V. Manifolds and Integration

5.1. Introduction

An n-dimensional manifold is a space which is not necessarily a Euclidean space nor is it a domain in a Euclidean space, but which, from the viewpoint of a short-sighted observer living in the space, looks just like such a domain of Euclidean space. A case in point is the two-sphere S^2. This cannot be considered a part of the Euclidean plane E^2. However our observer on S^2 sees that he can describe his immediate vicinity by two coordinates and so he fails to distinguish between this and a small domain on E^2.

We have the technical problem of describing an n-manifold with sufficient precision so that we can define functions, tensors, and differential forms on such a space. The definition which follows is motivated in this way. Each observer on the manifold has an immediate neighborhood (local coordinate neighborhood) described by n coordinates. Each point of the space must lie in at least one of these observed neighborhoods. Now if we consider simultaneously two observers, their immediate neighborhoods may overlap, and we must specify what happens in each such overlap. In the next three sections we go over these matters with some care.

After this is accomplished we tackle the problem of defining the integral of a differential form. In Sections 5 and 6 we lay the groundwork by defining chains, the geometrical sets over which forms are integrated, and in Section 7 we define the integral.

5.2. Manifolds

An n-dimensional manifold consists of a space M together with a collection of ***local coordinate neighborhoods*** $U_1, U_2, \ldots$ such that each point of M lies m at least one of these U. On each U is given a coordinate system

$$x^1, \ldots, x^n$$

so that the ***values*** of the coordinates

$$(x^1(P), \ldots, x^n(P)),$$

where P ranges over U, make up an open domain in Euclidean n-space E^n.

Suppose that U with coordinate system

$$x^1, \ldots, x^n$$

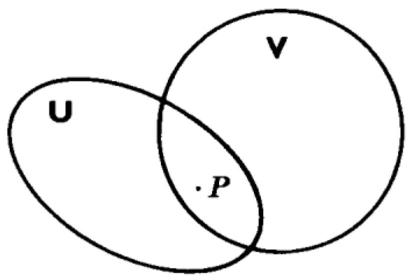
and V with coordinate system

$$y^1,\ldots,y^n$$

overlap (intersect). We may express the V coordinates $\boldsymbol{y}$ of a point P in terms of the U coordinates $\boldsymbol{x}$ of this point:

$$y^i = y^i(x^1,\ldots,x^n) \quad (i=1,\ldots,n).$$

As part of the definition, we assume that these functions are ***smooth*** (differentiable as often as we please).



Having this formal definition out of the way, we explore some consequences. First of all, on the overlap of U and V above we may interchange the role of U and V to write smooth functions

$$x^j = x^j(y^1,\ldots,y^n) \quad (j=1,\ldots,n).$$

Substituting yields

$$y^i = y^i(x^1(\boldsymbol{y}),\ldots,x^n(\boldsymbol{y}))$$

and we may differentiate by the chain rule:

$$\delta^i_k = \sum \frac{\partial y^i}{\partial x^j}\frac{\partial x^j}{\partial y^k},$$

which has the matrix interpretation

$$\left\|\frac{\partial y^i}{\partial x^j}\right\| \cdot \left\|\frac{\partial x^j}{\partial y^k}\right\| = I.$$

We take determinants by the product rule:

$$\frac{\partial(y^1,\ldots,y^n)}{\partial(x^1,\ldots,x^n)} \cdot \frac{\partial(x^1,\ldots,x^n)}{\partial(y^1,\ldots,y^n)} = 1.$$

It follows that the Jacobian

$$\frac{\partial(y^1,\ldots,y^n)}{\partial(x^1,\ldots,x^n)} \neq 0;$$

it is different from 0 at **each** point

A manifold is called ***orientable*** (two-sided) if it is possible to choose the local coordinates in the first place so that each such Jacobian (on an overlap of local coordinate neighborhoods) is positive.

Example. We make the two-sphere S^2 into a manifold by using six coordinate neighborhoods. We set

$$S^2 = \{(x, y, z) \quad \text{where} \quad x^2 + y^2 + z^2 = 1 \quad \}$$

The neighborhoods are

$U_1^+ = \{x > 0\}$, coordinate system *y, z*.

$U_1^- = \{x < 0\}$, coordinate system *z, y*.

$U_2^+ = \{y > 0\}$, coordinate system *z, x*.

$U_2^- = \{y < 0\}$, coordinate system *x, z*.

$U_3^+ = \{z > 0\}$, coordinate system *x, y*.

$U_3^- = \{z < 0\}$, coordinate system *y, x*.

In comparing the overlap of two of these, we shall not be pedantic and introduce different letters, hoping the reader will forgive this sloppy notation.

On the intersection of U_1^+ and U_2^+ we have the coordinate transformation

$$\begin{cases} y = \sqrt{1 - z^2 - x^2} \\ z = z \end{cases}, \quad (x > 0, y > 0)$$

and so

$$\frac{\partial(y, z)}{\partial(z, x)} = \begin{vmatrix} \frac{-z}{\sqrt{\ }} & \frac{-x}{\sqrt{\ }} \\ 1 & 0 \end{vmatrix} = \frac{x}{\sqrt{\ }} > 0 .$$

On the intersection of U_1^+ and U_3^- ,

$$\begin{cases} y = y \\ z = -\sqrt{1 - y^2 - x^2} \end{cases}, \quad (x > 0, z < 0),$$

$$\frac{\partial(y, z)}{\partial(y, x)} = \begin{vmatrix} 1 & 0 \\ \frac{y}{\sqrt{\ }} & \frac{x}{\sqrt{\ }} \end{vmatrix} = \frac{x}{\sqrt{\ }} > 0 .$$

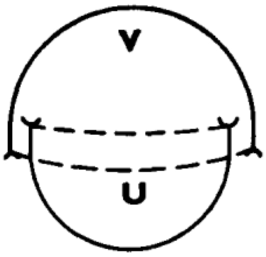
On the intersection of U_2^- and U_3^-,

$$\begin{cases} x = x \\ z = -\sqrt{1 - y^2 - x^2} \end{cases}, \quad (y < 0, z < 0),$$

$$\frac{\partial(x, z)}{\partial(y, x)} = \begin{vmatrix} 0 & 1 \\ \frac{y}{\sqrt{\ }} & \frac{x}{\sqrt{\ }} \end{vmatrix} = -\frac{y}{\sqrt{\ }} > 0 \text{, etc.}$$

Thus the two-sphere is a ***two-manifold***, and our choice of local coordinates proves it to be ***orientable***.

One could also cover the sphere S^2 with a system of only two local coordinate neighborhoods by taking two opposite hemispheres, each extended slightly to make open overlapping neighborhoods.



The sphere S^2 has two opposite ***orientations*** (outward or inward normal, corresponding to counterclockwise or clockwise sense of rotation). Similarly an orientable n-manifold has two opposite orientations. A definite one of these is determined by the order in which local coordinates $x^1,\ldots,x^n$ are given, up to an ***even permutation*** of this order. Making an ***odd permutation*** of local coordinates gives the opposite orientation.

Let M be an n-manifold. To say that a real-valued function f on M is ***smooth*** at a point P of M means the following. Let U be a local coordinate neighborhood containing P with coordinates $x^1,\ldots,x^n$. We require that $f(x^1,\ldots,x^n)$ be smooth near P. This restriction on f is **independent** of the particular U one chooses, since two coordinate systems whose neighborhoods overlap on a region including P are themselves related by smooth functions (from the definition of manifold). A real-valued function f is ***smooth*** on M if it is smooth at each point of M.

Similarly, if M and N are manifolds of dimensions m and n, respectively, one defines a ***smooth mapping***

$$\phi : M \to N$$

by the requirements that in local coordinates $x^1,\ldots,x^n$ on U in M and $y^1,\ldots,y^n$ on V in N, we have ϕ represented by smooth functions

$$y^i = y^i(x^1,\ldots,x^m) \quad (i=1,\ldots,n)$$

on that part of U which ϕ maps into V.

A manifold M is called a ***submanifold*** of a manifold N provided there is a one-to-one smooth mapping

$$j : M \to N$$

which has this ***regularity*** property: in local coordinates (as written above), the matrix

$$\left\| \frac{\partial y^i}{\partial x^j} \right\|$$

has (maximal) rank m at each point. We refer to j itself as an ***injection*** or ***imbedding*** of M in N.

This applies in particular when $N = E^n$ so that we may refer to submanifolds of Euclidean spaces. It is an established result of manifold theory that each m-dimensional manifold which is not too large may be imbedded in E^n with $n = 2m + 1$.

5.3. Tangent Vectors

We study a manifold M and a point P on M. Our job is to define the tangent space at P, an n-dimensional vector space whose elements are the tangent vectors at P. Because we are not within the simple terrain of Euclidean space we cannot merely draw arrows emanating at P. We need a way of considering ordinary Euclidean vectors which depends in no way on arrows, or directed line segments. The answer is simple. We may identify Euclidean vectors with directional differentiations. Thus in case P is a point of E^3 and $\boldsymbol{v} = (a, b, c)$ is a vector at P, we may identify $\boldsymbol{v}$ with the operator

$$\left.\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right)\right|_P .$$

This does the usual things to sums and products, which motivates the following definition.

First some notation. If M is a manifold, we denote by

$$F^0(M)$$

the ***space of all smooth real-valued functions*** on M.

Let P be a point on a manifold M. A ***tangent vector*** $\boldsymbol{v}$ at P is an operator

$$\boldsymbol{v} : F^0(M) \to R \text{ , the reals}$$

satisfying

(i) $\boldsymbol{v}(af + bg) = a\boldsymbol{v}(f) + b\boldsymbol{v}(g)$, a, b constant.

(ii) $\boldsymbol{v}(f \cdot g) = g(P) \cdot \boldsymbol{v}(f) + f(P) \cdot \boldsymbol{v}(g)$.

Thus $\boldsymbol{v}$ assigns to each smooth function f on M a real number $\boldsymbol{v}(f)$.

We shall first observe that if we take a constant function c, then $\boldsymbol{v}(c) = 0$. For setting $f = g = 0$ in (i) yields $\boldsymbol{v}(0) = 0$, setting $f = g = 1$ in (ii) yields $\boldsymbol{v}(1) = 0$, and setting $f=1$, $a = 0$, and $g = 0$ in (i) yields $\boldsymbol{v}(c) = 0$. Observe that

$$\boldsymbol{v}(cf)=c\boldsymbol{v}(f)$$

for any f and constant c.

Next, suppose $x^1, \dots, x^n$ is a local coordinate system, valid in

some neighborhood of P. Then each of the operators

$$\boldsymbol{v_i} = \left.\frac{\partial}{\partial x^i}\right|_P$$

(the vertical bar means "evaluated at P") is a tangent vector, as one easily verifies.

The totality of tangent vectors at P makes up a linear space, T_P, called the ***tangent space*** to M at P. We shall show that *these vectors* $\boldsymbol{v_i}$ *form a basis of this tangent space*. We set

$$(x^1,\ldots,x^n)\Big|_P = (c^1,\ldots,c^n)\,.$$

If $\boldsymbol{v}$ is any tangent vector at P, set

$$\boldsymbol{v}(x^i) = \boldsymbol{v}(x^i - c^i) = a^i\,.$$

Now if f is any smooth function on M, we expand f in a Taylor series up to first-order terms with the integral form of remainder:

$$f(\boldsymbol{x}) = f(\boldsymbol{c}) + \sum (x^i - c^i) g_i(\boldsymbol{x})\,,$$

$$g_i(c) = \left.\frac{\partial f}{\partial x^i}\right|_P .$$

Then

$$\begin{aligned} v(f) &= v[f(c)] + \sum g_i(c) v(x^i - c^i) + \sum (c^i - c^i) v(g_i) \\ &= 0 + \sum a^i \left.\frac{\partial f}{\partial x^i}\right|_P + 0 \\ &= \sum a^i \left.\frac{\partial f}{\partial x^i}\right|_P \end{aligned} ,$$

hence

$$\boldsymbol{v} = \sum a^i \left.\frac{\partial}{\partial x^i}\right|_P ,$$

which establishes the result. We refer to $a^1,\ldots,a^n$ as the ***components*** of $\boldsymbol{v}$ with respect to the coordinate system $\boldsymbol{x}$. If $\boldsymbol{y}$ is another coordinate system valid at P, and

$$\boldsymbol{v} = \sum b^i \left.\frac{\partial}{\partial y^i}\right|_P$$

we find, by the chain rule,

$$b^i = \sum a^j \left.\frac{\partial y^i}{\partial x^j}\right|_P ,$$

the usual transformation law for ***contravariant components*** of a vector. Note here that we are working at a single point so that $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

A ***vector field*** on M consists of a smooth assignment of a tangent vector to each point of M. In local coordinates,

$$\boldsymbol{v} = \sum a^i(\boldsymbol{x}) \frac{\partial}{\partial x^i}, \quad a^i(\boldsymbol{x}) \text{ smooth.}$$

On an overlap,

$$\boldsymbol{v} = \sum b^i(\boldsymbol{y}) \frac{\partial}{\partial y^i},$$

$$b^i(\boldsymbol{y}(\boldsymbol{x})) = \sum a^j(\boldsymbol{x}) \frac{\partial y^i}{\partial x^j}.$$

5.4. Differential Forms

The smooth functions on M will also be called ***0-forms***. They form a space $F^0(M)$, the space of forms of degree 0 on M.

We now define a ***one-form*** at a point P of M. We must have an expression

$$\sum a_i dx^i, \quad a_i \text{ constant,}$$

each local coordinate system (x^i) valid in a neighborhood U which includes P and such that any two such expressions

$$\sum a_i dx^i, \quad \sum b_i dy^i$$

at P are related by

$$\sum b_i \left. \frac{\partial y^i}{\partial x^j} \right|_P = a_j,$$

the usual transformation law for covariant vectors. Evidently this is completely consistent with our local study in Chapter III.

Having this, we may form sums of exterior products of one-forms at P to construct p-forms at P. Now we can define a ***p-form*** on M. This is a smooth assignment of a p-form to each point P of M. If U is given with local coordinates (x^i), then on the neighborhood U we have the representation

$$\omega = \sum a_H(\boldsymbol{x}) dx^H$$

with smooth functions $a_H(\boldsymbol{x})$ on U, $H = \{h_1, \ldots, h_p\}$.

If we have the representation

$$\omega = \sum b_K(\boldsymbol{y}) dy^K$$

with respect to a second coordinate system which overlaps the first,

then the relation between the b's and a's is given by substitution of $y^i = y^i(\boldsymbol{x})$ for y^i and

$$\sum \frac{\partial y^i}{\partial x^j} dx^j$$

for y^i. As a consequence of our study of coordinate changes in Section 3.4, we see that the space

$$F^p(M)$$

of p-forms on M is completely defined, that exterior multiplication

$$\omega \wedge \eta$$

of two-forms on M is accomplished by operating with one point at a time, and that the exterior derivative of a form on M is defined by working it out in each local coordinate system.

All the rules of Chapter III are readily verified,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{(\deg \omega)} \omega \wedge d\eta ,$$

for example.

If M and N are two manifolds and

$$\phi : M \to N$$

is a smooth mapping, then there is a natural induced mapping ϕ^*,

$$\phi^* : F^p(N) \to F^p(M)$$

which again is defined by applying the local construction in one local coordinate system at a time and piecing together the results. As in the local theory, we have the results

(i) $\phi^*(\omega + \eta) = \phi^* \omega + \phi^* \eta$.

(ii) $\phi^*(\lambda \wedge \mu) = (\phi^* \lambda) \wedge (\phi^* \mu)$.

(iii) $d(\phi^* \omega) = \phi^*(d\omega)$.

The last of these can be expressed by means of a **commutative diagram.**

$$\begin{tikzcd}
\mathbf{F}^p(\mathbf{M}) \arrow[d, "d"'] & \mathbf{F}^p(\mathbf{N}) \arrow[l, "\phi^*"'] \arrow[d, "d"] \\
\mathbf{F}^{p+1}(\mathbf{M}) & \mathbf{F}^{p+1}(\mathbf{N}) \arrow[l, "\phi^*"']
\end{tikzcd}$$

Each of the two possible paths from $F^p(N)$ to $F^{p+1}(M)$ leads to the same result.

In practice, one often constructs differential forms on a manifold this way. One knows in advance several smooth functions f, g,… on M. From these one constructs one-forms df, dg,… and from these in turn forms of higher degrees by taking exterior products.

Example 1. On the two sphere S2 considered in Section 5.2, the functions *x, y, z* are smooth **0-forms**. Thus *dx, dy, dz* are **one-forms** and *dxdy*, *dydz*, etc., are **two-forms**. On the neighborhood U_1^+ we have

$$x^2 = 1 - y^2 - z^2,$$

$$dx = \frac{-ydy - zdz}{x},$$

$$dxdy = \frac{-ydy - zdz}{x} dy = \frac{z}{x} dydz, \text{ etc.}$$

Example 2. The circle S^1 = $\{ x^2 + y^2 = 1 \}$. Here x and y are functions on S^1 and

$$xdx + ydy = 0 .$$

This means we can define a one-form α. At a point where $x \neq 0$,

$$\alpha = \frac{dy}{x} .$$

At a point where $x = 0$, we have $y \neq 0$ and

$$\alpha = -\frac{dx}{y} .$$

On any arc of S^1 which is not the complete circle, we can find a function θ such that

$$x = \cos\theta, \quad y = \sin\theta ,$$

hence

$$\alpha = d\theta .$$

It must be emphasized that no such function θ exists on all of S^1 — it would have to jump by 2π somewhere.

5.5. Euclidean Simplices

In this section we shall describe the standard building blocks which we later piece together to form fields of integration, *p*-dimensional spreads in a manifold over which we can integrate *p*-forms. These building blocks will be called ***Euclidean simplices*** of various dimensions—we shall omit repetition of the adjective Euclidean in this section, but we understand that everything takes place in Euclidean space.

A ***0-simplex*** is a single point (P_0).

A ***1-simplex*** is a directed closed segment on a straight line. It is completely determined by its ordered pair of vertices ($P_0 P_1$).

A ***2-simplex*** is a closed triangle with vertices taken in some

definite order. It is completely determined by its ordered order triple of vertices in the proper order,

$$(P_0, P_1, P_2).$$

Similarly one has a ***3-simplex*** based on an ordered quadruple

$$(P_0, P_1, P_2, P_3).$$

of four points, no three collinear. Geometrically it represents a tetrahedron. Finally, an ***n-simplex*** is the closed convex hull

$$(P_0, \ldots, P_n)$$

of (n+1) independent points taken in a definite order. The geometrical set so spanned consists of all points

$$P = t_0 P_0 + \cdots + t_n P_n\,, \quad t_i \geq 0\,, \quad \sum t_i = 1\,,$$

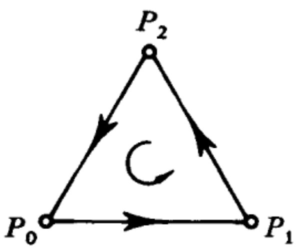
i.e., al possible centroids of systems of nonnegative masses $t_0, \ldots, t_n$ located at $P_0, \ldots, P_n$, respectively.

The ***boundary*** $\partial \boldsymbol{s}$ of a simplex $\boldsymbol{s}$ is a formal sum of simplices of one lower dimension with integer coefficients:

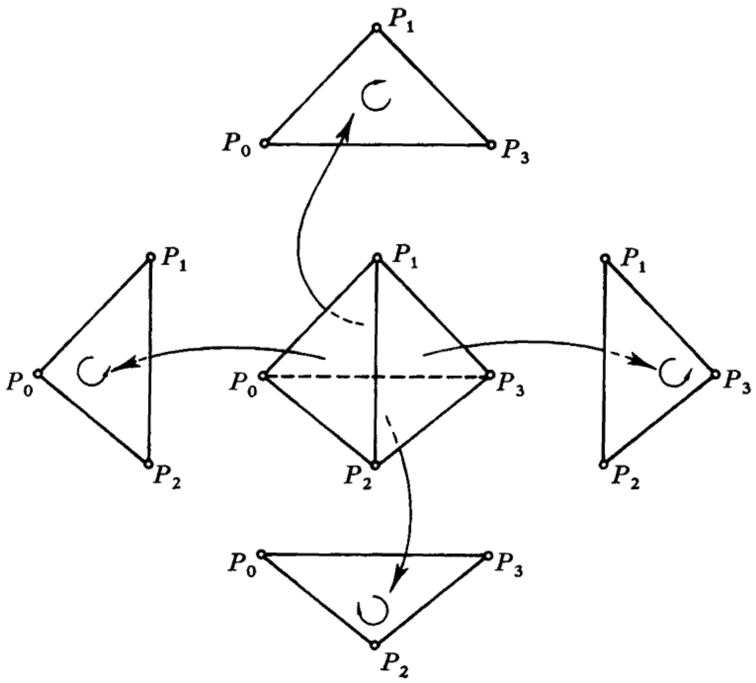
$$\partial(P_0, P_1, \ldots, P_n) = \sum_{i=0}^{n} (-1)^i (P_0, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)\,.$$

An examination of the lower dimensional cases convinces one that this is consistent with the customary ideas on boundaries of oriented regions

$$\partial(P_0, P_1) = (P_1) - (P_0)\,,$$
$$\partial(P_0, P_1, P_2) = (P_1, P_2) - (P_0, P_2) + (P_0, P_1)\,,$$
$$\partial(P_0, P_1, P_2, P_3) = (P_1, P_2, P_3) - (P_0, P_2, P_3) + (P_0, P_1, P_3) - (P_0, P_1, P_2)\,.$$



In the triangle, the ordering of the vertices gives a sense of rotation of the triangle. In the tetrahedron, the ordering of the vertices gives a right-handed screw sense in space and induces a positive sense of rotation in each triangular face (outward drawn normal). One thinks of each minus sign in $\partial \boldsymbol{s}$ as representing a reversal in this rotation sense. The result is that $\partial(P_0, \ldots, P_3)$ represents the oriented geometric boundary of the tetrahedron according to the outward drawn normal.



An ***n-chain*** is a formal sum

$$\boldsymbol{c} = \sum a^i \boldsymbol{s}_i \ ,$$

where the a^i are constants and the $\boldsymbol{s}_i$ are n-simplices. Its ***boundary*** is denned by

$$\partial \boldsymbol{c} = \sum a^i (\partial \boldsymbol{s}_i) \ .$$

A basic result is that the boundary of each chain itself has zero boundary:

$$\partial[\partial \boldsymbol{c}] = 0 \ .$$

It suffices to check this for simplices. Let us try low-dimensional cases:

$$\begin{aligned}\partial[\partial(P_0,P_1,P_2)] &= \partial(P_1,P_2) - \partial(P_0,P_2) + \partial(P_0,P_1) \\ &= [(P_2)-(P_1)] - [(P_2)-(P_0)] + [(P_1)-(P_0)] = 0\end{aligned} ,$$

$$\begin{aligned}\partial[\partial(P_0,\ldots,P_3)] = &[(P_2,P_3)-(P_1,P_3)+(P_1,P_2)] \\ &-[(P_2,P_3)-(P_0,P_3)+(P_0,P_2)] \\ &+[(P_1,P_3)-(P_0,P_3)+(P_0,P_1)] \\ &-[(P_1,P_2)-(P_0,P_2)+(P_0,P_1)] = 0\end{aligned} ,$$

which illustrates the general idea; each face occurs twice with opposite signs

More generally, in computing

$$\partial[\partial(P_0,\ldots,P_n)] ,$$

one obtains

$$(P_0,\ldots,P_{i-1},P_{i+1},\ldots,P_{j-1},P_{j+1},\ldots,P_n)$$

twice, with opposite signs, once each from

$$(P_0,\ldots,P_{i-1},P_{i+1},\ldots,P_n)$$

and

$$(P_0,\ldots,P_{j-1},P_{j+1},\ldots,P_n)$$

so that everything cancels.

Given two n-simplices $(P_0,\ldots,P_n)$, $(Q_0,\ldots,Q_n)$, there is a unique linear correspondence between them which preserves the ordering of the vertices. It is given by

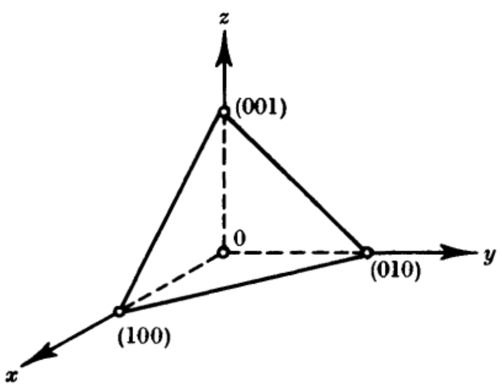
$$\sum_0^n t_i P_i \leftrightarrow \sum_0^n t_i Q_i \quad (t_i \ge 0,\ \sum_0^n t_i = 1).$$

It is convenient for defining integrals to have standard models of the simplices of each dimension. We define the ***standard n-simplex***

$$\bar{s}^n = (R_0,\ldots,R_n)$$

as the simplex in E^n based on

$$\begin{aligned} &R_0 = 0 \\ &R_1 = (10\cdots0) \\ &R_2 = (010\cdots0)\,. \\ &\cdots \\ &R_n = (00\cdots01) \end{aligned}$$



We must now agree on a certain convention for integration. Let ω be a n-form defined on a domain U of E^n which includes $\bar{s}^n$. We wish to define

$$\int_{\bar{s}^n} \omega\,.$$

We do this by writing ω in the unique way

$$\omega = A(x^1,\ldots,x^n)dx^1dx^2\cdots dx^n\,,$$

with the variables in their natural order, and then setting

$$\int_{\bar{s}^n}\omega = \int_{\bar{s}^n} A(\boldsymbol{x})dx^1 dx^2 \cdots dx^n \ ,$$

where the right-hand side is now the standard ordinary n-fold integration, which may be evaluated by any scheme of iteration, *regardless of what order in which the variables are taken*.

For example, if $\omega = dxdydz$, then

$$\int_{\bar{s}^n}\omega = \int_{\bar{s}^n} dxdydz = -\int_0^1 dy \int_0^{1-y} dx \int_0^{1-x-y} dz = -\frac{1}{6}.$$

5.6. Chains and Boundaries

Now we consider a manifold M and we shall define an ***n-simplex*** in M. As a preliminary definition, this consists of three things: a Euclidean n-simplex $\boldsymbol{s}^n$, an n-dimensional neighborhood U of $\boldsymbol{s}^n$ in Euclidean space, and a smooth mapping ϕ,

$$\phi : U \to M \ .$$

We denote this preliminary simplex by

$$(\boldsymbol{s}^n, U, \phi) \ .$$

If we are given a second one,

$$(\boldsymbol{t}^n, V, \psi) \ ,$$

it will be considered the same as the first provided

$$\phi\left(\sum_0^n t_i P_i\right) = \phi\left(\sum_0^n t_i Q_i\right) \quad (\, t_i \geq 0 \ , \ \sum_0^n t_i = 1\,).$$

where

$$\boldsymbol{s}^n = (P_0, P_1, \ldots, P_n) \ , \ \ \boldsymbol{t}^n = (Q_0, Q_1, \ldots, Q_n) \ .$$

In other words, if we set up the natural order-preserving linear equivalence between $\boldsymbol{s}^n$ and $\boldsymbol{t}^n$:

$$\boldsymbol{s}^n \leftrightarrow \boldsymbol{t}^n \ ,$$

then $\phi(P) = \psi(Q)$ whenever P and Q are corresponding points. This is also expressed by the commutative diagram.

$$\begin{tikzcd} \mathbf{s^n} \arrow[rr, leftrightarrow] \arrow[dr, "\phi"'] & & \mathbf{t^n} \arrow[dl, "\psi"] \\ & \mathbf{M} & \end{tikzcd}$$

The totality of these preliminary simplices $(\boldsymbol{s}^n, U, \phi)$ which in this way are identified with a single one make up an object which we call an ***n-simplex*** *in M*, denoted by a symbol $\boldsymbol{\sigma}^n$.

The open neighborhoods U we have introduced merely serve to eliminate difficulties with differentiability on the boundary.

If $\boldsymbol{\sigma}^n$ is a simplex represented by $(\boldsymbol{s}^n, U, \phi)$, then $\boldsymbol{s}^n$ has faces $\boldsymbol{t}_0, \ldots, \boldsymbol{t}_n$, each a Euclidean ($n$ - 1)-simplex, where

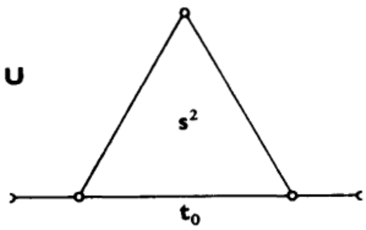
$$\partial \boldsymbol{s}^n = \sum \pm \boldsymbol{t}_i \ .$$

By restricting ϕ to the various $\boldsymbol{t}_i$, each extended a little in U to make open neighborhoods V_i, we define the faces of $\boldsymbol{\sigma}^n$, each represented by

$$\boldsymbol{\tau}_i = (\boldsymbol{t}_i, V^i, \phi)$$

and the corresponding ***boundary***

$$\partial \boldsymbol{\sigma}^n = \sum \pm \boldsymbol{\tau}_i \ .$$



This is an (n - 1)-chain in M. By an ***n-chain*** $\boldsymbol{c}$ of M we mean a formal sum

$$\boldsymbol{c} = \sum_i a_i \boldsymbol{\sigma}_i^n \ ,$$

with constant coefficients a_i and n-simplices $\boldsymbol{\sigma}_i^n$. Chains may be added and multiplied by constants. We denote by

$$\boldsymbol{C}_n(M)$$

the ***set of all n-chains*** on M. We set

$$\partial \boldsymbol{c} = \sum a_i \partial \boldsymbol{\sigma}_i^n \quad \text{for} \quad \boldsymbol{c} = \sum a_i \boldsymbol{\sigma}_i^n \ .$$

Thus

$$\partial : \boldsymbol{C}_n(M) \to \boldsymbol{C}_{n-1}(M) \quad (n=1,2,\ldots).$$

The basic property of the boundary operator ∂ follows readily from the corresponding Euclidean situation: for each n-chain $\boldsymbol{c}$,

$$\partial(\partial \boldsymbol{c}) = 0 \ .$$

A ***cycle*** is a chain z whose boundary vanishes, $dz = 0$.

A ***bounding cycle*** (or simply ***boundary***) $\boldsymbol{b}$ is a chain which is the boundary of a chain of one higher dimension, $\boldsymbol{b} = \partial \boldsymbol{c}$.

Each boundary is a cycle, for if $\boldsymbol{b} = \partial \boldsymbol{c}$, then

$$\partial(\boldsymbol{b}) = \partial(\partial \boldsymbol{c}) = 0 \ .$$

One further thing to be noted is this. In our preliminary definition of a simplex $(\boldsymbol{s}^n, U, \phi)$ we do ***not*** require that the smooth mapping ϕ on U into M be a one-to-one mapping. Indeed, it may happen that it takes all of $\boldsymbol{s}^n$ into a lower dimensional space, even into a single point! A close analysis shows that not only is there no harm in allowing such "bad" mappings but that there are very great technical difficulties involved in attempting to avoid them.

5.7. Integration of Forms

Our data is a manifold M of any dimension, a p-form ω on M and a p-chain $\boldsymbol{c}$ on M. We must define

$$\int_{c} \omega .$$

First we set

$$\boldsymbol{c} = \sum a_i \boldsymbol{\sigma_i} ,$$

where the a_i are constants and the $\boldsymbol{\sigma_i}$ are p-simplices and write

$$\int_{c} \omega = \sum a_i \int_{\sigma_i} \omega ,$$

so it remains to define the integral of ω over a **p**-simplex $\boldsymbol{\sigma}$. Now we can represent $\boldsymbol{\sigma}$ in the form

$$(\bar{\boldsymbol{s}}^p, U, \phi) ,$$

where $\bar{\boldsymbol{s}}^p$ is the standard p-simplex in E^p and ϕ is a smooth mapping of the neighborhood U of $\bar{\boldsymbol{s}}^p$ into M. Our definition is

$$\int_{\sigma} \omega = \int_{\bar{s}^p} \phi^* \omega .$$

Since $\phi^* \omega$ is a p-form on U, this is an ordinary p-fold integral, as discussed in the next to last section.

In application, one often does not bother to spell out in detail how a given geometrical region may be considered as a chain, but rather relies on the usual combination of experience and intuition, the latter an excellent guide in geometry. For example, suppose ω is a 2-form on $S^2 = \{ x^2 + y^2 + z^2 = 1 \}$ and one seeks $\int \omega$ taken over S^2. There will *usually* be a more effective procedure than using the coordinate planes to decompose the surface S^2 into eight spherical triangles, setting up mappings of the standard triangle onto each of these, etc.

What then is the value of this rather long story on chains, boundaries, and integrals? In this age, it hardly seems necessary to defend the placing on a logical and rigorous basis things which are only understood in an intuitive sense. In addition, we have here a powerful theoretical tool as we shall see immediately in the following section on the general Stokes' theorem.

As an exercise, one could check that each of the standard tricks used to evaluate surface integrals, etc., fits into the above scheme of things. It hardly seems worth our time here.

5.8. Stokes' Theorem

The general result we establish now includes all known formulas which transform an integral into one over a one-higher dimension spread.

Let ω be a p-form on a manifold M and $\boldsymbol{c}$ a $(p+1)$-chain. Then

$$\int_{\partial \boldsymbol{c}} \omega = \int_{\boldsymbol{c}} d\omega \, .$$

Since $\boldsymbol{c}$ is a sum of $(p+1)$-simplices with constant coefficients, it suffices to prove

$$\int_{\partial \sigma} \omega = \int_{\sigma} d\omega \, ,$$

where $\boldsymbol{\sigma}$ is a $(p+1)$-simplex. According to a representation

$$(\bar{\boldsymbol{s}}^{p+1}, U, \phi)$$

of $\boldsymbol{\sigma}$ we have from the definition

$$\int_{\sigma} d\omega = \int_{\bar{\boldsymbol{s}}^{p+1}} \phi^*(d\omega) = \int_{\bar{\boldsymbol{s}}^{p+1}} d(\phi^* \omega) \, .$$

This reduces the problem to a Euclidean one. Let η be a p-form on a neighborhood U of $\bar{\boldsymbol{s}}^{p+1}$ in E^{p+1}. To prove

$$\int_{\partial \bar{\boldsymbol{s}}^{p+1}} \eta = \int_{\bar{\boldsymbol{s}}^{p+1}} d\eta \, .$$

Now

$$\eta = \sum A_i(\boldsymbol{x}) dx^1 \cdots dx^{i-1} dx^{i+1} \cdots dx^{p+1} \, ,$$

so that it suffices to check the formula in case η is a monomial only. Since we may permute coordinates provided we are careful about signs, it suffices to take the case

$$\eta = A dx^1 \cdots dx^p \, .$$

Then

$$d\eta = (-1)^p \frac{\partial A}{\partial x^{p+1}} dx^1 \cdots dx^{p+1} .$$

We remember that $\bar{\boldsymbol{s}}^{p+1}$ consists of all points ($x^1,\dots,x^{p+1}$) satisfying

$$x^i \ge 0 , \quad \sum_1^{p+1} x^i \le 1 .$$

We have

$$\int_{\bar{s}^{p+1}} d\eta = (-1)^p \int_{\bar{s}^{p+1}} \frac{\partial A}{\partial x^{p+1}} dx^1 \cdots dx^{p+1}$$

$$= (-1)^p \int_{(x^i \ge 0, \sum_1^p x^i \le 1} dx^1 \cdots dx^p \left(\int_0^{(1-\sum_1^p x^i)} \frac{\partial A}{\partial x^{p+1}} dx^{p+1} \right)$$

$$= (-1)^p \int_{(x^i \ge 0, \sum_1^p x^i \le 1} \left[A(x^1,\dots,x^p,1-\sum_1^p x^i) - A(x^1,\dots,x^p,0) \right] dx^1 \cdots dx^p$$

We must next investigate $\partial \bar{\boldsymbol{s}}^{p+1}$. We write

$$\bar{\boldsymbol{s}}^{p+1} = (R_0, R_1, \dots, R_{p+1}) ,$$

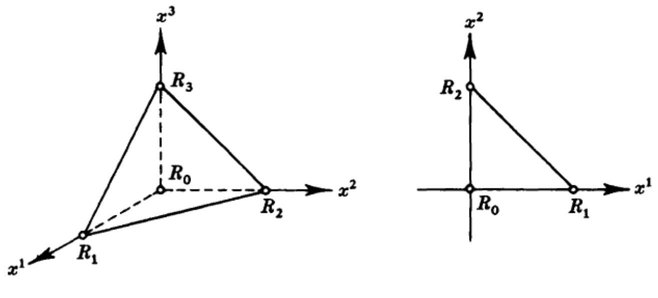
$$\left.\begin{aligned} R_0 &= 0 \\ R_1 &= (10\cdots0) \\ &\cdots \\ R_{p+1} &= (0\cdots01) \end{aligned}\right\} \text{ points in } E^{p+1} .$$

We have

$$\partial \bar{\boldsymbol{s}}^{p+1} = (R_1, \dots, R_{p+1}) + (-1)^{p+1} (R_0, R_1, \dots, R_p) + \text{other faces,}$$

where $\eta = 0$ on each of the other faces since some one of $x^1,\dots,x^p$ is constant there. Thus

$$\int_{\partial \bar{s}^{p+1}} \eta = \int_{(R_1,\dots,R_{p+1})} \eta + (-1)^{p+1} \int_{(R_0,R_1,\dots,R_p)} \eta .$$



The face ($R_0, R_1, \dots, R_p$) ***is*** the standard $\bar{\boldsymbol{s}}^p$. On it $x^{p+1} = 0$ and so

$$(-1)^{p+1} \int_{(R_0,\dots,R_{p+1})} \eta = (-1)^{p+1} \int_{\bar{s}^p} A(x^1, x^2, \dots, x^p, 0) dx^1 \cdots dx^p$$

which is precisely the second term in the expression for $\int d\eta$ above.

The first term is obtained by projecting downward in the x^{p+1} direction:

$$\int_{(R_1,\ldots,R_{p+1})} \eta = \int_{(R_1,\ldots,R_p,R_0)} A(x^1,\ldots,x^p,1-\sum_1^p x^i)dx^1\cdots dx^p$$

$$= (-1)^p \int_{(R_0,R_1\ldots,R_p)} A(x^1,\ldots,x^p,1-\sum_1^p x^i)dx^1\cdots dx^p \ ,$$

$$= (-1)^p \int_{\bar{s}^p} A(x^1,\ldots,x^p,1-\sum_1^p x^i)dx^1\cdots dx^p$$

and this is the first term in the expression for $\int d\eta$. The proof is completed.

5.9. Periods and De Rham's Theorems

We consider an example. The manifold M consists of E^3 with the origin removed,

$$M = E^3 - \{0\} .$$

Suppose ω is a one-form on M such that $d\omega = 0$. Then is ω exact ? That is, is it the differential of a function on M? The proof in Section 3.6 will not avail here because M cannot be shrunk to a point. Nonetheless, $\omega = df$, where

$$f(\boldsymbol{x}) = \int_{(1,0,0)}^{\boldsymbol{x}} \omega \ ,$$

the integral taken along any path $\boldsymbol{c}$ which avoids 0. That this is independent of the path follows from Stokes' theorem. For if $\boldsymbol{c'}$ is another path in M from (1, 0, 0) to $\boldsymbol{x}$, then the chain $\boldsymbol{c}$ - $\boldsymbol{c'}$ is the boundary of a piece of surface Σ (2-chain) in M and

$$\int_{\boldsymbol{c}} \omega - \int_{\boldsymbol{c}'} \omega = \int_{\boldsymbol{c}-\boldsymbol{c}} \omega = \int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega = 0 \ .$$

Next suppose α is a two-form on M such that $d\alpha = 0$. We seek a one- form λ on M such that $\alpha = d\lambda$. By the converse to Poincaré's lemma in Section 3.6, such a form λ exists ***locally***. But we are asking the ***global*** question: Is there such a form λ on all of M? The answer to this one is no in general, we shall have explicit examples later. For if there were such a one-form λ with $d\lambda = \alpha$ we would have

$$\int_{\boldsymbol{S}^2} \alpha = \int_{\boldsymbol{S}^2} d\lambda = \int_{\partial\boldsymbol{S}^2} \lambda = 0 \, ,$$

since the unit sphere S^2 has no boundary. But there is no reason a

priori for assuming that

$$\int_{S^2} \alpha = 0 \, .$$

The **correct result** is this. If α is a two-form on $M = E^3 - \{0\}$ with $d\alpha = 0$ and

$$\int_{S^2} \alpha = 0$$

then $\alpha = d\lambda$ for some one-form λ on *M*.

This result is contained in De Rham's theorems which we shall formulate now without proofs.

We deal with a fixed manifold *M* about which we assume only some mild limitation on its size, for example we may suppose it can be imbedded in a sufficiently high dimensional Euclidean space.

A ***closed form*** is a differential form won *M* satisfying $d\omega = 0$.

An ***exact form*** is a differential form o> on *M* satisfying $\omega = d\eta$ for some form η on *M*.

Each exact form is closed:

$$d\omega = d(d\eta) = 0 \, .$$

Let ω be a closed *p*-form. To each *p*-cycle ***z*** on *M* corresponds a ***period*** of ω ,

$$\int_z \omega$$

If ***z*** happens to be a boundary $\boldsymbol{b} = \partial \boldsymbol{c}$, the period vanishes,

$$\int_{\boldsymbol{b}} \omega = \int_{\partial \boldsymbol{c}} \omega = \int_{\boldsymbol{c}} d\omega = \int_{\boldsymbol{c}} 0 = 0 \, .$$

Because of this there is a relation between periods:

$$\begin{cases} Whenever\_cycles\_z_1, \ldots are\_related\_by \\ \sum a_i z_i = boundary \\ then \\ \sum a_i \int_{z_i} \omega = 0 \end{cases}$$

DE RHAM'S FIRST THEOREM. *A closed form is exact if and only if all of its periods vanish.*

DE RHAM'S SECOND THEOREM. *Suppose to each p-cycle **z** is assigned a number*, per(**z**), *subject to the consistency relations*

$$\begin{cases} whenever \\ \sum a_i z_i = boundary \\ then \\ \sum a_i \, per(z_i) = 0 \end{cases}$$

Then there is a closed form ω on M which has the assigned periods

$$\int_z \omega = \mathrm{per}(z) \quad \text{for each p-cycle } z.$$

On many spaces one is able to apply these results because there is a finite set of independent p-cycles which spans all p-cycles, up to boundaries. For example, on the n-sphere S^n it is known that each p-cycle is a boundary for $p > 0$, $p \neq n$, and that in dimension n there is a single n-cycle (S^n itself with outward normal for orientation) such that each n-cycle is a multiple of this one plus a boundary. These things are established by algebraic topology.

A complete analysis of De Rham's theorems reveals the following result, which has considerable attraction in itself.

Suppose we consider only chains $\boldsymbol{c} = \sum a_i \boldsymbol{\sigma_i}$ which are sums of simplices with integer coefficients. Then we may talk of these as ***integer-chains*** and have ***integer-cycles*** and ***integer-boundaries***. The ***integer-periods***

$$\int_z \omega$$

of a closed form ω are the periods taken over integer-cycles only.

Let ω and η be closed forms of degrees p and q, respectively. Suppose that the integer-periods of ω and η are all integers. Then the same is true of $\omega \wedge \eta$.

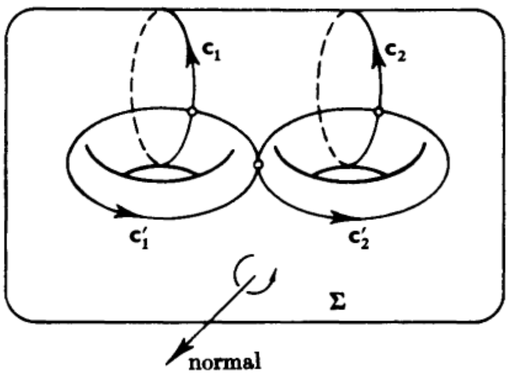
5.10. Surfaces; Some Examples

It is shown in topology that each closed surface in E^3 may be smoothly deformed into a sphere with h handles, or alternatively, a button with h holes. Let us consider the case $h = 2$ and orient this surface Σ with the outward drawn normal. The only significant two-cycle is Σ itself. By De Rham's First Theorem, a two-form α on this surface is an exact differential if and only if

$$\int_\Sigma \alpha = 0 .$$

There are four significant one-cycles, $\boldsymbol{c_1}, \boldsymbol{c'_1}, \boldsymbol{c_2}, \boldsymbol{c'_2}$. Here $\boldsymbol{c}_1$ and

c'_1 intersect once and cross, the same for c_2 and c'_2. But c'_1 and c'_2 intersect once without crossing. To see the geometric plausibility of the statement that each one-cycle c on Σ is a sum of multiples of the c_i and c'_i plus a boundary, one cuts the surface Σ along these basic cycles. Having done this, Σ may be smoothly deformed into a plane domain without holes.

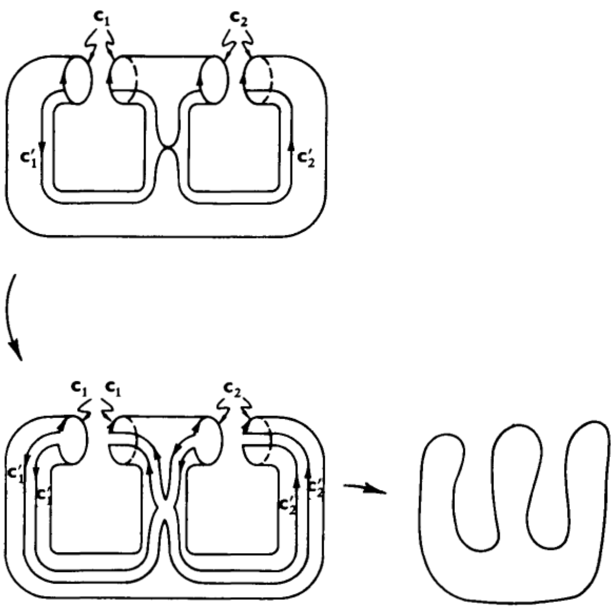


De Rham's First Theorem now asserts that if ω is a closed one-form on Σ, then ω is an exact differential if and only if

$$\int_{c_1}\omega = \int_{c'_1}\omega = \int_{c_2}\omega = \int_{c'_2}\omega = 0 .$$

Applied to dimension one, De Rham's Second Theorem asserts that if real numbers a_1, a'_1, a_2, a'_2 are given, there exists a closed one-form ω satisfying

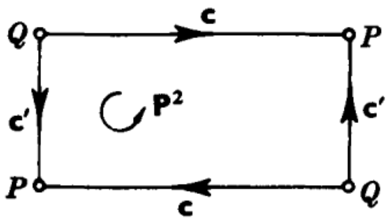
$$\int_{c_1}\omega = a_1 ,\quad \int_{c'_1}\omega = a'_1 ,\quad \int_{c_2}\omega = a_2 ,\quad \int_{c'_2}\omega = a'_2 .$$



It is also interesting to consider non-orientable closed surfaces. These of course cannot be realized in E^3. Perhaps the simplest is the ***projective plane*** P^2. This is defined by pasting the edge of a rectangle together in the order indicated. The boundary relations are

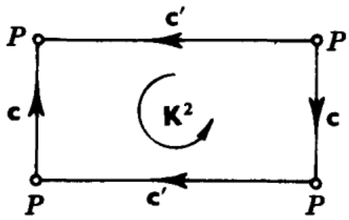
$$\partial(P^2) = 2\boldsymbol{c}' - 2\boldsymbol{c},$$
$$\partial c = (P) - (Q),$$
$$\partial c' = (P) - (Q).$$

This means first of all that there is no effective two-cycle, each two-form is exact. The only effective one-cycle is $\boldsymbol{c}' - \boldsymbol{c}$, and this actually bounds, $\boldsymbol{c}' - \boldsymbol{c} = \frac{1}{2}\partial P^2$. Thus each closed one-form is exact.



Another interesting example is the ***Klein bottle*** K^2, again defined by pasting edges together. The boundary relations are

$$\partial K^2 = -2\boldsymbol{c},$$
$$\partial \boldsymbol{c} = \partial \boldsymbol{c}' = 0.$$



The one independent one-cycle is $\boldsymbol{c}'$.

5.11. Mappings of Chains

Suppose M and **N** are manifolds and f is a smooth mapping:

$$f : M \to N.$$

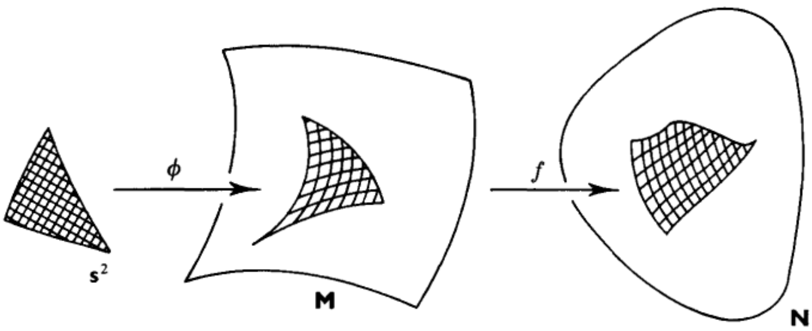
Then to each p-chain $\boldsymbol{c}$ on M there corresponds in a natural way a p-chain $f_*\boldsymbol{c}$ on N.

It suffices to explain this when $\boldsymbol{c}$ is a simplex $\boldsymbol{\sigma}^p$. Such a simplex is represented by $(\boldsymbol{s}^p, U, \phi)$ where U is a neighborhood of the Euclidean simplex $\boldsymbol{s}^p$ and $\phi : U \to M$. We merely compose f and ϕ so that $f_*\boldsymbol{c}$ is represented by

$$(\boldsymbol{s}^p, U, f \circ \phi).$$

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[dr, "f \circ \phi"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

We illustrate the process for the case of a two-simplex.



This induced map f_* takes the space of chains onto the space of chains:

$$M \xrightarrow[f]{} N ,$$

$$C_p(M) \xrightarrow[f_*]{} C_p(N) .$$

We observe that if $\boldsymbol{c}$ is a p-chain in M, then

$$f_*(\partial \boldsymbol{c}) = \partial(f_* \boldsymbol{c}) ,$$

which leads to the commutative diagram:

$$\begin{tikzcd} \mathbf{C}_p(\mathbf{M}) \arrow[r, "f_*"'] \arrow[d, "\partial"'] & \mathbf{C}_p(\mathbf{N}) \arrow[d, "\partial"'] \\ \mathbf{C}_{p-1}(\mathbf{M}) \arrow[r, "f_*"'] & \mathbf{C}_{p-1}(\mathbf{N}) \end{tikzcd}$$

which is certainly analogous to the corresponding diagram in Section 5.4 for f^* and d. The validity of the result is established by looking at individual simplices.

We now see what happens with two mappings. Let

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "g \circ f"'] & N \arrow[d, "g"] \\ & P \end{tikzcd}$$

Then the assertion is

$$(g \circ f)_* = g_* \circ f_*$$

which again follows for a simplex almost directly from the definition of f_*.

Finally we consider this situation. Let

$$f : M \to N \ .$$

Suppose that ω is a *p*-form on *N* and $\boldsymbol{c}$ is a *p*-chain on *M*. Then $f^*\omega$ is a *p*-form on *M* and $f_*\boldsymbol{c}$ is a *p*-chain on *N*. We have

$$\int_{\boldsymbol{c}} f^*\omega = \int_{f_*\boldsymbol{c}} \omega \ ..$$

This important result also follows directly from the definition for a simplex and is obtained for a general chain by summation.

5.12. Problems

VI. Applications in Euclidean Space

6.1. Volumes in E^n

We denote by

$$\omega = dx_1 \cdots dx_n$$

the element of volume in E^n, an n-form, and set

$$V_n = \int_{r \le 1} \omega , \quad r^2 = \sum x_i^{\,2} ,$$

so that V_n is the volume of the unit ball. Next we denote by σ' the element of $(n - 1)$-dimensional volume on the unit sphere $S^{n-1} = \{x \mid r = 1\}$, and set

$$A_{n-1} = \int_{S^{n-1}} \sigma' .$$

Thus $A_1 = 2\pi$, $A_2 = 4\pi$, $V_1 = 2$, $V_2 = \pi$, $V_3 = (4/3)\pi$. It is clear that the volume of the sphere of radius r is $r^{n-1} A_{n-1}$, hence

$$V_n = \int_0^1 r^{n-1} A_{n-1} dr = \frac{1}{n} A_{n-1} .$$

One may evaluate V_n by integrating over slabs:

$$V_n = \int_{-1}^{1} (1 - x^2)^{(n-1)/2} V_{n-1} dx = V_{n-1} J_n ,$$

where

$$J_n = \int_{-1}^{1} (1 - x^2)^{(n-1)/2} dx .$$

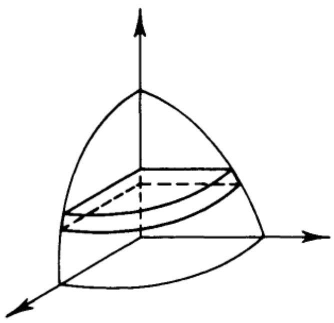
Integration by parts once leads to

$$J_n = \int_{-1}^{1} x(2x)\left(\frac{n-1}{2}\right)(1 - x^2)^{(n-3)/2} dx = (n-1)(-J_n + J_{n-2}) ,$$

$$J_n = \frac{n-1}{n} J_{n-2} .$$

These recursion formulae lead to the standard result

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\dfrac{n}{2} + 1\right)} .$$



Next we obtain an explicit formula for σ' in terms of the Euclidean coordinates $x_1,\dots,x_n$. We begin with the form

$$rdr = \sum x_i dx_i \ ,$$

a one-form in E^n which is invariant under rotations (orthogonal transformations) of E^n. Consequently

$$*\,rdr = \sum (-1)^{i-1} x_i dx_1 \cdots \overline{dx_i} \cdots dx_n \ ,$$

(the "hat" denotes a missing factor) is an (n - 1)-form in E^n which is invariant under rotations. It follows that on S^{n-1},

$$\sigma' = c * rdr \ ,$$

where c is a constant.

Next we note that

$$d(*rdr) = \sum (-1)^{i-1} dx_i dx_1 \cdots \overline{dx_i} \cdots dx_n = n\omega \,,$$

hence

$$A_{n-1} = \int_{S^{n-1}} \sigma' = c\int_{S^{n-1}} * rdr = c\int_{r\le 1} d(*rdr) = c\int_{r\le 1} n\omega = cnV_n = cA_{n-1} \,,$$

c=1, $\sigma' = *rdr$ on S^{n-1}.

Summarizing, we set

$$\sigma = *rdr = \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \cdots \overline{dx_i} \cdots dx_n \ ,$$

defining an (n - 1)-form σ in E^n. Then $d\sigma = n\omega$, and if σ is restricted to S^{n-1}, the result is the (n - 1)-dimensional volume form σ of S^{n-1}.

Next we consider the natural projection

$$\pi : E^n - \{0\} \to S^{n-1}$$

defined by $\pi(\boldsymbol{x}) = \boldsymbol{x}/|\boldsymbol{x}|$.

We seek $\pi^*\sigma'$, an (n - 1)-form on $E^n - \{0\}$ satisfying

$$d(\pi^*\sigma') = 0 \quad \text{since} \quad d(\pi^*\sigma') = \pi^*(d\sigma') = \pi^*(0) = 0 \ .$$

($d\sigma'$ is an n-form on S^{n-1}, hence 0.) We shall prove

$$\pi^*\sigma' = \frac{\sigma}{r^n} \ .$$

We could prove this by directly substituting

$$y_i = x_i / r \quad \text{in} \quad \sigma' = \sum (-1)^{i-1} y_i dy_1 \cdots \overline{dy_i} \cdots dy_n \ ,$$

but we prefer to proceed indirectly by exploiting the symmetries present. We set

$$\tau = \frac{\sigma}{r^n}.$$

Then

$$d\tau = \frac{1}{r^n} d\sigma - \frac{n}{r^{n+2}}(rdr)\sigma = \frac{n\omega}{r^n} - \frac{nr^2\omega}{r^{n+2}} = 0.$$

Now we observe that $*(\pi^*\sigma')$ is a one-form in $E^n - \{0\}$ which is invariant under rotations, hence dependent on r alone. We may write

$$*(\pi^*\sigma') = \frac{f(r)}{r^n}(rdr).$$

From this we have

$$\pi^*\sigma' = \frac{f(r)}{r^n}\sigma = f(r)\tau,$$

$$0 = d(\pi^*\sigma') = \frac{df}{dr}\tau,$$

$$\frac{df}{dr} = 0,$$

$$f = c,$$

a constant, $\pi^*\sigma' = c\tau$. To evaluate c, we simply note that on S^{n-1}, both $\pi^*\sigma'$ and τ collapse to σ, hence $c = 1$,

$$\pi^*\sigma' = \tau.$$

6.2. Winding Numbers, Degree of a Mapping

A basic result of topology (Seifert und Threlfall [20], p. 283) asserts that if M and N are closed oriented n-manifolds and $f: M \to N$, then the chain $f_* M$ is an integral multiple of N plus a boundary. This integer multiplier is called the ***degree*** of f and written deg f.

Now suppose that Σ is a closed oriented (n - 1)-manifold in $E^n - \{0\}$. Then by the Jordan-Brouwer theorem of topology, Σ decomposes E^n into exactly two regions. We assume Σ is oriented by the outward normal. The projection mapping π of Section 6.1 sends Σ into S^{n-1}. It is true that $\deg \pi = 0$ or 1; our point is that this can be determined by an integral. Let $\delta = \deg \pi$.

Then

$$\int_\Sigma \tau = \int_\Sigma \pi^*\sigma' = \int_{\pi(\Sigma)} \sigma' = \delta \int_{S^{n-1}} \sigma' = \delta A_{n-1},$$

hence

$$\delta = \frac{1}{A_{n-1}} \int_{\Sigma} \tau \, .$$

More generally, let M^{n-1} be a closed oriented manifold,

$$f : M^{n-1} \to E^n - \{0\} \, .$$

Essentially we are thinking of $f(M^{n-1})$ as a hypersurface in $E^n - \{0\}$ which may intersect itself. We look on this hypersurface as winding around the origin and we want to count how many times it encircles. This winding number is given by the ***Kronecker integral***

$$\frac{1}{A_{n-1}} \int_M f^* \tau \, .$$

We may justify this as follows. Set $g = \pi \circ f : M^{n-1} \to S^{n-1}$. What

$$\begin{tikzcd} \mathbf{M}^{n-1} \arrow[r, "f"] \arrow[dr, "g = \pi \circ f"'] & \mathbf{E}^n - \{0\} \arrow[d, "\pi"] \\ & \mathbf{S}^{n-1} \end{tikzcd}$$

we are after is $\deg g$. Now

$$g_*(M) = (\deg g) S^{n-1} + (\text{boundary}) \, ,$$

hence

$$\int_M g^* \sigma' = \int_{g_* M} \sigma' = (\deg g) \int_{S^{n-1}} \sigma' = A_{n-1} \deg g \, ,$$

$$\deg g = \frac{1}{A_{n-1}} \int_M g^* \sigma' \, .$$

But $g^* \sigma' = (f^* \circ \pi^*) \sigma' = f^* \tau$, so we have

$$\deg g = \frac{1}{A_{n-1}} \int_M f^* \tau \, .$$

The most general situation is this:

$$f : M^n \to N^n \, .$$

Let β be the volume form on N taken so that $\int_N \beta = 1$. Then

$$\deg f = \int_M f^* \beta \, .$$

For $f_* M = (\deg f) N$ +(boundary), hence

$$\int_M f^* \beta = \int_{f_* M} \beta = (\deg f) \int_N \beta = \deg f \, .$$

One interesting example: let T^n be the n-torus, $f : S^n \to T^n$ where $n \geq 2$. Then $\deg f$=0.

Because the integrals involved are integer-valued, they remain

constant when the mapping in question is subject to a deformation. Precisely, let $f_t : M \to N$ be a one-parameter family of maps. Then

$$\deg f_t = \int_M f_t^* \beta$$

is a smooth function of t, always an integer, hence constant. It follows that $\deg f_0 = \deg f_1$.

One other remark. Suppose we have

$f : M \to N$, $g : N \to P$ so that $h = g \circ f : M \to P$.

Then

$$\deg h = (\deg f) \cdot (\deg g) .$$

6.3. The Hopf Invariant

For each sphere S^n, let σ_n denote the element of area, normalized so that

$$\int_{S^n} \sigma_n = 1 .$$

Consider first a map $f : S^3 \to S^2$. Then $f^* \sigma_2$ is a 2-form on S^3. Also $d(f^* \sigma_2) = f^* (d\sigma_2) = 0$. Since S^3 has no nontrivial 2-dimensional cycles, we deduce that

$$f^* \sigma_2 = d\alpha_1$$

where the one-form α_1 on S^3 is unique up to the differential of a function. The 3-form $\alpha_1 \wedge f^* \sigma_2$ has an integral

$$\int_{S^3} \alpha_1 \wedge f^* \sigma_2 \ ,.$$

which has the remarkable property of being an integer, called the ***Hopf invariant*** of f. It is invariant under deformation of *f*. More generally, let

$$f : S^{2n-1} \to S^n .$$

Then $f^* \sigma_n = d\alpha_{n-1}$, and the Hopf invariant of *f* is

$$\int_{S^{2n-1}} \alpha_{n-1} \wedge f^* \sigma_n \ .$$

We may represent S^3 by pairs of complex numbers

$$(z,w), \ |z|^2 + |w|^2 = 1 .$$

The mapping $(z,w) \to z/w$ provides a mapping of S^3 into the closed complex plane, i.e., the Riemann sphere S^2. This map has Hopf invariant +1, hence it is ***essential*** in the sense that it cannot be deformed to a trivial map, everything going to a single point.

6.4. Linking Numbers, The Gauss Integral, Ampere's Law

Let M^r, N^s be oriented closed manifolds in E^n, where $r + s = n - 1$, and suppose these have no common point. (Best example: two disjoint closed curves in E^3.) We want to count how many times they link. To do this, we form the product space $M \times N$ which is an oriented manifold of dimension $r + s = n - 1$. We consider the map $f : M \times N \to E^n - \{0\}$ defined by

$$f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y} - \boldsymbol{x} .$$

Now we set

$$\operatorname{link}(M, N) = \deg f .$$

Thus if τ is the n-form in $E^n - \{0\}$ we considered above,

$$\operatorname{link}(M, N) = \frac{1}{A_{n-1}} \int_{M \times N} f^* \tau = \frac{1}{A_{n-1}} \int_M \int_N f^* \tau .$$

We shall work this out in E^3 for a pair of closed curves M, N:

$$\tau = \frac{1}{|\boldsymbol{z}|^3} \sum z_i dz_j dz_k = \frac{1}{|\boldsymbol{z}|^3} \frac{(\boldsymbol{z} \times d\boldsymbol{z}) \cdot d\boldsymbol{z}}{2} .$$

We let $\boldsymbol{x}$, $\boldsymbol{y}$ be the moving points on M, N, respectively. Then

$$\boldsymbol{z} = f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y} - \boldsymbol{x} ,$$

so that

$$\begin{aligned} f^* \tau &= \frac{1}{2|\boldsymbol{y} - \boldsymbol{x}|^3} [(\boldsymbol{y} - \boldsymbol{x}) \times (d\boldsymbol{y} - d\boldsymbol{x})] \cdot (d\boldsymbol{y} - d\boldsymbol{x}) \\ &= \frac{1}{2|\boldsymbol{y} - \boldsymbol{x}|^3} \{[-(\boldsymbol{y} - \boldsymbol{x}) \times d\boldsymbol{y}] \cdot d\boldsymbol{x} - [(\boldsymbol{y} - \boldsymbol{x}) \times d\boldsymbol{x}] \cdot d\boldsymbol{y}\} , \\ &= -\frac{1}{|\boldsymbol{y} - \boldsymbol{x}|^3} \{[(\boldsymbol{y} - \boldsymbol{x}) \times d\boldsymbol{y}] \cdot d\boldsymbol{x} \end{aligned}$$

$$\operatorname{link}(M, N) = \frac{-1}{4\pi} \int_{x \in M} d\boldsymbol{x} \cdot \int_{y \in N} \frac{(\boldsymbol{y} - \boldsymbol{x}) \times d\boldsymbol{y}}{|\boldsymbol{y} - \boldsymbol{x}|^3} .$$

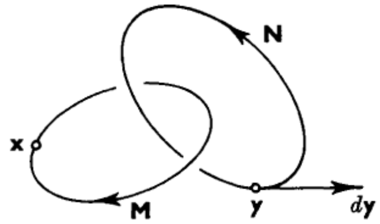
(In this computation $d\boldsymbol{y} \times d\boldsymbol{y} = 0$, etc., since $d\boldsymbol{y}$ involves only one variable.)

Imagine a steady unit electric current flowing around the closed loop N. By Ampère's law, the magnetic field at a point $\boldsymbol{x}$ due to the current in a segment $d\boldsymbol{y}$ is

$$-\frac{1}{4\pi} \frac{(\boldsymbol{y} - \boldsymbol{x}) \times d\boldsymbol{y}}{|\boldsymbol{y} - \boldsymbol{x}|^3} ,$$

hence the total magnetic field at $\boldsymbol{x}$ is

$$F(\boldsymbol{x}) = -\frac{1}{4\pi} \int_N \frac{(\boldsymbol{y} - \boldsymbol{x}) \times d\boldsymbol{y}}{|\boldsymbol{y} - x|^3} .$$



It follows that $\text{link}(M,N)=\int_M F(\boldsymbol{x})\cdot d\boldsymbol{x}$ is precisely the work done by this field on a unit magnetic pole which makes one circuit of M**.**

In the next example, $\text{link}(M,N)=0$, which seems surprising since the curves cannot really be separated.