X. Applications to Physics

10.1. Phase and State Space

We propose to study a ***holonomic*** mechanical system with a finite number of degrees of freedom, avoiding collision phenomena. In this section we formulate the geometry of such a system.

The ***position space*** is simply an n- dimensional manifold M.

We next define the ***phase space*** attached to M. This is the ***space of all covariant vectors*** at all points of $\boldsymbol{M}$. To make this precise, we consider a coordinate patch U on M with local coordinates

$$q^1,\dots,q^n.$$

At a point P of U, a **covariant vector** is simply a **one-form** at P, hence is given by its components

$$p_1,\dots,p_n, \; (p_i \text{ :real}),$$

(where the one-form itself is $\sum p_i dq^i$).

If

$$\bar{q}^1,\dots,\bar{q}^n$$

is another local coordinate system valid at P, then the components of the same covariant vector with respect to the $\bar{q}^i$ are

$$\bar{p}_i = \sum p_j \frac{\partial q^j}{\partial \bar{q}^i}.$$

The totality of all such covariant vectors at all points of M constitutes the ($2n$)-dimensional ***phase space*** P. To each coordinate neighborhood U on M with local coordinates $q^1,\dots,q^n$ corresponds the coordinate neighborhood $U \times E^n$ with local coordinates

$$q^1,\dots,q^n, p_1,\dots,p_n.$$

It follows that the one-form

$$\alpha = \sum p_i dq^i$$

is a one-form on P, entirely independent of local coordinates. We have

$$d\alpha = \sum dp_i dq^i,$$

so that the ***phase density*** (see Section 2.3)

$$dp_1 \cdots dp_n dq^1 \dots dq^n$$

is a $2n$-form on P, never zero, defined by

$$\pm (d\alpha)^n = (n!)(dp_1 \cdots dp_n dq^1 \cdots dq^n),$$

and serves us as a ***volume element*** on P.

We shall derive some useful relations from the transformation of

coordinates

$$\begin{cases} \bar{q}^i = \bar{q}^i(q^1,\ldots,q^n) \\ \bar{p}_i = \sum p_j \dfrac{\partial q^j}{\partial \bar{q}^i} \end{cases} \qquad (i=1,\ldots,n)$$

valid on the overlap of local coordinate neighborhoods U and $\bar{U}$[1].

We set

$$\boldsymbol{q} = (q^1,\ldots,q^n) \quad \text{and} \quad \boldsymbol{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

and define $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{p}}$ similarly. Then $\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}(\boldsymbol{q})$ implies

$$d\bar{\boldsymbol{q}} = d\boldsymbol{q}A\,, \quad A = \left(\frac{\partial \bar{q}^j}{\partial q^i} \right) = A(\boldsymbol{q})\,.$$

Since $\alpha = d\boldsymbol{q}\cdot\boldsymbol{p} = d\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{p}}$, we have

$$\boldsymbol{p} = A\bar{\boldsymbol{p}}\text{, that is, } \bar{\boldsymbol{p}} = \bar{A}\boldsymbol{p}\,,$$

where $\bar{A} = A^{-1}$ is also a Jacobean matrix. From

$$d\bar{\boldsymbol{p}} = d\bar{A}\boldsymbol{p} + \bar{A}\,d\boldsymbol{p} \quad \text{and} \quad d\boldsymbol{q} = d\bar{\boldsymbol{q}}\bar{A}$$

we deduce first that

$$\left(\frac{\partial \bar{p}_i}{\partial p_j} \right) = \bar{A} = \left(\frac{\partial q^j}{\partial \bar{q}^i} \right).$$

To continue, we note two relations. From $\bar{A} = A^{-1}$ we have

$$d\bar{A} = -A^{-1}dAA^{-1} = -\bar{A}\,dA\bar{A}\,.$$

From $d\bar{\boldsymbol{q}} = d\boldsymbol{q}A$ we have $0 = d\boldsymbol{q}dA$, which easily implies

$$\frac{\partial a^l{}_k}{\partial q^j} = \frac{\partial a^l{}_j}{\partial q^k}\,.$$

Now

$$d\bar{\boldsymbol{p}} = d\bar{A}\boldsymbol{p} + \bar{A}\,d\boldsymbol{p} = -\bar{A}\,dA\bar{A}\boldsymbol{p} + \bar{A}\,d\boldsymbol{p} = -\bar{A}\,dA\bar{\boldsymbol{p}} + \bar{A}\,d\boldsymbol{p}\,.$$

Also

$$d\boldsymbol{p} = d(A\bar{\boldsymbol{p}}) = dA\bar{\boldsymbol{p}} + Ad\bar{\boldsymbol{p}}\,.$$

Thus $\partial \bar{p}_i / \partial q^k$ is the coefficient of dq^k in the i-th row of $-\bar{A}\,dA\bar{\boldsymbol{p}}$ and $\partial p_k / \partial \bar{q}^i$ is the coefficient of $d\bar{q}^i$ in the k-th row of $dA\bar{\boldsymbol{p}}$. Now

$$\bar{A}\,dA = \left(\sum \bar{a}^j{}_i \frac{\partial a^l{}_j}{\partial q^k} dq^k \right)$$

and

$$dA = \left(\sum \frac{\partial a^l{}_k}{\partial q^i} dq^i \right) = \left(\sum \bar{a}^j{}_i \frac{\partial a^l{}_k}{\partial q^j} d\bar{q}^i \right) = \left(\sum \bar{a}^j{}_i \frac{\partial a^l{}_j}{\partial q^k} d\bar{q}^i \right).$$

Therefore the coefficient of dq^k in the i-th row of $-\bar{A}\,d\bar{A}\bar{\boldsymbol{p}}$ and the

[1] The proof of this Dover Edition is different from the original edition.

coefficient of $d\overline{q}^i$ in the k-th row of $dA\overline{\mathbf{p}}$ are, respectively

$$\frac{\partial \overline{p}_i}{\partial q^k} = \sum a^j{}_i \frac{\partial a^l{}_j}{\partial q^k}\overline{p}_l \quad \text{and} \quad \frac{\partial p_k}{\partial \overline{q}^i} = \sum \overline{a}^j{}_i \frac{\partial a^l{}_j}{\partial q^k}\overline{p}_l .$$

We conclude that $\partial \overline{p}_i / \partial q^k = -\partial p_k / \partial \overline{q}^i$, so we have proved

$$\begin{cases} \dfrac{\partial \overline{p}_i}{\partial q^k} = -\dfrac{\partial p_k}{\partial \overline{q}^i} \\ \dfrac{\partial \overline{p}_i}{\partial p_j} = \dfrac{\partial q^j}{\partial \overline{q}^i} \end{cases} .$$

Finally, the ***state space*** is the product

$$S = P \times E^1,$$

a $(2n + 1)$ dimensional space. We think of E^1 as the time axis. Local coordinates for S are

$$q^1, \ldots, q^n, p_1, \ldots, p_n, t .$$

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The original edition

We differentiate the relation

$$\sum \frac{\partial \overline{q}^i}{\partial q^j}\frac{\partial q^j}{\partial \overline{q}^k} = \delta^i_k ,$$

with respect to q^l to get

$$\sum \frac{\partial^2 \overline{q}^i}{\partial q^j \partial q^l}\frac{\partial q^j}{\partial \overline{q}^k} + \sum \frac{\partial \overline{q}^i}{\partial q^j}\frac{\partial^2 \overline{q}^j}{\partial \overline{q}^k \partial \overline{q}^r}\frac{\partial \overline{q}^r}{\partial q^l} = 0 .$$

Multiplying $\partial q^s / \partial \overline{q}^i$ and summing, we have

$$\sum \frac{\partial q^s}{\partial \overline{q}^i}\frac{\partial^2 \overline{q}^i}{\partial q^j \partial q^l}\frac{\partial q^j}{\partial \overline{q}^k} + \frac{\partial^2 q^s}{\partial \overline{q}^k \partial \overline{q}^r}\frac{\partial \overline{q}^r}{\partial q^l} = 0 .$$

Then we change of indeces by

$$\begin{pmatrix} s & r & k & l & i & j \\ j & r & i & k & l & s \end{pmatrix}$$

and get

$$\sum \frac{\partial^2 q^j}{\partial \overline{q}^i \partial \overline{q}^r}\frac{\partial \overline{q}^r}{\partial q^k} = -\sum \frac{\partial q^j}{\partial \overline{q}^l}\frac{\partial^2 \overline{q}^l}{\partial q^s \partial q^k}\frac{\partial q^s}{\partial \overline{q}^i} .$$

Next differentiate equation of $\overline{p}_i$ with respect to q^k and substituting we get

$$\begin{aligned} \frac{\partial \overline{p}_i}{\partial q^k} &= \sum p_j \frac{\partial^2 q^j}{\partial \overline{q}^i \partial \overline{q}^r}\frac{\partial \overline{q}^r}{\partial q^k} \\ &= -\sum p_j \frac{\partial q^j}{\partial \overline{q}^l}\frac{\partial^2 \overline{q}^l}{\partial q^s \partial q^k}\frac{\partial q^s}{\partial \overline{q}^l} . \\ &= -\sum \overline{p}_j \frac{\partial^2 \overline{q}^l}{\partial q^s \partial q^k}\frac{\partial q^s}{\partial \overline{q}^l} \end{aligned}$$

But

$$p_k = \sum \overline{p}_l \frac{\partial \overline{q}^l}{\partial q^k},$$

then

$$\frac{\partial p_k}{\partial \overline{q}^i} = \sum \overline{p}_l \frac{\partial^2 \overline{q}^l}{\partial q^k \partial q^s} \frac{\partial q^s}{\partial \overline{q}^l}.$$

So we have an important symmetric relation:

$$\frac{\partial \overline{p}_i}{\partial q^k} + \frac{\partial p_k}{\partial \overline{q}^i} = 0.$$

Also from

$$\overline{p}_i = \sum p_j \frac{\partial q^j}{\partial \overline{q}^i}$$

it is obvious that

$$\frac{\partial \overline{p}_i}{\partial p_j} = \frac{\partial q^j}{\partial \overline{q}^i}.$$

10.2. Hamiltonian Systems

We wish to consider a dynamical system in Hamiltonian form. We begin by tracing the evolution of this from Lagrange's equations of motion, which in Euclidean coordinates reduce to Newton's law of motion. We deal only with ***conservative holonomic systems***.

The treatment first of all is local. We deal with a coordinate patch in $q^1,\ldots,q^n$ space. For each instant of time, there is a point (position)

$$(q^1(t),\ldots,q^n(t)),$$

which represents the ***trajectory*** of the system. As is customary, we set $\dot{u} = du/dt$.

The ***kinetic energy*** is a function

$$T(q^1,\ldots,q^n,\dot{q}^1,\ldots,\dot{q}^n),$$

which is supposed to be a positive definite quadratic form in the variables $\dot{q}^i$.

The ***potential energy*** is a function

$$V = V(q^1,\ldots,q^n,t),$$

and the ***Lagrangian function****,* or ***kinetic potential****,* is

$$L = T - V.$$

The differential ***equations of motio***n are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \quad (i=1,\ldots,n).$$

For the first term we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right)=\frac{\partial^2 L}{\partial \dot q^i \partial t}+\frac{\partial^2 L}{\partial \dot q^i \partial q^j}\dot q^j+\frac{\partial^2 L}{\partial \dot q^i \partial \dot q^k}\ddot q^k$$

so that the Lagrange equations are a system of n second order ordinary equations for the unknowns $q^1,\ldots,q^n$. We now convert these to a system of $2n$ first order equations in $2n$ unknowns.

We introduce the ***generalized momentum components***

$$p_1,\ldots,p_n$$

by

$$p_i=\frac{\partial L}{\partial \dot q^i}=\frac{\partial T}{\partial \dot q^i}.$$

Because the quadratic form T is definite, the transformation of variables

$$(q^1,\ldots,q^n,\dot q^1,\ldots,\dot q^n)\leftrightarrow(q^1,\ldots,q^n,p_1,\ldots,p_n)$$

is a smooth one both ways.

To reach the Hamilton form, we shall follow tradition and use a rather confusing notation. The matter was better expressed in Section 3.5.

The function T is always considered as a function of the $2n$ variables

$$q^1,\ldots,q^n,\dot q^1,\ldots,\dot q^n.$$

The function V which involves the q^i (and t) alone may be considered as a function on the space of variables $q^1,\ldots,q^n$, $\dot q^1,\ldots,\dot q^n$, t or on the space of variables $q^1,\ldots,q^n$, $p_1,\ldots,p_n$, t.

We introduce the ***Hamiltonian***

$$H=H(q^1,\ldots,q^n,p_1,\ldots,p_n,t)=\sum p_i\dot q^i-L,$$

always considered as a function on the space of variables

$$q^1,\ldots,q^n,p_1,\ldots,p_n,t.$$

Since T is homogeneous quadratic in the $\dot q^i$ we have

$$2T=\sum \dot q^i\frac{\partial T}{\partial \dot q^i}=\sum p_i\dot q^i,$$

hence $H=2T-L=2T-(T-V)$,

$$H=T+V,$$

and so H represents ***total energy***.

From

$$2T=\sum p_i\dot q^i$$

we have

$$2dT = \sum p_i d\dot{q}^i + \sum \dot{q}^i dp_i \ .$$

But

$$dT = \sum \left(\frac{\partial T}{\partial q^i} dq^i + \frac{\partial T}{\partial \dot{q}^i} d\dot{q}^i \right) = \sum \frac{\partial T}{\partial q^i} dq^i + \sum p_i d\dot{q}^i \ .$$

Subtracting,

$$dT = -\sum \frac{\partial T}{\partial q^i} dq^i + \sum \dot{q}^i dp_i \ .$$

From this,

$$dH = dT + dV = \sum \left(-\frac{\partial T}{\partial q^i} + \frac{\partial V}{\partial q^i} \right) dq^i + \sum \dot{q}^i dp_i = -\sum \frac{\partial L}{\partial q^i} dq^i + \sum \dot{q}^i dp_i$$

Thus

$$\begin{cases} \dfrac{\partial H}{\partial q^i} = -\dfrac{\partial L}{\partial q^i} = -\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^i} \right) = -\dot{p}_i \\ \dfrac{\partial H}{\partial p_i} = \dot{q}^i \end{cases} .$$

This gives us the ***equations of motion in Hamilton***, or ***canonical***, form

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i} \end{cases} \quad (i=1,\ldots,n).$$

We next check what functions H qualify as Hamiltonians. We write

$$T = \frac{1}{2} \sum a_{ij}(q) \dot{q}^i \dot{q}^j \ ,$$

where $\left\| a_{ij}(q) \right\|$ is a symmetric positive definite matrix function of the position variables q. It is convenient to set

$$\left\| b^{ij}(q) \right\| = \left\| a_{ij}(q) \right\|^{-1},$$

also symmetric, positive definite. Then

$$p_i = \frac{\partial T}{\partial \dot{q}^i} = \sum a_{ij} \dot{q}^j \ ,$$

which we invert to

$$\dot{q}^i = \sum b^{ik} p_k \ .$$

This gives us

$$T = \frac{1}{2}\sum a_{ij} b^{ik} b^{jl} p_k p_l = \frac{1}{2}\sum b^{jl} p_j p_l \ .$$

From this,

$$H(q,p,t) = \frac{1}{2}\sum b^{ij}(q) p_i p_j + V(q,t)\ .$$

This shows us the ***form of any Hamiltonian function***.

We now wish to formulate Hamiltonian mechanics ***globally***. To discover the correct approach, we compare two Hamiltonian systems

$$\begin{bmatrix} H = T - V \\ T = \frac{1}{2}\sum b^{ij}(q) p_i p_j \\ V = V(q,t) \\ \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \end{bmatrix}, \begin{bmatrix} \overline{H} = \overline{T} - \overline{V} \\ \overline{T} = \frac{1}{2}\sum \overline{b}^{ij}(\overline{q}) \overline{p}_i \overline{p}_j \\ \overline{V} = \overline{V}(\overline{q},t) \\ \dot{\overline{q}}^i = \frac{\partial \overline{H}}{\partial \overline{p}_i} \\ \dot{\overline{p}}_i = -\frac{\partial \overline{H}}{\partial \overline{q}^i} \end{bmatrix}$$

which are supposed to be defined on intersecting regions U, $\overline{U}$ and be equivalent on the common part $U \cup \overline{U}$ of U and $\overline{U}$. The coordinate transformation

$$(q,p,t) \leftrightarrow (\overline{q},\overline{p},t)$$

is given as in Section 10.1:

$$\begin{cases} \overline{q}^i = \overline{q}^i(q^1,\ldots,q^n) \\ \overline{p}_i = \sum p_j \frac{\partial q^j}{\partial \overline{q}^i} \end{cases} .$$

We have

$$\sum \overline{b}^{ij}\overline{p}_j = \frac{\partial \overline{H}}{\partial \overline{p}_i} = \dot{\overline{q}}^i = \sum \frac{\partial \overline{q}^i}{\partial q^k}\dot{q}^k = \sum \frac{\partial \overline{q}^i}{\partial q^k}\frac{\partial H}{\partial p_k} = \sum \frac{\partial \overline{q}^i}{\partial q^k} b^{kl} p_l \ .$$

Hence

$$\overline{T} = \frac{1}{2}\sum \overline{p}_i \overline{b}^{ij}\overline{p}_j = \frac{1}{2}\sum \overline{p}_i \frac{\partial \overline{q}^i}{\partial q^k} b^{kl} p_l = \frac{1}{2}\sum p_k b^{kl} p_l = T\ ,$$

and so

$$\overline{T} = T$$

on the common part of U and $\overline{U}$, hence also

$$\overline{H} - H = \overline{V} - V\ .$$

Using the symmetry relation,

$$\frac{\partial \overline{p}_i}{\partial q^k} + \frac{\partial p_k}{\partial \overline{q}^i} = 0\ ,$$

derived in the last section, and remaining equations of motion:

$$-\frac{\partial \overline{H}}{\partial \overline{q}^i} = \dot{\overline{p}}_i = \sum \frac{\partial \overline{p}_i}{\partial q^j}\dot{q}^j + \sum \frac{\partial \overline{p}_i}{\partial p_k}\dot{p}_k = -\sum \frac{\partial p_j}{\partial \overline{q}^i}\frac{\partial H}{\partial p_j} - \sum \frac{\partial q^k}{\partial \overline{q}^i}\frac{\partial H}{\partial q^k} = -\frac{\partial H}{\partial \overline{q}^i}$$

$$\frac{\partial \overline{H}}{\partial \overline{q}^i} = \frac{\partial H}{\partial \overline{q}^i}.$$

From this,

$$\frac{\partial}{\partial \overline{q}^i}(\overline{V} - V) = 0 \quad (i=1,\ldots,n).$$

It follows that $\overline{V} - V$ is a function of t alone,

$$\overline{V} = V + f(t),$$

$$\overline{H} = H + f(t),$$

this taking place on the intersection of U and $\overline{U}$.

Having this, we can formulate what we mean by a ***global Hamiltonian system***.

We begin with a position space M and form its derived spaces, ***phase space*** P, and ***state space*** S. We are give a function T on P such that over any local coordinate neighborhood U on M with coordinates $q^1,\ldots,q^n$ we have

$$T = \frac{1}{2}\sum b^{ij}(q) p_i p_j,$$

a positive definite quadratic form. Here $(q^1,\ldots,q^n, p_1,\ldots,p_n)$ are the derived coordinates on the neighborhood $U \times E^n$ of P lying over U.

Now let $U_\alpha, U_\beta, \ldots$ denote the various local coordinate neighborhoods on M. For each one of these U_α we have a function

$$V_\alpha = V(q,t)$$

defined on

$$U_\alpha \times E^1.$$

Whenever U_α overlaps U_β, then on the common part $U_\alpha \cap U_\beta$ we have

$$V_\alpha - V_\beta = f_{\alpha\beta}(t),$$

a function of t alone. (Clearly $f_{\alpha\beta} + f_{\beta\alpha} = 0$, and

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.)$$

On the part of state space S lying over U_α, i.e., on $U_\alpha \times E^n \times E^1$, set

$H_\alpha = T + V_\alpha$. The ***equations of motion*** are given by

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i} \end{cases}$$

on $U_\alpha \times E^n \times E^1$, where (q^i) are local coordinates on U_α. These are independent of the local coordinate systems (consistent) and define a ***motion*** (or ***flow***) on all of *S*, always moving forward in the *t*, or time, direction.

Remark. It is customary in mechanics to exhibit the potential function so we have set down the functions V_α in our formulation. Actually the equations of motion merely require the ***gradient*** of the potential which is an **intrinsic** quantity on *S*. Precisely, there is a differential form ϖ on *S* so that locally

$$\varpi = \sum \frac{\partial V_\alpha}{\partial q^i} dq^i + \sum \frac{\partial V_\alpha}{\partial p_i} dp_i \ .$$

We can free ourselves altogether of reference to the functions V_α by requiring that there be given a one-form ϖ on *S* satisfying

(1) ϖ is free of *dt*,

(2) $d\varpi\big|_{dt=0} = 0$.

By the converse of the Poincaré Lemma (Sections 3.6 and 3.7, especially the last remark on p. 31) this implies the existence of the functions V_α.

A ***trajectory*** of the motion is any particular solution of this system of differential equations. From the theory of ordinary differential equations we know that there is a unique trajectory through each point, so that state space *S* is smoothly filled with these curves. Along each one, *t* steadily increases, but not necessarily to arbitrarily large values. (For example, a particle may run off to infinity in finite time.)

Finally we note the ***energy law***:

Along any trajectory,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \ .$$

For

$$\frac{dH}{dt} = \dot{H} = \sum \frac{\partial H}{\partial q^i} \dot{q}^i + \sum \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial t} = \sum (-\dot{p}_i \dot{q}^i) + \sum \dot{q}^i \dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

Remark. Whether or not the functions $f_{\alpha\beta}(t)$ can be removed altogether by redefining each V_α so that there will be a single

potential function V on all of $M \times E^1$ depends on two things; the manner in which the applied forces vary with time, and the topology of M.

The topological difficulties are easily seen from the standard example of the steady magnetic field in the manifold M, which consists of E^3 minus the z-axis, due to a steady electric current in the z-axis. The trouble is that there are closed loops in M which are not boundaries of surfaces. (Cf. the situation in De Rham's theorems, Section 5.9.)

10.3. Integral-invariants

Over a local coordinate neighborhood $U = U_\alpha$ in **position space** M we consider the one-form

$$\omega = \omega_\alpha = \sum p_i dq^i - Hdt \,.$$

This is defined on the portion $U \times E^n \times E^1$ of **state space** which lies over U. These forms ω_α do not necessarily fit together to make a one-form on S because on an intersection $U_\alpha \cap U_\beta$ we have

$$H_\alpha - H_\beta = V_\alpha - V_\beta = f_{\alpha\beta}(t) \,.$$

If the V_α can be chosen so that all $f_{\alpha\beta}(t) = 0$, then we do have such a one-form on all of S. This is exactly the case for a globally conservative system, the case in which the external forces are derived from a single potential function V. While this cannot be expected in general, we do see that

$$d\omega = d\omega_\alpha = \sum dp_i dq^i - dHdt$$

is a 2-form on all of S, independent of local coordinates. This simply means that

$$d\omega_\alpha = d\omega_\beta$$

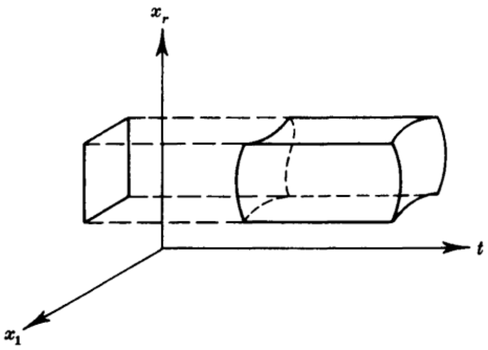
on $U_\alpha \cap U_\beta$, which is true because

$$dH_\beta dt = d(H_\alpha - f_{\alpha\beta})dt = (dH_\alpha - \dot{f}_{\alpha\beta} dt)dt = dH_\alpha dt \,.$$

We shall call this 2-form $d\omega$ even though there is no one-form ω on all of S which it is the "d" of.

Suppose we have on some portion of S an r-parameter family of solutions of the equations of motion. Let us denote by $x_1, \ldots, x_r$ the parameters. What this means is that we have a mapping ϕ on a region W of (t, x) space of the sort indicated, a cylinder in the t direction with top and bottom curved r-chains,

$$\phi : W \to S$$



where in local coordinates

$$\phi : \begin{cases} q^i = f^i(t, x_1, \ldots, x_r) \\ p_i = g_i(t, x_1, \ldots, x_r) \\ t = t \end{cases} .$$

For each (x_i), this represents a trajectory, hence

$$\begin{cases} \dfrac{\partial f^i}{\partial t} = \dfrac{\partial H}{\partial p_i}(\boldsymbol{f}, \boldsymbol{g}, t) \\ \dfrac{\partial g_i}{\partial t} = -\dfrac{\partial H}{\partial q^i}(\boldsymbol{f}, \boldsymbol{g}, t) \end{cases} .$$

The mapping ϕ is supposed smooth and one-to-one, so that an (r +1)-dimensional region in S is evenly filled up by these trajectories.

Now we compute $\phi^*(d\omega)$. We have

$$\begin{aligned} d\omega &= \sum dp_i dq^i - \sum \frac{\partial H}{\partial q^i} dq^i dt - \sum \frac{\partial H}{\partial p_i} dp_i dt \\ &= \sum \left(dp_i + \frac{\partial H}{\partial q^i} dt \right)\left(dq^i + \frac{\partial H}{\partial p_i} dt \right) \end{aligned} .$$

Now

$$\phi^*\left(dq^i - \frac{\partial H}{\partial p_i} dt \right) = \left(\frac{\partial f^i}{\partial t} dt + \sum \frac{\partial f^i}{\partial x_j} dx_j \right) - \frac{\partial f^i}{\partial t} dt = \sum \frac{\partial f^i}{\partial x_j} dx_j$$

and similarly

$$\phi^*\left(dp_i + \frac{\partial H}{\partial q^i} dt \right) = \sum \frac{\partial q^i}{\partial x_k} dx_k \ ,$$

so that

$$\phi^*(d\omega) = \frac{1}{2} \sum \frac{\partial (g_i, f^i)}{\partial (x_j, x_k)} dx_j dx_k = \sum A^{jk}(\boldsymbol{x}, t) dx_j dx_k \ ,$$

which establishes our first point, $\phi^*(d\omega)$ ***is independent of dt***.

Since $d(d\omega) = 0$, we have $d[\phi^*(d\omega)] = 0$. But

$$d[\phi^*(d\omega)] = \sum \frac{\partial A^{jk}}{\partial t} dt dx_j dx_k + (\text{terms\_in\_}dx_i dx_j dx_k) = 0 .$$

We conclude that

$$\frac{\partial A^{jk}}{\partial t} = 0 ,$$

each $A^{jk} = A^{jk}(\boldsymbol{x})$ ***is independent of t***, and we may write

$$\phi^*(d\omega) = \sum A^{jk}(\boldsymbol{x}) dx_j dx_k .$$

A differential form α of degree s on the state space S is called an (***absolute***) ***integral-invariant*** (historical terminology) if for each r-parameter family of trajectories given by such a mapping ϕ, $\phi^*(\alpha)$ is an s-form on the $\boldsymbol{x}$-space alone (no t nor dt terms) and if in addition $d\alpha = 0$.

Each of the forms

$$d\omega, (d\omega)^2, \ldots, (d\omega)^n$$

is an integral-invariant. For

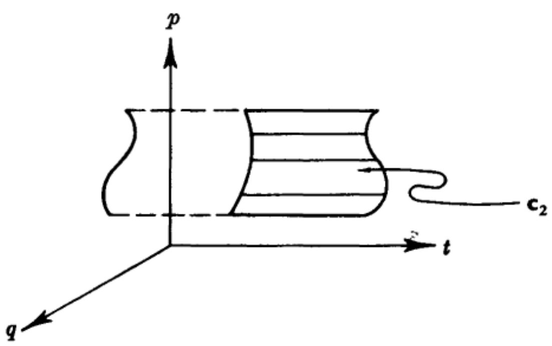
$$d(d\omega)^s = 0$$

and

$$\phi^*(d\omega)^s = [\phi^*(d\omega)]^s = [\sum A^{jk}(x) dx_j dx_k]^s$$

is independent of t and dt.

Consider in state space S a small piece $\boldsymbol{c_2}$ of surface which is filled by a one-parameter family of trajectories. We may describe $\boldsymbol{c_2}$ analytically by



$$\begin{cases} q^i = q^i(t, y) \\ p_i = p_i(t, y) \end{cases}$$

$$\begin{cases} a(y,z) \le t \le b(y,z) \\ y_0 \le y \le y_1 \end{cases} .$$

Our reasoning above shows that the two-form we get by substituting

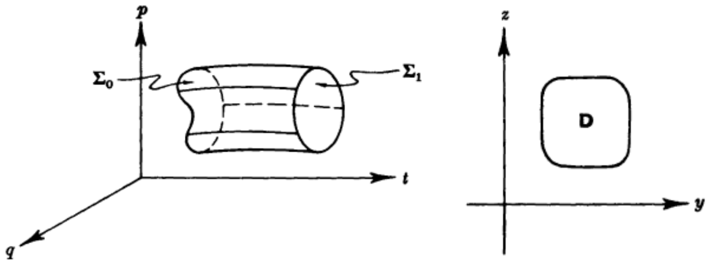
these expressions for p and q in $d\omega$ is a two-form in y and dy only, hence vanishes. This means in particular

$$\int_{c_2} d\omega = 0 .$$

Next, suppose one has a piece of volume, or three-chain $\boldsymbol{c_3}$ in S which is the span of a two-parameter family of trajectories:

$$\begin{cases} q^i = q^i(t, y, z) \\ p_i = p_i(t, y, z) \end{cases}$$

$$\begin{cases} a(y,z) \le t \le b(y,z) \\ (y,z)\text{\_in\_a\_domain\_}D \end{cases}.$$



Then

$$\partial \boldsymbol{c_3} = \Sigma_1 - \Sigma_0 + \boldsymbol{c_2} ,$$

where Σ_1, Σ_0 are the terminal and initial surfaces, respectively and where the lateral surface $\boldsymbol{c_2}$ is spanned by a one-parameter family of trajectories corresponding to the parameter point (y, z) on ∂D . Now

$$\int_{\partial c_3} d\omega = \int_{c_3} d(d\omega) = 0 ,$$

and we showed above that

$$\int_{c_2} d\omega = 0 ,$$

hence

$$\int_{\Sigma_1} d\omega = \int_{\Sigma_0} d\omega .$$

We have established the very striking property of the form $d\omega$:

If Σ_0 is any 2-chain in S *transversal to the trajectories, and if one displaces each point of Σ_0 any amount along its trajectory to form a new surface Σ_1, then*

$$\int_{\Sigma_1} d\omega = \int_{\Sigma_0} d\omega .$$

It should be clear that we have not used any special properties of $d\omega$ in proving this result so that any integral-invariant satisfies a corresponding property.

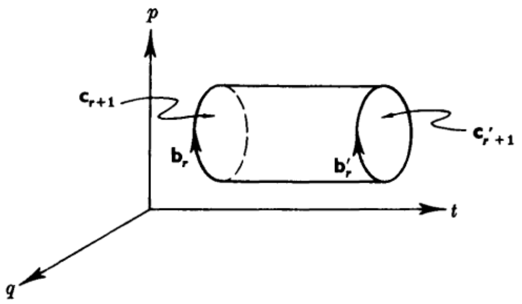
Let a *be an integral-invariant of degree r on S. Let* $\boldsymbol{c}_r$ *be any r-chain on S transversal to the trajectories. Let* $\boldsymbol{c'}_r$ *be a second such r-chain so that the points of* $\boldsymbol{c}_r$ *and* $\boldsymbol{c'}_r$ *may be put into one-one correspondence with corresponding points on the same trajectory. Then*

$$\int_{c_r} \alpha = \int_{c'_r} \alpha \, .$$

It is possible to reverse our steps to prove that the property expressed in this result actually characterizes integral-invariants. This is done in E. Cartan [8].

We pass on to relative integral-invariants.

An r-form α on S is a ***relative integral-invariant*** provided $d\alpha$ is an ***integral-invariant***. The basic result about relative integral-invariants is this.



Let α *be a relative integral-invariant of degree r. Let* $\boldsymbol{b}_r$ *and* $\boldsymbol{b}'_r$ *be two r-dimensional boundaries which are in one-one correspondence in such a way that corresponding points are on the same trajectory. Then*

$$\int_{b_r} \alpha = \int_{b'_r} \alpha \, .$$

To prove this, we select $(r+1)$-chains $\boldsymbol{c}_{r+1}$, $\boldsymbol{c'}_{r+1}$ so that

$$\boldsymbol{b}_r = \partial \boldsymbol{c}_{r+1} , \ \ \boldsymbol{b}'_r = \partial \boldsymbol{c}'_{r+1}$$

and do this in such a way that $\boldsymbol{c}_{r+1}$ and $\boldsymbol{c}'_{r+1}$ correspond one-one with corresponding points on the same trajectory. Then

$$\int_{b'_r} \alpha = \int_{\partial c'_{r+1}} \alpha = \int_{c'_{r+1}} d\alpha = \int_{c_{r+1}} d\alpha = \int_{\partial c_{r+1}} \alpha = \int_{b_r} \alpha \, .$$

The differential form

$$\omega = \sum p_i dq^i - Hdt \, ,$$

which is defined locally, is a relative integral-invariant of degree one

where it is defined. Consequently so are the forms

$$\omega d\omega, \omega(d\omega)^2, \ldots, \omega(d\omega)^n,$$

since

$$d[\omega(d\omega)^r] = (d\omega)^{r+1}.$$

We shall specialize by considering chains which exist at a single instance of time. Let α be any differential r-form and $\boldsymbol{c}$ an r-chain in S lying in a hyperplane t = constant. Then clearly

$$\int_{\boldsymbol{c}} \alpha = \int_{\boldsymbol{c}} \alpha\Big|_{dt=0}.$$

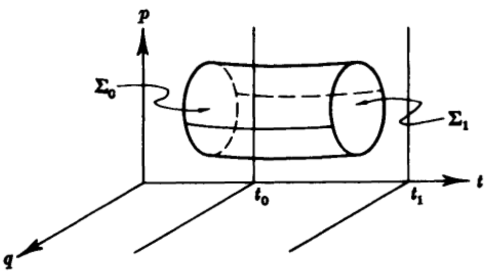
Let us apply this to $d\omega$ in particular. We have

$$d\omega\Big|_{dt=0} = \sum dp_i dq^i ,$$

which leads to this result:

Let Σ_0 be a 2-chain in S at $t = t_0$ and Σ_1 the 2-chain obtained by moving each point Σ_0 along its trajectory to time $t = t_1$. Then

$$\int_{\Sigma_0} \sum dp_i dq^i = \int_{\Sigma_1} \sum dp_i dq^i .$$



In this result we may think of Σ_0 and Σ_1 as 2-chains in phase space.

If we apply the procedure to the $(2n)$-form

$$(d\omega)^n = \pm n!(dp_1 \cdots dp_n dq^1 \cdots dq^n) + \lambda dt$$

which gives us the ***phase density***

$$\mu = dp_1 \cdots dp_n dq^1 \cdots dq^n ,$$

we have the

LIOUVILLE THEOREM. *If a 2n-dimensional region D_0 in phase space at time t_0 moves to a region D_1 at time t_1, then*

$$\int_{D_1} \mu = \int_{D_0} \mu .$$

We shall close this section with a result which will be needed in Section 10.5. It shows that the integral-invariant $d\omega$ completely

determines the equations of motion, in itself an important mechanical principle.

Let

$$\begin{cases} \dot{q}^i = A^i(t, \boldsymbol{q}, \boldsymbol{p}) \\ \dot{p}_i = B_i(t, \boldsymbol{q}, \boldsymbol{p}) \end{cases}$$

be a system of equations on a region of state space S which has

$$d\omega = \sum dp_i dq^i - dHdt$$

as an integral-invariant. Then

$$\begin{cases} A^i = \dfrac{\partial H}{\partial p_i} \\ B_i = -\dfrac{\partial H}{\partial qhi} \end{cases} /$$

For let

$$\begin{cases} q^i = f^i(t, x_1, \ldots, x_{2n}) \\ p_i = g_i(t, x_1, \ldots, x_{2n}) \end{cases}$$

be a general solution, so that $d\omega$ must be expressible in the terms of dx_i alone. From this, and the differential equations

$$\begin{cases} \dfrac{\partial f^i}{\partial t} = A^i \\ \dfrac{\partial g_i}{\partial t} = B_i \end{cases},$$

we deduce

$$\begin{cases} dq^i = A^i dt + \lambda^i \\ dp_i = B_i dt + \mu_i \end{cases},$$

where λ^i, μ_i are one-forms in the dx_j alone, and so

$$\begin{aligned} d\omega &= \sum dp_i dq^i - dHdt \\ &= \left(\sum A^i \mu_i - \sum B_i \lambda^i\right) dt + \sum \mu_i \lambda^i - \left(\sum \frac{\partial H}{\partial q^i} \lambda^i + \sum \frac{\partial H}{\partial p_i} \mu_i\right) dt \end{aligned}.$$

Since $d\omega$ is free of dt,

$$\sum A^i \mu_i - \sum B_i \lambda^i - \sum \frac{\partial H}{\partial q^i} \lambda^i - \sum \frac{\partial H}{\partial p_i} \mu_i = 0,$$

$$\sum \left(A^i - \frac{\partial H}{\partial p_i}\right) \mu_i - \sum \left(B_i + \frac{\partial H}{\partial q^i}\right) \lambda^i = 0,$$

$$\sum \left(A^i - \frac{\partial H}{\partial p_i}\right) (dp_i - B_i dt) - \sum \left(B_i + \frac{\partial H}{\partial q^i}\right) (dq^i A^i dt) = 0,$$

$$\sum\left(A^i - \frac{\partial H}{\partial p_i}\right)dp_i - \sum\left(B_i + \frac{\partial H}{\partial q^i}\right)dq^i + (\cdots)dt = 0 \, .$$

We conclude that

$$\begin{cases} A^i - \dfrac{\partial H}{\partial p_i} = 0 \\ B_i + \dfrac{\partial H}{\partial q^i} = 0 \end{cases}$$

as asserted.

10.4. Brackets

In the transformation theory of classical mechanics one uses the bracket expressions of Poisson and Lagrange. In this section we shall show how these expressions relate to differential forms.

Before doing this, it is a good idea to digress on the subject of ***Lie brackets***. We take any differentiate manifold N and recall the definition in Section 5.3 of **tangent vector** and the definition of **vector field** at the end of that section. We may consider a vector field $\boldsymbol{v}$ on N as an operator which takes each function on N to another function on N:

$$\boldsymbol{v} : F^0(N) \to F^0(N) \, .$$

If $x^1, \ldots, x^n$ x" is a local coordinate system in which $\boldsymbol{v}$ has the representation

$$\boldsymbol{v} = \sum a^i(\boldsymbol{x}) \frac{\partial}{\partial x^i} \, ,$$

then

$$\boldsymbol{v}(f) = \sum a^i(\boldsymbol{x}) \frac{\partial f}{\partial x^i}$$

shows, locally, what $\boldsymbol{v}$ does to a function f. One sees from this formula, or from the very definition of $\boldsymbol{v}$ as the assignment of a tangent vector (directional differentiation) at P to each point P of N, that

$$\boldsymbol{v}(f \cdot g) = \boldsymbol{v}(f) \cdot g + f \cdot \boldsymbol{v}(g)$$

for any two functions f and g on N. If $\boldsymbol{v}$ and w are two vector fields on N, we define the ***Lie bracket*** of $\boldsymbol{v}$ and $\boldsymbol{w}$ by

$$[\boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{v} \circ \boldsymbol{w} - \boldsymbol{w} \circ \boldsymbol{v} \, .$$

This is another vector field on N. If in local coordinates

$$\boldsymbol{v} = \sum a^i \frac{\partial}{\partial x^i} \, , \quad \boldsymbol{w} = \sum b^j \frac{\partial}{\partial x^j} \, ,$$

then a computation shows that

$$[\boldsymbol{v}, \boldsymbol{w}] = \sum\left(\sum a^i \frac{\partial b^j}{\partial x^i} - \sum b^i \frac{\partial a^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

(The main point to notice is that the second partials cancel each other.)

The following algebraic identities are easily established:

(i) $[\boldsymbol{v}, \boldsymbol{v}] = 0$.

(ii) $[\boldsymbol{w}, \boldsymbol{v}] + [\boldsymbol{v}, \boldsymbol{w}] = 0$.

(iii) $[\boldsymbol{v_1} + \boldsymbol{v_2}, \boldsymbol{w}] = [\boldsymbol{v_1}, \boldsymbol{w}] + [\boldsymbol{v_2}, \boldsymbol{w}] = 0$.

(iv) $[[\boldsymbol{u}, \boldsymbol{v}], \boldsymbol{w}] + [[\boldsymbol{v}, \boldsymbol{w}], \boldsymbol{u}] + [[\boldsymbol{w}, \boldsymbol{u}], \boldsymbol{v}] = 0$ (**Jacobi identity**).

Now we return to mechanical systems. As before, let P be the **phase space** associated to a **position space** M. We denote by α the differential form

$$\alpha = \sum p_i dq^i$$

on P so that

$$d\alpha = \sum dp_i dq^i$$

as in Section 10.1.

POISSON BRACKETS. To each pair f, g of real functions on phase space P we associate a new function (f, g) defined by

$$n(dfdg) \wedge (d\alpha)^{n-1} = (f, g)(d\alpha)^n.$$

In local coordinates

$$(d\alpha)^{n-1} = [(n-1)!]\sum (dp_1 dq^1)\cdots\overline{(dp_i dq^i)}\cdots(dp_n dq^n)$$

and

$$(d\alpha)^n = (n!) dp_1 dq^1 \cdots dp_n dq^n,$$

from which we deduce the local expression for (f, g)

$$(f, g) = \sum \frac{\partial(f, g)}{\partial(p_i, q^i)}.$$

From the definition one derives these relations

(i) $(f, f) = 0$.

(ii) $(f, g) + (g, f) = 0$.

(iii) $(f, g_1 + g_2) = (f, g_1) + (f, g_2)$.

Using $(df)d(g_1 g_2) = g_1(dfdg_2) + g_2(dfdg_1)$, one has

(iv) $(f, g_1 g_2) = g_1 \cdot (f, g_2) + g_2 \cdot (f, g_1)$.

The identities (iii) and (iv) taken together may be expressed by saying that for fixed f,

$$g \to (f, g)$$

is a vector field $\boldsymbol{v}_f$ on P:

$$\boldsymbol{v}_f(g) = (f, g).$$

The basic connection between the **Lie** and **Poisson** brackets is given by

(v) $\boldsymbol{v}_{(f,g)} = [\boldsymbol{v}_f, \boldsymbol{v}_g]$.

One has

$$\boldsymbol{v}_{(f,g)}(h) = \big((f,g),h\big)$$

and

$$\begin{aligned}\boldsymbol{v}_{(f,g)}(h) &= \boldsymbol{v}_f[\boldsymbol{v}_g(h)] - \boldsymbol{v}_g[\boldsymbol{v}_f(h)]\\ &= \boldsymbol{v}_f\big((g,h)\big) - \boldsymbol{v}_g\big((f,h)\big)\\ &= \big(f,(g,h)\big) - \big(g,(f,h)\big)\\ &= -\big((g,h),f\big) - \big((h,f),g\big)\end{aligned}$$

so that (v) is equivalent to ***Jacobi's relation***

(vi) $\big((f,g),h\big) + \big((g,h),f\big) + \big((h,f),g\big) = 0$.

One proves this relation [(v) or (vi)] either by a lengthy direct calculation, or by the following more sophisticated argument based on the fact that a vector field is completely determined locally by its effect on each member of a local coordinate system.

First of all,

$$(f,q^i) = \sum \frac{\partial(f,q^i)}{\partial(p_j,q^j)} = \sum \begin{vmatrix} \dfrac{\partial f}{\partial p_j} & \dfrac{\partial f}{\partial q^j} \\ 0 & \delta^i_j \end{vmatrix} = \frac{\partial f}{\partial p_i},$$

which may be interpreted as

$$\boldsymbol{v}_{q^i} = -\frac{\partial}{\partial p^i}.$$

Similarly

$$(f,p_i) = -\frac{\partial f}{\partial q^i} \quad \text{and} \quad \boldsymbol{v}_{p_i} = \frac{\partial}{\partial q^i}.$$

Because of these relations (vi) easily follows when **both** g and h are taken from the set of coordinate functions $\{q^1, \ldots, p_n\}$. This means that

$$\boldsymbol{v}_{(f,x)} = [\boldsymbol{v}_f, \boldsymbol{v}_x]$$

when x is one of these coordinate functions since the vector fields on both sides agree when applied to any coordinate function q^j or p_j.

Hence for <u>any</u> h,

$$\big((f,x),h\big) + \big((x,h),f\big) + \big((h,f),x\big) = 0.$$

This is now established for any functions f and h and any coordinate

function x. But this may be interpreted as saying

$$\boldsymbol{v}_{(h,f)}x = [\boldsymbol{v}_h, \boldsymbol{v}_f]x$$

so that the vector field

$$\boldsymbol{v}_{(h,f)} \text{ and } [\boldsymbol{v}_h, \boldsymbol{v}_f]$$

must agree since they agree on all of the coordinate functions. Hence

$$\boldsymbol{v}_{(h,f)}(g) = [\boldsymbol{v}_h, \boldsymbol{v}_f]g$$

for all f,g,h. This completes the proof of (v) and (vi).

One applies brackets to a pair of functions on the state space S by simply treating t as a parameter.

A function f on S is called a ***first integral*** of the equations of motion if it is constant on each trajectory. If this is the case, then

$$\begin{aligned} 0 = \frac{df}{dt} &= \sum \frac{\partial f}{\partial q^i}\dot{q}^i + \sum \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} \\ &= \sum \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \sum \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i} + \frac{\partial f}{\partial t} \quad , \\ &= (H,f) + \frac{\partial f}{\partial t} \end{aligned}$$

partial differential equation for first integral is

$$\frac{\partial f}{\partial t} = (f, H)\,.$$

LAGRANGE BRACKETS. Let

$$\phi : E^2 \to P\,,$$

where E^2 is the Euclidean plane with rectangular coordinates u, v. Then $\phi^*(d\alpha)$ is a 2-form on E^2 (where as before $d\alpha = \sum dp_i dq^i$) and we write

$$\phi^*(d\alpha) = [u,v]dudv$$

defining the ***Lagrange brackets*** which have the local expression

$$[u,v] = \sum \frac{\partial(p_i, q^i)}{\partial(u,v)}\,.$$

10.5. Contact Transformations

Here we can but touch briefly on an extensive topic. For simplicity we shall only treat the **local problem** and shall look on contact transformations as **coordinate changes**.

First we take the case of a time independent change. As usual, let q^i, p_i denote local coordinates in **phase space** P. We consider new coordinates $\bar{q}^i, \bar{p}_i$,

$$\begin{cases} \bar{q}^i = \bar{q}^i(q^1,\dots,q_n,p_1,\dots,p_n) \\ \bar{p}_i = \bar{p}_i(q^1,\dots,q^n,p_1,\dots,p_n) \end{cases}$$

which are unrelated to the old except for one requirement. We set

$$\alpha = \sum p_i dq^i \ , \ \ \bar{\alpha} = \sum \bar{p}_i d\bar{q}^i$$

and require

$$d\bar{\alpha} = d\alpha \ .$$

This defines what we shall call a ***homogeneous contact transformation***.

Since $d(\bar{\alpha} - \alpha) = 0$ and we are only working locally, the condition may be expressed as

$$\bar{\alpha} = \alpha + d\phi \,,$$

where ϕ is a real function on P. In the relation $d\bar{\alpha} = d\alpha$,

$$\sum \bar{p}_i d\bar{q}^i = \sum p_i dq^i \ ,$$

one replaces $d\bar{p}_i$ and $d\bar{q}^i$ by their expressions in terms of dp_i and dq^i to obtain

$$\begin{cases} \sum \dfrac{\partial(\bar{p}_i,\bar{q}^i)}{\partial(q^j,q^k)} = 0 \\ \sum \dfrac{\partial(\bar{p}_i,\bar{q}^i)}{\partial(p_j,p_k)} = 0 \\ \sum \dfrac{\partial(\bar{p}_i,\bar{q}^i)}{\partial(p_j,q^k)} = \delta^j_k \end{cases} .$$

(These relations may be expressed in terms of Lagrange brackets.) Similarly,

$$\begin{cases} \sum \bar{p}_i \dfrac{\partial \bar{q}^i}{\partial q^j} = p_j + \dfrac{\partial \phi}{\partial q^j} \\ \sum \bar{p}_i \dfrac{\partial \bar{q}^i}{\partial p_j} = \dfrac{\partial \phi}{\partial p_j} \end{cases} .$$

If we have equations of motion

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i} \end{cases} ,$$

we shall show that they transform into equations of the same type. For suppose after changing the coordinates according to our contact transformation we arrive at

$$\begin{cases} \dot{\overline{q}}^i = A^i \\ \dot{\overline{p}}_i = -B_i \end{cases},$$

so that

$$\begin{cases} A^i = \sum \dfrac{\partial \overline{q}^i}{\partial q^j} \dfrac{\partial H}{\partial p_j} - \sum \dfrac{\partial \overline{q}^i}{\partial p_j} \dfrac{\partial H}{\partial q^j} \\ B_i = -\sum \dfrac{\partial \overline{p}_i}{\partial q^j} \dfrac{\partial H}{\partial p_j} + \sum \dfrac{\partial \overline{p}_i}{\partial p_j} \dfrac{\partial H}{\partial q^j} \end{cases}.$$

If we multiply the first by

$$d\overline{p}_i = \sum \frac{\partial \overline{p}_i}{\partial q^k} dq^k + \sum \frac{\partial \overline{p}_i}{\partial p_k} dp_k$$

and the second by $d\overline{q}^i$ similarly and sum, we find after some simplification

$$\sum A^i d\overline{p}_i + \sum B_i d\overline{q}^i = \sum \frac{\partial H}{\partial q^j} dq^j + \sum \frac{\partial H}{\partial p_k} dp_k = dH - \frac{\partial H}{\partial t} dt .$$

Hence

$$\begin{cases} A^i = \dfrac{\partial H}{\partial \overline{p}_i} \\ B_i = \dfrac{\partial H}{\partial \overline{q}^i} \end{cases},$$

so that the equations of motion in the new coordinates are precisely

$$\begin{cases} \dot{\overline{q}}^i = \dfrac{\partial H}{\partial \overline{p}_i} \\ \dot{\overline{p}}_i = -\dfrac{\partial H}{\partial \overline{q}^i} \end{cases}.$$

Now we pass to the general situation. We begin with a mechanical system

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i} \end{cases}$$

on the **state space** S and consider a coordinate change on S:

$$\begin{cases} \overline{q}^i = \overline{q}^i(t, q^1, \ldots, q_n, p_1, \ldots, p_n) \\ \overline{p}_i = \overline{p}_i(t, q^1, \ldots, q^n, p_1, \ldots, p_n) \\ t = t \end{cases}.$$

We set

$$\omega = \sum p_i dq^i - H dt$$

as usual. The coordinate change is called a ***contact transformation*** if

there is a

function $\overline{H}$ and a function ϕ so that if

$$\overline{\omega} = \sum \overline{p}_i d\overline{q}^i - \overline{H}dt ,$$

then

$$\overline{\omega} = \omega + d\phi ,$$

or what is the same thing,

$$d\overline{\omega} = d\omega .$$

The first basic result is that the equations of motion in the new coordinates are precisely

$$\begin{cases} \dot{\overline{q}}^i = \dfrac{\partial \overline{H}}{\partial \overline{p}_i} \\ \dot{\overline{p}}_i = -\dfrac{\partial \overline{H}}{\partial \overline{q}^i} \end{cases} .$$

For whatever the new equations of motion are, they admit

$$d\overline{\omega} = \sum d\overline{p}_i d\overline{q}^i - d\overline{H}dt$$

as an ***integral-invariant***. But the final result in Section 10.3 tells us that the only equations with this integral-invariant are the stated ones. (Compare this slick proof with a direct computation!)

A particular type of contact transformation is obtained as follows. Let

$$\phi(t, x^1, \ldots, x^n, y^1, \ldots, y^n)$$

be a function of *2n*+1 variables satisfying the independence condition

$$\left| \frac{\partial^2 \phi}{\partial x^i \partial y^j} \right| \neq 0 .$$

Set $\phi = \phi(t, q^1, \ldots, q^n, \overline{q}^1, \ldots, \overline{q}^n)$,

$$p_i = \frac{\partial \phi}{\partial q^i}, \quad \overline{p}_i = -\frac{\partial \phi}{\partial \overline{q}^i} .$$

For fixed q and p, the $\overline{q}$ are determined by the first set of equations (since the determinant above does not vanish) and then the $\overline{p}$ are determined by the second set of equations.

We have

$$\begin{aligned} d\phi &= \frac{\partial \phi}{\partial t} dt + \sum \frac{\partial \phi}{\partial q^i} dq^i + \sum \frac{\partial \phi}{\partial \overline{q}^i} d\overline{q}^i \\ &= \frac{\partial \phi}{\partial t} dt + \sum p_i dq^i - \sum \overline{p}_i d\overline{q}^i \end{aligned} ,$$

$$\omega - \overline{\omega} = -\left(\sum \overline{p}_i d\overline{q}^i - \overline{H}dt\right) + \left(\sum p_i dq^i - Hdt\right)$$
$$= d\phi - \frac{\partial \phi}{\partial t} dt + \overline{H}dt - Hdt$$

and so

$$\overline{\omega} = \omega - d\phi$$

provided we set

$$\overline{H} = H + \frac{\partial \phi}{\partial t}.$$

The most important case is that in which ϕ is a solution of the ***Hamilton-Jacobi equation***

$$\frac{\partial \phi}{\partial t}(t, q, \overline{q}) + H\left(t, q, \ldots, \frac{\partial \phi}{\partial q^i}, \ldots\right) = 0.$$

In this situation $\overline{H} = 0$ and the new equations of motion are simply

$$\begin{cases} \dot{\overline{q}}^i = 0 \\ \dot{\overline{p}} = 0 \end{cases},$$

with solutions

$$\begin{cases} \overline{q}^i = \text{constant} \\ \overline{p}_i = \text{constant} \end{cases}.$$

The original system is said to be ***transformed to equilibrium***.

One other point we shall notice is this. If a contact transformation is **stationary**, i.e., independent of time, then it is equivalent to a **homogeneous contact transformation**. For suppose

$$\begin{cases} \overline{q}^i = \overline{q}^i(\mathrm{q}, \mathrm{p}) \\ \overline{p}_i = \overline{p}_i(\mathrm{q}, \mathrm{p}) \end{cases}$$

and

$$\left(\sum \overline{p}_i d\overline{q}^i - \overline{H}dt\right) = \left(\sum p_i dq^i - Hdt\right) + d\phi.$$

Equating coefficients:

$$\begin{cases} \sum \overline{p}_i \dfrac{\partial \overline{q}^i}{\partial q^j} = p_j + \dfrac{\partial \phi}{\partial q^j} \\ \sum \overline{p}_i \dfrac{\partial \overline{q}^i}{\partial p_k} = \dfrac{\partial \phi}{\partial p_k} \\ \overline{H} = H - \dfrac{\partial \phi}{\partial t} \end{cases}.$$

Since $q, p, \overline{q}, \overline{p}$ are independent of t, we deduce from the first two equations that

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t \partial q^j} = 0 \\ \dfrac{\partial^2 \phi}{\partial t \partial p_k} = 0 \end{cases} .$$

Hence $\partial\phi / \partial t$ is a function of *t* alone. This evidently implies

$$\phi(t, q, p) = f(t) + \psi(q, p),$$

and so

$$\sum \overline{p}_i d\overline{q}^i = \sum p_i dq^i + d\psi,$$

which means we have a homogeneous contact transformation as asserted.

Let us briefly examine what are called ***infinitesimal contact transformations***.

We ask when

$$\begin{cases} \overline{q}^i = q^i + \varepsilon f^i \\ \overline{p}_i = p_i + \varepsilon g_i \end{cases}$$

is a contact transformation up to first order terms in ε. We restrict attention to the **homogeneous** (**stationary**) case. We have

$$\sum \overline{p}_i d\overline{q}^i - \sum p_i dq^i = \varepsilon d\phi,$$

$$\sum (p_i + \varepsilon g_i)(dq^i + \varepsilon df^i) - \sum p_i dq^i = \varepsilon d\phi,$$

$$\varepsilon\left(\sum g_i dq^i + \sum p_i dq^i\right) = \varepsilon d\phi,$$

so the condition is

$$\sum g_i dq^i + \sum p_i df^i = d\phi.$$

If we set

$$\psi = \phi - \sum p_i f^i,$$

this becomes

$$\sum g_i dq^i - \sum f^i dp_i = d\psi,$$

or

$$\begin{cases} g_i = \dfrac{\partial \psi}{\partial q^i} \\ f^i = -\dfrac{\partial \psi}{\partial p_i} \end{cases} .$$

We finally have

$$\begin{cases} \bar{q}^i = q^i - \varepsilon \dfrac{\partial \psi}{\partial p_i} \\ \bar{p}_i = p_i + \varepsilon \dfrac{\partial \psi}{\partial q^i} \end{cases}.$$

Here ψ is called a ***generating function*** for the infinitesimal contact transformation.

10.6. Fluid Mechanics

We consider a fluid moving in a region of E^3. The **position vector** as usual is

$$\boldsymbol{x} = (x, y, z) = (x^1, x^2, x^3).$$

At each time t, the **velocity** at $\boldsymbol{x}$ is $\boldsymbol{v}$,

$$\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x}) = (u, v, w) = (v^1, v^2, v^3).$$

The density of the fluid is a scalar

$$\rho = \rho(t, \boldsymbol{x}).$$

In this section we shall denote the **vectorial area element** of a surface by $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} = (dtdz, dzdx, dxdy).$$

(See Section 4.5, p. 43.)

If $\boldsymbol{c_3}$ is a three-dimensional region which is fixed in space, the change in mass at each point $\boldsymbol{x}$ of $\boldsymbol{c_3}$ per unit time is

$$\frac{\partial \rho}{\partial t} dxdydz$$

and so the total time derivative of mass in $\boldsymbol{c_3}$ is

$$\int_{c_3} \frac{\partial \rho}{\partial t} dxdydz.$$

We assume **conservation of matter**, so this must result from flux of fluid over the boundary, hence

$$\int_{c_3} \frac{\partial \rho}{\partial t} dxdydz = -\int_{\partial c_3} \rho \boldsymbol{v} \cdot \boldsymbol{\sigma}.$$

By Gauss' theorem,

$$\int_{\partial c_3} \rho \boldsymbol{v} \cdot \boldsymbol{\sigma} = \int_{c_3} div(\rho \boldsymbol{v}) dxdydz.$$

By taking $\boldsymbol{c_3}$ arbitrary we deduce from the equality of these integrals the ***continuity equation***

$$\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{v}) = 0,$$

a necessary condition the flow must satisfy. We shall deduce some con sequences of this. We set

$$\Omega = \rho(dx^1 - v^1 dt)(dx^2 - v^2 dt)(dx^3 - v^3 dt) .$$

To compute $d\Omega$, we set

$$\beta = (dx^1 - v^1 dt)(dx^2 - v^2 dt)(dx^3 - v^3 dt) ,$$

so that

$$d\beta = -\frac{\partial v^1}{\partial x^1} dx^1 dt dx^2 dx^3 + dx^1 \left(\frac{\partial v^2}{\partial x^2} dx^2 dt \right) dx^3 - dx^1 dx^2 \left(\frac{\partial v^3}{\partial x^3} dx^3 dt \right),$$

$$= (\operatorname{div} \boldsymbol{v}) dt dx^1 dx^2 dx^3$$

$$\begin{aligned} d\Omega &= d(\rho\beta) = d\rho \wedge \beta + \rho d\beta \\ &= \left(\frac{\partial \rho}{\partial t} dt + \sum \frac{\partial \rho}{\partial x^i} dx^i \right) \wedge \beta + \rho (\operatorname{div} \boldsymbol{v})(dt dx^1 dx^2 dx^3) . \\ &= \left[\frac{\partial \rho}{\partial t} + \sum v^i \frac{\partial \rho}{\partial x^i} + \rho(\operatorname{div} \boldsymbol{v}) \right] (dt dx^1 dx^2 dx^3) \end{aligned}$$

Thus the continuity equation is equivalent to the relation

$$d\Omega = 0 .$$

Suppose we express the flow in terms of initial conditions (or other parameters) by

$$\boldsymbol{x} = \boldsymbol{x}(t, \alpha^1, \alpha^2, \alpha^3) ,$$

so that the α^i are the parameters and

$$\frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{v} .$$

Thus

$$(dx^i - v^i dt) = \left(\frac{\partial x^i}{\partial t} dt + \sum \frac{\partial x^i}{\partial \alpha^j} d\alpha^j \right) - v^i dt = \sum \frac{\partial x^i}{\partial \alpha^j} d\alpha^j ,$$

so that

$$\Omega = \rho \frac{\partial (x^1, x^2, x^{3)}}{\partial(\alpha^1, \alpha^2, \alpha^3)} d\alpha^1 d\alpha^2 d\alpha^3 = A(t, \boldsymbol{\alpha}) d\alpha^1 d\alpha^2 d\alpha^3 .$$

Since $d\Omega = 0$, we deduce that $\partial A / \partial t = 0$,

$$\Omega = A(\boldsymbol{\alpha}) d\alpha^1 d\alpha^2 d\alpha^3 .$$

This means that Ω is an ***integral-invariant*** for the flow as explained in Section 10.3. We consequently have the following result.

*Let $\boldsymbol{c_3}, \boldsymbol{c'_3}$ be three-chains in the four-dimensional (t, **x**) space which are in one-one correspondence in such a way that corresponding points lie on the same trajectory of the flow. Then*

$$\int_{c_3} \Omega = \int_{c'_3} \Omega .$$

In particular, if all the points of $\boldsymbol{c_3}$ exist simultaneously, i.e., at a

fixed time t_0, then

$$\int_{c_3} \Omega = \int_{c_3} \Omega\Big|_{t=t_0} = \int_{c_3} \rho dxdydz \; .$$

Thus if we take a region $c_3^{(0)}$ at time t_0 and follow it to $c_3^{(1)}$ at time t_1, we have

$$\int_{c_3^{(0)}} \rho dxdydz \;=\; \int_{c_3^{(0)}} \rho dxdydz \quad ,$$

which says that ***mass is preserved*** in the flow, another form of the conservation of mass.

We now proceed to the **dynamic situation**. We suppose our fluid is nonviscous so that the pressure is a force per unit area at each point acting normal to any surface element through the point, always with the same magnitude. Let

$p = p(t,\boldsymbol{x})$ = pressure,

$\boldsymbol{F} = \boldsymbol{F}(t,\boldsymbol{x})$ = body force per unit mass.

Let c_3 be a fixed region in space. The total acceleration of all matter in c_3 is

$$\int_{c_3} \rho \frac{d\boldsymbol{v}}{dt} dxdydz \; .$$

At the instant of time in question, this must equal the total force on the matter in c_3 which is

$$\int_{c_3} \rho \boldsymbol{F} dxdydz - \int_{\partial c_3} p\boldsymbol{\sigma} \; .$$

By a variant of Gauss' theorem,

$$\int_{\partial c_3} p\boldsymbol{\sigma} = \int_{c_3} (\operatorname{grad} p) dxdydz \; ,$$

hence

$$\int_{c_3} \left(\rho \frac{d\boldsymbol{v}}{dt} - \rho F + \operatorname{grad} p \right) dxdydz = 0 \, .$$

We conclude that

$$\frac{d\boldsymbol{v}}{dt} = F - \frac{1}{\rho} \operatorname{grad} p \; ,$$

the ***Euler equation of motion***.

Here the interpretation is

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \sum \frac{\partial \boldsymbol{v}}{\partial x^i} v^i \; .$$

Let us suppose that the body force $\boldsymbol{F}$ is conservative,

$$\boldsymbol{F} = -\operatorname{grad} V,$$

where

$$V = V(t, \boldsymbol{x})$$

is the force potential.

We shall add the hypothesis that p and ρ are functionally related, i.e.,

$$d\rho \wedge dp = 0,$$

as is the case, for example, with an isothermal motion. In this case we can define a function $q = q(t, \boldsymbol{x})$ by

$$q = \int_0^{(t,\boldsymbol{x})} \frac{dp}{\rho},$$

so that

$$dq = \frac{dp}{\rho}.$$

The equations of motion may be written

$$\frac{d\boldsymbol{v}}{dt} = -\operatorname{grad}(V+q)$$

or

$$\frac{du}{dt} = -\frac{\partial}{\partial x}(V+q), \text{ etc.,}$$

i.e.,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial}{\partial x}(V+q), \text{ etc.}$$

We set

$$\begin{aligned} E &= \frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{v}) + V + q \\ &= \frac{1}{2}(u^2+v^2+w^2) + V + q \end{aligned},$$

the ***energy per unit mass***.

Finally we define the ***vorticity***

$$\xi = (\xi, \eta, \zeta) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

We compute

$$\begin{aligned} \frac{\partial E}{\partial x} + \frac{\partial u}{\partial t} &= u\frac{\partial u}{\partial x} + \left(v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) + \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(V+q) \\ &= \left(v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) - \left(v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \\ &= v\zeta - w\eta \end{aligned},$$

and similarly

$$\begin{cases} \dfrac{\partial E}{\partial x} + \dfrac{\partial u}{\partial t} = v\zeta - w\eta \\ \dfrac{\partial E}{\partial y} + \dfrac{\partial v}{\partial t} = w\xi - u\zeta \\ \dfrac{\partial E}{\partial z} + \dfrac{\partial w}{\partial t} = u\eta - v\xi \end{cases}.$$

(These are equivalent to the vector formula

$$\operatorname{grad} E + \frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{v} \times \boldsymbol{\xi} \ .)$$

Now we consider the differential form

$$\omega = udx + vdy + wdz - Edt \, .$$

We have

$$\begin{aligned} d\omega = &(\xi dydz + \eta dzdx + \zeta dxdy) \\ &+ dt(u_t dx + v_t dy + w_t dz) \\ &- (E_x dx + E_y dy + E_z dz)dt \end{aligned},$$

$$\begin{aligned} d\omega = &(\xi dydz + \eta dzdx + \zeta dxdy) \\ &+ dt[(v\zeta - w\eta)dx + (w\xi - u\zeta)dy + (u\eta - v\xi)dz] \end{aligned}.$$

Actually, $d\omega$ is an ***integral-invariant*** so that ω is a ***relative*** *integral-invariant* One sees this indirectly by making a comparison to Hamiltonian systems. For if one thinks for the moment of x, y, z, u, v, w, t as independent variables, the equations of motion are

$$\begin{cases} \dot{x} = u \\ \dot{y} = v \\ \dot{z} = w \\ \dot{u} = -\dfrac{\partial}{\partial x}(V+q) \\ \dot{v} = -\dfrac{\partial}{\partial y}(V+q) \\ \dot{w} = -\dfrac{\partial}{\partial z}(V+q) \end{cases},$$

or

$$\begin{cases} \dot{x} = \partial E / \partial u \\ \dot{y} = \partial E / \partial v \\ \dot{z} = \partial E / \partial w \\ \dot{u} = -\partial E / \partial x \\ \dot{v} = -\partial E / \partial y \\ \dot{w} = -\partial E / \partial z \end{cases},$$

which makes our assertion clear. It follows that if $\boldsymbol{c_2}, \boldsymbol{c'_2}$ are two-chains in $(t, \boldsymbol{x})$ space which are in one-one correspondence so that corresponding points are on the same trajectory, then

$$\int_{\boldsymbol{c}_2} d\omega = \int_{\boldsymbol{c}'_2} d\omega .$$

In particular if $\boldsymbol{c_2}^{(0)}$ is a 2-chain in E^3 at time t_0 and it moves to $\boldsymbol{c_2}^{(1)}$ at time t_1 according to the motion, then

$$\int_{\boldsymbol{c_2}^{(0)}} (\xi dydz + \eta dzdx + \zeta dxdy) = \int_{\boldsymbol{c_2}^{(1)}} (\xi dydz + \eta dzdx + \zeta dxdy) .$$

This is the Helmholtz theorem on ***conservation of vorticity***. In the first integral we must understand $\xi = \xi(t_0, \boldsymbol{x})$, etc., and in the second, $\xi = \xi(t_1, \boldsymbol{x})$, etc. An important consequence is this further result of Helmholtz.

Suppose at a fixed time t_0*, the vorticity vanishes identically. Then it always vanishes.*

For by the formula above,

$$\int_{\boldsymbol{c_2}^{(1)}} (\xi dydz + \cdots) = 0$$

for each 2-chain $\boldsymbol{c_2}^{(1)}$ at each time t_1, which evidently means that the integrand must vanish.

10.7. Problems