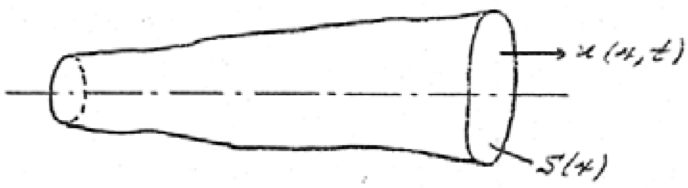
II. PROPAGATION OF WAVES IN TUBES

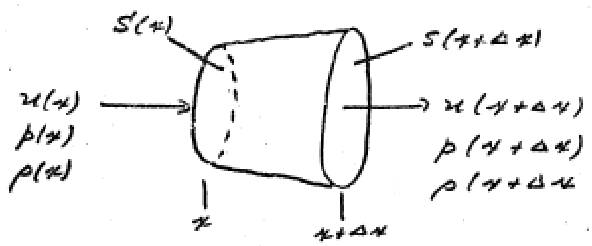
There are many practical problems involving the propagation of waves in tubes, including musical instruments as well as technical devices. Very often it is a realistic approximation greatly simplifying the results, that spatial variations occur only in one dimension, in the direction coinciding with the axis of the tube. Several examples of this kind of motion will be examined. A discussion of the conditions to be satisfied, if this is to be a valid approximation, will be given later together with contrary examples (Sect. 2.6).

2.1 Wave Propagation In Tubes Having Variable Cross-Section

The description of one-dimensional wave motions is based from the beginning on the assumption that the motions depend on a single space dimension. There is no completely satisfactory way of deducing the one-dimensional equations of motion from the general conservation equations. The first step, then, is to deduce the equations of motion for one-dimensional flow and then linearize them. Let $S(x)$ denote the cross-sectional area of the tube.



Consider an element between x and $x+\Delta x$:



The appropriate balance for conservation of mass is

$$\frac{\partial}{\partial t}(\rho S\Delta x)=[\rho uS]_x-[\rho uS]_{x+\Delta x}$$

Divide by Δx and take the limit $\Delta x\to 0$ to find the continuity equation:

$$\frac{\partial(\rho S)}{\partial t}+\frac{\partial}{\partial x}(\rho uS)=0 \tag{2.1}$$

Similarly, the momentum equation may be constructed, with account taken of the pressure exerted on the lateral boundary:

$$\frac{\partial}{\partial t}(\rho uS\Delta x)=[\rho u^2S]_x-[\rho u^2S]_{x+\Delta x}+[pS]_x-[pS]_{x+\Delta x}+p\frac{\Delta S}{\Delta x}\Delta x$$

Again divide by Δx and let $\Delta x \to 0$ to find

$$\frac{\partial(\rho u S)}{\partial t}+\frac{\partial}{\partial x}\left(\rho u^2 S\right)=-\frac{\partial pS}{\partial x}+p\frac{\partial S}{\partial x}$$

or

$$\frac{\partial(\rho u S)}{\partial t}+\frac{\partial}{\partial x}\left(\rho u^2 S\right)=-S\frac{\partial p}{\partial x} \tag{2.2}$$

Finally, the balance of energy for the volume element is

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{u^2}{2}\right)S\Delta x\right]=\left[\rho u\left(e+\frac{u^2}{2}\right)S\right]_x-\left[\left[\rho u\left(e+\frac{u^2}{2}\right)S\right]\right]_{x+\Delta x}+[puS]_x-[puS]_{x+\Delta x}$$

from which one finds

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{u^2}{2}\right)S\right]+\frac{\partial}{\partial x}\left[\rho u\left(e+\frac{u^2}{2}\right)S\right]+\frac{\partial}{\partial x}\left(puS\right)=0 \tag{2.3}$$

By use of the continuity equation, the momentum equation may be simplified, and both may be used to re-write the energy equation. The three equations are then:

$$\frac{\partial(\rho S)}{\partial t}+\frac{\partial}{\partial x}\left(\rho u S\right)=0 \tag{2.4}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0 \tag{2.5}$$

$$S\left[\rho\frac{\partial e}{\partial t}+\rho u\frac{\partial e}{\partial x}\right]+p\frac{\partial(uS)}{\partial x}=0 \tag{2.6}$$

Detail derivation.

The momentum equation will be

$$\frac{\partial(\rho u S)}{\partial t}+\frac{\partial}{\partial x}\left(\rho u^2 S\right)=-S\frac{\partial p}{\partial x}$$

$$u\frac{\partial(\rho S)}{\partial t}+\rho S\frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}\left(\rho u S\right)+\rho u S\frac{\partial}{\partial x}\left(u\right)=-S\frac{\partial p}{\partial x}$$

$$u\left[\frac{\partial(\rho S)}{\partial t}+\frac{\partial}{\partial x}\left(\rho u S\right)\right]+\rho S\frac{\partial u}{\partial t}+\rho u S\frac{\partial}{\partial x}\left(u\right)=-S\frac{\partial p}{\partial x}$$

The first term on LHS vanishes by continuity equation. Thus

$$\rho S\frac{\partial u}{\partial t}+\rho u S\frac{\partial}{\partial x}\left(u\right)=-S\frac{dp}{dx}$$

Dividing by S, we have

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial}{\partial x}(u)+\frac{dp}{dx}=0$$

Next the energy equation will be

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{u^2}{2}\right)S\right]+\frac{\partial}{\partial x}\left[\rho u\left(e+\frac{u^2}{2}\right)S\right]+\frac{\partial}{\partial x}(puS)=0$$

$$\frac{\partial}{\partial t}(\rho eS)+\frac{1}{2}\frac{\partial}{\partial t}(\rho u^2S)+\frac{\partial}{\partial x}(\rho ueS)+\frac{1}{2}\frac{\partial}{\partial x}(\rho u^3S)+\frac{\partial}{\partial x}(puS)=0$$

$$\left[e\frac{\partial}{\partial t}(\rho S)+\rho S\frac{\partial}{\partial t}(e)\right]+\frac{1}{2}\left[u\frac{\partial}{\partial t}(\rho uS)+\rho uS\frac{\partial}{\partial t}(u)\right]$$
$$+\left[e\frac{\partial}{\partial x}(\rho uS)+\rho uS\frac{\partial}{\partial x}(e)\right]+\frac{1}{2}\left[u\frac{\partial}{\partial x}(\rho u^2S)+\rho u^2S\frac{\partial}{\partial x}(u)\right]$$
$$+\left[p\frac{\partial}{\partial x}(uS)+uS\frac{\partial}{\partial x}(p)\right]=0$$

$$e\frac{\partial}{\partial t}(\rho S)+e\frac{\partial}{\partial x}(\rho uS)$$
$$+\frac{1}{2}\left[u\frac{\partial}{\partial t}(\rho uS)+\rho uS\frac{\partial}{\partial t}(u)+u\frac{\partial}{\partial x}(\rho u^2S)+\rho u^2S\frac{\partial}{\partial x}(u)\right]$$
$$+\rho S\frac{\partial}{\partial t}(e)+\rho uS\frac{\partial}{\partial x}(e)+p\frac{\partial}{\partial x}(uS)+uS\frac{\partial}{\partial x}(p)=0$$

$$e\left[\frac{\partial}{\partial t}(\rho S)+\frac{\partial}{\partial x}(\rho uS)\right]$$
$$+\frac{1}{2}u\left[\frac{\partial}{\partial t}(\rho uS)+\frac{\partial}{\partial x}(\rho u^2S)\right]+\frac{1}{2}uS\left[\rho\frac{\partial}{\partial t}(u)+\rho u\frac{\partial}{\partial x}(u)\right]$$
$$+\rho S\frac{\partial}{\partial t}(e)+\rho uS\frac{\partial}{\partial x}(e)+p\frac{\partial}{\partial x}(uS)+uS\frac{\partial}{\partial x}(p)=0$$

The parentheses in the second and the third terms can be replaced by using the momentum equation as follows.

$$e\left[\frac{\partial}{\partial t}(\rho S)+\frac{\partial}{\partial x}(\rho uS)\right]$$
$$+\frac{1}{2}u\left[-S\frac{\partial p}{\partial x}\right]+\frac{1}{2}uS\left[-\frac{\partial p}{\partial x}\right]$$
$$+\rho S\frac{\partial}{\partial t}(e)+\rho uS\frac{\partial}{\partial x}(e)+p\frac{\partial}{\partial x}(uS)+uS\frac{\partial}{\partial x}(p)=0$$

The first term vanish by continuity equation while the second and the third terms cancel with the last term. Thus we have

$$\rho S\frac{\partial}{\partial t}(e)+\rho uS\frac{\partial}{\partial x}(e)+p\frac{\partial}{\partial x}(uS)=0$$

The linearized forms are readily established, for the case of average flow at rest:

$$\frac{\partial(\rho' S)}{\partial t}+\frac{\partial}{\partial x}(\rho_0 u' S)=0 \tag{2.7}$$

$$\rho_0\frac{\partial u'}{\partial t}+\frac{\partial p'}{\partial x}=0 \tag{2.8}$$

$$\rho_0 C_V S\frac{\partial}{\partial t}(T' S)+p_0\frac{\partial(u' S)}{\partial x}=0 \tag{2.9}$$

Details. (Take terms of order ε.)

Continuity:

$$\frac{\partial((\rho_0+\varepsilon\rho')S)}{\partial t}+\frac{\partial}{\partial x}((\rho_0+\varepsilon\rho')\varepsilon u' S)=0$$

$$\frac{\partial\rho_0 S}{\partial t}+\varepsilon\frac{\partial(\rho' S)}{\partial t}+\varepsilon\frac{\partial}{\partial x}(\rho_0 u' S)+\varepsilon^2\frac{\partial}{\partial x}(\rho' u' S)=0$$

Momentum:

$$(\rho_0+\varepsilon\rho')\frac{\partial\varepsilon u'}{\partial t}+(\rho_0+\varepsilon\rho')\varepsilon u'\frac{\partial\varepsilon u'}{\partial x}+\frac{\partial(p_0+\varepsilon p')}{\partial x}=0$$

$$\varepsilon\rho_0\frac{\partial u'}{\partial t}+\varepsilon^2\rho'\frac{\partial u'}{\partial t}+\varepsilon^2(\rho_0+\varepsilon\rho')u'\frac{\partial u'}{\partial x}+\frac{\partial p_0}{\partial x}+\varepsilon\frac{\partial p'}{\partial x}=0$$

Energy:

$$(\rho_0+\varepsilon\rho')S\frac{\partial}{\partial t}\{C_V(T_0+\varepsilon T')\}+(\rho_0+\varepsilon\rho')\varepsilon u' S\frac{\partial}{\partial x}\{C_V(T_0+\varepsilon T')\}$$
$$+(p_0+\varepsilon p')\frac{\partial}{\partial x}(\varepsilon u' S)=0$$

$$\rho_0 S\frac{\partial}{\partial t}\{C_V(T_0+\varepsilon T')\}+\varepsilon\rho' S\frac{\partial}{\partial t}\{C_V(T_0+\varepsilon T')\}$$
$$+\rho_0\varepsilon u' S\frac{\partial}{\partial x}\{C_V(T_0+\varepsilon T')\}+\varepsilon\rho'\varepsilon u' S\frac{\partial}{\partial x}\{C_V(T_0+\varepsilon T')\}$$
$$+p_0\frac{\partial}{\partial x}(\varepsilon u' S)+\varepsilon p'\frac{\partial}{\partial x}(\varepsilon u' S)=0$$

$$\rho_0 S\frac{\partial C_V T_0}{\partial t}+\rho_0 S\frac{\partial C_V \varepsilon T'}{\partial t}+\varepsilon\rho' S\frac{\partial C_V T_0}{\partial t}+\varepsilon\rho' S\frac{\partial C_V \varepsilon T'}{\partial t}$$
$$+\rho_0\varepsilon u' S\frac{\partial C_V T_0}{\partial x}+\rho_0\varepsilon u' S\frac{\partial C_V \varepsilon T'}{\partial x}+\varepsilon\rho'\varepsilon u' S\frac{\partial C_V T_0}{\partial x}+\varepsilon\rho'\varepsilon u' S\frac{\partial C_V \varepsilon T'}{\partial x}$$
$$+p_0\frac{\partial}{\partial x}(\varepsilon u' S)+\varepsilon p'\frac{\partial}{\partial x}(\varepsilon u' S)=0$$

$$\rho_0 S\frac{\partial C_V T_0}{\partial t}+\varepsilon\rho_0 S\frac{\partial C_V T'}{\partial t}+\varepsilon\rho' S\frac{\partial C_V T_0}{\partial t}+\varepsilon^2\rho' S\frac{\partial C_V T'}{\partial t}$$
$$+\rho_0\varepsilon u' S\frac{\partial C_V T_0}{\partial x}+\rho_0\varepsilon^2 u' S\frac{\partial C_V T'}{\partial x}+\varepsilon^2\rho' u' S\frac{\partial C_V T_0}{\partial x}+\varepsilon^3\rho' u' S\frac{\partial C_V T'}{\partial x}$$
$$+\varepsilon p_0\frac{\partial}{\partial x}(u' S)+\varepsilon^2 p'\frac{\partial}{\partial x}(u' S)=0$$

The terms in order ε are:

$$\varepsilon\rho_0 S\frac{\partial C_V T'}{\partial t}+\varepsilon\rho' S\frac{\partial C_V T_0}{\partial t}+\rho_0\varepsilon u' S\frac{\partial C_V T_0}{\partial x}+\varepsilon p_0\frac{\partial}{\partial x}(u' S)=0$$

Therefore we have

$$\rho_0 C_V\frac{\partial T' S}{\partial t}+p_0\frac{\partial}{\partial x}(u' S)=0$$

As in earlier discussion, the motions are assumed to be isentropic, so ρ' may be replaced by p'/a_0^2 in (2.7). Then differentiate (2.7) with respect to t and use S times (2.8) to eliminate $\rho_0\,\partial u'/\partial t$; the wave equation is

$$\frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial p'}{\partial x}\right)-\frac{1}{a_0^2}\frac{\partial^2 p'}{\partial t^2}=0 \qquad (2.10)$$

Details.

While S times (2.8) is

$$\rho_0 S\frac{\partial u'}{\partial t}+S\frac{\partial p'}{\partial x}=0$$

Then Equation (2.7) will be

$$\frac{\partial(\rho' S)}{\partial t}+\frac{\partial}{\partial x}(\rho_0 u' S)=0 \qquad (2.7)$$

Differentiate (2.7) with respect to t is

$$\frac{\partial^2(\rho' S)}{\partial t^2}+\frac{\partial^2}{\partial t \partial x}(\rho_0 u' S)=0$$

$$\frac{\partial^2(\rho' S)}{\partial t^2}+\frac{\partial}{\partial x}\left\{u'\frac{\partial}{\partial t}(\rho_0 S)+\rho_0 S\frac{\partial}{\partial t}(u')\right\}=0$$

Substitution gives

$$\frac{\partial^2(\rho' S)}{\partial t^2}+\frac{\partial}{\partial x}\left\{u'\frac{\partial}{\partial t}(\rho_0 S)-S\frac{\partial p'}{\partial x}\right\}=0$$

Taking into consideration of time dependence of S and ρ_0 we have

$$S\frac{\partial^2 \rho'}{\partial t^2}+\frac{\partial}{\partial x}\left\{-S\frac{\partial p'}{\partial x}\right\}=0$$

Since

$$\rho'=\frac{p'}{a_0^2}$$

we have

$$\frac{S}{a_0^2}\frac{\partial^2 p'}{\partial t^2}-\frac{\partial}{\partial x}\left\{S\frac{\partial p'}{\partial x}\right\}=0$$

Finally

$$\frac{1}{a_0^2}\frac{\partial^2 p'}{\partial t^2}-\frac{1}{S}\frac{\partial}{\partial x}\left\{S\frac{\partial p'}{\partial x}\right\}=0$$

Boundary conditions in x and initial conditions must be set. These will be treated in examples.

2.2 Waves in Uniform Tubes; Normal Modes; the Impedance Tube

If the tube is uniform, S is independent of x and Eq. (2.10) reduces to the simple equation for plane waves. While this can be treated with a certain amount of generality, it is more useful at this point to restrict attention to harmonic waves. With $k=\omega/a_0$ the pressure amplitude satisfies

$$\frac{d^2\hat{p}}{dx^2}+k^2\hat{p} \qquad (2.11a)$$

which has solutions of the form

$$\hat{p}=e^{\pm ikx},\ \cos(kx),\ \sin(kx) \qquad (2.11b)$$

The trick is to combine the right choices in the right way to obtain the results desired for a particular

problem.

2.2.1 Normal Modes.

A simple problem is that of determining the normal modes and frequencies for a straight tube. A normal mode is defined to be a wave motion, within a finite boundary not necessarily closed, which has the pure harmonic dependence, $\exp(i\omega t)$. The amplitude is supposed constant in time. Consequently, the solutions must be chosen from (2.12); which ones to use depends on the boundary conditions.

For example, consider a tube closed at both ends. Because u' is zero, then $d\hat{p}/dx = 0$. This is satisfied at $x = 0$ if $\hat{p} = \cos(kx)$; and if the length is L, the other boundary condition requires

$$\left(\frac{d\hat{p}}{dx}\right)_L = -k\sin(kL) = 0 \qquad (2.12)$$

which means

$$k = \frac{l\pi}{L} \qquad (2.13)$$

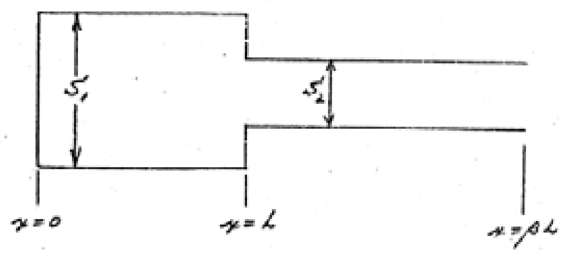
If the tube is open at both ends, $\hat{p} = 0$ at $x = 0, L$ and the normal modes and frequencies are

$$\hat{p} = \sin(kL)\,, \quad k = \frac{l\pi}{L} \qquad (2.14)$$

Finally, if the tube is closed at $x = 0$ and open at $x = L$, one finds

$$\hat{p} = \cos(kx) \quad k = (2l+1)\frac{\pi}{2L} \qquad (2.15)$$

Now suppose that the tube has a discontinuity in the cross section as, for example, in the sketch following.



In each of the straight tubes, the solutions (2.11b) must be valid, and the problem comes down to matching the solutions at the discontinuity and making sure that the boundary conditions are satisfied. Obviously, for the case sketched, the boundary conditions are

$$\left[\frac{d\hat{p}}{dx}\right]_{x=0} = 0\,, \quad \hat{p}(\beta L) = 0 \qquad (2.16)$$

The first can be satisfied by

$$\hat{p}(x) = A\cos(kx) \qquad (0 \le x \le L) \tag{2.17}$$

and the second by

$$\hat{p}(x) = B\sin k(\beta L - x) \qquad (L < x \le \beta L) \tag{2.18}$$

Because $k = \omega/a_0$, it is the same k in both (2.17) and (2.18), but it is still undetermined. Both k and the ratio A/B are found by matching the solutions for the two straight sections. Clearly, one condition for matching is continuity of pressure:

$$\hat{p}(L-\varepsilon) = \hat{p}(L+\varepsilon) \tag{2.19}$$

which means for $\varepsilon \to 0$,

$$A\cos(kL) = B\sin(\beta - 1)kL \tag{2.20}$$

A second condition may be deduced from Eq. (2.10). For harmonic motions, (2.10) can be written

$$\frac{d}{dx}\left(S\frac{d\hat{p}}{dx}\right) + k^2 S\hat{p} = 0 \tag{2.21}$$

Integrate (2.21) over the interval $L - \varepsilon < x < L + \varepsilon$ to find

$$\left[S\frac{d\hat{p}}{dx}\right]_{L-\varepsilon}^{L+\varepsilon} + k^2 \int_{L-\varepsilon}^{L+\varepsilon} S\hat{p}dx = 0 \tag{2.22}$$

Now let $\varepsilon \to 0$; the second term vanishes because the product $S\hat{p}$ remains finite. Thus, (2.22) provides the continuity condition

$$\left[S\frac{d\hat{p}}{dx}\right]_{L-\varepsilon} = \left[S\frac{d\hat{p}}{dx}\right]_{L+\varepsilon} \tag{2.23}$$

Because $d\hat{p}/dx$ is proportional to the acoustic velocity, (2.23) is merely the statement that the mass flow is continuous. Note that the statement applies really to two surfaces just upstream and just downstream of the geometrical discontinuity. In the real flow, there is a transition region which must, if the approximation is to work, be thin. This works quite well in practice so long as S_1/S_2 is not too large. Some experimental results are cited below.

Well, anyway, if (2.17) and (2.18) are matched according to (2.23), the result is

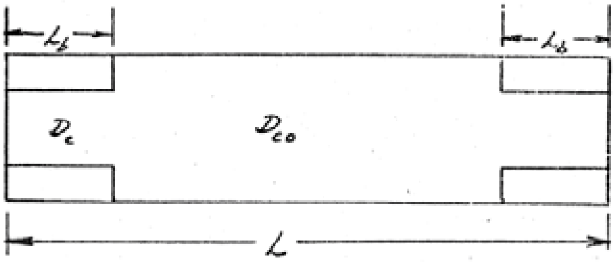
$$-AS_1\sin(kL) = -BS_2\cos k(\beta - 1)L \tag{2.24}$$

Divide (2.24) by (2.20) to find the transcendental equation for k:

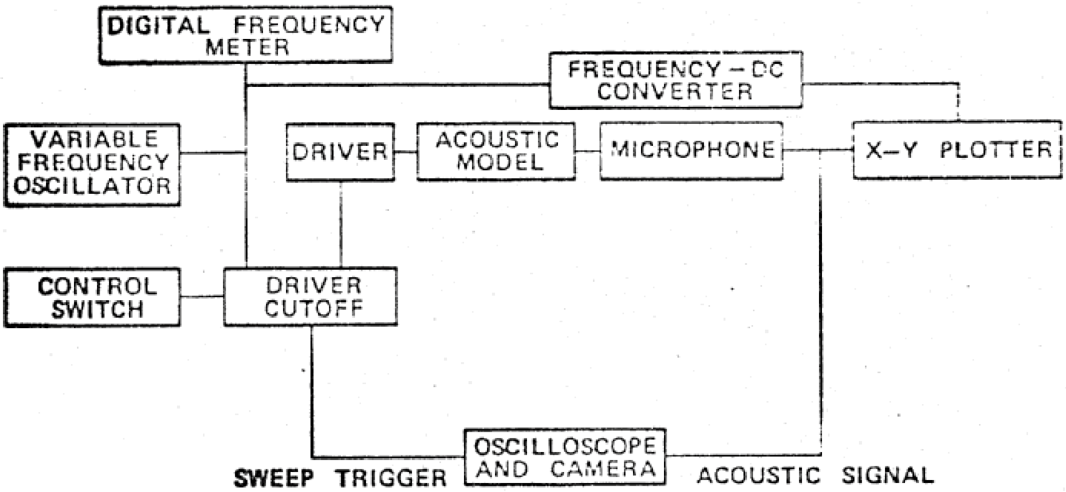
$$\frac{S_1}{S_2}\tan(kL) = \cot k(\beta - 1)L \tag{2.25}$$

After the values of k have been found, the ratio A/B can be determined from either (2. 20) or (2. 24). And if one is so inclined, one has various choices for normalizing; for example, set $A = 1$ or $B = 1$, or set the integral of $\hat{p}^2 = 1$, etc., etc.

In any case, the extension to more than one discontinuity can be done without essential difficulty, though with increased effort. For example, consider the closed tube sketched.



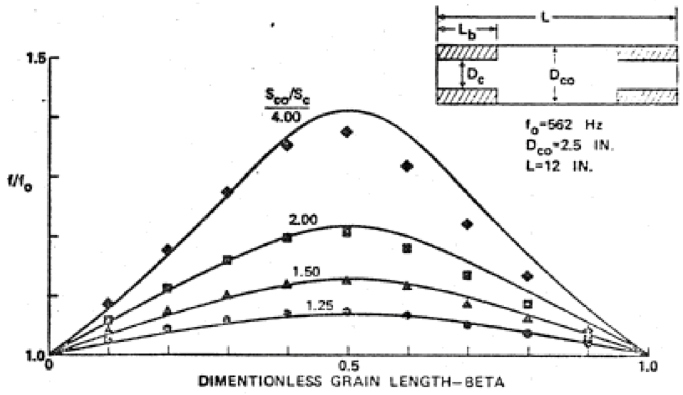
Experiments have been conducted to measure the natural frequencies for various combinations of the geometrical variables and to measure the decay of waves in the chamber. A block diagram of the apparatus is shown below.



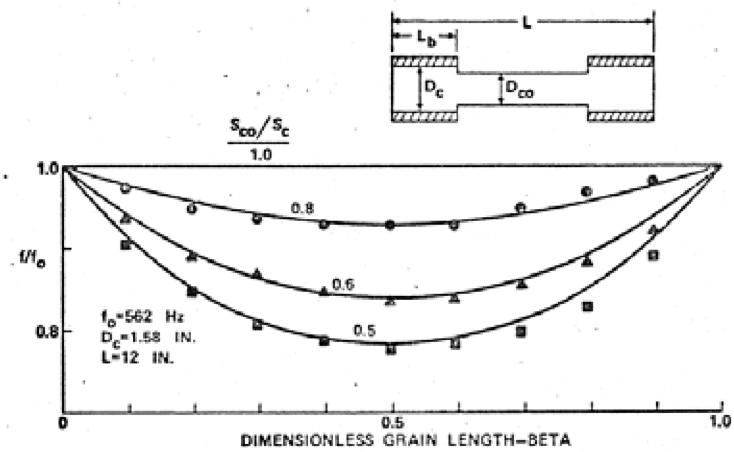
Some of the results are given in the following figures for the natural frequencies of the fundamental and second mode. The solid lines are the results obtained by the procedure outlined above. Solutions in the uniform sections of the tube were matched at the discontinuities, eventually producing a transcendental equation for the frequency. For the odd modes the result is

$$\tan K_l(1-\beta)\tan K_l\beta = \frac{S_{co}}{S_c} \tag{2.26}$$

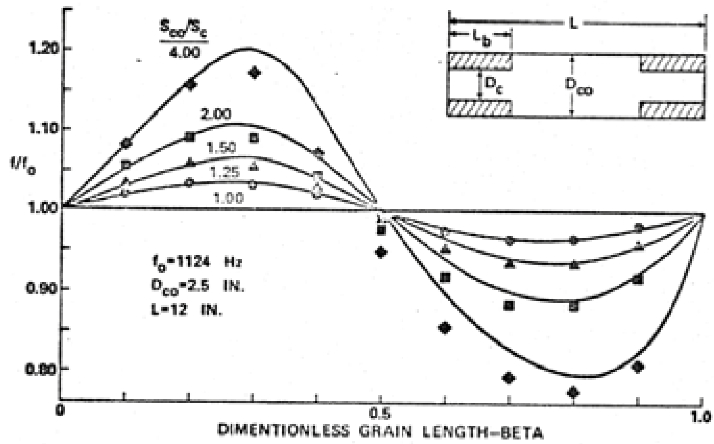
where $K_l = Lk_l/2$ and $\beta = 2L_b/L$.



Experimental and Analytical First Mode Frequencies for Protruding Grains.



Experimental and Analytical First Mode Frequencies for Recessed Grains.



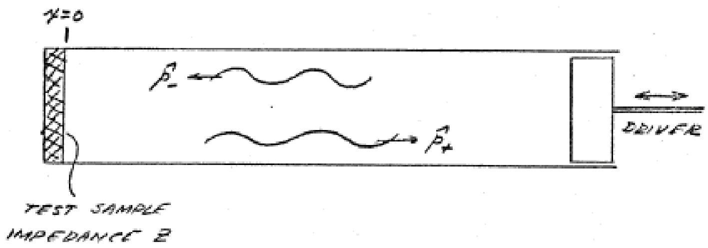
Experimental and Analytical Second Mode Frequencies for Protruding Grains.

2.2.2 The Impedance Tube.

The normal modes described above, and as usually defined, are for conservative systems. If

there are losses of energy -- as in any real system -- motions of constant amplitude exist only if they are driven; the power in must precisely compensate the power loss due to dissipation of energy. When a system is driven, a motion of constant amplitude can be sustained at any frequency. In the neighborhood of a natural frequency, the response of the system is relatively greater because of favorable interference of the traveling waves.

The impedance tube is an example of a forced system which has been widely used to obtain the impedance function for surfaces. As suggested by the sketch, the device is really quite simple.



A sample of the surface to be tested is placed at one end of the tube and a suitable driver is placed at the other. The amplitude of the leftward traveling wave is less than that of the rightward traveling wave because energy is absorbed by the sample. A standing wave system is sustained within the tube because the power input by the piston precisely compensates the energy losses in the tube.

If energy losses along the tube are ignored the analysis is quite easy. The rightward and leftward traveling waves are represented by

$$
\begin{aligned}
\hat{p}_+ &= Be^{ikx} \\
\hat{p}_- &= Be^{-ikx} \\
\hat{u}_+ &= \frac{B}{\rho_0 a_0} e^{ikx} \\
\hat{u}_- &= -\frac{B}{\rho_0 a_0} e^{-ikx}
\end{aligned}
\tag{2.27}
$$

Because there is a change of both amplitude and phase at $z = 0$, the constant B can be written

$$B = -Ae^{2\psi} \tag{2.28}$$

where ψ is in general a complex number

$$\psi = \pi\alpha_0 + i\pi\beta_0 \tag{2.29}$$

With these definitions, the total pressure and velocity at any point along the tube are

$$\hat{p} = A\left[e^{-ikx} - e^{ikx+2\psi}\right] \tag{2.30a}$$

$$\hat{u} = -\frac{A}{\rho_0 a_0}\left[e^{-ikx} + e^{ikx+2\psi}\right] \tag{2.30b}$$

The acoustic impedance for a surface has been defined by (1.109) as $z = p'/\vec{u}' \cdot \hat{n}$. Here, at $x = 0$, $\vec{u}' \cdot \hat{n} = -\hat{u}(0)$, so to satisfy the boundary condition at the test sample:

$$z = \left[\frac{\hat{p}}{-\hat{u}}\right]_{x=0} = \rho_0 a_0 \frac{1-e^{2\psi}}{1+e^{2\psi}} \tag{2.31}$$

This can be solved to give

$$e^{2\psi} = \frac{1 - z/\rho_0 a_0}{1 + z/\rho_0 a_0} \tag{2.32}$$

from which the real and imaginary parts of ψ can be related to the real and imaginary parts of the surface impedance z .

Now the waveform along the tube obviously depends very much on ψ. One might expect that complete knowledge of the waveform -- by measurement of the pressure field, for example -- would provide values for ψ and hence z . That is true, but what makes the impedance tube useful is that only a few measurements are required., namely the position and relative magnitude of the maxima and minima of pressure along the tube. This can be shown quite easily by first re-writing (2.30a) in the form

$$\hat{p}(x) = -Ae^{\psi}\left[e^{ikx+\psi} - e^{-ikx-\psi}\right] = -2Ae\psi \sinh(\psi + ikx) \tag{2.33}$$

Similarly, the velocity is

$$\hat{u}(x) = -\frac{A}{\rho_0 a_0} e^{\psi}\left[e^{ikx+\psi} + e^{-ikx-\psi}\right] = -\frac{2A}{\rho_0 a_0} e^{\psi} \cosh(\psi + ikx) \tag{2.34}$$

The experimental technique rests on the behavior of (2.33) as a function of x, i.e., the variation of the pressure amplitude as a function of position from the sample. Define

$$\psi + ikx = \pi(\alpha + i\beta) \tag{2.35}$$

so

$$\begin{aligned} \alpha &= \alpha_0 \\ \beta &= \beta_0 + \frac{2x}{\lambda} \end{aligned} \tag{2.36}$$

and

$$\hat{p} = 2A\exp(\psi)\sinh \pi(\alpha + i\beta)$$

The magnitude of the pressure amplitude is

$$|\hat{p}| = 2A\left|e^{\psi}\right|\left|\sinh \pi(\alpha + i\beta)\right| = 2Ae^{\pi\alpha}\sqrt{\cosh^2 \pi\alpha - \cos^2 \pi\beta} \tag{2.37}$$

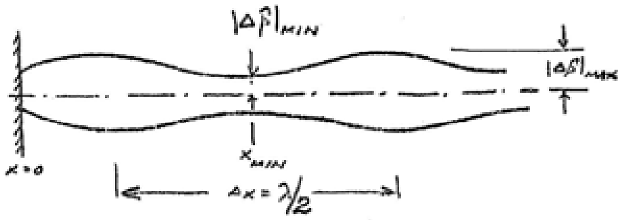
Note that α is independent of x: this would not be true if losses along the tube were accounted for, so $|\hat{p}|$ varies only because β is a function of x .

Now $\cos^2 \pi\beta$ vanishes where $\beta = \pm 1/2, \pm 3/2, \ldots$ and has the value 1 for $\beta = \pm 1, \pm 2, \pm 3, \ldots$

The envelope of the pressure has, therefore, the following characteristics:

$$|\hat{p}|_{\max} = 2Ae^{\pi\alpha} \cosh \pi\alpha \qquad \left(\beta_0 + 2\frac{x}{\lambda} = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots\right)$$

$$|\hat{p}|_{\min} = 2Ae^{\pi\alpha} \sqrt{\cosh^2 \pi\alpha - 1} \qquad \left(\beta_0 + 2\frac{x}{\lambda} = \pm 1, \pm 2, \ldots\right) \tag{2.38}$$



The ratio of maxima and minima is

$$\frac{|\hat{p}|_{\max}}{|\hat{p}|_{\min}} = \frac{\cosh \pi\alpha}{\sqrt{\cosh^2 \pi\alpha - 1}} = \coth \pi\alpha \tag{2.39}$$

from which $\alpha = \alpha_0$ can be determined.

The value of β at the sample is β_0, and the value of $|\hat{p}|$ there is

$$|\hat{p}|_0 = 2Ae^{\pi\alpha} \sqrt{\cosh^2 \pi\alpha - \cosh^2 \pi\beta_0}$$

The first minimum away from the sample occurs as

$$\beta + \frac{2}{\lambda} x_{\min} = 1$$

so

$$\beta_0 = 1 - \frac{2}{\lambda} x_{\min} \tag{2.40}$$

Thus, both α_0 and β_0 can be determined. In practice, more complicated procedures are generally used, particularly to take account of energy losses at the walls (See Morse, “Vibration and Sound”, pp.240ff).

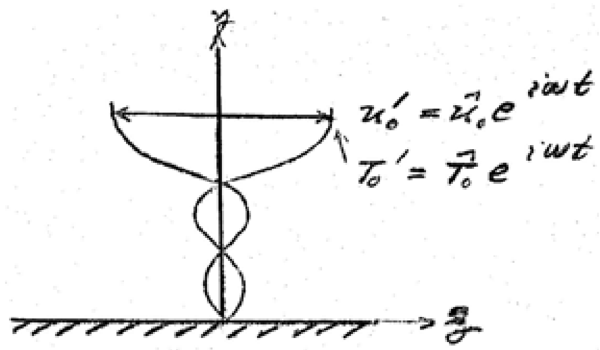
Impedance tubes have also been used to measure acoustic admittances with mean flow.

2.3 Viscous Losses at a Surface and Their Influence on Standing Waves in a Tube

In Section 1.3, it was shown that heat conduction and viscous stresses have very little influence on the motion of a plane wave. The computation was directed to the interior of the acoustic field and the same result is not valid for acoustic waves in the vicinity of a boundary. For simplicity, a rigid flat surface will be treated. If there is a fluctuating velocity parallel to the wall, an acoustic boundary layer develops. If the temperature fluctuates, there is as well a thermal boundary layer. First, the boundary layer calculations will be done, including computation of the average energy loss per unit area of surface. Then the way in which the result may be incorporated in analysis of the behavior of standing waves in a tube will be examined in two different ways.

2. 3. 1 The Acoustic Boundary Layer.

Consider two-dimensional flow in the vicinity of a flat surface, with both the velocity and temperature, far from the surface, varying sinusoidally in time. Because the flow reverses direction periodically, a boundary layer does not develop along the surface as in steady flow. Locally one may treat a one-dimensional flow parallel to the surface, with variations in the direction normal to the surface.



The equations for this boundary layer flow, linearized and for constant density, are

$$\rho_0\left(\frac{\partial v'}{\partial y}+\frac{\partial u'}{\partial z}\right)=0 \tag{2.41}$$

$$\rho_0\frac{\partial u'}{\partial t}=\mu\frac{\partial^2 u'}{\partial y^2}+\rho_0\frac{\partial u_0'}{\partial t} \tag{2.42}$$

$$\rho_0 C_p\frac{\partial T'}{\partial t}=\lambda\frac{\partial^2 T'}{\partial y^2}+\frac{\partial p'}{\partial t} \tag{2.43}$$

The momentum equation (2.42) may be deduced by considering first the flow near an oscillating plate, so $u'\to 0$ for $y\to\infty$ and $u'=u_0'$ at the plate. Then bring the plate to rest by superposing the velocity $-u_0'$ on the whole field, so $u'\to u'-u_0'$. This produces (2.42).

Equation (2.43) follows directly from the energy equation written for the enthalpy.

It is clear from the form of equation (2.42) and (2.43) that $\hat{u}_0$ and $\hat{T}_0$ can be taken as function of z, without affecting the form of the solutions. The boundary conditions are

$$\left.\begin{array}{l} u' = 0 \\ T' = 0 \end{array}\right\} \qquad y = 0 \tag{2.44a}$$

$$\left.\begin{array}{l} u' \to u_0' = \hat{u}_0 e^{i\omega t} \\ T' \to T_0' = \hat{T}_0 e^{i\omega t} \end{array}\right\} y \to \infty \tag{2.44b}$$

It is assumed that the wall temperature remains constant, so $T' = 0$ as shown.

Because for $y \to \infty$, $\hat{p} = (\gamma/\gamma - 1)p_0$, $\hat{T}_0 / T_0 = \rho_0 C_p \hat{T}_0$, the last term in (2.43) is $\partial\hat{p}/\partial t = \rho_0 C_p \, \partial\hat{T}_0/\partial t$. Thus, appropriate solutions to (2.42) and (2.43) have the form

$$u' = \hat{u}_0 e^{i\omega t} + \hat{u} e^{i\omega t}$$
$$T' = T_0' e^{i\omega t} + \hat{T} e^{i\omega t}$$

and the equations for $\hat{u}$ and $\hat{T}$ are

$$\frac{\partial^2 \hat{u}}{\partial y^2} - 2\frac{i}{\delta^2}\hat{u} = 0 \tag{2.45}$$

$$\frac{\partial^2 \hat{T}}{\partial y^2} - 2\frac{i}{\delta^2}\Pr \hat{T} = 0 \tag{2.46}$$

where

$$\delta = \sqrt{2\nu/\omega} \tag{2.47}$$

and $\Pr = C_v \mu / \lambda_c$ is the Prandtl number. Thus, $\hat{u}$ and $\hat{T}$ have the forms

$$\hat{u} = Ae^{-\lambda y}$$
$$\hat{T} = Be^{-\lambda\sqrt{\Pr}y}$$

with

$$\lambda = \frac{\sqrt{2}}{\delta}\sqrt{i} = \frac{1}{\delta}(1+i) \tag{2.48}$$

The boundary conditions (2.44a) and (2.44b) are satisfied by taking $A = \hat{u}_0$, $B = \hat{T}_0$, so the velocity and temperature within the acoustic boundary layer are given by

$$\hat{u} = \hat{u}_0\left[1 - e^{-\lambda y}\right] \tag{2.49}$$

$$\hat{T} = \hat{T}_0\left[1 - e^{-\lambda\sqrt{\Pr}y}\right] \tag{2.50}$$

One way to calculate the losses associated with the acoustic boundary layer is to find the time-averaged total energy dissipation, defined as

$$\left\langle \int_0^\infty \left[\mu\left(\frac{d\hat{u}}{dy}\right)^2 + \frac{k}{T_0}\left(\frac{d\hat{T}}{dy}\right)^2 \right] dy \right\rangle \tag{2.51}$$

To evaluate the integral, either the real or imaginary parts of (2.49) and (2.50) must be used. The integrals are easy and are left as part of a homework problem. The result, "time-averaged energy loss per unit area", is

$$\frac{1}{2\gamma p_0}\sqrt{\frac{\omega\nu}{2}}\left[\gamma p_0 \rho_0 \hat{u}_0^2 + \frac{\gamma - 1}{\sqrt{\Pr}}\hat{p}^2\right] \tag{2.52}$$

The influence of surface losses on a standing wave can be found by finding the total energy loss for the system and dividing by the total time-averaged energy in the chamber. As shown in the next section, this ratio equals twice the decay constant, α.

2. 3. 2 The Relation Between the Decay Constant and the Time-Averaged Loss of Acoustic Energy.

As shown in Section 1.4, the energy density for an acoustic field is

$$E = \frac{1}{2}\rho_0 u'^2 + \frac{1}{2}\frac{p'^2}{\rho_0 a_0^2} \tag{2.53}$$

and the time-averaged value for steady harmonic waves is

$$\langle E \rangle = \frac{1}{4}\rho_0 \hat{u}^2 + \frac{1}{4}\frac{\hat{p}^2}{\rho_0 a_0^2} \tag{2.54}$$

Now suppose that the waves are varying in time according to

$$\begin{aligned} p' &= \hat{p}e^{-\alpha t}e^{i\omega t} \\ u' &= \hat{u}e^{-\alpha t}e^{i\omega t} \end{aligned} \tag{2.55}$$

The exponential decay will always be found for freely decaying linear waves.

For computing the time average of (2.53), the real or imaginary parts of (2.55) must be used. Thus, for example,

$$E = \left[\frac{1}{2}\rho_0 \hat{u}^2 + \frac{1}{2}\frac{\hat{p}^2}{\rho_0 a_0^2} \right] e^{-2\alpha t} \cos\omega t \tag{2.56}$$

If the amplitude does not change much from cycle to cycle, the time average of (2.56) can be taken over one cycle with $\exp(-\alpha t)$ approximately constant. Hence, instead of (2.54), one has

$$\langle E \rangle \approx e^{-2\alpha t} \left[\frac{1}{4}\rho_0 \hat{u}^2 + \frac{1}{4}\frac{\hat{p}^2}{\rho_0 a_0^2} \right]$$

and

$$\frac{d}{dt}\langle E \rangle \approx -2\alpha \langle E \rangle \tag{2.57}$$

Therefore, the decay constant is approximately given by

$$2\alpha \approx -\frac{\langle \dot{E} \rangle}{\langle E \rangle} \tag{2.58}$$

Consider the case of standing waves in a closed tube, so except in the acoustic boundary layer,

$$\hat{p} = A\cos kz$$

$$\hat{u} = \frac{A}{\rho_0 a_0}\sin kz$$

The total time-averaged energy in a tube of length L and radius R_c is:

$$\langle E \rangle = \frac{A^2}{4}\int_0^L \left[\frac{\sin^2 kz}{\rho_0 a_0^2} + \frac{\cos^2 kz}{\rho_0 a_0^2} \right] \pi R_c^2 dz = \frac{A^2}{4\rho_0 a_0^2}\left(\pi R_c^2 L\right) \tag{2.59}$$

With (2.52), the total time-averaged rate of loss of energy is

$$\begin{aligned} \langle \dot{E} \rangle &= \frac{1}{2\gamma p_0}\sqrt{\frac{\omega\nu}{2}}\left[1 + \frac{\gamma - 1}{\sqrt{\Pr}}\right] A^2 \int_0^L \cos^2 kz \, 2\pi R_c dz \\ &= \frac{1}{2\gamma p_0}\sqrt{\frac{\omega\nu}{2}}\left[1 + \frac{\gamma - 1}{\sqrt{\Pr}}\right] A^2 (\pi R_c L) \end{aligned} \tag{2.60}$$

Therefore, (2.58) gives

$$-2\alpha = -\frac{2\rho_0 a_0^2}{\gamma p_0}\sqrt{\frac{\omega\nu}{2}}\left[1+\frac{\gamma-1}{\sqrt{\Pr}}\right]\frac{\pi R_c L}{\pi R_c^2 L}A^2\int_0^L \cos^2 kz 2\pi R_c dz$$

or

$$\alpha = -\frac{2}{D}\sqrt{\frac{\omega\nu}{2}}\left[1+\frac{\gamma-1}{\sqrt{\Pr}}\right] \qquad (2.61)$$

where $D = 2R_c$ is the diameter.

A number of measurements have been made of the decay of waves in a tube. It is easy (i.e., not much trouble need be taken) to verify that α is proportional to $\sqrt{\omega}/D$, but the values are inevitably high. The best results give values about 10 per cent greater than those predicted by (2.61). [Henderson and Donnelly, J. Acoust. Soc. of Amer., Vol. 34, No. 6 (1962).]

2.4 Another Way of Computing the Decay Constant; the Dollar Rule

The problem treated here is a fairly simple example which can be used to illustrate a general and powerful means of determining the consequences of various perturbations to a problem of inviscid classical acoustics. Suppose that viscous stresses and heat transfer are included in the equations of motion, which then become, in linearized form,

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{u}' = 0 \tag{2.62}$$

$$\rho_0 \frac{\partial \vec{u}'}{\partial t} + \nabla p' = \vec{F}' \tag{2.63}$$

$$\rho_0 \frac{\partial T'}{\partial t} + \frac{p_0}{C_v} \nabla \cdot \vec{u}' = \frac{1}{C_v} \nabla \cdot \vec{q}' \tag{2.64}$$

For viscous stresses, $\vec{F}'$ stands for the divergence of the stress tensor. The equation for the pressure is formed by adding T_0 times (2.62) to (2.64) and noting that from the perfect gas law, $p' = R(\rho_0 T' + T_0 \rho')$:

$$\frac{\partial p'}{\partial t} + \gamma p_0 \nabla \cdot \vec{u}' = \frac{R}{C_v} \nabla \cdot \vec{q}' \tag{2.65}$$

Now combine this equation with the momentum equation (2.63) to find the wave equation:

$$\frac{\partial^2 p'}{\partial t^2} - a_0^2 \nabla^2 p' = \frac{R}{C_v} \frac{\partial}{\partial t}\left(\nabla \cdot \vec{q}'\right) - a_0^2 \nabla^2 \vec{F}' \tag{2.66}$$

with the boundary condition:

$$\hat{n} \cdot \nabla p' = -\rho_0 \frac{\partial \vec{u}'}{\partial t} \cdot \hat{n} + \hat{n} \cdot \vec{F}' \tag{2.67}$$

For harmonic motions,

$$p' = \hat{p} e^{i a_0 k t} \tag{2.68}$$

and the last two equations become

$$\nabla^2 \hat{p} + k^2 \hat{p} = \hat{h} \tag{2.69a}$$

$$\hat{n} \cdot \nabla \hat{p} = -\hat{f} \tag{2.69b}$$

where

$$\hat{h} = \nabla \cdot \dot{\vec{F}} - \frac{i\omega}{a_0^2} \frac{R}{C_v} \nabla \cdot \vec{q}' \tag{2.70}$$

$$\hat{f} = i\omega\rho_0\vec{u}' \cdot \hat{n} - \hat{n} \cdot \hat{\vec{F}} \tag{2.71}$$

Now the wave number introduced in (2.68) must be regarded as a complex quantity,

$$k = \frac{1}{a_0}(\omega - i\alpha) \tag{2.72}$$

and the problem is essentially to find k.

The functions $\hat{h}$ and $\hat{f}$ represent perturbations (here due to viscous effects only) from the classical problem for normal modes defined by

$$\nabla^2 \hat{p}_n + k_n^2 \hat{p}_n = 0 \tag{2.73}$$

$$\hat{n} \cdot \nabla \hat{p}_n = 0 \tag{2.74}$$

The perturbations cause k to be different from k_n and $\hat{p}$ to be different from $\hat{p}_n$. Thus, the actual problem $(\hat{p}, \hat{k})$ can be considered as one of harmonic motion perturbed from one of the normal modes $(\hat{p}_n, \hat{k}_n)$. The fact that the two cases are only slightly different can be used as the basis for a simple iterative solution.

Multiply (2.69a) by $\hat{p}_n$, (2.73) by $\hat{p}$, subtract the two equations and integrate over the entire closed volume:

$$\int \left[\hat{p}_n \nabla^2 \hat{p} - \hat{p} \nabla^2 \hat{p}_n\right] dV + \left(k^2 - k_n^2\right) \int \hat{p}\hat{p}_n dV = \int \hat{h}\hat{p}_n dV \tag{2.75}$$

Use Green's theorem and the boundary conditions (2.69b) and (2.74) to rewrite the first integral as

$$\int \left[\hat{p}_n \nabla^2 \hat{p} - \hat{p} \nabla^2 \hat{p}_n\right] dV = \oiint \left[\hat{p}_n \nabla^2 \hat{p} - \hat{p} \nabla^2 \hat{p}_n\right] \cdot \hat{n} dS = -\oiint \hat{f}\hat{p}_n dS$$

Thus, (2.75) becomes a formula for k^2:

$$k^2 = k_n^2 + \frac{1}{\int \hat{p}\hat{p}_n dV} \left\{ \int \hat{h}\hat{p}_n dV + \oiint \hat{f}\hat{p}_n dS \right\}$$

Now because $\hat{p}$ does not differ much from $\hat{p}_n$, set $\hat{p} \approx \hat{p}_n$ in the denominator, and later in the numerator as well, so

$$k^2 = k_n^2 + \frac{1}{E_n^2}\left\{\int \hat{h}\hat{p}_n dV + \oiint \hat{f}\hat{p}_n dS\right\} \tag{2.76}$$

with.

$$E_n^2 = \int \hat{p}_n^2 dV \tag{2.77}$$

Equation (2.77) is a general result for the problem defined by (2.69a) and (2.69b). It is exceedingly useful and will be used many times in these notes. In fact, it is so useful for obtaining results for practical problems that it will be called “The Dollar Rule.” Later the “Time Dependent Dollar Rule” will be derived.

For the present, the special case of viscous losses in the acoustic boundary layer will be treated. The heat source in the energy equation and the force in the momentum equation then represent heat conduction and the viscous stress within the boundary layer, and are significantly non-zero only in thin regions adjacent to the wall. Let y denote the coordinate normal to the wall, measured positive inward, so the volume element in the integral over $\hat{h}$ in (2.77) is $dV = dydS$, and

$$\nabla \cdot \hat{\vec{q}} = \frac{\partial \hat{q}_y}{\partial y} = \lambda_c \frac{\partial^2 \hat{T}}{\partial y^2} \tag{2.78}$$

$$\hat{\vec{F}} = \nabla \cdot \hat{\vec{\tau}} = \frac{\partial \hat{\vec{\tau}}_y}{\partial y} = \mu \frac{\partial^2 \hat{\vec{u}}_p}{\partial y^2} \tag{2.79}$$

where $\hat{\vec{u}}_p$ is the velocity component parallel to the wall.

First combine the two terms arising in (2.77):

$$\int \nabla \cdot \hat{\vec{F}}\hat{p}_n dV - \oiint \hat{n} \cdot \hat{\vec{F}}\hat{p}_n dS = \int \nabla \cdot (\hat{\vec{F}}\hat{p}_n) dV - \int \hat{\vec{F}} \cdot \nabla \hat{p}_n dV - \oiint \hat{n} \cdot \hat{\vec{F}}\hat{p}_n dS$$
$$= -\int \hat{\vec{F}} \cdot \nabla \hat{p}_n dV$$

Then (2.77) is, with $dV = dydS$:

$$E_n^2(k^2 - k_n^2) = -\mu \int \frac{\partial^2 \hat{\vec{u}}_p}{\partial y^2} \cdot \nabla \hat{p}_n dydS - \frac{i\omega}{a_0^2}\frac{R}{C_v}\lambda_c \int \frac{\partial^2 \hat{T}}{\partial y^2}\hat{p}_n dydS$$

Now both $\hat{p}_n$ and $\nabla \hat{p}_n$ are essentially independent of y, and the integrals over y may be carried out; because both $\partial \hat{\vec{u}} / \partial y$ and $\partial \hat{T} / \partial y$ vanish for $y \to \infty$, one has

$$E_n^2(k^2-k_n^2) = -\mu\int dS\nabla\hat{p}_n\cdot\left[\frac{\partial\hat{\vec{u}}_p}{\partial y}\right]_0^\infty - \frac{i\omega}{a_0^2}\frac{R}{C_v}\lambda_c\oint\!\!\oint dS\,\hat{p}_n\left[\frac{\partial\hat{T}}{\partial y}\right]_0^\infty$$
$$= \mu\oint\!\!\oint\left(\frac{\partial\hat{\vec{u}}_p}{\partial y}\right)_w\cdot\nabla\hat{p}_n dS + \frac{i\omega}{a_0^2}\frac{R}{C_v}\lambda_c\oint\!\!\oint\left(\frac{\partial\hat{T}}{\partial y}\right)_w\hat{p}_n dS \tag{2.80}$$

The derivatives at the wall can be calculated from (2.49) and (2.50), $\hat{\vec{u}}_p$ being the velocity $\hat{u}$ in (2.49), directed in the z-direction for the standing wave treated above; the integrands become

$$\left(\frac{\partial\hat{\vec{u}}_p}{\partial y}\right)_w\cdot\nabla\hat{p}_n = \lambda\hat{u}_0\frac{\partial\hat{p}_n}{\partial z}$$

$$\left(\frac{\partial\hat{T}}{\partial y}\right)_w\hat{p}_n = \lambda\sqrt{\Pr}\hat{T}_0\hat{p}_n \tag{2.81}$$

Now $\hat{u}_0$ and $\hat{T}_0$ represent fluctuations associated with the unperturbed acoustic field $\hat{p}_n$ (this is really a good first approximation, but see the refinement computed at the end of this section), so according to the acoustic momentum equation, and the relation for isentropic motions,

$$\hat{u}_0 = \frac{i}{\rho_0\omega}\frac{\partial\hat{p}_n}{\partial z}$$
$$\hat{T}_0 = T_0\frac{\gamma-1}{\gamma}\frac{\hat{p}_n}{p_0} \tag{2.82}$$

Substitution of all these formulas, (2.81) and (2.82), into (2.80) gives

$$E_n^2(k^2-k_n^2) = \frac{i\lambda\mu}{\rho_0\omega}\oint\!\!\oint\left(\frac{\partial\hat{p}_n}{\partial x}\right)^2 dS + i\omega\lambda\left(\frac{\sqrt{\Pr}R}{a_0^2C_v}\right)\left(\frac{\gamma-1}{\gamma}\right)\frac{\lambda_c T_0}{p_0}\oint\!\!\oint\hat{p}_n^2 dS$$

For the longitudinal standing wave in a tube of length L, $\hat{p}_n = \cos(n\pi x/L)$ and with

$dS = 2\pi R_c dx$

$$E_n^2 = \int\hat{p}_n^2 dV = \pi R_c^2\int\cos^2\frac{n\pi z}{L}dz = \frac{\pi R_c^2}{2}$$

$$\oiint \left(\frac{\partial \hat{p}_n}{\partial x}\right)^2 dS = 2\pi R k_n^2 \int_0^L \sin^2 \frac{n\pi z}{L} dz = \pi R_c k_n^2 L$$

$$\oiint \hat{p}_n^2 dS = 2\pi R_c \int_0^L \cos^2 \frac{n\pi z}{L} dz = \pi R_c k_n L$$

Thus,

$$k^2 = k_n^2 + i\lambda\nu \frac{2}{R_c}\left(\frac{k_n}{\omega}\right) + i\omega\lambda\left(\frac{\sqrt{\mathrm{Pr}}}{a_0^2}\right)\left(\frac{R}{C_v}\right)(\gamma-1)\frac{\lambda_c}{\rho_0}\left(\frac{\rho_0}{\gamma p_0}\right)T_0 \frac{2}{R_c}$$

Make the following substitutions;

$$k_n = \frac{\omega}{a_0}$$

$$\frac{\rho_0}{\gamma p_0} = \frac{1}{a_0^2} = \frac{1}{\gamma R T_0}$$

$$\frac{\lambda_c}{\mu C_p} = \frac{1}{\mathrm{Pr}}$$

to give

$$k^2 = k_n^2 + i\lambda \frac{\omega\nu}{a_0^2}\frac{2}{R_c}\left[1 + \frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right] \tag{2.83}$$

From the definition (2.72) of k,

$$k^2 = \frac{1}{a_0^2}\left(\omega^2 - 2i\alpha\omega + \alpha^2\right) \approx \frac{\omega^2}{a_0^2} - i\left(\frac{2\alpha\omega}{a_0^2}\right)$$

because $\alpha << \omega$. The real and imaginary parts of (2.83) therefore give the formulas

$$\begin{aligned}\left(\frac{\omega}{a_0^2}\right)^2 &= \left(\frac{\omega_n}{a_0^2}\right)^2 - \frac{1}{\delta}\frac{\omega\nu}{a_0^2}\frac{2}{R_c}\left[1 + \frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right] \\ &= \left(\frac{\omega_n}{a_0^2}\right)^2 \left\{1 - \frac{2}{D}\sqrt{\frac{2\nu}{\omega}}\left[1 + \frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right]\right\}\end{aligned} \tag{2.84}$$

and

$$\begin{aligned}\alpha &= -\frac{a_0^2}{2\omega}\frac{1}{\delta}\frac{\omega\nu}{a_0^2}\frac{2}{R_c}\left[1 + \frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right] \\ &= -\frac{2}{D}\sqrt{\frac{\omega\nu}{2}}\left[1 + \frac{\gamma-1}{\sqrt{\mathrm{Pr}}}\right]\end{aligned} \tag{2.85}$$

Equation (2.85) is identical with (2.61), computed in an entirely different way.

In the next section, still another way of calculating the decay constant is given, based on the analytical technique leading to (2.77), but with a different view of the action of the acoustic boundary layer.

2.5 Still Another Way of Computing the Decay Constant for Viscous Losses at a Rigid Surface

THIS SECTION IS NOT COMPLETED.

It is an essential assumption, in both of the preceding calculations of the viscous losses, that the shear stress and heat conduction are confined to the thin acoustic boundary layers. Yet the energy losses presumably affect the entire acoustic field in the tube. If the device sustaining a standing wave in a tube is suddenly switched off, it is a simple experimental matter to verify that the amplitude of the acoustic pressure decays throughout the tube and not merely within the acoustic boundary layers. One might reasonably inquire -- how is the influence of the acoustic boundary layer communicated to the bulk of the field within the tube?

The answer is that the boundary layer, while thin, nevertheless exerts an influence extending out into the bulk of the flow -- the flow variables do not suddenly attain the free-stream values at the edge of the boundary layer. In particular, if the velocity $\hat{u}_0$, representing the free-stream velocity parallel to the surface, varies in the direction parallel to the surface, then so also does the velocity $\hat{u}$ within the boundary layer. But then, according to the continuity equation (2.41), there must be a non-zero fluctuation of the velocity v' in the direction normal to the surface, even at the edge of the boundary layer. So far as the analysis of the boundary layer is concerned, that fluctuation extends to infinity far from the wall. For the acoustic boundary layer, v' represents a periodic pumping of gas normal to the wall and therefore normal to the acoustic motions outside the boundary layer. Because of this pumping action – which is really mass addition to the bulk of the flow – a drag force is exerted on the main acoustic field. In this way, the presence of the boundary layer is communicated to the external flow. It is the purpose of this section to show how that picture of the flow forms the basis for a calculation which will produce exactly the same result for the decay constant.

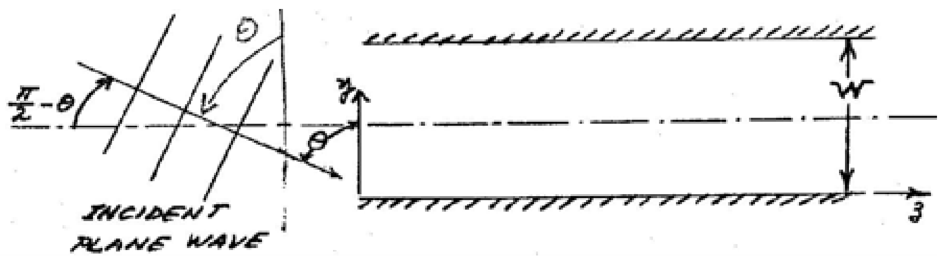
(to be continued)

2.6 Propagation of Higher Order (Two- or Three-Dimensional) Modes in Tubes; The Cut-off Frequency

2.6.1 Traveling Waves and Reflections in a Duct.

In the preceding sections the motions in a tube were assumed *ab initio* to be one-dimensional, except for the small correction considered in Section 2.5.1. Now one must expect, in any problem involving acoustic waves within finite boundaries, that if the wavelength (roughly estimated as the speed of sound divided by the frequency) becomes comparable with a particular dimension of the volume treated, then vibrations in that direction may occur. For the case of wave propagation in tubes, the motions treated cannot be restricted to the direction parallel to the axis of the tube if the frequency is (approximately) equal to or greater than the speed of sound divided by the largest lateral dimension of the tube. This Ii referred to as the "cut-off" frequency, f_c. For frequencies greater than f_c, modes other than the simple one-dimensional mode will propagate in the duct.

The simplest case is a two-dimensional channel, having width L, say, in the y-direction and extending infinitely far in the x-direction. One-dimensional waves of the sort already discussed are supposed to propagate in the z-direction.



For this case the critical or cut off frequency is exactly $a_0/2W$, a result to be demonstrated shortly. The wavelength of plane waves having this frequency is twice the width of the tube, which is the wavelength for the fundamental standing wave for motions normal to the axis. Although purely one-dimensional waves having frequencies higher than this critical value will propagate, it should be at least appealing that in the range $f > a_0/2L$ more interesting things may happen.

Suppose, as sketched above, that a plane wave is incident, at an angle $\pi/2-\theta$ to the axis, upon the entrance to such a tube: what gets through the tube? Before examining the solutions to the two-dimensional wave equation, consider again the reflection of plane waves from a surface (because this must be related to what is happening inside the tube). Assume that the surface is perfectly rigid, so $\vec{u}' \cdot \hat{n} = 0$; then the surface impedance is infinitely large, and according to (1.117),

the reflection coefficient $\beta = 1$. Thus, from (1.115), $g_i = g_r$, and the velocity component in the direction normal to the wall is

$$
\begin{aligned}
u_y &= \frac{\cos\theta}{\rho_0 a_0}\left[-g_i(\xi_i) + g_r(\xi_r)\right] \\
&= -\frac{\cos\theta}{\rho_0 a_0}\left[g\{k(z\sin\theta - y\cos\theta) - \omega t\} - g\{k(z\sin\theta + y\cos\theta) - \omega t\}\right]
\end{aligned}
\qquad (2.124)
$$

where x in (1.121) is replaced by z here, to agree with the sketch given above. For the case of sinusoidal waves, $g(\xi) = A\exp(i\xi)$ and (2.124) is

$$
\begin{aligned}
u_y &= -A\frac{\cos\theta}{\rho_0 a_0} e^{i(kz\sin\theta - \omega t)}\left[e^{-iky\cos\theta} - e^{iky\cos\theta}\right] \\
&= 2iA\frac{\cos\theta}{\rho_0 a_0}\sin(ky\cos\theta) e^{i(kz\sin\theta - \omega t)}
\end{aligned}
\qquad (2.125)
$$

The term $\exp i(kz\sin\theta - \omega t)$ represents a disturbance traveling in the z-direction, having wave number $k\sin\theta$. But it has the variation in the y-direction, $\sin(ky\cos\theta)$, which does not change with time. Thus, superposition of the incident and reflected waves produces a standing wave pattern in the direction normal to the surface. In particular, the velocity in the y-direction vanishes, for all z, on all planes y such that

$$\sin(ky\cos\theta) = 0$$

Consider a special value $y = W$; the velocity normal to that plane vanishes if the frequency $\omega = a_0 k$ has values given by

$$\sin\left(\frac{\omega W}{a_0}\cos\theta\right) = 0 \qquad (2.126)$$

If this condition is satisfied, the plane $y = W$ can be taken to represent a rigid reflecting surface, the other side of the duct. For $\theta = \pi/2$, the wave propagates parallel to the wall, the velocity is everywhere parallel to the wall, and (2. 126) is satisfied for all W and ω. This corresponds to the case of one-dimensional wave propagation in a duct.

But if $\theta < \pi/2$, only particular combinations of ω and W will satisfy (2.126). This condition defines what are called "higher order modes" for propagation in the tube or channel. To see what is meant by the "cut-off frequency," suppose that W is fixed, and consider the relation between $\theta < \pi/2$ and ω demanded by (2.126):

$$\frac{\omega W}{a_0}\cos\theta = l\pi$$

or

$$\cos\theta = l\pi\frac{a_0}{\omega W} = l\frac{\lambda}{2W} \tag{2.127}$$

where X is the wavelength of the incident wave. Note that θ must lie in the range $0 \le \theta \le \pi/2$ because by supposition here, the wave is incident, from the left, on the entrance to the channel. The values of l define different modes of propagation. For $l = 0$, $\theta = \pi/2$ and the motions are purely one-dimensional; the wavenumber in the z-direction is $k\sin\theta$, $k = \omega/a_0$, where, as always, ω is the frequency of the incident wave.

The lowest propagating mode having motions in the y-direction occurs for $l = 1$, and

$$\cos\theta = \pi\frac{a_0}{\omega W} \le 1 \tag{2.128}$$

The frequency must therefore satisfy the condition $\omega \ge \pi a_0/W$ and the critical or limiting value is

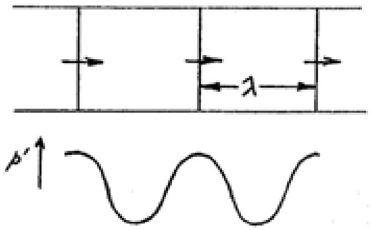
$$\omega_c = \pi\frac{a_0}{W} \tag{2.129}$$

which implies a wavelength

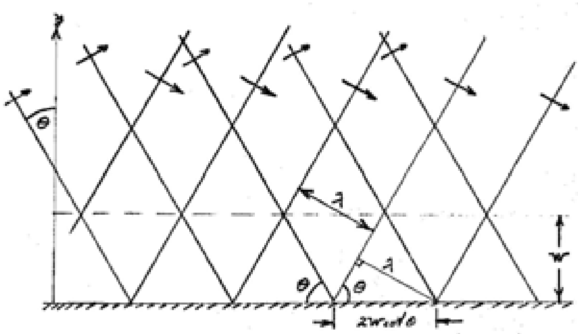
$$\lambda_c = 2\pi\frac{a_0}{\omega_c} = 2W \tag{2.130}$$

For any frequency less than that given by (2.129), θ is not real (because $\cos\theta > 1$), which means that there is no solution -- a wave having this frequency and the form prescribed by (2.125) cannot exist if the velocity must vanish on $y = W$. Equation (2.130) shows that just at the critical frequency the wavelength is that for the fundamental standing wave between the two surfaces $y = 0$ and $y = W$.

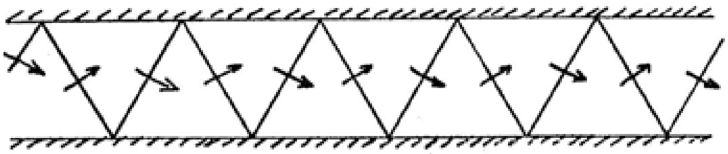
What the preceding discussion demonstrates is that for a plane wave reflecting from a rigid surface, there are planes, parallel to the reflecting surface, on which the normal velocity vanishes. One may imagine that any such plane is replaced by a second rigid surface, thereby forming a channel, within which a wave is traveling in the z-direction. Reference to the following sequence of sketches may help. The solid lines represent wave fronts -- e.g. planes where the pressure fluctuation is maximum. In the second sketch, the first intersections of the incident and reflected wave fronts are shown a distance W from the reflecting surface. At these points, the pressure is a maximum and the velocity vanishes. By simple geometry for the figure, one finds $\lambda = 2W\cos\theta$, the result given by (2.127) for $l = 1$. If one then imagines that a rigid surface is placed at $y = W$, the motion between the planes appears as a plane wave propagating with successive reflections from the two surfaces.



One- dimensional Wave in a Tube

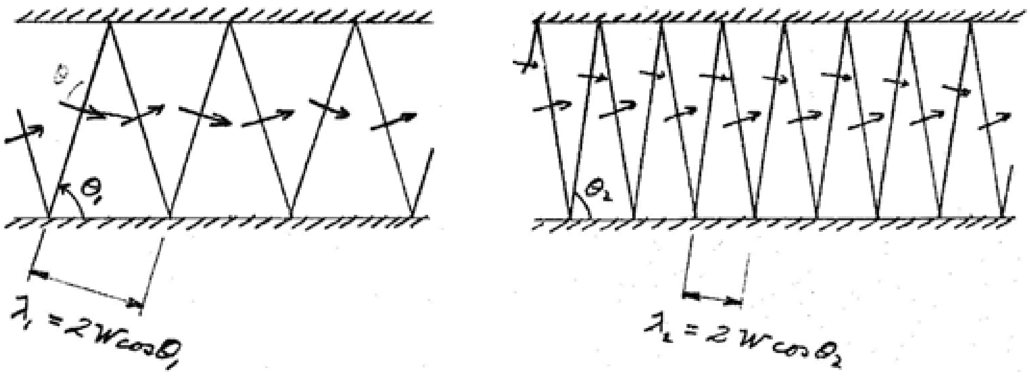


Plane Wave Reflected from a Rigid Surface



Propagation in a Tube, with Internal Reflections

Now imagine that the planes are kept a fixed distance apart, but θ is increased; it is evident that at $\theta = 0$ ($\cos\theta = 1$), the plane waves reflect to-and-fro between the planes, always propagating in the y-direction. For this motion, the correct boundary condition can be satisfied simultaneously at the two surfaces only for one wavelength, namely $\lambda = 2W$. If the wavelength is decreased, the boundary conditions on $y = 0$, W are still satisfied if θ is increased, so the condition $\lambda = 2W\cos\theta$ is satisfied; see Eq. (2.127) and the second of the sketches above. The situations for two values of the wavelength look like:



$\lambda_2 < \lambda_1$

All waves having $\lambda < \lambda_c$ will propagate, becoming closer to the one-dimensional limit as $\lambda \to 0$.

On the other hand, if W is fixed, waves with $\lambda > \lambda_c$ will not propagate in the first mode.

If a wave having $\lambda > \lambda_c$ is incident at some angle $\theta \neq \pi/2$, on the entrance to a tube, the motions penetrate for some distance into the tube, but the reflections from the surfaces cause destructive interference. The amplitude decays with distance into the tube, and the wave will be totally reflected.

2.6.2 Higher Order Modes as Solutions to the Wave Equation in Three Dimensions.

In Cartesian coordinates, the wave equation is

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2} + \frac{\partial^2 \hat{p}}{\partial z^2} + k^2 \hat{p} = 0 \qquad (2.131)$$

The frequency ω, and hence the wavenumber k, are fixed quantities. Now look for a solution having the form of a wave traveling through a rectangular duct, in the z-direction:

$$\hat{p} = P(x,y)e^{i(k_z z - \omega t)} \qquad (2.132)$$

for which P must satisfy the boundary conditions

$$\frac{\partial P}{\partial x} = 0 \qquad y = 0, W \qquad (2.133a)$$

$$\frac{\partial P}{\partial y} = 0 \qquad x = 0, V \qquad (2.133b)$$

It is easy to verify that the problem is solved by the form

$$\hat{p} = A\cos(k_x x)\cos(k_y y)e^{i(k_z z - \omega t)} \tag{2.134}$$

with

$$k_x = l_x \frac{\pi}{V} \qquad l_x = 0,1,2,3,... \tag{2.135a}$$

$$k_y = l_y \frac{\pi}{W} \qquad l_y = 0,1,2,3,... \tag{2.135b}$$

and

$$k_z^2 = k^2 - (k_x^2 + k_y^2) \tag{2.136}$$

First, for the particular case of the two-dimensional duct treated above, the wavenumber for the motions m the z direction is

$$k_z = \sqrt{\left(\frac{\omega}{a_0}\right)^2 - l^2\left(\frac{\pi}{W}\right)^2} \tag{2.137}$$

Only if k_z is real will the solution (2.134) represent a traveling wave for all $z > 0$; this condition, with (2.137), once again produces a lower limit, or cut-off frequency:

$$\omega_c = l\frac{\pi a_0}{W} \tag{2.138}$$

which is exactly (2.127) written for $\cos\theta = 1$. The discussion in Section 2.6.1 showed that for $\cos\theta = 1$, or $\theta = 0$, the mode is a standing wave with motions normal to the axis of the duct.

For $\omega > \omega_c$, (2.137) can be written

$$k_z = k\sqrt{1 - \left(\frac{l\pi a_0}{\omega W}\right)^2} = k\sin\theta \tag{2.139}$$

which defines $\cos\theta = (l\pi a_0/\omega W)$, Eq. (2.127) again. In a rectangular duct, the critical frequency is set by (2.136) with (2.135a, b).

$$k_z = \sqrt{\left(\frac{\omega}{a_0}\right)^2 - \left[\left(l_x\frac{\pi}{V}\right)^2 + \left(l_y\frac{\pi}{W}\right)^2\right]}$$

and

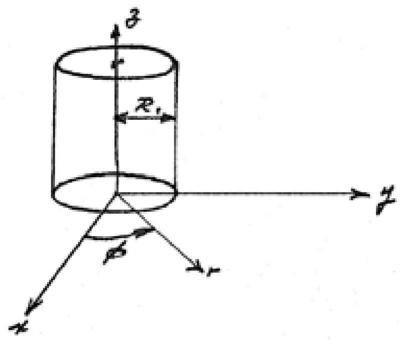
$$\omega_c = a_0\sqrt{\left(l_x\frac{\pi}{V}\right)^2 + \left(l_y\frac{\pi}{W}\right)^2} \tag{2.140}$$

It should be noted that the phase speed for propagation of waves along the tube is a function of frequency. Thus, the presence of the lateral boundaries introduces dispersion, even though the medium itself is not dispersive.

Finally, it must be emphasized that the failure of higher modes to propagate at frequencies less than cut-off, and the associated decay of amplitude along the axis, has nothing to do with viscous effects or any other dissipative processes. It is entirely a consequence of destructive interference due to reflections from the lateral boundaries.

2.6.3 Cut-off Frequency for Cylindrical Tubes.

It is convenient in practice to work with round uniform tubes. The appropriate coordinates to describe motions with such tubes are circular cylindrical, (r,ϕ,z):



For steady traveling waves -- traveling or standing -- the wave equation for the pressure is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{p}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \hat{p}}{d\phi^2}+\frac{\partial^2 \hat{p}}{dz^2}+k^2\hat{p}=0 \qquad (2.141)$$

where

$$p'(r,\phi,z;t)=\hat{p}(r,\phi,z)e^{-i\omega t} \qquad (2.142)$$

The question to be examined is again: under what conditions will a wave propagate along the tube? Then $\hat{p}$ must be proportional to a function of r and ϕ times $\exp(ik_z z)$ where $k_z = 2\pi/\lambda_z$ represents the wavenumber in the z-direction. Let $\hat{p}=F(r,\phi)\exp(ik_z z)$, and substitution into (2.141) gives the equation for F:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F}{d\phi^2}+\left(k^2-k_z^2\right)F=0 \qquad (2.143)$$

This equation is separable; set

$$F(r,\phi)=R(r)\Phi(\phi)$$

and (2.143) can be put in the form

$$\frac{r}{R}\left(rR'\right)+r^2\left(k^2-k_z^2\right)=-\frac{\Phi''}{\Phi}=m^2$$

so that R and Φ satisfy the equations

$$\Phi'' + m^2\Phi = 0 \tag{2.144}$$

$$\frac{1}{r}\left(rR'\right)' + \left(k^2 - k_z^2 - \frac{m^2}{r^2}\right)R = 0 \tag{2.145}$$

The solutions to (2. 144) are

$$\Phi = \begin{cases} \sin m\phi \\ \cos m\phi \end{cases} \qquad m = 1,2,3,... \tag{2.146}$$

where m must take on integral values in order that the pressure be single valued in a cross section; that is if the azimuthal angle is changed by a multiple of 2π the value of pressure must be unchanged.

The solutions to (2.145) are the Bessel functions,

$$R(r) = J_m(\alpha_{mn} r) \tag{2.147}$$

with

$$\alpha_{mn}^2 = k^2 - k_z^2 \tag{2.148}$$

Now the gradient of the pressure in the radial direction is

$$\frac{\partial \hat{p}}{\partial r} = \frac{dR}{dr}\Phi e^{ik_z z}$$

This must vanish in order that the component of velocity normal to the tube wall should vanish, so

$$\left[\frac{dJ_m(\alpha_{mn} r)}{dr}\right]_{r=R_c} = 0 \tag{2.149}$$

For each value of m, this equation determines the values of α_{mn} $(n = 1,2,3,...)$.

From (2.148), the wavenumber for the oscillations in the z-direction is

$$k_z = \sqrt{\frac{\omega^2}{a_0^2} - \alpha_{mn}^2} \tag{2.150}$$

If a wave is to have the form $\exp\{i(k_z z - \omega t)\}$ and represent a traveling wave, k_z must be real. Thus, for a given value of α_{mn}, associated with a particular radial/azimuthal mode shape, the condition that k_z be real sets a lower limit to the frequency,

$$\omega > a_0 \alpha_{mn} = \omega_c \tag{2.151}$$

This is the cut-off frequency for propagation of the (m,n) mode. Waves having frequencies less than this value will not propagate. Their amplitudes decay down the tube according to $\exp(-|k_z|z)$.

If a wave having frequency less than ω_c is introduced in the tube, it will be reflected. In a steady-state condition, the amplitude of the reflected wave equals that of the incident wave and there is a stationary exponential decrease of the amplitude down the tube away from the entrance. It should be noted that Eq. (2.149) really determines values of $\alpha_{mn}R$, so that the values of α_{mn} are inversely proportional to the radius of the tube. The cut-off frequencies are therefore lower for larger diameter tubes. A few of the values of $\alpha_{mn}R$ are given in the table below.

m \ n	0	1	2	3	4
0	0	1.2197	2.2331	3.2383	4.2411
1	0.5861	1.6970	2.7140	3.7261	4.7312
2	0.9722	2.1346	3.1734	4.1923	5.2036
3	1.3373	2.9547	4.0368	4.6428	5.6624
4	2.0421	3.3486	4.4523	5.0815	6.1103

Values of $\alpha_{mn}R/\pi$ for the Roots of $\left[\dfrac{dJ_m(\alpha_{mn}r)}{dr}\right]_{R=0} = 0$.

2.6.4 The Cut-off Frequency for Horns Having Variable Cross-section.

The equation for propagation of one-dimensional waves in a tube having variable cross section is (2.21):

$$\frac{1}{S}\frac{d}{dx}\left(S\frac{d\hat{p}}{dx}\right)+k^2\hat{p}=0 \qquad (2.152)$$

Expand this to the form

$$\frac{d^2\hat{p}}{dx^2}+\frac{d\ln S}{dx}\frac{d\hat{p}}{dx}+k^2\hat{p}=0 \qquad (2.153)$$

Questions related to the cut-off frequency arise in the study of wind instruments which emit sound through a horn. The kind of horn is of course defined by the variation of the cross-section area along the axis. Three common types are the following:

Conical horn: $\quad S = S_0\left[1+\left(\dfrac{x}{x_0}\right)^2\right]$

Exponential horn: $S = S_0 e^{\kappa x}$

Catenoidal horn: $S = S_0 \cosh^2\left(\dfrac{x}{x_0}\right)$

Both the exponential and catenoidal horns have cut-off frequencies and are very poor radiators of sound for frequencies below the cut-off value. However, even though it exhibits cut-off frequency, the conical horn is not very effective at low frequencies. The input impedance, $\hat{p}/\hat{n}$ at $z = 0$, becomes infinitely large as $\omega \to \infty$, and as a consequence, the conical horn is not, in practice, much better than the other two kinds of horn.