frac{\lambda_c}{\rho_0 c_v c_0} \omega \cong 4 \times 10^{-10} \omega$$

For sound waves, the angular frequency for audible sound waves varies roughly from 60 - 100.000 rad/sec, and the inequality (1. 77) is clearly satisfied.

The preceding constitutes a more formal and quantitative justification for assuming that sound waves propagate isentropically. If the frequency becomes high enough, the gradients become

large enough that heat conduction becomes significant. This condition arises in shock waves, but otherwise, unless one is interested in the decay of waves, viscous effects can be safely ignored.

1.4 Energy and Intensity Associated with Acoustic Waves; Time Averages; the Decibel Scale

The energy of any system really originates in the formal description, although subsequently much physical content and reality is often associated with the notion. It is perfectly reasonable, then, to use the equations to determine what combination of properties should be regarded as the "energy" of an acoustic wave. Begin at the beginning: integrate the energy equation (1.3) over an arbitrary closed stationary volume. In the gas, after using the continuity equation (1.1);

$$\begin{aligned} &\frac{\partial}{\partial t}\int \rho\left(e+\frac{u^2}{2}\right)dV = -\int \nabla\cdot\left[\rho\vec{u}\left(e+\frac{u^2}{2}\right)\right]dV - \int \nabla\cdot\left(\rho\vec{u}\right)dV \\ &= -\oiint \left(e+\frac{u^2}{2}\right)\rho\vec{u}\cdot d\vec{s} - \oiint p\vec{u}\cdot d\vec{s} \end{aligned} \tag{1.85}$$

In the integral on the left hand side, $\rho u^2/2$ is the kinetic energy, and to second order in the acoustic fluctuations,

$$\rho u^2/2 \approx \rho_0 \frac{1}{2}\vec{u}'^2 \tag{1.86}$$

The internal energy may be expanded in Taylor series as

$$\rho e = \rho_0 e_0 + \rho'\left[\frac{\partial}{\partial\rho}(\rho e)\right]_{s_0} + \frac{1}{2}\rho'^2\left[\frac{\partial^2}{\partial\rho^2}(\rho e)\right]_{s_0}$$

where, for isentropic motions, the derivatives must be evaluated at constant entropy. The energy is given by the combined First and Second Laws of Thermodynamics as

$$de = Tds + \frac{p}{\rho^2}d\rho$$

so

$$\left(\frac{\partial e}{\partial\rho}\right)_s = \frac{p}{\rho^2}$$

and therefore

$$\left[\frac{\partial}{\partial\rho}(\rho e)\right]_s = e + \rho\left(\frac{\partial e}{\partial\rho}\right)_s = e + \frac{p}{\rho}$$

For a perfect gas, the second derivative is

$$\left[\frac{\partial^2}{\partial\rho^2}(\rho e)\right]_{s_0} = \left(\frac{\partial e}{\partial\rho}\right)_{s_0} + \left[\frac{\partial}{\partial\rho}\frac{p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\right]_{s_0}$$

$$= \frac{p_0}{\rho_0^2} + \frac{p_0}{\rho_0}\left[\frac{\gamma-1}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-2}\right]_{s_0}$$

$$= \frac{p_0}{\rho_0^2} + (\gamma-1)\frac{p_0}{\rho_0}\left[\frac{1}{\rho_0}\left(\frac{p}{p_0}\right)\left(\frac{\rho}{\rho_0}\right)^{-2}\right]_{s_0}$$

$$= \gamma\frac{p_0}{\rho_0^2} = \frac{a_0^2}{\rho_0}$$

Hence,

$$\begin{aligned}\rho e &\approx \rho_0 e_0 + \rho'\left(e_0 + \frac{p_0}{\rho_0}\right) + \frac{1}{2}\frac{a_0^2}{\rho_0}\rho'^2 \\ &= \rho_0 e_0 + \rho' h_0 + \frac{1}{2}\frac{p'^2}{\rho_0 a_0^2}\end{aligned} \tag{1.87}$$

where h_0 is the enthalpy of the undisturbed fluid.

Another way of obtaining this result for a perfect gas is to note first that for c_v constant, and with the reference energy ignored, $e = c_v T = (c_v/R)(p/\rho)$, so $\rho e = (c_v/R)p$ and the fluctuation of ρe is proportional to the fluctuation of pressure:

$$\rho e = \frac{c_v}{R}(p_0 + p')$$

But the pressure fluctuation is required to be isentropic, and can be expanded in terms of the density fluctuations as follows:

$$p' = \rho'\left(\frac{\partial p}{\partial\rho}\right)_s + \frac{1}{2}\rho'^2\left(\frac{\partial^2 p}{\partial\rho^2}\right)_s$$

After the derivatives are evaluated as above, Eq. (1.87) can be recovered.

The first term in (1.87) can be ignored because it is the uninteresting (here) energy of the undisturbed fluid. Substitution of (1.86) and (1.87) into (1. 85) gives:

$$\begin{aligned}&\frac{\partial}{\partial t}\int\left[\rho' h_0 + \frac{1}{2}\frac{p'^2}{\rho_0 a_0^2} + \frac{1}{2}\rho_0 u'^2 + \rho_0 e_0\right]dV \\ &= -\oiint\left[\rho' h_0 + \frac{1}{2}\frac{p'^2}{\rho_0 a_0^2} + \frac{1}{2}\rho_0 u'^2 + \rho_0 e_0\right]\vec{u}'\cdot d\vec{S} - \oiint(p_0 + p')\vec{u}'\cdot d\vec{S}\end{aligned} \tag{1.87a}$$

The first order terms and the term containing $\rho' h_0 \vec{u}'$ on the right hand side are

$$h_0\left[\frac{\partial}{\partial t}\int \rho' dV+\left(\frac{\rho_0 e_0+p_0}{h_0}\right)\oiint \vec{u}'\cdot d\vec{S}+\oiint \rho'\vec{u}'\cdot d\vec{S}\right]$$

But $h_0 = e_0 + p_0/\rho_0$ so the terms in brackets are

$$\int \frac{\partial \rho'}{\partial t}dV+\int \rho_0\nabla\cdot\vec{u}'dV+\int \nabla\cdot(\rho'\vec{u}')dV$$

which vanish in virtue of the continuity equation, carried out to second order.

The term with $\rho_0 e_0$ on the left hand side is zero, so (1.87a) is now

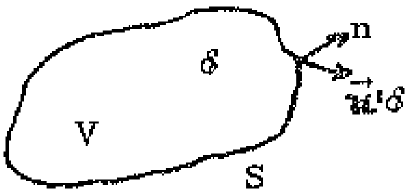
$$\begin{aligned}&\frac{\partial}{\partial t}\int\left[\frac{1}{2}\frac{p'^2}{\rho_0 a_0^2}+\frac{1}{2}\rho_0 u'^2\right]dV\\&=-\oiint\left[\frac{1}{2}\frac{p'^2}{\rho_0 a_0^2}+\frac{1}{2}\rho_0 u'^2\right]\vec{u}'\cdot d\vec{S}-\oiint p'\vec{u}'\cdot d\vec{S}\end{aligned} \tag{1.88}$$

Let E denote the combination

$$E=\frac{1}{2}\frac{p'^2}{\rho_0 a_0^2}+\frac{1}{2}\rho_0 u'^2 \tag{1.89}$$

and Eq. (1.88) is

$$\frac{\partial}{\partial t}\int EdV=-\oiint E\vec{u}'\cdot d\vec{S}-\oiint p'\vec{u}'\cdot d\vec{S} \tag{1.90}$$



To interpret this equation, consider an arbitrary volume V within the fluid, enclosed by a (closed) surface S: The last integral in (1.90) represents the total work done by the fluid within the volume, on the fluid outside due to the fluctuating pressure acting on the fluctuating flow through the surface. Then E is reasonably interpreted as the density of acoustic energy. For then the first integral on the right hand side represents the convection of energy out of the volume due to the fluctuating flow itself, and the entire right hand side is the total rate of change of acoustic energy within the volume. That is exactly what the left hand side stands for.

Now the first integral on the right hand side of (1.90) is third order in small quantities and therefore must be dropped. The integral formulation for the balance of acoustic energy is correctly (to second order)

$$\frac{\partial}{\partial t}\int EdV=-\oiint p'\vec{u}'\cdot d\vec{S}=-\int\nabla\cdot(p'\vec{u}')dV \tag{1.91}$$

which implies the differential form

$$\frac{\partial E}{\partial t} + \nabla \cdot (p'\vec{u}') = 0 \tag{1.92}$$

To be more specific, consider a plane wave travelling to the right, for which (1.35) holds.

$$u' = \frac{p'}{\rho_0 a_0}$$

The energy density is

$$\begin{aligned} E &= \frac{1}{2}\frac{(\rho_0 a_0 u')^2}{\rho_0 a_0^2} + \frac{1}{2}\rho_0 u'^2 \\ &= \rho_0 u'^2 \end{aligned} \tag{1.93}$$

showing that the potential and kinetic energies are equal. Also,

$$p'u' = \rho_0 a_0 u'^2 = a_0 E \tag{1.94}$$

and (1.92) becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(a_0 E) = 0 \tag{1.95}$$

This result, of course, holds for a wave travelling to the left as well. The flux of energy associated with a plane wave travelling in the direction defined by the wave vector $\vec{k}$ is

$$\vec{I} = a_0 E \frac{\vec{k}}{|k|} \tag{1.96}$$

which is the intensity of the wave, having magnitude

$$I = \left|\vec{I}\right| = p'\left|\vec{u}'\right| = a_0 E \tag{1.97}$$

The case of harmonic or sinusoidal waves is particularly important. Let the amplitudes of waves travelling to the right and left be denoted by p_+, p_-, respectively:

$$\begin{aligned} p'_+ &= p_+ e^{-i(\omega t - kx)} \\ p'_- &= p_- e^{-i(\omega t + kx)} \\ u'_+ &= u_+ e^{-i(\omega t - kx)} \\ u'_- &= u_- e^{-i(\omega t + kx)} \end{aligned} \tag{1.98}$$

Equations (1.35) and (1.36) for plane waves show that

$$\begin{aligned} u_+ &= \frac{p_+}{\rho_0 a_0} \\ u_- &= \frac{-p_-}{\rho_0 a_0} \end{aligned} \tag{1.99}$$

The energy densities are

$$E_+ = \frac{p_+'^2}{\rho_0 a_0^2}$$
$$E_- = \frac{p_-'^2}{\rho_0 a_0^2} \tag{1.100}$$

and the intensities are

$$I_+ = p_+' u_+' = \frac{p_+'^2}{\rho_0 a_0}$$
$$I_- = p_-' u_-' = -\frac{p_-'^2}{\rho_0 a_0} \tag{1.101}$$

The negative sign appended to I_- shows that the flux is in the negative x-direction.

It is often convenient to work with time-averaged values, for example,

$$\langle E_+ \rangle = \frac{\langle p_+'^2 \rangle}{\rho_0 a_0^2} \tag{1.102}$$

Because squares of the acoustic quantities are involved, it is necessary to take the real or imaginary parts of (1.98) before the time averages are taken; thus,

$$\langle p_+'^2 \rangle = p_+^2 \frac{1}{\tau} \int_t^{t+\tau} \cos^2(\omega t' - kx) dt' \tag{1.103}$$

Let $\xi = \omega t' - kx$, so $dt' = \frac{1}{\omega} d\xi = \frac{\tau}{2\pi} d\xi$, and because the integrand has period 2π, the limits can be changed from $(t, t+\tau)$ to $(0, 2\pi)$, giving

$$\langle p_+'^2 \rangle = p_+^2 \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \xi d\xi = \frac{1}{2} p_+^2 \tag{1.104}$$

The symbol p_{rms} is often used to denote $\langle p'^2 \rangle^{1/2}$; thus, $p_{rms} = p_{\max} / \sqrt{2}$ for a sinusoidal wave.

Consequently, one obtains the results

$$\left\langle E_+ \right\rangle = \frac{p_+'^2}{2\rho_0 a_0^2}$$
$$\left\langle E_- \right\rangle = \frac{p_-'^2}{2\rho_0 a_0^2}$$
$$\left\langle I_+ \right\rangle = \frac{p_+'^2}{2\rho_0 a_0} \tag{1.104a}$$
$$\left\langle I_- \right\rangle = -\frac{p_-'^2}{2\rho_0 a_0}$$

These formulas for energy density and intensity may be usefully obtained in more general form for later use. Let the pressure and velocity fields be

$$\begin{aligned} p' &= \hat{p}e^{-i\omega t} = \left|\hat{p}\right| e^{-i(\omega t+\phi)} \\ \vec{u}' &= \hat{\vec{u}}e^{-i\omega t} = \hat{u}\left|\hat{\vec{u}}\right| e^{-i(\omega t+\psi)} \end{aligned} \tag{1.105}$$

The unit vector $\hat{u}$ lies in the direction of the velocity vector $\hat{\vec{u}}$. To form the energy density and intensity, take the real parts:

$$p'^{(r)} = \left|\hat{p}\right|\cos(\omega t+\phi)$$
$$\vec{u}'^{(r)} = \hat{u}\left|\hat{\vec{u}}\right|\cos(\omega t+\psi)$$

Thus the instantaneous energy density (1.89) is

$$E = \frac{1}{2}\frac{\left|\hat{p}\right|^2}{\rho_0 a_0^2}\cos^2(\omega t+\phi) + \frac{1}{2}\rho_0\left|\hat{\vec{u}}\right|^2\cos^2(\omega t+\psi) \tag{1.106}$$

and the time average is

$$\left\langle E \right\rangle = \frac{1}{4}\left(\frac{\left|\hat{p}\right|^2}{\rho_0 a_0^2} + \rho_0\left|\hat{\vec{u}}\right|^2\right) \tag{1.107}$$

The intensity is the flux of energy $p'\vec{u}'$ (if Eq. (1.92)); again taking real parts when the complex notation is used.

$$I = \hat{v}\left|\hat{p}\right|\left|\hat{\vec{u}}\right|\cos(\omega t+\phi)\cos(\omega t+\psi) \tag{1.108}$$

To take the time average, the cosines must first be expanded; the result is

$$\left\langle \vec{I} \right\rangle = \hat{v}\frac{1}{2}\left|\hat{p}\right|\left|\hat{\vec{u}}\right|\cos(\phi-\psi)$$

Thus, in order that there be a flux of energy, there must be a component of velocity in phase with the pressure $\phi-\psi \neq \beta/2$. Suppose that $\hat{p}$ is real, so (1.105) can be written

$$\vec{u}' = \hat{v}\left|\hat{\vec{u}}\right| e^{-i\omega t} e^{-i\psi} = \hat{v}\frac{\left|\hat{\vec{u}}\right|}{\left|\hat{p}\right|} p' e^{-i\psi} = \hat{v}\frac{\left|\hat{\vec{u}}\right|}{\left|\hat{p}\right|} p'(\cos\psi - \sin\psi)$$

For $\psi = \pi/2$, $\vec{u}'$ and p' are out of phase and therefore $\vec{I} = 0$.

Another way of writing these formulas is to note that $\left|\hat{f}\right|^2 = ff^*$, so (1.106) is

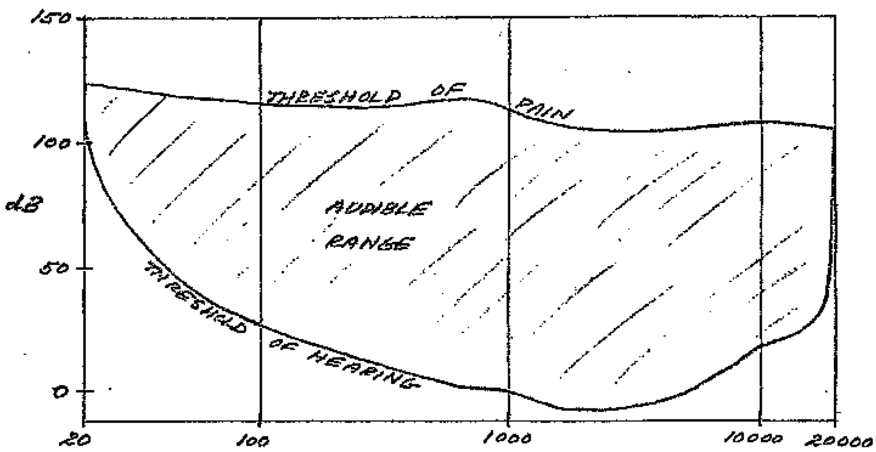
$$\left\langle E \right\rangle = \frac{1}{4}\left(\frac{p'p'^*}{\rho_0 a_0^2} + \rho_0 \vec{u}' \cdot \vec{u}'^*\right) \tag{1.109}$$

Also, the real part of fg is $f^*g + fg^*$, and the time average of (1.108) may be written

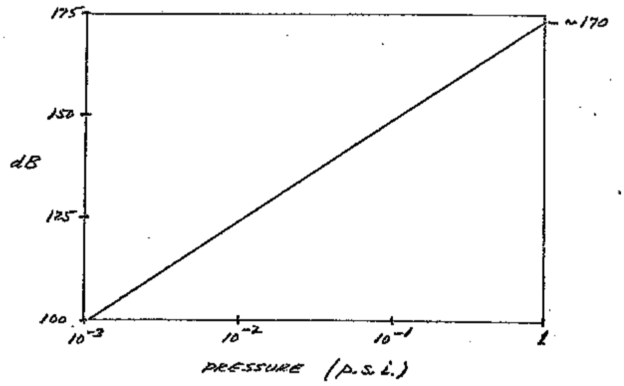
$$\left\langle \vec{I} \right\rangle = \frac{1}{4}\left(p'^*\vec{u} + p'\vec{u}'^*\right) \tag{1.101}$$

Obviously if p' is real and $\vec{u}'$ is pure imaginary, i.e., out of phase with p', the intensity vanishes.

Finally, it is interesting to see the frequency response for the human ear, sketched below.



With the numbers given above, the relation between dB and pressure is that shown.



Note that $1\_dyne/cm^2 = 74\_dB$ and one atmosphere is $10^6\_dyne/cm^2$. From these two figures it is clear that any steady wave which can be heard without pain has sufficiently small amplitude that linearization of the equations of motion is reasonable.

l.5 Boundary Conditions; Impedance and Admittance; Reflection from a Plane Surface

For inviscid flow at a rigid impermeable surface, the correct boundary condition for inviscid flow is that the normal component of velocity vanishes:

$$\vec{u}' \cdot \hat{n} = 0 \qquad (1.113)$$

If the surface is not rigid, or allows flow into it, then the velocity will not be zero, and according to the momentum equation, the normal gradient of the pressure fluctuation is

$$\hat{n} \cdot \nabla p' = \rho_0 \frac{\partial \vec{u}'}{\partial t} \cdot \hat{n} \qquad (1.114)$$

It is very often a reasonable approximation that the local motion at a surface element depends only on local conditions. Moreover, it is a reasonable assumption - at least as a first try - that the normal component of the velocity is proportional to the pressure:

$$\vec{u}' \cdot \hat{n} = \frac{1}{z} p' \qquad (1.115)$$

The quantity z is called the **acoustic impedance** of the surface, and the inverse, $y = 1/z$ is called the **acoustic admittance**.

The impedance and admittance are in practice defined for harmonic motions, and are generally complex functions of frequency. With (1.110) the boundary condition for harmonic motions is

$$\hat{n} \cdot \nabla p' \rho_0 \frac{i\omega}{z} p' \rho_0 i\omega y p' \qquad (1.116)$$

The dimensions of z are density times velocity; a quantity having the same dimensions for the medium is $\rho_0 a_0$, called the **characteristic acoustic impedance**. For air at ambient conditions,

$$\rho_0 a_0 = 42\_gm/cm^2 \cdot \text{sec}\ .$$

Often the dimensionless functions are used:

$$\varsigma = \frac{z}{\rho_0 a_0} \qquad \text{acoustic impedance ratio,}$$

$$\eta = \frac{1}{\varsigma} \qquad \text{acoustic admittance ratio.} \qquad (1.117)$$

These quantities have been defined, and named, by analogy with quantities used to characterize electrical circuits. Thus, the real part of z is called the **acoustic resistance**, and the imaginary part is called the **acoustic reactance**.

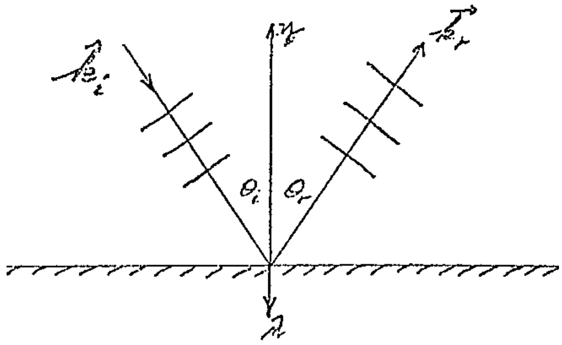
Consider now reflection of plane waves from a surface. According to the results (1.33) and (1.35), a plane wave progressing to the right in the x-direction can be represented by

$$p'(x,t) = g(x - a_0 t)$$
$$u'(x,t) = \frac{1}{\rho_0 a_0} g(x - a_0 t)$$

These can be generalized for propagation in the direction $\vec{k}$:

$$p'(\vec{r},t) = g(\vec{k} \cdot \vec{r} - \omega t)$$
$$u'(\vec{r},t) = \frac{1}{\rho_0 a_0} g(\vec{k} \cdot \vec{r} - \omega t) \tag{1.118}$$

Suppose that a plane wave is incident upon a plane surface, at an angle θ_i to the normal to the surface; a plane reflected wave leaves the surface at angle θ_r to the normal as sketched.



Because $\left|\vec{k}_i\right| = \omega/a_0 = \left|\vec{k}_r\right|$, these two waves can be represented by the following formulas:

$$p_i' = g_i(\xi_i)$$
$$p_r' = g_r(\xi_r)$$
$$\vec{u}_i' = \frac{1}{\rho_0 a_0} \frac{\vec{k}_i}{\left|\vec{k}\right|} g_i(\xi_i) \tag{1.119}$$
$$\vec{u}_r' = \frac{1}{\rho_0 a_0} \frac{\vec{k}_r}{\left|\vec{k}\right|} g_r(\xi_r)$$

where

$$\xi_i = \vec{k}_i \cdot \vec{r} - \omega t = k(x \sin\theta_i - y \cos\theta_i) - \omega t$$
$$\xi_r = \vec{k}_r \cdot \vec{r} - \omega t = k(x \sin\theta_r + y \cos\theta_r) - \omega t \tag{1.120}$$

Now here $\vec{u}' \cdot \hat{n} = -u_y$ for use in the definition (1.111) of the surface impedance. The component of velocity in the y-direction follows immediately from (1.119), evaluated at $y = 0$:

$$u_y = -\frac{\cos\theta_i}{\rho_0 a_0} g_i(kx \sin\theta_i - \omega t) + \frac{\cos\theta_r}{\rho_0 a_0} g_r(kx \sin\theta_r - \omega t) \tag{1.121}$$

The ratio $p' / -u_y'$ evaluated at $y = 0$ is the surface impedance z:

$$-z = \rho_0 a_0 \frac{g_i(kx\sin\theta_i - \omega t) + g_r(kx\sin\theta_r - \omega t)}{-\cos\theta_i \cdot g_i(kx\sin\theta_i - \omega t) + \cos\theta_r \cdot g_r(kx\sin\theta_r - \omega t)}$$

In general, z may be a function of position along the surface, but it is simplest to take care of this later; for the present, assume that z is independent of x. This may be satisfied only if

$$\theta_i = \theta_r \qquad g_r(\xi) = \beta g_i(\xi) \tag{1.122}$$

where β is the reflection coefficient. Then

$$-z = \rho_0 a_0 \frac{1+\beta}{(-1+\beta)\cos\theta}$$

which can be solved for β to give

$$\beta = \frac{z\cos\theta - \rho_0 a_0}{z\cos\theta + \rho_0 a_0} \tag{1.123}$$

Note, for example, that if $\theta = 0$, there is no reflection if $z = \rho_0 a_0$ a result that must obviously be true because the surface y =0 is then an imaginary surface within a homogeneous medium.

But if $z = \rho_0 a_0$ and $\theta \neq 0$, the reflection coefficient does not vanish. The reason for this nonsensical result is that the transmitted wave has not been accounted for. A complete treatment produces a result which does reduce correctly for a homogeneous medium.