I. Introduction

1.1. Exterior Differential Forms

The objects which we shall study are called ***exterior differential forms***. These are the things which occur under integral signs. For example, a line integral

$$\int Adx + Bdy + Cdz$$

leads us to the ***one-form***

$$\omega = Adx + Bdy + Cdz\ ;$$

a surface integral

$$\iint Pdydz + Qdzdx + Rdxdy$$

leads us to the ***two-form***

$$\alpha = Pdydz + Qdzdx + Rdxdy\ ;$$

and a volume integral

$$\iiint Hdxdydz$$

leads us to the ***three-form***

$$\lambda = Hdxdydz\ .$$

These are all examples of differential forms which live in the space E^3 of three variables. If we work in an n-dimensional space, the quantity under the integral sign in an r-fold integral (integral over an r-dimensional variety) is an ***r-form in n variables***.

In the expression α above, we notice the absence of terms in $dzdy$, $dxdzx$, $dydx$, which suggests symmetry or ***skew-symmetry***. The further absence of terms $dxdx, \ldots$ strongly suggests the latter.

We shall set up a calculus of differential forms which will have certain inner consistency properties, one of which is the rule for changing variables in a multiple integral. Our integrals are always ***oriented integrals***, hence we ***never*** take absolute values of Jacobians.

Consider

$$\iint A(x, y)dxdy$$

with the change of variable

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}.$$

We have

$$\iint A(x,y)dxdy = \iint A[x(u,v),y(u,v)]\frac{\partial(x,y)}{\partial(u,v)}dudv\ ,$$

which leads us to write

$$dxdy = \frac{\partial(x,y)}{\partial(u,v)}dudv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv\ .$$

If we set $y = x$, the determinant has equal rows, hence vanishes. Also if we interchange x and y, the determinant changes sign. This motivates the rules

$$\begin{cases} dxdx = 0 \\ dydx = -dxdy \end{cases}$$

for multiplication of differentials in our calculus.

In general, an (***exterior***) ***r-form in n variables*** $x^1,\cdots,x^n$ will be an expression

$$\omega = \frac{1}{r!}\sum A_{i_1,\dots,i_r}dx^{i_1}\cdots dx^{i_r}\ ,$$

where the coefficients A are smooth functions of the variables and skew-symmetric in the indices.

We shall associate with each r-form ω an $(r + 1)$-form $d\omega$ called the ***exterior derivative*** of ω. Its definition will be given in such a way that validates the general Stokes' formula

$$\int_{\partial\Sigma}\omega = \int_{\Sigma}d\omega\ .$$

Here Σ is an $(r+1)$-dimensional oriented variety and $\partial\Sigma$ is its boundary.

A basic relation is the ***Poincaré Lemma***:

$$d(d\omega) = 0\ .$$

In all cases this reduces to the equality of mixed second partials

1.2. Comparison with Tensors

At the outset we can assure our readers that we shall not do away with tensors by introducing differential forms. Tensors are here to stay; in a great many situations, particularly those dealing with symmetries, tensor methods are very natural and effective. However, in many other situations the use of the exterior calculus, often combined with the method of moving frames of E Cartan, leads to decisive results in a way which is very difficult with tensors alone. Sometimes a combination of techniques is in order. We list several points of contrast.

(a) Tensor analysis *per se* seems to consist only of techniques for calculations with indexed quantities. It lacks a body of substantial or deep results

established once and for all within the subject and then available for application. The exterior calculus does have such a body of results.

If one takes a close look at Riemannian geometry as it is customarily developed by tensor methods one must seriously ask whether the geometric results cannot be obtained more cheaply by other machinery.

(b) In classical tensor analysis, one never knows what is the range of applicability simply because one is never told what the space is. Everything seems to work in a coordinate patch, but we know this is inadequate for most applications. For example, if a particle is constrained to move on the sphere S^2, a single coordinate system cannot describe its position space, let alone its phase or state spaces.

This difficulty has been overcome in modern times by the theory of ***differentiable manifolds*** (varieties) which we discuss in Chapter V.

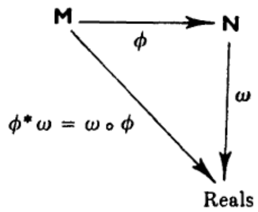
(c) Tensor fields do not behave themselves under mappings. For example, given a **contravariant vector field** a^i on x-space and a mapping ϕ on x-space to y-space, there is **no** naturally induced field on the y-space. [Try the map $t \to (t^2, t^3)$ on E^1 into E^2.]

With exterior forms we have a really attractive situation in this regard. If

$$\phi : M \to N$$

and if ω is a p-form on N, there is naturally induced a p-form $\phi^* \omega$ on M.

Let us illustrate this for the simplest case in which ω is a 0-form, or scalar, i.e., a real-valued function on N. Here $\phi^* \omega = \omega \circ \phi$, the composition of the mapping ϕ followed by ω.



(d) In tensor calculations the maze of indices often makes one lose sight of the very great differences between various types of quantities which can be represented by tensors, for example, vectors tangent to a space, mappings between such vectors, geometric structures on the tangent spaces.

(e) It is often quite difficult using tensor methods to discover the deeper invariants in geometric and physical situations, even the local ones. Using exterior forms, they seem to come naturally according to these principles:

(i) All local geometric relations arise one way or another from the

equality of mixed partials, i.e., Poincaré's Lemma.

(ii) Local invariants themselves usually appear as the result of applying exterior differentiation to everything in sight.

(iii) Global relations arise from integration by parts, i.e., Stokes' theorem.

(iv) Existence problems which are not genuine partial differential equations (boundary value or Cauchy problems) generally are of the type of Frobenius-Cartan-Kähler system of exterior differential forms and can be reduced thereby to systems of ordinary equations.

(f) In studying geometry by tensor methods, one is invariably restricted to the natural frames associated with a local coordinate system. Let us consider a Riemannian geometry as a case in point. This consists of a manifold in which a Euclidean geometry has been imposed in each of the tangent spaces. A natural frame leads to an oblique coordinate system in each tangent space. Now who in his right mind would study Euclidean geometry with oblique coordinates? Of course the orthonormal coordinate systems are the natural ones for Euclidean geometry, so they must be the correct ones for the much harder Riemannian geometry. We are led to introduce moving frames, a method which goes hand-in-glove with exterior forms.

We conclude the case by stating our opinion, that exterior calculus is here to stay, that it will gradually replace tensor methods in numerous situations where it is the more natural tool, that it will find more and more applications because of its inner simplicity, body of substantial results begging for further use, and because it simply is there wherever integrals occur. There is generally a time lag of some fifty years between mathematical theories and their applications. The mathematicians H. Poincaré, E. Goursat, and E. Cartan developed the exterior calculus in the early part of this century; in the last twenty years it has greatly contributed to the rebirth of differential geometry, now part of the mathematical main stream. Physicists are beginning to realize its usefulness; perhaps it will soon make its way into engineering.

II. Exterior Algebra

2.1. The Space of p-Vectors

Notation:

R = field of real numbers a, b, c,…

L = an n-dimensional vector space over R with elements $\alpha, \beta, ...$

For each $p = 0,1,2,\dots,n$ we shall construct a new vector space

$$\wedge^p L$$

over R, called the ***space of p-vectors on*** L. We begin with

$$\wedge^0 L = R, \quad \wedge^1 L = L.$$

Next we shall work out $\wedge^2 L$ in some detail. This space consists of all sums

$$\sum a_i(\alpha_i \wedge \beta_i)$$

subject only to these constraints, or reduction rules, and no others:

$$\begin{cases} (a_1\alpha_1 + a_2\alpha_2) \wedge \beta - a_1(\alpha_1 \wedge \beta) - a_2(\alpha_2 \wedge \beta) = 0 \\ \alpha \wedge (b_1\beta_1 + b_2\beta_2) - b_1(\alpha \wedge \beta_1) - b_2(\alpha \wedge \beta_2) = 0 \\ \alpha \wedge \alpha = 0 \\ \alpha \wedge \beta + \beta \wedge \alpha = 0 \end{cases},$$

Here α, β, etc, are vectors in L and a, b, etc., are real numbers; $\alpha \wedge \beta$ is called the ***exterior product*** of the vectors α and β. If α and β are dependent, say $\beta = c\alpha$, then

$$\alpha \wedge \beta = \alpha \wedge (c\alpha) = c(\alpha \wedge \beta) = c \cdot 0 = 0$$

according to our reductions. Otherwise $\alpha \wedge \beta \neq 0$.

Suppose $\sigma^1, \dots, \sigma^n$ is a basis of L. Then

$$\alpha = \sum a_i \sigma^i, \quad \beta = \sum b_j \sigma^j,$$

$$\alpha \wedge \beta = \left(\sum a_i \sigma^i\right) \wedge \left(\sum b_j \sigma^j\right) = \sum a_i b_j (\sigma^i \sigma^j).$$

We rearrange this as follows. Each term $\sigma^i \wedge \sigma^i = 0$ and each $\sigma^j \wedge \sigma^i = -\sigma^i \wedge \sigma^j$ for $i < j$. Hence

$$\alpha \wedge \beta = \sum_{i<j} (a_i b_j - a_j b_i) \sigma^i \wedge \sigma^j.$$

The typical element of $\wedge^2 L$ is a linear combination of such exterior products, hence the 2-vectors

$$\sigma^i \wedge \sigma^j, \quad 1 \le i < j \le n,$$

form a basis of $\wedge^2 L$. We conclude

$$\dim \wedge^2 L = \frac{n(n-1)}{2} = \binom{n}{2}.$$

In general, we form $\wedge^p L$ $(2 \le p \le n)$ by the same idea. It consists of

all formal sums (***p-vectors***, or ***vectors of degree p***)

$$\sum a(\alpha_1 \wedge \cdots \wedge \alpha_p)$$

subject only to these constraints:

(i) $(a\alpha + b\beta) \wedge \alpha_2 \wedge \cdots \wedge \alpha_p = a(\alpha \wedge \alpha_2 \wedge \cdots \wedge \alpha_p) + b(\beta \wedge \alpha_2 \wedge \cdots \wedge \alpha_p)$

and the same if any α_i is replaced by a linear combination.

(ii) $\alpha_1 \wedge \cdots \wedge \alpha_p = 0$ if for some pair of indices $i \neq j$, $\alpha_i = \alpha_j$

(iii) $\alpha_1 \wedge \cdots \wedge \alpha_p$ changes sign if any two α_i are interchanged.

It follows easily from (i) that $\alpha_1 \wedge \cdots \wedge \alpha_p$ is linear in each variable; we may replace any variable by a linear combination of any number (not just two) of other vectors and compute the value by distributing, for example

$$\alpha \wedge (b_1\beta_1 + b_2\beta_2 + b_3\beta_3) \wedge \gamma \wedge \delta = b_1(\alpha \wedge \beta_1 \wedge \gamma \wedge \delta) + b_2(\alpha \wedge \beta_2 \wedge \gamma \wedge \delta) + b_3(\alpha \wedge \beta_3 \wedge \gamma \wedge \delta)$$

It follows from (iii) that if π is any permutation of the $\{1, 2, \ldots, p\}$, then

$$\alpha_{\pi(1)} \wedge \cdots \wedge \alpha_{\pi(p)} = (\operatorname{sgn} \pi)\alpha_1 \wedge \cdots \wedge \alpha_p .$$

Exactly as in the case $p = 2$, we can show that if

$$\sigma^1, \ldots, \sigma^n$$

is a basis of L, then a basis of $\wedge^p L$ is made up as follows: for each set of indices

$$H = \{h_1, h_2, \ldots, h_p\}, \;\; 1 \leq h_1 < h_2 < \cdots < h_p \leq n ,$$

we set

$$\sigma^H = \sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p} .$$

Then the totality of σ^H is a basis of $\wedge^p L$. We conclude that

$$\dim \wedge^p L = \binom{n}{p},$$

the number of combinations of n things taken p at a time. In particular

$$\dim \wedge^n L = 1 .$$

If λ is in $\wedge^p L$, then

$$\lambda = \sum_H \sigma_H \sigma^H ,$$

summed over all of these ordered sets H. One can also sum over all p-tuples of indices by introducing skew-symmetric coefficients:

$$\lambda = \frac{1}{p!} \sum_{h_1, \ldots, h_p} b_{h_1, \ldots, h_p} \sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p}$$

where the $b_{h_1, \ldots, h_p}$ is a skew-symmetric tensor and

$$b_{h_1, \ldots, h_p} = \sigma_H \;\; \text{for} \;\; H = \{h_1, h_2, \ldots, h_p\}, \;\; h_1 < h_2 < \cdots < h_p .$$

This skew-symmetric representation is often quite useful.

Let us note why we do not define $\wedge^p L$ for $p > n$. (Sometimes it is

convenient to simply set $\wedge^p L = 0$ for $p > n$.) We express each α in a product $\alpha_1 \wedge \cdots \wedge \alpha_p$ as a linear combination of the basis vectors $\sigma^1, \ldots, \sigma^n$ and completely distribute according to Rule (i). This leads to

$$\alpha_1 \wedge \cdots \wedge \alpha_p = \sum a_{h_1,\ldots,h_p} \sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p} .$$

Each term $\sigma^{h_1} \wedge \cdots \wedge \sigma^{h_p}$ is a product of $p > n$ vectors taken from the set $\sigma^1, \ldots, \sigma^n$ so there must be a repetition; by Rule (ii) it vanishes. We are left with $\alpha_1 \wedge \cdots \wedge \alpha_p = 0$ as the only possibility.

We close with a very important property of the spaces $\wedge^p L$.

In order to define a linear mapping *f* on $\wedge^p L$ it suffices to present a function g of p variables on L such that (i) g is ***linear*** in each variable separately, (ii) g is ***alternating*** in the sense that g vanishes when two of its variables are equal and g changes sign when two of its variables are interchanged. Then

$$f(\alpha_1 \wedge \cdots \wedge \alpha_p) = g(\alpha_1, \ldots, \alpha_p)$$

defines *f* on the generators of $\wedge^p L$.

It can be shown that this property provides an axiomatic characterization of $\wedge^p L$. In the next section we apply this property to define the determinant of a linear transformation.

2.2. Determinants

As above L is a fixed linear space of dimension *n*. Let A be a linear transformation on L into itself. We define a function $g = g_A$ of *n* variables on L as follows:

$$g_A(\alpha_1, \ldots, \alpha_n) = A\alpha_1 \wedge \cdots \wedge A\alpha_n ,$$

$$g_A : \times^n L \to \wedge^n L$$

where $\times^n L$ denotes the cartesian product. Since g is multilinear and alternating, there is a linear functional $f = f_A$,

$$f_A : \wedge^n L \to \wedge^n L$$

satisfying

$$f_A(\alpha_1 \wedge \cdots \wedge \alpha_n) = g_A(\alpha_1, \ldots, \alpha_n) = A\alpha_1 \wedge \cdots \wedge \alpha_n .$$

But $\wedge^n L$ is one-dimensional so the only linear transformation on this space is multiplication by a scalar. We denote the particular one here by $|A|$ and have

$$A\alpha_1 \wedge \cdots \wedge A\alpha_n = |A|(\alpha_1 \wedge \cdots \wedge \alpha_n) .$$

This serves to define the ***determinant*** $|A|$ of *A*. We must not fail to note definition is completely independent of a matrix representation of *A*.

We observe next

$$
\begin{aligned}
|AB|(\alpha_1 \wedge \cdots \wedge \alpha_n) &= (AB\alpha_1) \wedge \cdots \wedge (AB\alpha_n) \\
&= |A|(B\alpha_1 \wedge \cdots \wedge B\alpha_n) \\
&= |A| \cdot |B|(\alpha_1 \wedge \cdots \wedge \alpha_n)
\end{aligned}
$$

hence

$$|AB| = |A| \cdot |B| .$$

We can relate this to the determinant of a matrix as follows. Let $\sigma^1, \ldots, \sigma^n$ be a basis of L and $\|a_{ij}\|$ a $n \times n$ matrix. Set

$$\alpha_i = \sum a_{ij} \sigma^j .$$

Then

$$\alpha_1 \wedge \cdots \wedge \alpha_n = |a_{ij}| \sigma^1 \wedge \cdots \wedge \sigma^n .$$

In particular, if one obtains the matrix representation of A with respect to the basis (σ^i) by

$$A\sigma^i = \sum a^i{}_j \sigma^j ,$$

then

$$A\sigma^1 \wedge \cdots \wedge A\sigma^n = |a^i{}_j| \sigma^1 \wedge \cdots \wedge \sigma^n , \quad |A| = |a^i{}_j| .$$

2.3. Exterior Products

We now observe that our spaces $\wedge^p L$ have a built-in multiplication process called ***exterior multiplication*** and denoted by $\wedge$ for obvious reasons. We multiply a p-vector μ by a q-vector ν to obtain a (p+q)-vector $\mu \wedge \nu$ (which is 0 by definition if $p + q > n$):

$$\wedge : \left(\wedge^p L\right) \times \left(\wedge^q L\right) \to \wedge^{p+q} L .$$

It suffices to define $\wedge$ on generators and use the basic principle at the end of Section 1 to extend it to all p- and q-vectors:

$$(\alpha_1 \wedge \cdots \wedge \alpha_p) \wedge (\beta_1 \wedge \cdots \wedge \beta_q) = \alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q .$$

The basic properties of this exterior product are

(1) $\lambda \wedge \mu$ is ***distributive***,

(2) $\lambda \wedge (\mu \wedge \nu) = (\lambda \wedge \mu) \wedge \nu$, the ***associative*** law,

(3) $\mu \wedge \lambda = (-1)^{pq} \lambda \wedge \mu$.

Property (3) simply says that any two vectors of odd degrees ***anticommute***, otherwise vectors ***commute***. The following will illustrate why this is the case:

$$
\begin{aligned}
(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \wedge \beta &= -(\alpha_1 \wedge \alpha_2 \wedge \beta \wedge \alpha_3) \\
&= (-1)^2 (\alpha_1 \wedge \beta \wedge \alpha_2 \wedge \alpha_3)\,, \\
&= (-1)^3 \beta \wedge (\alpha_1 \wedge \alpha_2 \wedge \alpha_3)
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \wedge (\beta_1 \wedge \beta_2) &= (-1)^3 \beta_1 \wedge (\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \wedge \beta_2 \\
&= (-1)^3 (-1)^3 (\beta_1 \wedge \beta_2) \wedge (\alpha_1 \wedge \alpha_2 \wedge \alpha_3)\,. \\
&= (-1)^{3\cdot 2} (\beta_1 \wedge \beta_2) \wedge (\alpha_1 \wedge \alpha_2 \wedge \alpha_3)
\end{aligned}
$$

Examples. We take for L the linear space based on the differentials $dx, dy, \ldots$ and, as is customary, omit the exterior multiplication sign $\wedge$ between *dx's.* Thus *dxdy* denotes $dx \wedge dy$.

1.

$$(Adx + Bdy + Cdz) \wedge (Edx + Fdy + Gdz) = (BG - CF)dydz + (CE - AG)dzdx + (AF - BE)dxdy$$

illustrating the ***vector-***, or ***cross-product*** of two ordinary vectors.

2.

$$(Adx + Bdy + Cdz) \wedge (Pdydz + Qdzdx + Rdxdy) = (AP + BQ + CR)dxdydz$$

illustrating the ***dot-***, or ***inner-product*** of vector algebra.

3. Let α be any form of odd degree. Then

$$\alpha^2 = \alpha \wedge \alpha = 0\,.$$

For if α and β are of odd degree *p,* then

$$\beta \wedge \alpha = -\alpha \wedge \beta\,.$$

We set $\beta = \alpha$ to have

$$\alpha \wedge \alpha = -\alpha \wedge \alpha\,,\;\; 2(\alpha \wedge \alpha) = 0\,,\;\; \alpha \wedge \alpha = 0\,.$$

4. Here we take

$$\omega = dp_1 dq^1 + \cdots + dp_n dq^n\,,$$

a form arising in ***mechanics***. The two-forms $dp_i dq^i$ all commute, hence

$$
\begin{aligned}
\omega^n &= (n!) dp_1 dq^1 dp_2 dq^2 \cdots dp_n dq^n \\
&= (-1)^{n(n-1)/2} (n!) dp_1 \cdots dp_n dq^1 \cdots dq^n\,.
\end{aligned}
$$

The product $dp_1 \cdots dq^n$ is called ***phase-density***. We shall discuss this further in Chapter X.

We apply the exterior product to obtain the ***Laplace expansion*** of a determinant by complementary minors.

Let $\left\| a_{ij} \right\|$ be an $n \times n$ matrix. For $H = \{h_1, \ldots, h_p\}$, set

$$b_H = \begin{vmatrix} a_{1,h_1} & \cdots & a_{1,h_p} \\ \vdots & & \vdots \\ a_{p,h_1} & \cdots & a_{p,h_p} \end{vmatrix}.$$

Set $p + q = n$. For $K = \{k_1, \ldots, k_q\}$, set

$$c_K = \begin{vmatrix} a_{p+1,k_1} & \cdots & a_{p+1,k_q} \\ \vdots & & \vdots \\ a_{n,k_1} & \cdots & a_{n,k_q} \end{vmatrix}.$$

Thus if $K = H'$, the **complementary set of indices** to H (always arranged in natural order), then b_H and c_K are **complementary minors** of $\|a_{ij}\|$.

Now set

$$\alpha_i = \sum a_{ij}\sigma^j$$

where (σ^j) is a basis of L. We easily see that

$$\alpha_1 \wedge \cdots \wedge \alpha_p = \sum b_H \sigma^H ,$$

$$\alpha_{p+1} \wedge \cdots \wedge \alpha_n = \sum c_K \sigma^K ,$$

hence

$$\begin{aligned} \alpha_1 \wedge \cdots \wedge \alpha_n &= (\alpha_1 \wedge \cdots \wedge \alpha_p) \wedge (\alpha_{p+1} \wedge \cdots \wedge \alpha_n) \\ &= \sum b_H c_K \sigma^H \wedge \sigma^K \end{aligned}.$$

But

$$\alpha_1 \wedge \cdots \wedge \alpha_n = |a_{ij}|(\sigma^1 \wedge \cdots \wedge \sigma^n)$$

and

$$\sigma^H \wedge \sigma^K = \begin{cases} 0\_\text{if}\_K \neq H' \\ \varepsilon^{\mathrm{H,H'}}(\sigma^1 \wedge \cdots \wedge \sigma^n)\_\text{if}\_K = H' \end{cases},$$

hence

$$|a_{ij}| = \sum \varepsilon^{H,H'} b_H c_{H'} .$$

If $H = \{h_1, \ldots, h_p\}$, $H' = \{k_1, \ldots, k_q\}$, then

$$\varepsilon^{H,H'} = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ h_1 & h_2 & \cdots & k_q \end{pmatrix}.$$

2.4 Linear Transformations

In this section we deal with two linear spaces M and N with

$$\dim M = m , \quad \dim N = n .$$

Let us agree that when we need bases, $\sigma^1, \ldots, \sigma^m$ will denote a basis of M and $\tau^1, \ldots, \tau^n$ a basis of N.

Let A be a linear transformation,

$$A : M \to N .$$

The mapping

$$(\alpha_1, \ldots, \alpha_p) \to A\alpha_1 \wedge \cdots \wedge A\alpha_p$$

sends

$$\times^p M \to \wedge^p N .$$

It is alternating multilinear, hence defines a linear transformation, denoted $\wedge^p A$ on $\wedge^p M$ to $\wedge^p N$. This ***exterior p-th power*** of A is defined on generators by

$$(\wedge^p A)(\alpha_1 \wedge \cdots \wedge \alpha_p) = A\alpha_1 \wedge \cdots \wedge A\alpha_p .$$

Suppose A is represented by the $m \times n$ matrix $\left\| a^i{}_j \right\|$ according to

$$A\sigma^i = \sum a^i{}_j \tau^j .$$

The σ^H and τ^K form bases of $\wedge^p M$ and $\wedge^p N$, respectively, where H and K are ordered sets of p indices. We have

$$\begin{aligned}(\wedge^p A)\sigma^H &= A\sigma^{h_1} \wedge \cdots \wedge A\sigma^{h_p} \\ &= \sum a^{h_1}{}_{k_1} \cdots a^{h_p}{}_{k_p} \tau^{k_1} \wedge \cdots \wedge \tau^{k_p} . \\ &= \sum a^H{}_K \tau^K \end{aligned}$$

Hence $\wedge^p A$ is represented by the matrix

$$\left\| a^H{}_K \right\|$$

of all $p \times p$ minors of $\left\| a^i{}_j \right\|$. This is sometimes called the ***p-th compound*** of $\left\| a^i{}_j \right\|$.

Suppose one has three spaces L, M, N and this situation:

$$\begin{tikzcd} L \arrow[r, "B"] \arrow[dr, "AB"'] & M \arrow[d, "A"] \\ & N \end{tikzcd}$$

We compute $\wedge^p (AB)$:

$$\begin{aligned}\wedge^p (AB)(\alpha_1 \wedge \cdots \wedge \alpha_p) &= (AB\alpha_1) \wedge \cdots \wedge (AB\alpha_p) \\ &= (\wedge^p A)[(B\alpha_1) \wedge \cdots \wedge (B\alpha_p)] \\ &= (\wedge^p A)[(\wedge^p B)(\alpha_1 \wedge \cdots \wedge \alpha_p)] \\ &= [(\wedge^p A)(\wedge^p B)](\alpha_1 \wedge \cdots \wedge \alpha_p)\end{aligned},$$

hence

$$\wedge^p (AB) = (\wedge^p A)(\wedge^p B).$$

It follows that the *p*th compound of the product of two matrices is the product of their *p*th. compounds, a nontrivial result.

We must consider one other matter. Again let $A: M \to N$. Suppose ω is in $\wedge^p M$ and η is in $\wedge^q M$. Then

$$(\wedge^{p+q} A)(\omega \wedge \eta) = (\wedge^p A)(\omega) \wedge (\wedge^q A)(\eta).$$

For if we take monomials, $\omega = \alpha_1 \wedge \cdots \wedge \alpha_p$, $\eta = \beta_1 \wedge \cdots \wedge \beta_q$, then

$$\begin{aligned}(\wedge^{p+q} A)(\omega \wedge \eta) &= (\wedge^{p+q} A)(\alpha_1 \wedge \cdots \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q) \\ &= A\alpha_1 \wedge \cdots \wedge A\beta_q \\ &= (A\alpha_1 \wedge \cdots \wedge A\alpha_p) \wedge (A\beta_1 \wedge \cdots \wedge A\beta_q) \\ &= (\wedge^p A)(\omega) \wedge (\wedge^q A)(\eta)\end{aligned}.$$

2.5. Inner Product Spaces

In the remainder of the chapter we shall study a space *L* which has an ***inner product*** (α, β). This is a real-valued function on $L \times L$ which is

(i) Linear in each variable,

(ii) Symmetric: $(\alpha, \beta) = (\beta, \alpha)$,

(iii) Nondegenerate: if for fixed α, $(\alpha, \beta) = 0$ for all β, then $\alpha = 0$.

Example 1. The ***Euclidean*** inner product on E^n is given by

$$\alpha = (a_1, \ldots, a_n), \quad \beta = (b_1, \ldots, b_n),$$
$$(\alpha, \beta) = a_1 b_1 + \cdots + a_n b_n.$$

Example 2. The ***Lorentz*** inner product in four-space:

$$\alpha = (a_1, \ldots, a_4), \quad \beta = (b_1, \ldots, b_4),$$
$$(\alpha, \beta) = a_1 b_1 + a_2 b_2 + a_3 b_3 - c^2 a_4 b_4$$

where *c* is the speed of light.

Condition (iii) is equivalent to the following. If $\sigma^1, \ldots, \sigma^n$ is a basis of *L*, then

$$\left|(\sigma^i, \sigma^j)\right| \neq 0.$$

(The left-hand member is the Gram determinant, or ***Grammian***.) For this determinant vanishes if and only if there is a nontrivial solution $(a_1, \ldots, a_n)$ of the homogeneous system

$$\sum a_i(\sigma^i,\sigma^j)=0\,.$$

But this is the same as having the vector

$$\alpha=\sum a_i\sigma^i$$

satisfy the relation $(\alpha,\beta)=0$ for all β.

An ***orthonormal basis*** of L consists of a basis $\sigma^1,\ldots,\sigma^n$ such that

$$(\sigma^i,\sigma^j)=\pm\delta^{ij}\,.$$

If there are r plus signs and s minus signs, then $r+s=n$, and $t=r-s$ is the ***signature*** of the inner product. It does not depend on the choice of basis.

It is a basic fact that ***each inner product space*** L ***has an orthonormal basis***. This is proved in several steps.

1. If dim $L>0$, there is a vector σ in L such that $(\sigma,\sigma)\neq 0$. For if $(\alpha,\alpha)=0$ for all α, then

$$0=(\alpha+\beta,\alpha+\beta)=(\alpha,\alpha)+2(\alpha,\beta)+(\beta,\beta)=2(\alpha,\beta)\,,$$

$$(\alpha,\beta)=0 \text{ for all } \alpha,\beta\,,$$

a contradiction to nondegeneracy.

2. Pick a ***maximal*** sequence $\sigma^1,\ldots,\sigma^r$ of vectors satisfying

$$(\sigma^i,\sigma^j)=\pm\delta^{ij}\,.$$

Let M be the subspace of L these vectors span. Then $\dim M=r$. [The σ^i are independent since $\sum a_i\sigma^i=0$ implies $\sum a_i(\sigma^i,\sigma^j)=0$, $\pm a_j=0$.]

We suppose $r<n$.

3. Let N be the orthogonal complement of M, i.e., N is the space of all vectors β such that $(\alpha,\beta)=0$ for all α in M. Since N is determined by the r relations $(\sigma^i,\beta)=0$, $\dim N\geq n-r$. But obviously $M\cap N=0$ (i.e., the only vector common to M and N is 0), hence $\dim N=n-r$, M and N together span L, $M+N=L$.

4. N itself is an inner product space relative to the inner product of L. Only the property (iii) of nondegeneracy must be checked. Suppose β is in N and $(\gamma,\beta)=0$ for all γ in N. But $(\alpha,\beta)=0$ for all α in M, hence $(\alpha,\beta)=0$ for all α in L since M and N together span L. Hence $\beta=0$.

5. By (1), there is a vector α in N such that $(\alpha,\alpha)\neq 0$. We set

$$\sigma^{r+1}=\alpha/|(\alpha,\alpha)|^{1/2}$$

and see that we have constructed a sequence $\sigma^1,\ldots,\sigma^{r+1}$ longer than a maximal one. Since this is impossible we conclude that we must have had $r=n$ in the first place, which completes the proof.

There is another basic property of inner product spaces which we shall need below.

Let f be a linear functional on L. Then there is a unique vector β *in L such that*

$$f(\alpha) = (\alpha, \beta).$$

This is easily established by taking an orthonormal basis $\sigma^1, \ldots, \sigma^n$. We set $b_i = f(\sigma^i)$ and for β simply take

$$\beta = \sum \pm b_j \sigma^j = \sum (\sigma^j, \sigma^j) b_j \sigma^j .$$

For then

$$(\sigma^i, \beta) = \sum_j (\sigma^j, \sigma^j) b_j (\sigma^i, \sigma^j) = b_j = f(\sigma^i).$$

2.6. Inner Products of *p*-Vectors

Again we start with an n-dimensional vector space L with an inner product (α, β). We shall define an ***induced inner product*** on each of the spaces $\wedge^p L$. We set

$$(\lambda, \mu) = |(\alpha_i, \beta_i)|$$

for $\lambda = \alpha_1 \wedge \cdots \wedge \alpha_p$, $\mu = \beta_1 \wedge \cdots \wedge \beta_p$. This definition works because the determinant on the right is an alternating multilinear function of the $\alpha's$, ditto the $\beta's$. This means the formula defines a scalar-valued function on $(\wedge^p L) \times (\wedge^p L)$ which is linear in each variable. Next $(\mu, \lambda) = (\lambda, \mu)$ because interchanging the rows and columns of a matrix (transposing) does not change its determinant.

The nondegeneracy of this inner product is most easily seen by computing with respect to an orthonormal basis $\sigma^1, \ldots, \sigma^n$ of *L*. As usual the σ^H, $H = \{h_1 < h_2 < \cdots < h_p\}$, form a basis of $\wedge^p L$. We have

$$(\sigma^H, \sigma^K) = \left|(\sigma^{h_i}, \sigma^{k_j})\right|.$$

If $H \neq K$, this is zero since the determinant has a row (also a column) of zeros. If $H = K$, all but the diagonal elements vanish and these are ± 1, hence

$$(\sigma^H, \sigma^K) = \pm \delta^{H,K}.$$

In other words; the σ^H form an orthonormal basis of $\wedge^p L$, nondegeneracy follows free of charge.

In particular $\sigma = \sigma^1 \wedge \cdots \wedge \sigma^n$ is an orthonormal basis of $\wedge^p L$ and

$$(\sigma,\sigma) = (\sigma^1,\sigma^1)\cdots(\sigma^n,\sigma^n) = (-1)^{(n-t)/2},$$

where t is the **signature** of L.

For another example, set

$$\alpha^i = \sigma^1 \wedge \cdots \wedge \sigma^{i-1} \wedge \sigma^{i+1} \wedge \cdots \wedge \sigma^n,$$

forming a basis of $\wedge^{n-1} L$. Clearly

$$(\alpha^i,\alpha^i) = (\sigma,\sigma)/(\sigma^i,\sigma^i) = (\sigma,\sigma)(\sigma^i,\sigma^i),$$

hence

$$\begin{aligned}\left(\sum a_i\alpha^i, \sum b_i\alpha^i\right) &= (\sigma,\sigma)\sum(\sigma^i,\sigma^i)a_ib_i \\ &= (\sigma,\sigma)\left(\sum a_i\sigma^i, \sum b_j\sigma^j\right).\end{aligned}$$

2.7. The Star Operator

Again let L have inner product (α,β). We shall take a definite orientation of L which will remain fixed. (This simply means we take one basis for L and only consider other bases which are expressed in terms of this one by a matrix with positive determinant. The space L has two orientations and we take one of them.) We only use bases coherent to the orientation.

We shall define an operation *, called the (***Hodge***) ***star operator***. This will be a linear transformation on $\wedge^p L$ onto $\wedge^{n-p} L$. This operator depends, of course, on the inner product and also depends on the orientation. Reversing orientation will change its sign.

We note that the orientation of L determines a definite orthonormal basis σ of $\wedge^n L$.

Now fix λ in $\wedge^p L$. The mapping

$$\mu \to \lambda \wedge \mu$$

is a linear transformation on $\wedge^{n-p} L$ into the one-dimensional space $\wedge^n L$. We may write

$$\lambda \wedge \mu = f_\lambda(\mu)\sigma,$$

where f_λ is a linear functional on $\wedge^{n-p} L$. By our result at the end of Section 2.5, there is a unique (n - p)-vector, which we denote $*\lambda$ to indicate its dependence on λ, such that

$$\lambda \wedge \mu = (*\lambda,\mu)\sigma.$$

This equation defines the * map which is evidently linear on $\wedge^p L$ into $\wedge^{n-p} L$.

In order to compute $*\lambda$ for generators of $\wedge^p L$, in view of the linearity, it is enough to compute $*\lambda$ where $\lambda = \sigma^1 \wedge \cdots \wedge \sigma^p$ and where $\sigma^1,\ldots,\sigma^n$ is an orthonormal basis. Let K run over sets of q = n - p indices. Then

$$\lambda \wedge \sigma^K = (*\lambda, \sigma^K)\sigma .$$

The left-hand side vanishes unless $K = \{p+1, p+2, \ldots, n\}$, hence

$$*\lambda = c\sigma^{p+1} \wedge \cdots \wedge \sigma^n ,$$

and the constant c is determined taking $K = \{p+1, \ldots, n\}$:

$$\sigma = \lambda \wedge \sigma^K = c(\sigma^K, \sigma^K)\sigma ,$$

$$c = (\sigma^K, \sigma^K) = \pm 1 ,$$

$$*\lambda = (\sigma^K, \sigma^K)\sigma^K .$$

For definiteness, set $H = \{1, \ldots, p\}$, $K = \{p+1, \ldots, n\}$. We have proved

$$*\sigma^H = (\sigma^K, \sigma^K)\sigma^K .$$

Since $\sigma^K \wedge \sigma^H = (-1)^{p(n-p)}\sigma^H \wedge \sigma^K$, we deduce, taking orientation into account,

$$*\sigma^K = (-1)^{p(n-p)}(\sigma^H, \sigma^H)\sigma^H ,$$

hence

$$*(*\sigma^H) = (-1)^{p(n-p)}(\sigma^H, \sigma^H)(\sigma^K, \sigma^K)\sigma^H ,$$

or

$$\begin{aligned} *(*\sigma^H) &= (-1)^{p(n-p)}(\sigma, \sigma)\sigma^H \\ &= (-1)^{p(n-p)+(n-t)/2}\sigma^H \end{aligned}$$

where t is the **signature**.

It follows that if α is any p-vector, then

$$**\alpha = (-1)^{p(n-p)+(n-t)/2}\alpha .$$

Another consequence of these formulas is this result.

If α, β are p-vectors, then

$$\alpha \wedge *\beta = \beta \wedge *\alpha = (-1)^{(n-t)/2}(\alpha, \beta)\sigma .$$

For when $\beta = \sigma^H$ as above, the only generator $\alpha = \sigma^J$ for which both sides do not vanish is $\alpha = \sigma^H$, and then

$$\begin{aligned} \alpha \wedge *\beta &= \sigma^H \wedge (\sigma^K, \sigma^K)\sigma^K = (\sigma^K, \sigma^K)\sigma \\ &= (\sigma^H, \sigma^H)(-1)^{(n-t)/2}\sigma \\ &= (-1)^{(n-t)/2}(\alpha, \beta)\sigma \end{aligned} .$$

Example 1. We take 4-space with coordinates so normalized that dx^1, dx^2, dx^3, dt is an orthonormal basis with $(dx^i, dx^i) = 1$, $(dt, dt) = -1$. We have n=4, t=2, $(-1)^{(n-t)/2} = -1$. We shall study certain two-forms. For p=2, $p(n\text{-}p)$=4. Thus

$$*(dx^i dt) = dx^j dx^k ,$$

where (i, j, k) is cyclic order,

$$*(dx^j dx^k) = -dx^i dt .$$

Let E_i be the components of electric field strength, H_i the components of magnetic field strength (all in free space) and consider the form

$$\omega = (E_1 dx^1 + E_2 dx^2 + E_3 dx^3)dt + (H_1 dx^2 dx^3 + H_2 dx^3 dx^1 + H_3 dx^1 dx^2) .$$

Then

$* \omega = -(H_1 dx^1 + H_2 dx^2 + H_3 dx^3) dt + (E_1 dx^2 dx^3 + E_2 dx^3 dx^1 + E_3 dx^1 dx^2)$.

We shall see the use of these forms in Maxwell's equations later.

Example 2. E^3 with the ordinary metric. If *f* and *g* are functions,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz ,$$

$$* df = \frac{\partial f}{\partial x} dydz + \frac{\partial f}{\partial y} dzdx + \frac{\partial f}{\partial z} dxdy ,$$

and we have

$$df \wedge * dg = \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} \right) dxdydz .$$

2.8 Problems

III. Exterior Derivative

3.1. Differential Forms

Let P be a point in E^n. The ***one-forms*** at P are the expressions

$$\sum_1^n a_i s x^i \ , \quad (a_i \ \text{constants}).$$

These form an n-dimensional linear space $L = L_P$. The ***p-forms*** at P are the elements of

$$\wedge^P L = \wedge^P L_P ,$$

i.e., expressions

$$\sum a_H dx^{h_1} \cdots dx^{h_p} \ , \quad a_H \ \text{constants.}$$

Note that we are dropping the notation "$\wedge$" so that differentials dx^i juxtaposed will always be multiplied by exterior multiplication.

Now let U denote an (open) domain in E^n. A ***p-form*** on U is obtained by choosing at each point P of U a p-form at that point, and doing this smoothly. Thus a p-form ω has the representation

$$\omega = \sum a_H (x^1, \ldots, x^n) dx^H \ ,$$

where the functions $a_H(\boldsymbol{x})$ are smooth functions on U, differentiate as often as we please.

The exterior algebra applies at each point of U and so may be interpreted on the differential forms on U itself. Thus if ω is a p-form and η is a q-form on U, then $\omega \wedge \eta$ is a $(p + q)$-form on U. (Of course $\omega \wedge \eta = 0$ if $p + q > n$.) If

$$\omega = \sum a_H dx^H \ , \quad \eta = \sum b_K dx^K \ ,$$

then

$$\omega \wedge \eta = \sum a_H b_K dx^H dx^K \ ,$$

so that the coefficients of $\omega \wedge \eta$ are again smooth functions, being polynomials in the coefficients of ω and η.

For example a **one-form**

$$\omega = Pdx + Qdy + Rdz$$

may be identified with an ordinary ***vector field*** (P, Q, R) in E^3, a **two-form**

$$\alpha = Adydz + Bdzdx + Cdxdy$$

may be identified with a ***polar vector field*** in E^3.

3.2. Exterior Derivatives

We denote by

$$F^P(U)$$

the ***totality of p-forms*** on *U*. In particular $F^0(U)$ is simply the set of all smooth functions on *U*.

We shall now set up an operation *d* which takes each *p*-form ω to a (*p*+l)-form $d\omega$. In E^3 it will work this way. For a **0-form** *f*,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \,.$$

For the **one-form** ω above,

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \,,$$

while for the **two-form** α above,

$$d\alpha = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)dxdydz \,.$$

Thus the operator *d* subsumes the ordinary ***gradient***, ***curl*** or ***rotation***, and ***divergence***.

It will turn out that *d* is completely independent of coordinate systems. This will be more or less clear when we axiomatize *d*.

We shall establish the existence and uniqueness of an operator

$$d : F^P(U) \to F^{P+1}(U)$$

such that

(i) $d(\omega + \eta) = d\omega + d\eta$,

(ii) $d(\lambda \wedge \mu) = d\lambda \wedge \mu + (-1)^{(\deg \lambda)} \lambda \wedge d\mu$,

(iii) For each ω, $d(d\omega) = 0$,

(iv) For each function *f*, $df = \sum \frac{\partial f}{\partial x^i} dx^i$.

Let us note the consistency of (iv) as it applies to the coordinate functions. For example x^1 is a function on *U* and $d(x^1)$ the effect of *d* on this function x^1 is the symbol dx^1. Thus from (iii), $d(dx^1) = 0$ once we have *d.*

First we prove there is only one such operation *d.* Suppose we are given such a *d.* We first show that

$$d(dx^{h_1} \cdots dx^{h_p}) = 0$$

by induction on *p*. We have just noted this for *p* = 1. If it is true for *p*-1, then by (ii),

$$d[x^{h_1}(dx^{h_2} \cdots dx^{h_p})] = dx^{h_1} \cdots dx^{h_p} \,,$$

$$d(dx^{h_1} \cdots dx^{h_p}) = d\left\{d(x^{h_1} dx^{h_2} \cdots dx^{h_p})\right\} = 0$$

by (iii). Now if ω is a p-form,

$$\omega = \sum a_H(\boldsymbol{x}) dx^H \ ,$$

$$\begin{aligned} d\omega &= \sum d(a_H dx^H) \\ &= \sum (da_H) dx^H \qquad , \\ &= \sum \frac{\partial a_H}{\partial x^j} dx^j dx^H \end{aligned}$$

which shows that the recipe (i-iv) completely determines $d\omega$. To prove that there exists such an operator d, we simply set

$$d\omega = \sum \frac{\partial a_H}{\partial x^j} dx^j dx^H$$

for $\omega = \sum a_H dx^H$ and check that the properties are satisfied. Properties (i) and (iv) are fairly clear; let us look at (ii) and (iii). Evidently if we can establish these for monomials, by summation they will follow generally.

Suppose

$$\lambda = a dx^H \ , \quad \mu = b dx^K \ .$$

Then

$$\begin{aligned} & d(\lambda \wedge \mu) = d(ab dx^H dx^K \\ & = \sum \frac{\partial (ab)}{\partial x^i} dx^i dx^H dx^K \\ & = \sum \frac{\partial a}{\partial x^i} b dx^i dx^H dx^K + \sum a \frac{\partial b}{\partial x^i} dx^i dx^H dx^K \qquad . \\ & = \sum \left(\frac{\partial a}{\partial x^i} dx^i dx^H \right) \wedge (b dx^K + (-1)^{(\deg \lambda)} \sum (a dx^H) \wedge \left(\frac{\partial b}{\partial x^i} dx^i dx^K \right) \\ & = (d\lambda) \wedge \mu + (-1)^{(\deg \lambda)} \lambda \wedge d\mu \end{aligned}$$

The sign results from

$$dx^i dx^H = (-1)^{(\deg \lambda)} dx^H dx^i \ .$$

This proves (ii).

Again, let $\omega = a dx^H$. Then

$$\begin{aligned} d(d\omega) &= d\left(\sum \frac{\partial a}{\partial x^i} dx^i dx^H \right) \\ &= \sum \frac{\partial^2 a}{\partial x^i \partial x^j} dx^j dx^i dx^H \\ &= \frac{1}{2} \sum \left(\frac{\partial^2 a}{\partial x^i \partial x^j} - \frac{\partial^2 a}{\partial x^j \partial x^i} \right) dx^j dx^i dx^H \\ &= 0 \end{aligned}$$

which verifies (iii).

Property (iii) is nothing more than the equality of mixed second partial derivatives. It is the source of most "***integrability conditions***" in partial differential equations and differential geometry. It is usually referred to as the ***Poincaré Lemma***.

3.3. Mappings

We study the following situation: U is a domain in E^m, V is a domain in E^n and ϕ is a smooth mapping on U into V. We write
$\phi : U \to V$.
Also, we denote by $x^1,\ldots,x^m$ the coordinates of E^m and by $y^1,\ldots,y^n$ the coordinates of E^n. Then we can write

$$y^i = y^i(x^1,\ldots,x^m)$$

to show that the point with coordinates $\boldsymbol{x}$ is transformed by ϕ to the point with coordinates $\boldsymbol{y}$. The functions $y^i(\boldsymbol{x})$ are smooth.

As before, R denotes the reals. If g is any real-valued function on V,

$$g : V \to R$$

then we may combine this with ϕ to obtain a function on U to R which we write

$$\phi^* g = g \circ \phi .$$

Thus

$$\phi^* : F^0(V) \to F^0(U) .$$

From the mapping ϕ on U to V we have constructed a new (induced) mapping ϕ^* on $F^0(V)$ to $F^0(V)$.

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[dr, "\phi^* g = g \circ \phi"'] & V \arrow[d, "g"] \\ & R \end{tikzcd}$$

We are now going to define a map ϕ^* taking p-forms on V to p-forms on U:

$$\phi^* : F^P(V) \to F^P(U) .$$

(Strictly speaking we should index ϕ^* and write $\phi_P{}^*$, p = 0, 1,…, but we shall skip this.) We have taken care of p = 0 already. The crucial case is p=1; after we do that, the algebraic considerations of Chapter II do the rest of the work.

The basic idea is ***substitution of coordinate functions***, replacing dy^i by

$$\sum \frac{\partial y^i}{\partial x^j} dx^j \ .$$

Thus if $\omega = \sum a_i(\boldsymbol{y}) dy^i$ is a one-form on V, we set

$$\phi^* \omega = \sum a_i(\boldsymbol{y}(\boldsymbol{x})) \frac{\partial y^i}{\partial x^j} dx^j \ .$$

We now have

$$\phi^* : F^1(V) \to F^1(U) .$$

By the method of Section 2.4, we extend this mapping to the exterior products to obtain

$$\phi^* : F^P(V) \to F^P(U) .$$

As an example,

$$\begin{aligned}
\phi^*(dy^1 dy^2) &= (\phi^* dy^1)(\phi^* dy^2) \\
&= \left(\sum \frac{\partial y^1}{\partial x^i} dx^i\right)\left(\sum \frac{\partial y^2}{\partial x^j} dx^j\right) \\
&= \sum \frac{\partial y^1}{\partial x^i}\frac{\partial y^2}{\partial x^j} dx^i dx^j \\
&= \frac{1}{2}\sum \left(\frac{\partial y^1}{\partial x^i}\frac{\partial y^2}{\partial x^j} - \frac{\partial y^1}{\partial x^j}\frac{\partial y^2}{\partial x^i}\right) dx^i dx^j \\
&= \frac{1}{2}\sum \frac{\partial(y^1, y^2)}{\partial(x^i, x^j)} dx^i dx^j
\end{aligned} \ .$$

We now list the basic properties of ϕ^*.

(i) $\phi^*(\omega + \eta) = \phi^* \omega + \phi^* \eta$.

(ii) $\phi^*(\lambda \wedge \mu) = (\phi^* \lambda) \wedge (\phi^* \mu)$.

(iii) If ω is a p-form on V, $d(\phi^* \omega) = \phi^*(d\omega)$.

(iv) If $\phi : U \to V$ and $\psi : V \to W$, then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

The first property is evident and the second follows from the final formula of Section 2.4.

Property (iii) is essentially the chain rule for partial derivatives. First we take a 0-form g on V.

$$dg = \sum \frac{\partial g}{\partial y^j} dy^j \ ,$$

$$\begin{aligned}
\phi^* dg &= \sum \frac{\partial g(\boldsymbol{y}(\boldsymbol{x}))}{\partial y^j}\frac{\partial y^j}{\partial x^i} dx^i \\
&= \sum \frac{\partial(\phi^* g)}{\partial x^i} dx^i \\
&= d\phi^* g
\end{aligned} \ .$$

We proceed inductively, supposing we have verified (iii) for (p-1)-forms. It

suffices to verify (iii) for p-forms ω which are monomial since each p-form is a sum of such. Suppose then that

$$\omega = g dy^H = g d\eta ,$$

where $\eta = y^{h_1} dy^{h_2} \cdots dy^{h_p}$ is a (p- 1)-form. Then

$$\begin{aligned}\phi^* \omega = (\phi^* g)(\phi^* d\eta) = (\phi^* g) \wedge (d\phi^* \eta) ,\\ d(\phi^* \omega) = d(\phi^* g) \wedge d(\phi^* \eta) ,\end{aligned}$$

and

$$d\omega = dg \wedge d\eta ,$$

$$\begin{aligned}\phi^* d\omega &= (\phi^* dg) \wedge (\phi^* d\eta)\\ &= d(\phi^* g) \wedge d(\phi^* \eta) ;\\ &= d\phi^* \omega\end{aligned}$$

we have pushed through the next case.

We now look at the final property (iv).

For a **0-form** (**function**) h on W we have

$$\begin{aligned}[(\psi \circ \phi)^* h](\boldsymbol{x}) &= h[(\psi \circ \phi)(\boldsymbol{x})] = h\{\psi[\phi(\boldsymbol{x})]\}\\ &= [\psi^* h][\phi(\boldsymbol{x})] = \{\phi^* [\psi^* h]\}(\boldsymbol{x}) ,\\ &= [(\phi^* \circ \psi^*) h](\boldsymbol{x})\end{aligned}$$

hence

$$(\psi \circ \phi)^* h = (\phi^* \circ \psi^*) h .$$

An induction similar to that above establishes the property in general. All it means is that one can substitute directly the expressions for the coordinates z^k on W in terms of the coordinates x^i on U, or indirectly by first going through the coordinates y^j of V; the results are the same.

$$\begin{tikzcd} \mathbf{U} \arrow[r, "\phi"] \arrow[dr, "\psi \circ \phi"'] & \mathbf{V} \arrow[d, "\psi"] \\ & \mathbf{W} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{F}^p(\mathbf{U}) & \mathbf{F}^p(\mathbf{V}) \arrow[l, "\phi^*"'] \\ & \mathbf{F}^p(\mathbf{W}) \arrow[u, "\psi^*"'] \arrow[ul, "{(\psi \circ \phi)^* = (\phi^* \circ \psi^*)}"] \end{tikzcd}$$

What has really been seen in this section is that one can carry on fearlessly with the most obvious kind of calculations with differential forms.

Examples. Consider the map $\phi : t \to (x, y)$ on $E^1 \to E^2$ given by $x = t^2$, $y = t^3$. If $\omega = x dy$, a one-form on E^2,

$$\phi^* \omega = (t^2) \frac{\partial y}{\partial t} dt = 3t^4 dt .$$

Take the map $\psi : (x, y) \to t = x - y$.

$$\psi^* (dt) = dx - dy .$$

One final remark. Suppose $m < n$ and ϕ is a map on the domain U of

E^m into the domain V of E^n. If ω is a p-form on V and $p > m$, then necessarily $\phi^* \omega = 0$.

3.4 Change of Coordinates

We apply the results of the last section to the special case in which U and V are both domains in E^n and ϕ is a one-to-one mapping on U onto V with both ϕ and $\psi = \phi^{-1}$ smooth. (Note the map $x \to y = x^3$ on $E^1 \to E^1$ is one-to-one and smooth. But the inverse map $y \to x = y^{1/3}$ is not smooth - no derivative at y=0.) In each figure, ι is the identity map, $\iota(\boldsymbol{x}) = \boldsymbol{x}$. It follows that ϕ^* is a one-one map on $F^P(V)$ onto $F^P(U)$ and its inverse is ψ^*. If we interpret the coordinates $\boldsymbol{y}$ of V as new coordinates on U, the result

$$d\phi^* \omega = \phi^* d\omega$$

means that the ***exterior derivative of a differential form is independent of the coordinate system in which it is computed***.

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[dr, "\iota"'] & V \arrow[d, "\psi"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\psi"] \arrow[dr, "\iota"'] & U \arrow[d, "\phi"] \\ & V \end{tikzcd}$$

This inner consistency of the differential form calculus is most important. Later we shall base the global theory (forms on manifolds) on this.

We note in passing that with a proper formulation this independence of d on the coordinate system can be obtained as a consequence of the four basic defining properties (i-iv) of the exterior derivatives in Section 3.2.

3.5 An Example from Mechanics

The following problem is taken from E. Goursat [15, p.85]. We work in a region with coordinates $(\boldsymbol{x},\boldsymbol{u})$ = ($x_1,\ldots,x_n,u_1,\ldots,u_n$). We are given a function

$$\phi = \phi(\boldsymbol{x},\boldsymbol{u})$$

which is supposed homogeneous of degree 2 in the variable $\boldsymbol{u}$. (For example, a kinetic energy form $\sum a_{ij}(\boldsymbol{x})u_i u_j$.) Define

$$p_i = \partial\phi / \partial u_i \,.$$

We assume that the mapping $(\boldsymbol{x},\boldsymbol{u}) \to (\boldsymbol{x},\boldsymbol{p})$ defines a regular change of variables. We then write

$$\phi(\boldsymbol{x},\boldsymbol{u})=\psi(\boldsymbol{x},\boldsymbol{p}) .$$

The problem is to prove the relations

$$\frac{\partial\psi}{\partial x_i}=-\frac{\partial\phi}{\partial x_i}\,,\quad \frac{\partial\psi}{\partial p_k}=u_k .$$

The proof depends on two things, the Euler formula for homogeneous functions which in our case implies

$$\sum\frac{\partial\phi}{\partial u_k}u_k=2\phi$$

i.e.,

$$\sum p_k u_k=2\phi ,$$

and the fact that exterior relations are independent of how they are derived. We have

$$d\phi=\sum\frac{\partial\phi}{\partial x_i}dx_i+\sum\frac{\partial\phi}{\partial u_k}du_k=\sum\frac{\partial\phi}{\partial x_i}dx_i+\sum p_k du_k \;,$$

and

$$2d\phi=\sum p_k du_k+\sum u_k dp_k \;,$$

hence by subtracting

$$d\phi=-\sum\frac{\partial\phi}{\partial x_i}dx_i+\sum u_k dp_k \;.$$

Now everything follows from $\phi=\psi$ and

$$d\psi=\sum\frac{\partial\psi}{\partial x_i}dx_i+\sum\frac{\partial\psi}{\partial p_k}dp_k \;.$$

3.6. Converse of the Poincaré Lemma

The Poincaré Lemma, $d(d\omega)=0$, has these interpretations in 3-space:

$$\operatorname{curl}(\operatorname{grad} f)=0\,,$$
$$\operatorname{div}(\operatorname{curl}\boldsymbol{v})=0\,,$$

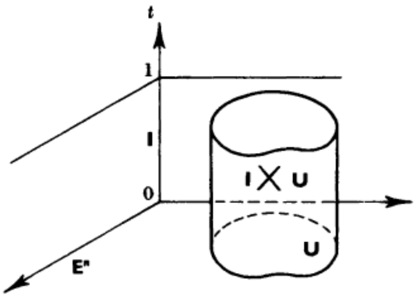
according to the examples at the beginning of Section 3.2. In vector analysis one proves that a curl-free vector field is a ***gradient*** by line integrals and that a divergence-free vector field is a ***curl***, usually by a brute-force method. We are now going to prove a general result. If ω is a p-form ($p\geq 1$) and $d\omega=0$, then there is a (p - 1)-form α such that $\omega=d\alpha$. The result is hard if p > 1 because there are many solutions. Also the result is valid only in domains which are not too complicated topologically.

The demonstration is based on a "***cylindrical construction***." We begin with a domain U in E^n. We denote by $I=[0,1]$ the ***unit interval*** on the

t-axis and consider the ***cylinder*** or ***product space***.

$$I \times U \,.$$

This consists of all pairs $(t, \boldsymbol{x})$ where $0 \le t \le 1$ and $\boldsymbol{x}$ runs over points of U.



We single out the two maps which identify U with the top and bottom of the cylinder, namely,

$$j_1 : U \to I \times U\,,\quad j_1(\boldsymbol{x}) = (1, \boldsymbol{x})\,,$$
$$j_0 : U \to I \times U\,,\quad j_0(\boldsymbol{x}) = (0, \boldsymbol{x})$$

Thus

$$j_i{}^* : F^P(I \times U) \to F^P(U)\,,\quad (i = 0,1)\,.$$

For example, to form $j_1{}^*\omega$ where ω is a form on $I \times U$ simply replace t by 1 wherever it occurs in ω (and dt by 0 correspondingly)

We now form a new operation K,

$$K : F^{P+1}(I \times U) \to F^P(U)\,;$$

K is defined on monomials by the formulas

$$K(a(t,\boldsymbol{x})dx^H) = 0\,,$$

$$K(a(t,\boldsymbol{x})dtdx^J) = \left(\int_0^1 a(t,x)dt\right)dx^J\,,$$

and on general differential forms by summing the results on the monomial parts. Here is the basic property of K: If ω is any $(p + 1)$-form on $I \times U$, then

$$K(d\omega) + d(K\omega) = j_1{}^*\omega - j_0{}^*\omega\,.$$

It is enough to check this for monomials

Case 1. $\omega = a(t,\boldsymbol{x})dx^H$.

We have $K\omega = 0$, $dK\omega = 0$,

$$d\omega = \frac{\partial a}{\partial t}dtdx^H + [\text{terms\_free\_of\_}dt]\,,$$

$$Kd\omega = \left(\int_0^1 \frac{\partial a}{\partial t}dt\right)dx^H = [a(1,\boldsymbol{x}) - a(0,\boldsymbol{x})]dx^H\,.$$

But $j_1{}^*\omega = a(1,\boldsymbol{x})dx^H$, $j_0{}^*\omega = a(0,\boldsymbol{x})dx^H$ so the formula is valid.

Case 2. $\omega = a(t,\boldsymbol{x})dtdx^J$.

First $j_1{}^*\omega = j_0{}^*\omega = 0$. Next

$$
\begin{aligned}
Kd\omega &= K\left[-\sum \frac{\partial a}{\partial x^i} dt dx^i dx^J\right] \\
&= -\sum \left(\int_0^1 \frac{\partial a}{\partial x^i} dt\right) dx^i dx^J
\end{aligned},
$$

$$
\begin{aligned}
dK\omega &= d\left[\left(\int_0^1 a(t, \boldsymbol{x}) dt\right) dx^J\right] \\
&= \sum \frac{\partial}{\partial x^i}\left[\int_0^1 a(t, \boldsymbol{x}) dt\right] dx^i dx^J, \\
&= \sum \left(\int_0^1 \frac{\partial a}{\partial x^i} dt\right) dx^i dx^J
\end{aligned}
$$

so the formula again works.

Definition. A domain *U* is ***deformable to a point*** *P* if there is a mapping

$$\phi : I \times U \to U$$

such that

$$\phi(1, \boldsymbol{x}) = \boldsymbol{x}, \quad \phi(0, \boldsymbol{x}) = P.$$

The boundary conditions may be interpreted in terms of the j_i as follows:

$$\phi \circ j_1 = \iota, \quad \phi \circ j_0 = P.$$

For a *(p* + 1)-form ω on *U* we have as a consequence

$$j_1 * [\phi * \omega] = \omega, \quad j_0 * [\phi * \omega] = 0.$$

Now we can state and prove the main result.

Let U be a domain in E^n *which can be deformed to a point P. Let* ω *be a (p* + 1)*-form on U such that* $d\omega = 0$. *Then there is a p-form* α *on U such that*

$$\omega = d\alpha.$$

We merely substitute $\phi * \omega$ in the formula above to have

$$K[d(\phi * \omega)] + d[K(\phi * \omega)] = \omega.$$

But $d(\phi * \omega) = \phi * (d\omega) = 0$, hence $\omega = d\alpha$ with $\alpha = K(\phi * \omega)$.

It is interesting to see how far the solution of the equation $d\alpha = \omega$ is determined. If β is another solution, then $d\beta = \omega = d\alpha$, $d(\alpha - \beta) = 0$. If $p \geq 1$, we conclude by the main result again that $\alpha - \beta = d\lambda$ where λ is a *(p* - 1)-form. In other words, given one solution α, the general solution is $\alpha - d\lambda$ where λ is absolutely arbitrary. (When *p = 0,* α *and* β are functions and we conclude that $\alpha - \beta$ is constant.)

3.7. An Example

We shall illustrate this whole method in the case *n* = 3, *p* = 2. Thus we take a **two-form**

$$\omega = Adydz + Bdzdx + Cdxdy$$

in E^3 for which $d\omega = 0$, i.e.,

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0 .$$

The space E^3 can be deformed to 0 by the map

$$\phi(t, x, y, z) = (tx, ty, tz) .$$

The assertion is that $\omega = d\alpha$ where

$$\alpha = K\phi^* \omega .$$

First we compute $\phi^* \omega$:

$$\begin{aligned} \phi^* \omega &= A(tx, ty, tz) d(ty) d(tz) + \cdots \\ &= A(tx, ty, tz)(tdy + ydt)(tdz + zdt) + \cdots \\ &= A(tx, ty, tz)(ytdtdz - ztdtdy) + \cdots + (\text{terms\_free\_of\_}dt) \end{aligned} .$$

Now we have

$$\begin{aligned} \alpha = K(\phi^* \omega) = &\left(\int_0^1 A(tx, ty, tz) t dt \right)(ydz - zdy) \\ &+ \left(\int_0^1 B(tx, ty, tz) t dt \right)(zdx - xdz) \\ &+ \left(\int_0^1 C(tx, ty, tz) t dt \right)(xdy - ydx) \end{aligned} .$$

One verifies after some calculation that indeed $d\alpha = \omega$.

3.8. Further Remarks

For

$$\omega = Adydz + Bdzdx + Cdxdy$$

the problem of finding

$$\alpha = Pdx + Qdy + Rdz$$

so that

$$d\alpha = \omega$$

is that of finding three unknown functions *P, Q, R* of the three variables *x, y, z* so that the system

$$\begin{cases} \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z} = A \\ \dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x} = B \\ \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = A \end{cases}$$

of three partial differential equations is satisfied, the given functions *A, B, C* being subject to the necessary condition

$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z}=0\,.$$

It is remarkable that this system (and the more general ones covered in Section 3.6) can be solved by an explicit formula involving quadratures. In general, the ***theory of exterior differential forms*** exposes many types of systems of **partial** differential equations which are reducible to systems of **ordinary** differential equations and often solved by quadratures.

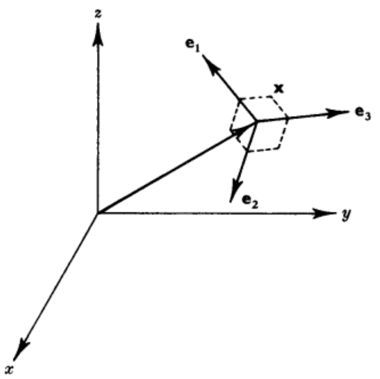
Another point to be noted is this. If we are dealing with a (p + 1)-form ω such that $d\omega=0$ and ω happens to depend on several parameters smoothly, then we can find an α such that $d\alpha=\omega$ and α depends on the same parameters just as smoothly. This again follows from the explicit formulas of Section 3.6.

3.9. Problems

IV. Applications

4.1. Moving Frames in E^3

We first point out that in dealing with vectors in Euclidean space, no matter where we draw them for picturesque purposes, when we deal with them analytically, they always start at the origin.



We attach to each point **x** of E^3 a right-handed orthonormal frame $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ and suppose that the vector fields $\boldsymbol{e_i}$ are smooth fields.

What we shall do is express everything in sight in terms of the $\boldsymbol{e_i}$, apply d to these relations to derive further ones, and continue until we obtain no further results.

First of all, $d\boldsymbol{x}$ is a vector with one-form coefficients, for example,

$$d\boldsymbol{x} = (dx, dy, dz) = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k} \ .$$

We express $d\boldsymbol{x}$ in terms of the frame $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ at the point **x**, which we certainly may do, say, by first expanding ***i, j, k*** in terms of the $\boldsymbol{e_i}$; and then collecting terms:

$$d\boldsymbol{x} = \sigma_1\boldsymbol{e_1} + \sigma_2\boldsymbol{e_2} + \sigma_3\boldsymbol{e_3} \ ,$$

where the σ_i are **one-forms**. We do the same with each $\boldsymbol{e_i}$;

$$d\boldsymbol{e_i} = \omega_{i1}\boldsymbol{e_1} + \omega_{i2}\boldsymbol{e_2} + \omega_{i3}\boldsymbol{e_3} \quad (i = 1, 2, 3)$$

where the ω_{ij} are **one-forms**.

Since $\boldsymbol{e_i} \cdot \boldsymbol{e_k} = \delta_{ik}$, we have

$$d\boldsymbol{e_i} \cdot \boldsymbol{e_k} + \boldsymbol{e_i} d\boldsymbol{e_k} = 0 \ ,$$

that is,

$$\omega_{ik} + \omega_{ki} = 0 \ .$$

In particular, $\omega_{ii} = 0$.

It will be convenient to introduce some matrix notation. We set

$$\boldsymbol{e} = \begin{pmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \\ \boldsymbol{e_3} \end{pmatrix}, \quad \sigma = (\sigma_1, \sigma_2, \sigma_3) \ , \quad \Omega = \|\omega_{ij}\| \ ,$$

and have these ***structure equations***:

$$d\boldsymbol{x} = \sigma\boldsymbol{e} ,$$

$$d\boldsymbol{e} = \Omega\boldsymbol{e} ,$$

$$\Omega + {}^t\Omega = 0 .$$

Here applying d to a matrix means simply applying it to each element. In the last equation, the left-hand superscript t denotes transpose of the matrix, i.e., interchange of rows and columns, so this equation expresses the skew-symmetry of Ω.

From $d(d\boldsymbol{x}) = 0$ we have

$$d\sigma\boldsymbol{e} - \sigma d\boldsymbol{e} = 0 ,$$

$$d\sigma\boldsymbol{e} - \sigma\Omega\boldsymbol{e} = 0 ,$$

$$(d\sigma - \sigma\Omega)\boldsymbol{e} = 0 .$$

Because the $\boldsymbol{e}_i$ are linearly independent, this means

$$d\sigma = \sigma\Omega .$$

Similarly, from $d(d\boldsymbol{e}) = 0$, we have

$$0 = d\Omega\boldsymbol{e} - \Omega d\boldsymbol{e} = (d\Omega - \Omega^2)\boldsymbol{e} ,$$

$$d\Omega = \Omega^2 .$$

In summary, then we have

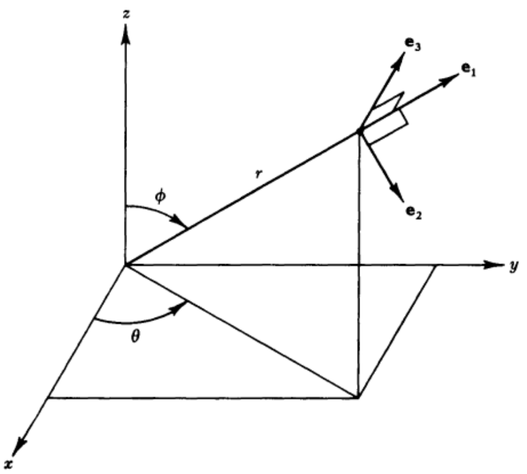
Structure equations	**Integrability conditions**
$\begin{cases} d\boldsymbol{x} = \sigma\boldsymbol{x} \\ d\boldsymbol{e} = \Omega\boldsymbol{e} \\ \Omega + {}^t\Omega = 0 \end{cases}$	$\begin{cases} d\sigma = \sigma\Omega \\ d\Omega = \Omega^2 \end{cases}$

Further differentiation does not lead to new results. We shall see in our study of Riemannian geometry that the equation $d\Omega - \Omega^2 = 0$ expresses the ***lack of curvature of Euclidean space***.

A point to be noticed is that the **three-form** $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ is precisely the ***element of volume*** in E^3:

$$\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = dxdydz .$$

We shall verify this in the next section.



It will be observed that the calculations of this section work equally well in E^n.

Example. Spherical coordinates. The orthonormal unit vectors $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ are taken in the directions of increasing r, ϕ, θ, respectively. From

$$\boldsymbol{x} = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\theta)$$

we have

$$\begin{aligned} d\boldsymbol{x} &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\theta)dr \\ &\quad + (r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi)d\phi \\ &\quad + (-r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0)d\theta \\ &= (dr)\boldsymbol{e_1} + (rd\phi)\boldsymbol{e_2} + (r\sin\phi d\theta)\boldsymbol{e_3} \end{aligned},$$

with

$$\begin{cases} \boldsymbol{e_1} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \\ \boldsymbol{e}_2 = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) \\ \boldsymbol{e}_3 = (-\sin\theta, \cos\theta, 0) \end{cases},$$

and so

$$\sigma_1 = dr, \quad \sigma_2 = rd\phi, \quad \sigma_3 = r\sin\phi d\theta.$$

Differentiating,

$$\begin{cases} d\boldsymbol{e_1} = (d\phi)\boldsymbol{e_2} + (\sin\phi d\theta)\boldsymbol{e_3} \\ d\boldsymbol{e}_2 = (-d\phi)\boldsymbol{e}_1 + (\cos\phi d\theta)\boldsymbol{e_3} \end{cases},$$

hence since Ω is skew-symmetric,

$$\Omega = \begin{pmatrix} 0 & d\phi & \sin\phi d\theta \\ -d\phi & 0 & \cos\phi d\theta \\ -\sin\phi d\theta & -\cos\phi d\theta & 0 \end{pmatrix}.$$

The volume element is

$$\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = r^2 \sin\phi dr d\phi d\theta.$$

4.2. Relation between Orthogonal and Skew-symmetric Matrices

It is no accident that Ω turns out to be skew-symmetric. This is a consequence of the principle that the first-order approximation to an orthogonal transformation is a skew-symmetric one. We shall look at this from several viewpoints.

A matrix B is orthogonal if its transpose equals its inverse, ${}^tB = B^{-1}$, or $B{}^tB = {}^tBB = I$. Suppose A is skew-symmetric, $A + {}^tA = 0$. Then for small ε we set $B = I + \varepsilon A$ and have

$$B{}^tB = (I + \varepsilon A)(I - \varepsilon A) = I + O(\varepsilon^2)$$

that B is orthogonal up to first-order terms.

Here is another approach. Let A be skew-symmetric. Since the

characteristic roots of A are pure imaginary, $I + A$ and $I - A$ are non-singular. Set

$$B = \frac{I + A}{I - A}.$$

Then

$$B^t B = \left(\frac{I + A}{I - A}\right)\left(\frac{I - A}{I + A}\right) = I\,,$$

so that B is orthogonal.

Next we re-examine the calculations of the last section. Let

$$\boldsymbol{i} = \begin{pmatrix} \boldsymbol{i}_1 \\ \boldsymbol{i}_2 \\ \boldsymbol{i}_3 \end{pmatrix},$$

where the $\boldsymbol{i}_j$ are the fixed unit vectors in the x, y, z directions, respectively (***i, j, k*** in usual vector notation). Then

$$\boldsymbol{e}_i = \sum b_{ij}\boldsymbol{i}_j\,,\quad \boldsymbol{e} = B\boldsymbol{i}$$

leading to a matrix $B = \|b_{ij}\|$ which is clearly orthogonal:

$$I = \boldsymbol{e}^t\boldsymbol{e} = B\boldsymbol{i}^t\boldsymbol{i}^t B = BI^t B = B^t B\,.$$

(Now we can prove the fact $dxdydz = \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ mentioned at the end of the last section. We have

$$d\boldsymbol{x} = (dx, dy, dz)\boldsymbol{i} = \sigma\boldsymbol{e} = \sigma B\boldsymbol{i}\,,$$

$$(dx, dy, dz) = \sigma B\,,$$

hence

$$dxdydz = |B|\sigma_1 \wedge \sigma_2 \wedge \sigma_3\,.$$

But from ${}^t BB = I$ we have $|B|^2 = 1$, $|B| = \pm 1$. Since we are supposing $\boldsymbol{e}$ is a right-handed system, $|B| = +1$,

$$dxdydz = \sigma_1 \wedge \sigma_2 \wedge \sigma_3\,.)$$

Then we have

$$d\boldsymbol{e} = dB\boldsymbol{i} = (dB)B^{-1}\boldsymbol{e}$$

so that

$$\Omega = (dB)B^{-1}\,.$$

We note this general result: *If A is an orthogonal matrix whose elements are functions of any number of variables, then*

$$(dA)A^{-1}$$

is a skew-symmetric matrix of one-forms.

For we have

$$
\begin{aligned}
{}^{t}AA &= I \\
{}^{t}dAA + {}^{t}AdA &= 0 \\
{}^{t}A^{-1}\,{}^{t}dA + dAA^{-1} &= 0 \\
{}^{t}(dAA^{-1}) + dAA^{-1} &= 0
\end{aligned}.
$$

There is also a converse which is important. *Suppose A is a matrix of functions defined on a domain U. Suppose A is orthogonal at a single point of U and that*

$$dA = \Lambda A,$$

where Λ is a skew-symmetric matrix of one-form. Then A is orthogonal on all of U.

We set $C = {}^{t}AA$ and have

$$dC = ({}^{t}dA)A + {}^{t}A(dA) = (-{}^{t}A\Lambda)A + {}^{t}A(\Lambda A) = 0,$$

hence C is a constant matrix on U. But we are assuming $C = I$ at one point of U, hence $C = I$ on U, ${}^{t}AA = I$ on U, A is orthogonal.

Another point is this. If A is a variable orthogonal matrix (transformation), each point $\boldsymbol{v_0}$ of space is sent by the general A to

$$\boldsymbol{v} = A\boldsymbol{v_0}.$$

We then have

$$d\boldsymbol{v} = dA\boldsymbol{v_0} = (dA)A^{-1}\boldsymbol{v},$$

so that one passes from $\boldsymbol{v}$ to the "infinitely near" vector $\boldsymbol{v} + d\boldsymbol{v}$ under the action of the general A of our family by means of

$$d\boldsymbol{v} \to \boldsymbol{v} + d\boldsymbol{v} = [I + (dA)A^{-1}]\boldsymbol{v}$$

with the skew-symmetric $(dA)A^{-1}$ representing this "***infinitesimal transformation***."

All of these considerations work equally well in E^n.

4.3. The 6-dimensional Frame Space

We consider the space of **all** right-handed orthonormal frames $\boldsymbol{E_1}$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ at **all** points $\boldsymbol{x}$ of E^3. This space is ***6-dimensional*** because we have three degrees of freedom in choosing $\boldsymbol{x}$, two degrees of freedom in choosing the unit vector $\boldsymbol{E_1}$, one degree of freedom in choosing the unit vector $\boldsymbol{E}_2$, perpendicular to $\boldsymbol{E_1}$ and then $\boldsymbol{E}_3$ is determined.

We write

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{E_1} \\ \boldsymbol{E_2} \\ \boldsymbol{E_3} \end{pmatrix},$$

and have

$$\boldsymbol{E} = A\boldsymbol{e},$$

where A is a variable (three parameter) orthogonal matrix and $\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{x})$ is a definite ***moving frame***.

Then

$$d\boldsymbol{x} = \sigma\boldsymbol{e} = \sigma A^{-1}\boldsymbol{E} \,,$$

$$d\boldsymbol{E} = (dA)\boldsymbol{e} + Ad\boldsymbol{e} = [dA + A\Omega]\boldsymbol{e} = [dA + A\Omega]A^{-1}\boldsymbol{E} \,.$$

We set

$$\tilde{\sigma} = \sigma A^{-1}, \quad \tilde{\Omega} = (dA)A^{-1} + A\Omega A^{-1} \,.$$

These are matrices of one-forms on the 6-dimensional frame space and we have

Structure equations	**Integrability conditions**
$\begin{cases} dx = \tilde{\sigma}E \\ dE = \tilde{\Omega}E \\ \tilde{\Omega} + {}^t\tilde{\Omega} = 0 \end{cases}$	$\begin{cases} d\tilde{\sigma} = \tilde{\sigma}\tilde{\Omega} \\ d\tilde{\Omega} = \tilde{\Omega}^2 \end{cases}.$

To check the integrability conditions we note

$$0 = d(d\boldsymbol{x}) = d\tilde{\sigma}\boldsymbol{E} - \tilde{\sigma}d\boldsymbol{E} = (d\tilde{\sigma} - \tilde{\sigma}\tilde{\Omega})\boldsymbol{E}$$

$$d\tilde{\sigma} = \tilde{\sigma}\tilde{\Omega} \text{, etc.}$$

In making a penetrating study of the differential geometry of E^3 one is necessarily led to this ***6-dimensional frame space*** and its differential forms $\tilde{\sigma}^i$, $\tilde{\omega}_{ij}$ which, it will be noted, are entirely independent of the choice of the moving frame $\boldsymbol{e}$ on E^3.

4.4. The Laplacian, Orthogonal Coordinates

We continue the considerations of Sections 4.1 and 4.2. The forms dx, dy, dz make up an orthonormal basis for the Euclidean geometry of the space of one-forms at each point; these are related to the fixed (absolute) frame $\boldsymbol{i}$. From

$$\boldsymbol{e} = B\boldsymbol{i} \,, \quad d\boldsymbol{x} = \sigma\boldsymbol{e} = (dx, dy, dz)\boldsymbol{i}$$

we have

$$\sigma B = (dx, dy, dz)$$

as already noted. As B is orthogonal, we see that $\sigma_1, \sigma_2, \sigma_3$ is an ***orthonormal basis for one-forms*** at each point.

Let f be a function on E^3. Then we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \,,$$

$$*\,df = \frac{\partial f}{\partial x}dydz + \frac{\partial f}{\partial y}dzdx + \frac{\partial f}{\partial z}dxdy \,,$$

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dxdydz = (\Delta f) dxdydz\,.$$

The Laplacian Δf of f is known as soon as the three-form $d\ {*df}$ is known, for this has turned out to be the Laplacian multiplied by the volume element $dxdydz$.

Now we know that the * operator can be computed equally well in any orthonormal coordinate system. Also $\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = dxdydz$, so our procedure is this. We express df in terms of the σ_i ,

$$df = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3\,.$$

Then

$$*df = a_1\sigma_2\sigma_3 + a_2\sigma_3\sigma_1 + a_3\sigma_1\sigma_2\,,$$

$$d * df = (\Delta f)\sigma_1\sigma_2\sigma_3\,.$$

A coordinate system *u, v, w* in a domain in $E3$ is called an ***orthogonal coordinate system*** if the vectors

$$\frac{\partial \boldsymbol{x}}{\partial u}, \ \frac{\partial \boldsymbol{x}}{\partial v}, \ \frac{\partial \boldsymbol{x}}{\partial w}$$

are mutually perpendicular. This means that for suitable functions λ, μ, ν , the vectors

$$\boldsymbol{e_1} = \frac{1}{\lambda}\frac{\partial \boldsymbol{x}}{\partial u}, \ \ \boldsymbol{e_2} = \frac{1}{\mu}\frac{\partial \boldsymbol{x}}{\partial v}, \ \ \boldsymbol{e_3} = \frac{1}{\nu}\frac{\partial \boldsymbol{x}}{\partial w}$$

form an orthonormal, or moving frame. We shall presuppose that this is a right-handed one. (Otherwise we merely permute *w* and *v*.) We have

$$\begin{aligned} d\boldsymbol{x} &= du\frac{\partial \boldsymbol{x}}{\partial u} + dv\frac{\partial \boldsymbol{x}}{\partial v} + dw\frac{\partial \boldsymbol{x}}{\partial w} \\ &= (\lambda du)\boldsymbol{e_1} + (\mu dv)\boldsymbol{e_2} + (\nu dw)\boldsymbol{e_3} \end{aligned},$$

so that

$$\sigma_1 = \lambda du\,, \ \ \sigma_2 = \mu dv\,, \ \ \sigma_3 = \nu dw$$

build an orthonormal frame for one-forms. Now we compute the Laplacian:

$$\begin{aligned} df &= f_u du + f_v dv + f_w dw \\ &= (f_u / \lambda)\sigma_1 + (f_v / \mu)\sigma_2 + (f_w / \nu)\sigma_3 \end{aligned},$$

$$\begin{aligned} *df &= (f_u / \lambda)\sigma_2\sigma_3 + (f_v / \mu)\sigma_3\sigma_1 + (f_u / \nu)\sigma_1\sigma_2 \\ &= (\mu\nu f_u / \lambda)dvdw + (\lambda\nu f_v / \mu)dwdu + (\lambda\mu f_u / \nu)dudv \end{aligned}.$$

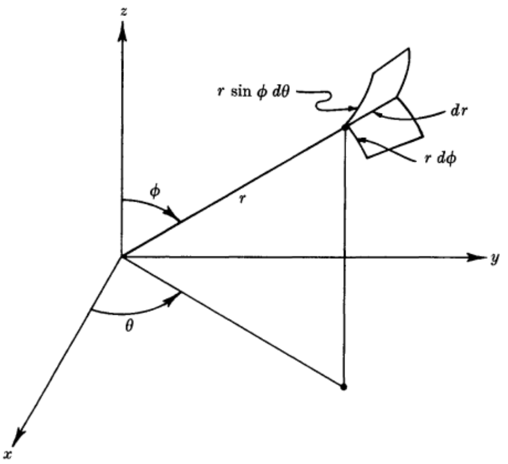
We compare this to

$$d * df = (\Delta f)\sigma_1\sigma_2\sigma_3 = \lambda\mu\nu(\Delta f)dudvdw:$$

$$\Delta f = \frac{1}{\lambda\mu\nu}\left[\frac{\partial}{\partial u}\left(\frac{\mu\nu}{\lambda}\frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v}\left(\frac{\lambda\nu}{\mu}\frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w}\left(\frac{\lambda\mu}{\nu}\frac{\partial f}{\partial w} \right) \right].$$

Let us apply this to spherical coordinates r, ϕ, θ :

$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \, . \\ r\cos\phi \end{cases}$$



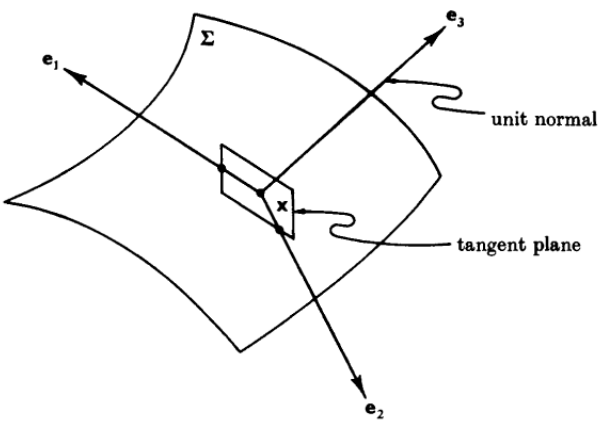
The orthogonality is easily checked (it is obvious geometrically) and we have

$$\sigma_1 = dr\,, \quad \sigma_2 = rd\phi\,, \quad \sigma_3 = r\sin\phi d\theta\,,$$

$$\Delta f = \frac{1}{r^2\sin\phi}\left[\frac{\partial}{\partial r}\left(r^2\sin\phi\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}\right)\right].$$

4.5. Surfaces

We study a smooth surface Σ in E^3. We choose a moving frame $\boldsymbol{e}$ at each point **x** of Σ in such a way that $\boldsymbol{e_3}$ is the normal to the surface. Then $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ span the tangent plane at each point. We shall see how the equations of Section 4.1 specialize.



Since $\boldsymbol{x}$ is constrained to move in the surface, $d\boldsymbol{x}$ must lie in the tangent plane, $\sigma_3 = 0$:

$$d\boldsymbol{x} = \sigma_1\boldsymbol{e_1} + \sigma_2\boldsymbol{e_2}\,.$$

It is clear that the two-form $\sigma_1\delta_2$ represents the element of area of Σ .

We exploit the skew-symmetry of Ω by writing

$$\Omega = \begin{pmatrix} 0 & \varpi & -\omega_1 \\ -\varpi & 0 & -\omega_2 \\ \omega_1 & \omega_2 & 0 \end{pmatrix}.$$

The structure and integrability conditions now reduce to

Structure equations	**Integrability conditions**
$\begin{cases} d\boldsymbol{x} = \sigma_1 \boldsymbol{e_1} + \sigma_2 \boldsymbol{e_2} \\ d\boldsymbol{e_1} = \varpi \boldsymbol{e}_2 - \omega_1 \boldsymbol{e}_3 \\ d\boldsymbol{e}_2 = -\varpi \boldsymbol{e}_1 - \omega_2 \boldsymbol{e}_3 \\ d\boldsymbol{e}_3 = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 \end{cases}$,	$\begin{cases} d\sigma_1 = \varpi\sigma_2 \\ d\sigma_2 = -\varpi\sigma_1 \\ \sigma_1\omega_1 + \sigma_2\omega_2 = 0 \\ d\varpi + \omega_1\omega_2 = 0 \\ d\omega_1 = \varpi\omega_2 \\ d\omega_2 = -\varpi\omega_1 \end{cases}$.

In a certain sense, all of local surface theory is contained in these equations. It remains to interpret them in terms of curvatures, curves on the surface, etc. We illustrate a little of this.

As already remarked, $\sigma_1\sigma_2$ is the element of area on Σ. As $\boldsymbol{x}$ moves over Σ, $\boldsymbol{e_3}$ moves over a region on the unit sphere S^2, called the ***normal***, or ***spherical***, image of Σ. Since $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal to $\boldsymbol{e_3}$, they lie in the tangent plane to the spherical image and form a frame there. We see that the equation $d\boldsymbol{e_3} = \omega_1 \boldsymbol{e_1} + \omega_2 \boldsymbol{e_2}$ plays the same role for the spherical image as $d\boldsymbol{x} = \sigma_1 \boldsymbol{e_1} + \sigma_2 \boldsymbol{e_2}$ does for Σ, hence $\omega_1\omega_2$ represents the element of area of the spherical image.

Since there is only one linearly independent 2-form on the 2-dimensional space Σ, we have

$$\omega_1\omega_2 = K\sigma_1\sigma_2,$$

where K is a scalar called the ***Gaussian curvature***. We shall see shortly that it is entirely independent of the choice of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

Similarly $\sigma_1\omega_2 - \sigma_2\omega_1$ is a 2-form on Σ, and so

$$\sigma_1\omega_2 - \sigma_2\omega_1 = 2H\sigma_1\sigma_2$$

defines a scalar H called the ***mean curvature*** of Σ.

The one-forms ω_1, ω_2 are linear combinations of σ_1 and σ_2. Because of the relation

$$\sigma_1\omega_1 + \sigma_2\omega_2 = 0,$$

we have a symmetry in the coefficients:

$$\begin{cases} \omega_1 = p\sigma_1 + q\sigma_2 \\ \omega_2 = q\sigma_1 + r\sigma_2 \end{cases}.$$

We easily have from this

$$2H = p + r, \quad K = pr - q^2.$$

The characteristic roots of the symmetric matrix

$$\begin{pmatrix} p & q \\ q & r \end{pmatrix}$$

are called the ***principal curvatures*** κ_1, κ_2 of Σ. We consequently have

$$2H = \kappa_1 + \kappa_2, \quad K = \kappa_1 \kappa_2 .$$

From the relation $d\varpi + \omega_1 \omega_2 = 0$ we have

$$d\varpi + K\sigma_1\sigma_2 = 0 .$$

This relation gives us K once we know σ_1, σ_2 and ϖ. But the relations

$$d\sigma_1 = \varpi\sigma_2, \quad d\sigma_2 = -\varpi\sigma_1$$

suffice to determine ϖ once σ_1 and σ_2 are given. (For then $d\sigma_1 = a\sigma_1\sigma_2$ and $d\sigma_2 = b\sigma_1\sigma_2$ are determined and we must have $\varpi = a\sigma_1 + b\sigma_2$.) In total then, K is **completely determined analytically** from σ_1 and σ_2. This contains the ***theorem of Gauss*** that the curvature K is an ***intrinsic invariant*** of Σ, independent of how Σ is imbedded in E^3, so long as the distance between points of Σ measured along Σ (on geodesies, or shortest paths) is preserved locally.

When we apply vector operations to vectors with differential form coefficients, we must always combine the coefficients according to the rules of exterior algebra and pay strict attention to the ordering of the factors. With this we form ***vector*** (***cross***) ***products***:

$$\begin{aligned} d\boldsymbol{x} \times d\boldsymbol{x} &= (\sigma_1\boldsymbol{e_1} + \sigma_2\boldsymbol{e_2}) \times (\sigma_1\boldsymbol{e_1} + \sigma_2\boldsymbol{e_2}) \\ &= \sigma_1{}^2(\boldsymbol{e_1} \times \boldsymbol{e}_1) + \sigma_2{}^2(\boldsymbol{e}_2 \times \boldsymbol{e_2}) + \sigma_1\sigma_2(\boldsymbol{e_1} \times \boldsymbol{e_2}) + \sigma_2\sigma_1(\boldsymbol{e}_2 \times \boldsymbol{e}_1) \end{aligned} .$$

Now $\sigma_1{}^2 = 0$ (and $\boldsymbol{e_1} \times \boldsymbol{e_1} = 0$), etc. Also

$$\sigma_2\sigma_1(\boldsymbol{e_2} \times \boldsymbol{e}_1) = (-\sigma_1\sigma_2)(-\boldsymbol{e}_1 \times \boldsymbol{e}_2) = (\sigma_1\sigma_2)\boldsymbol{e}_3 ,$$

so finally

$$d\boldsymbol{x} \times d\boldsymbol{x} = 2(\sigma_1\sigma_2)\boldsymbol{e_3}$$

and we have obtained the ***vectorial area element***. Precisely, the vectorial area element is

$$(\sigma_1\sigma_2)\boldsymbol{e_3} ,$$

a vector directed along the normal with magnitude $\sigma_1\sigma_2$, the element of area of Σ. Since

$$d\boldsymbol{x} \times d\boldsymbol{x} = (dx, dy, dz) \times (dx, dy, dz) = 2(dydz, dzdx, dxdy)$$

we have

$$(dydz, dzdx, dxdy) = (\sigma_1\sigma_2)\boldsymbol{e_3} .$$

If $\boldsymbol{v} = (P, Q, R)$ is a vector field, then

$$\int_\Sigma (Pdydz + Qdzdx + Rdxdy) = \int \boldsymbol{v} \cdot (\sigma_1\sigma_2\boldsymbol{e_3}) = \int_\Sigma (\boldsymbol{v} \cdot \boldsymbol{e_3})(\sigma_1\sigma_2)$$

is the flux of $\boldsymbol{v}$ through Σ.

Similarly we have

$$d\boldsymbol{x} \times d\boldsymbol{x} = 2(\sigma_1\sigma_2)\boldsymbol{e_3}$$

$$d\boldsymbol{x} \times d\boldsymbol{e_3} = 2H(\sigma_1\sigma_2)\boldsymbol{e_3}$$

$$d\boldsymbol{e_3} \times d\boldsymbol{e_3} = 2K(\sigma_1\sigma_2)\boldsymbol{e_3}$$

which shows the independence of H and K on the tangent vectors $\boldsymbol{e_1}, \boldsymbol{e_2}$.

If f s a function on Σ with

$$df = a_1\sigma_1 + a_2\sigma_2,$$

then on Σ,

$$*df = -a_2\sigma_1 + a_1\sigma_2,$$

$$d * df = d - a_2\sigma_1 + a_1\sigma_2) = (\Delta f)\sigma_1\sigma_2$$

defines the ***Laplacian of f on the surface*** or the ***second Beltrami operator*** Δ. The same works for vectors and we have

$$d\boldsymbol{x} = \sigma_1\boldsymbol{e_1} + \sigma_2\boldsymbol{e_2},$$

$$*d\boldsymbol{x} = \sigma_2\boldsymbol{e_1} - \sigma_1\boldsymbol{e_2}.$$

We notice that

$$d\boldsymbol{x} \times \boldsymbol{e}_3 = (\sigma_1\boldsymbol{e_1} + \sigma_2\boldsymbol{e_2}) \times \boldsymbol{e}_3 = \sigma_2\boldsymbol{e}_1 - \sigma_1\boldsymbol{e_2}$$

hence

$$*d\boldsymbol{x} = d\boldsymbol{x} \times \boldsymbol{e}_3,$$

$$d * d\boldsymbol{x} = -d\boldsymbol{x} \times d\boldsymbol{e}_3 = 2H(\sigma_1\sigma_2)\boldsymbol{e}_3$$

and so

$$\Delta\boldsymbol{x} = (\Delta x, \Delta y, \Delta z) = -2H\boldsymbol{e_3}.$$

A ***minimal surface*** (surface of stationary area) is one for which the mean curvature vanishes, $H = 0$. We have proved: *The coordinate functions x, y, z are harmonic on each minimal surface*. (That is, they satisfy $\Delta x = \Delta y = \Delta z = 0$.)

In this section we have given a sample of how the exterior calculus fits into the classical differential geometry of surfaces. Further material will be found in Sections 8.1 and 8.2, but there is much of the subject that we cannot cover in this text. A treatment from this point of view of exterior calculus which is not quite completely satisfactory and which unfortunately is embellished with historical comments often in bad taste is found in Blaschke [3].

4.6. Maxwell's Field Equations

In classical electromagnetic field theory one deals with the following quantities:

E = electric field **H** = magnetic field

B = magnetic induction **J** = electric current density

D = dielectric displacement ρ = charge density

These are all functions of the space variables x^1, x^2, x^3 and the time *t*. The basic Maxwell equations in ordinary vector language are

(i) $\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$ (Faraday's law of induction)

(ii) $\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}$ (Ampère's law)

(iii) $\operatorname{div} \boldsymbol{D} = 4\pi\rho$ (continuity)

(iv) $\operatorname{div} \boldsymbol{B} = 0$ (nonexistence of true magnetism)

Here *c* is the speed of light. We shall put these equations into the language of exterior forms. To this end, we set

$\alpha = (E_1 dx^1 + E_2 dx^2 + E_3 dx^3)(cdt) + (B_1 dx^2 dx^3 + B_2 dx^3 dx^1 + B_3 dx^1 dx^2)$,

$\beta = -(H_1 dx^1 + H_2 dx^2 + H_3 dx^3)(cdt) + (D_1 dx^2 dx^3 + D_2 dx^3 dx^1 + D_3 dx^1 dx^2)$,

$\gamma = (J_1 dx^2 dx^3 + J_2 dx^3 dx^1 + J_3 dx^1 dx^2)dt - \rho dx^1 dx^2 dx^3$.

Equations (i) and (iv) become

$$d\alpha = 0 .$$

Equations (ii) and (iii) become

$$d\beta + 4\pi\gamma = 0 .$$

Applying *d* to this last equation yields

$$d\gamma = 0 ,$$

in vector notation

$$\operatorname{div} \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 .$$

From the equation $d\alpha = 0$ one concludes, at least in any region of space-time which can be shrunken to a point, that there is a one-form λ such that

$$d\lambda = \alpha .$$

We introduce the vector potential $\boldsymbol{A}$ and a scalar A_0 by writing

$$\lambda = A_1 dx^1 + A_2 dxh2 + A_3 dx_3 + A_0 cdt .$$

The equation $d\lambda = \alpha$ in vector form is

$$\begin{cases} \operatorname{curl} \boldsymbol{A} = \boldsymbol{B} \\ \operatorname{grad} A_0 - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{E} \end{cases} .$$

In ***free space***, everything simplifies according to

$$\boldsymbol{E} = \boldsymbol{D},\ \boldsymbol{H} = \boldsymbol{B},$$

$$\boldsymbol{J} = 0,\ \ \rho = 0,$$

so that the Maxwell equations become

$$\begin{cases} \operatorname{curl} \boldsymbol{E} = -\dfrac{1}{c}\dfrac{\partial H}{\partial t} & \operatorname{div} \boldsymbol{E} = 0 \\ \operatorname{curl} \boldsymbol{H} = \dfrac{1}{c}\dfrac{\partial \boldsymbol{E}}{\partial t} & \operatorname{div} \boldsymbol{H} = 0 \end{cases}.$$

We introduce the ***Lorentz metric*** into 4-space whereby

$$dx^1, dx^2, dx^3, cdt$$

is an orthonormal basis:

$$(dx^i, dx^j) = \delta^{ij}, \quad (dx^i, cdt) = 0, \quad (cdt, cdt) = -1.$$

The **signature** is 3 - 1 = 2.

According to the formulas of Section 2.7,

$$*(dx^1 dx^2) = -dx^3(cdt), \text{ etc.},$$
$$*(dx^1 cdt) = dx^2 dx^3, \text{ etc.}$$

We see that

$$\alpha = (E_1 dx^1 + E_2 dx^2 + E_3 dx^3)(cdt) + (H_1 dx^2 dx^3 + H_2 dx^3 dx^1 + H_3 dx^1 dx^2),$$

$$\begin{aligned} \beta &= -(H_1 dx^1 + H_2 dx^2 + H_3 dx^3)(cdt) + (E_1 dx^2 dx^3 + E_2 dx^3 dx^1 + E_3 dx^1 dx^2) \\ &= *\alpha \end{aligned}.$$

Consequently Maxwell's equations in free space are simply

$$\begin{cases} d\alpha = 0 \\ d*\alpha = 0 \end{cases}.$$

We return to the general situation and refine our analysis by introducing one-forms:

$$\begin{aligned} \omega_1 &= E_1 dx^1 + E_2 dx^2 + E_3 dx^3, \\ \omega_2 &= B_1 dx^2 dx^3 + B_2 dx^3 dx^1 + B_3 dx^1 dx^2, \\ \omega_3 &= H_1 dx^1 + H_2 dx^2 + H_3 dx^3, \\ \omega_4 &= D_1 dx^2 dx^3 + D_2 dx^3 dx^1 + D_3 dx^1 dx^2, \\ \omega_5 &= J_1 dx^2 dx^3 + J_2 dx^3 dx^1 + J_3 dx^1 dx^2. \end{aligned}$$

These involve space variable differentials only. Now we interpret d' to denote the ***exterior derivative with respect to space variables only***. We introduce $\partial / \partial t$ in this form

$$\frac{\partial}{\partial t}(\omega_1) = \dot{\omega}_1 = \dot{E}_1 dx^1 + \cdots, \text{ etc.}$$

Now the ***Maxwell equations*** are

$$\begin{cases} d'\omega_1 = -\dfrac{1}{c}\dot{\omega}_2 \\ d'\omega_3 = \dfrac{4\pi}{c}\omega_5 + \dfrac{1}{c}\dot{\omega}_4 \\ d'\omega_2 = 0 \\ d'\omega_4 = 4\pi\rho dx^1 dx^2 dx^3 \end{cases}.$$

The ***Poynting energy-flux vector S*** is introduced by

$$\boldsymbol{S} = \left(\frac{c}{4\pi}\right)\boldsymbol{E} \times \boldsymbol{H} \,,$$

that is

$$\left(\frac{c}{4\pi}\right)\omega_1 \wedge \omega_3 = S_1 dx^2 dx^3 + S_2 dx^3 dx^1 + S_3 dx^1 dx^2 \,.$$

Poynting's theorem,

$$\left(\frac{c}{4\pi}\right)\dot{\boldsymbol{B}} \cdot \boldsymbol{H} + \boldsymbol{E} \cdot \boldsymbol{J} + \left(\frac{1}{4\pi}\right)\boldsymbol{E} \cdot \dot{\boldsymbol{D}} + \operatorname{div} \boldsymbol{S} = 0 \,,$$

follows from

$$\begin{aligned} d'(\omega_1 \wedge \omega_3) &= d'\omega_1 \wedge \omega_3 - \omega_1 \wedge d'\omega_3 \\ &= \left(-\frac{1}{c}\dot{\omega}_2\right) \wedge \omega_3 - \omega_1 \wedge \left(\frac{4\pi}{c}\omega_5 + \frac{1}{c}\dot{\omega}_4\right). \\ &= -\frac{1}{c}\dot{\omega}_2 \wedge \omega_3 - \frac{4\pi}{c}\omega_1 \wedge \omega_5 - \frac{1}{c}\omega_1 \wedge \dot{\omega}_4 \end{aligned}$$

For bodies at rest, one assumes $\boldsymbol{D} = \kappa \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$ where the ***dielectric constant*** κ and the ***permeability*** μ are constant in time. Then ***Poynting's theorem*** becomes

$$-\frac{\partial u}{\partial t} = \operatorname{div} \boldsymbol{S} + \boldsymbol{E} \cdot \boldsymbol{J} \,,$$

where

$$u = \frac{1}{8\pi}(\kappa \boldsymbol{E}^2 + \mu \boldsymbol{H}^2)$$

is the energy density of the field. The quantity $\boldsymbol{E} \cdot \boldsymbol{J}$ is called the ***thermochemical activity***.

4.7. Problems