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CONTENTS

1 Review of Acoustics of Moving Media

1.1 INTRODUCTION

1.2 DERIVATION OF BASIC EQUATIONS

1.3 ELEMENTARY SOLUTIONS OF ACOUSTIC EQUATIONS

1.4 INTEGRAL FORMULAS FOR SOLUTIONS TO THE WAVE EQUATION

1.5 SOURCE DISTRIBUTION IN FREE SPACE: MULTIPOLE EXPANSION

1.6 RADIATION FIELD

1.7 ENERGY RELATIONS

1.8 MOVING SOUND SOURCES

APPENDIX 1. A - FOURIER REPRESENTATION OF FUNCTIONS

APPENDIX I. B - CLEBSCH POTENTIAL

APPENDIX I. C - COMMONLY USED SYMBOLS

REFERENCES

2 Aerodynamic Sound

2.1 INTRODUCTION

2.2 LIGHTHILL'S ACOUSTIC ANALOGY

2.3 SOLUTION TO LIGHTHILL'S EQUATION WHEN NO SOLID BOUNDARIES ARE PRESENT

2.4 APPLICATION OF LIGHTHILL'S THEORY TO TURBULENT FLOWS

2.5 PHYSICS OF JET NOISE

APPENDIX - TRANSFORMATION OF SOURCE CORRELATION FUNCTION

REFERENCES

3 Effect of Solid Boundaries

3 1 INTRODUCTION

3.2 DERIVATION OF FUNDAMENTAL EQUATION

3.3 FFOWCS WILLIAMS - HAWKINGS EQUATION

3.4 CALCULATION OF AERODYNAMIC FORCES

3. 5 CALCULATION OF SOUND FIELD FROM SPECIAL FLOWS

APPENDIX 3. A - REDUCTION OF VOLUME DISPLACEMENT TERM TO DIPOLE AND QUADRUPOLE TERMS
APPENDIX 3. B - SOLUTION TO TWO-DIMENSIONAL UNSTEADY AIRFOIL PROBLEM
APPENDIX 3. C - LIFT SPECTRA
REFERENCES

4 Effect of Uniform Flow
4.1 INTRODUCTION
4.2 DERIVATION OF BASIC EQUATION
4.3 APFLICATION TO FAN NOISE
REFERENCES

5 Theories Based on Solution of Linearized Vorticity - Acoustic Field Equations
5.1 INTRODUCTION
5.2 DECOMPOSITION OF LINEARIZED SOLUTIONS INTO ACOUSTICAL AND VORTICAL MODES: SPLITTING THEOREM
5.3 SOUND GENERATED BY A BLADE ROW
APPENDIX 5. A - SOLUTION TO CASCADE PROBLEM
APPENDIX 5. B - EVALUATION OF SINGLE AIRFOIL INTEGRAL
APPENDIX 5. C - EVALUATION OF TERMS IN DUCT COORDINATES
REFERENCES

6 Effects of Nonuniform Mean Flow on Generation of Sound
6.1 INTRODUCTION
6.2 DERIVATION OF PHILLIPS' EQUATION
6.3 DERIVATION OF LILLEY'S EQUATION
6.4 INTERPRETATION OF EQUATIONS
6. 5 SIMPLIFICATION OF PHILLIPS' AND LILLEY'S EQUATIONS
6.6 EQUATION BASED ON SEPARATION OF ACOUSTICAL AND VORTICAL MOTIONS
6.7 APPLICATION TO MIXING REGION OF A SUBSONIC JET
6.8 SOLUTIONS OF PHILLIPS' AND LILLEY'S EQUATIONS
APPENDIX 6. A - DERIVATION OF EQUATION (6-26).
APPENDIX 6. B - ASYMPTOTIC SOLUTIONS TO STURM-LIOUVILLE EQUATION
REFERENCES

CHAPTER 1 Review of Acoustics of Moving Media

1.1 INTRODUCTION

In order to make the material in this book available to as broad an audience as possible, portions of the first chapter are devoted to a review of those aspects of classical acoustics and the acoustics of moving media which are necessary for understanding the theory of aerodynamic sound. In addition, a number of the mathematical techniques needed in the succeeding chapters on aerodynamic sound theory are developed. It is assumed that the reader is familiar with basic fluid mechanics.

A vector quantity is denoted by an arrow ($\vec{A}$) and the magnitude of the vector by the same letter (*A*). The components of the vector A are denoted by A_i with *i* equal to 1, *2,* or 3. An asterisk (*) denotes complex conjugates. Whenever possible, the capital and lower case of the same letter are used to denote Fourier transform pairs with respect to the time variable. Overbars (-) denote time averages, and brackets $\langle \ \rangle$ denote space averages. The letter T (without subscripts) denotes a large time interval. Other commonly used symbols are defined in appendix 1. C.

1.2 DERIVATION OF BASIC EQUATIONS

We shall now consider an inviscid non-heat-conducting flow whose motion is governed by Euler's equation (i.e., the momentum equation for inviscid flow)

$$\rho\left(\frac{\partial \vec{v}}{\partial \tau} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \vec{f}\ , \qquad (1\text{-}1)$$

the continuity equation

$$\frac{\partial \rho}{\partial \tau} + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v} = \rho q\ , \qquad (1\text{-}2)$$

and the energy equation (which we write in the form)

$$\frac{\partial S}{\partial \tau} + \vec{v}\cdot\nabla S = 0\ , \qquad (1\text{-}3)$$

where ∇ is the vector operator

$$\hat{i}\frac{\partial}{\partial y_1} + \hat{j}\frac{\partial}{\partial y_2} + \hat{k}\frac{\partial}{\partial y_3}\ .$$

$\vec{v} = \{v_1, v_2, v_3\}$ is the velocity of the fluid, ρ is its density, *p* is its pressure, and *S* is its entropy. The time is denoted by τ , $\{y_1, y_2, y_3\}$ are Cartesian spatial coordinates, *q* denotes the volume flow being emitted per unit volume by any source of fluid within the flow, and $\vec{f}$ denotes an externally applied volume force (which produces no entropy).

Now, in general, any thermodynamic property can be expressed as a function of any two others. Thus, in particular,

$$\rho = \rho(p, S) .$$

Hence,

$$d\rho = \frac{1}{c^2} dp + \left(\frac{\partial \rho}{\partial S}\right)_p dS , \qquad (1\text{-}4)$$

where

$$c^2 = \frac{1}{\left(\frac{\partial \rho}{\partial p}\right)_S} . \qquad (1\text{-}5)$$

Consequently,

$$\frac{\partial \rho}{\partial \tau} + \vec{v} \cdot \nabla \rho = \frac{1}{c^2}\left(\frac{\partial p}{\partial \tau} + \vec{v} \cdot \nabla p\right). \qquad (1\text{-}6)$$

For a steady flow with velocity $\vec{v}_0$, pressure p_0, density ρ_0, entropy $S_0 = S(p_0, \rho_0)$, and $c_0 = c(p_0, \rho_0)$, equations (1-1) to (1-3) and (1-6) become

$$\left.\begin{aligned} \rho_0 \vec{v}_0 \cdot \nabla \vec{v}_0 &= -\nabla p_0 \\ \nabla \cdot \rho_0 \vec{v}_0 &= 0 \\ \vec{v}_0 \cdot \nabla S_0 &= 0 \\ \vec{v}_0 \cdot \nabla p_0 &= c_0^2 \vec{v}_0 \cdot \nabla \rho_0 \end{aligned}\right\} \qquad (1\text{-}7)$$

provided there are no external forces or mass addition.

Consider an unsteady disturbance with characteristic length λ traveling at a propagation speed whose typical value is $\widetilde{C}$ through a fluid in which the velocity, pressure, and density are otherwise determined by equations (1-7). This disturbance introduces changes in velocity, pressure, density, entropy, and c^2 ($\vec{u} \equiv \vec{v} - \vec{v}_0$, $p' \equiv p - p_0$, $\rho' \equiv \rho - \rho_0$, $S' \equiv S - S_0$, $c'^2 \equiv c^2 - c_0^2$, respectively) as it passes by a fixed observer[1]. These changes

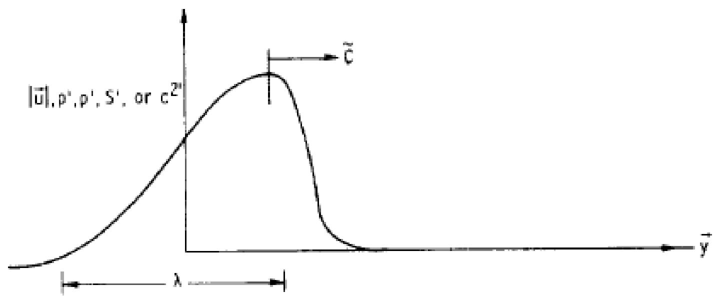

Figure 1-1. - Propagating disturbance.

[1] The flow velocity $\vec{u}$ induced by the passage of the disturbance is called the ***acoustic particle velocity***. It is entirely distinct from the propagation speed $\widetilde{C}$ of the disturbance.

all occur on time scale $T_p = 1/f$, where $f = \widetilde{C}/\lambda$ is the characteristic frequency of the disturbance. The propagating disturbance is shown schematically in figure 1-1.

The amplitude of the disturbance is measured by the magnitude of the fluctuations $\vec{u}, p', \rho', S'$ and c'^2. We shall consider only those flows for which this amplitude is so small that not only is

$$|\vec{u}| << \widetilde{C} = \frac{\lambda}{T_p}, \tag{1-8}$$

but also[2] $p' << \langle p_0 \rangle$, $\rho' << \langle \rho_0 \rangle$, $S' << \langle S_0 \rangle$ and $c'^2 << \langle c_0^2 \rangle$. Then the amplitude of the disturbance can be characterized by a dimensionless variable ε such that

$$0 < \varepsilon << 1, \tag{1-9}$$

and

$$\left.\begin{aligned} |\vec{u}|/\widetilde{C} &= O(\varepsilon) \\ p'/\langle p_0 \rangle &= O(\varepsilon) \\ \rho'/\langle \rho_0 \rangle &= O(\varepsilon) \\ S'/\langle S_0 \rangle &= O(\varepsilon) \\ c'^2/\langle c_0^2 \rangle &= O(\varepsilon) \end{aligned}\right\}. \tag{1-10}$$

Inequality (1-8) involves the assumption (to be verified subsequently for specific cases) that for sufficiently small disturbances the propagation speed is independent of the amplitude of the disturbance.

We allow $|\vec{v}_0|$ to be of the same order as $\widetilde{C}$. Then since the changes of time and length associated with the disturbance occur on the scale of T_p and λ, respectively, it is reasonable to introduce the nondimensional variables[3]

$$\begin{aligned} \widetilde{\tau} &= \tau / T_p = f\tau & \widetilde{\rho}_0 &= \rho_0 / \langle \rho_0 \rangle \\ \widetilde{y}_i &= y_i / \lambda & \widetilde{S}_0 &= S_0 / \langle S_0 \rangle \\ \widetilde{\vec{v}}_0 &= \vec{v}_0 / \widetilde{C} & \widetilde{c}_0^2 &= c_0^2 / \langle c_0^2 \rangle \\ \widetilde{p}_0 &= (p_0 - \langle p_0 \rangle)/(\langle \rho_0 \rangle \langle v_0^2 \rangle), & \widetilde{\vec{u}} &= \vec{u} / \widetilde{C}\varepsilon \\ \widetilde{p}' &= p' / \langle p_0 \rangle \varepsilon & \widetilde{S}' &= S' / \langle S_0 \rangle \varepsilon \\ \widetilde{\rho}' &= \rho' / \langle \rho_0 \rangle \varepsilon & \widetilde{c}'^2 &= c'^2 / \langle c_0^2 \rangle \varepsilon \end{aligned}.$$

When these quantities are substituted into equations (1-1) to (1-3) and (1-6),

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[2] The first inequality requires that the velocity induced by the disturbance be small compared with its propagation speed. The remaining inequalities ensure that the fluctuation in thermodynamic properties are small relative to their mean background values.

[3] Recall that the pressure variations in a steady inviscid flow are of order $\langle \rho_0 \rangle$, $\langle v_0^2 \rangle$.

we obtain after subtracting out equations (1-7)

$$(\tilde{\rho}_0+\varepsilon\tilde{\rho}')\left[\frac{\partial\tilde{\vec{u}}}{\partial\tilde{\tau}}+\tilde{\vec{v}}_0\cdot\tilde{\nabla}\tilde{\vec{u}}+\tilde{\vec{u}}\cdot\tilde{\nabla}(\tilde{\vec{v}}_0+\varepsilon\tilde{\vec{u}})\right]+\tilde{\rho}'\tilde{\vec{v}}_0\cdot\tilde{\nabla}\tilde{\vec{v}}_0=-\frac{\langle p_0\rangle}{\tilde{C}^2\langle\rho_0\rangle}\tilde{\nabla}\tilde{p}'+\frac{\vec{f}}{\varepsilon f\langle\rho_0\rangle\tilde{C}}$$

$$\frac{\partial\tilde{\rho}'}{\partial\tilde{\tau}}+\tilde{\nabla}\cdot\left[(\tilde{\rho}_0+\varepsilon\tilde{\rho}')\tilde{\vec{u}}+\tilde{\rho}'\tilde{\vec{v}}_0\right]=\frac{(\tilde{\rho}+\varepsilon\tilde{\rho}')q}{\varepsilon f},$$

$$\frac{\partial\tilde{S}'}{\partial\tilde{\tau}}+\tilde{\vec{v}}_0\cdot\tilde{\nabla}\tilde{S}'+\tilde{\vec{u}}\cdot\tilde{\nabla}\tilde{S}_0+\varepsilon\tilde{\vec{u}}\cdot\tilde{\nabla}\tilde{S}'=0,$$

$$(\tilde{c}_0^2+\varepsilon\tilde{c}'^2)\left[\frac{\partial\tilde{\rho}}{\partial\tilde{\tau}}+\tilde{\vec{v}}_0\cdot\tilde{\nabla}\tilde{\rho}'+\tilde{\vec{u}}\cdot\tilde{\nabla}(\tilde{\rho}_0+\varepsilon\tilde{\rho}')\right]+\tilde{c}'^2\tilde{\vec{v}}_0\cdot\tilde{\nabla}\tilde{\rho}_0=\frac{\langle p_0\rangle}{\langle c_0^2\rangle\langle\rho_0\rangle}\left[\frac{\partial\tilde{p}'}{\partial\tilde{\tau}}+\tilde{\vec{v}}_0\cdot\tilde{\nabla}\tilde{p}'+\tilde{\vec{u}}\cdot\tilde{\nabla}(\tilde{p}_0+\varepsilon\tilde{p}')\right]$$

But since the nondimensionalization has been specifically chosen to make the dimensionless variables of order 1, the inequality (1-9) shows that the terms multiplied by ε in these equations can be neglected to obtain, upon reverting to dimensional quantities,

$$\left.\begin{array}{c}\rho_0\left(\frac{\partial\vec{u}}{\partial\tau}+\vec{v}_0\cdot\nabla\vec{u}+\vec{u}\cdot\nabla\vec{v}_0\right)+\rho'\vec{v}_0\cdot\nabla\vec{v}_0=-\nabla p'+\vec{f}\\ \frac{\partial\rho'}{\partial\tau}+\nabla\cdot(\rho_0\vec{u}+\rho'\vec{v}_0)=\rho_0 q\\ \frac{\partial S'}{\partial\tau}+\vec{v}_0\nabla S'+\vec{u}\cdot\nabla S_0=0\\ c_0^2\left(\frac{\partial\rho'}{\partial\tau}+\vec{v}_0\cdot\nabla\rho'+\vec{u}\cdot\nabla\rho_0\right)+c'^2\vec{v}_0\cdot\nabla\rho_0=\frac{\partial p'}{\partial\tau}+\vec{v}_0\cdot\nabla p'+\vec{u}\cdot\nabla p_0\end{array}\right\},\qquad(1\text{-}11)$$

These equations are frequently referred to as ***linearized gas-dynamic equations***. We have shown that they govern the propagation of small disturbances through a steady flow.

Perhaps the simplest nontrivial solution to equations (1-7) is provided by a unidirectional, transversely sheared mean flow wherein

$$\vec{v}_0=\hat{i}U(y_2),\quad \rho_0=\text{Constant},\quad p_0=\text{Constant}\qquad(1\text{-}12)$$

and $\hat{i}$ denotes the unit vector in the u_1 direction. This velocity field is illustrated in figure 1-2. For several reasons the main emphasis will be on cases where the background flows are of this type.[4] The first is the relative simplicity of this flow. Since the equations governing the

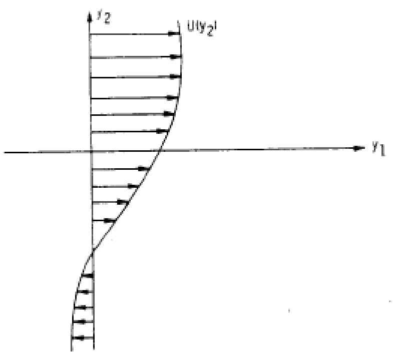

Figure 1-2. - Unidirectional, transversely sheared, mean flow.

[4] A more complete treatment of the acoustics of moving media from a different point of view can be found in Blokhintsev (ref. 1).

propagation of sound in a moving medium are, in general, quite complex, it is helpful to consider one of the simplest cases. The second reason results from the fact that in the following chapters only the effects of **velocity gradients** on aerodynamic sound generation are considered and not the effects of *gradients in thermodynamic variables*. Since the flow field given by equations (1-12) has only velocity gradients and no pressure or density gradient, it is particularly suitable for illustrating the effect of the former. Finally, it turns out that in many of the cases for which the study of aerodynamic sound is important the mean flow field is, to a first approximation, of the type given by equation (1-12).

Inserting equations (1-12) into equations (1-11) and eliminating ρ' between the first and last equation shows that

$$\left.\begin{aligned} \rho_0\left(\frac{D_0\vec{u}}{D\tau}+i\frac{dU}{dy_2}u_2\right)=-\nabla p+\vec{f} \\ \frac{1}{\rho_0 c_0^2}\frac{D_0 p}{D\tau}+\nabla\cdot\vec{u}=q \\ \frac{D_0 S}{D\tau}=0 \end{aligned}\right\}, \qquad (1\text{-}13)$$

where

$$\frac{D_0}{D\tau}\equiv\frac{\partial}{\partial\tau}+U\frac{\partial}{\partial y_1},$$

and we have dropped the prime on p so that it now denotes the fluctuating pressure. This will be done whenever no confusion is likely to result.

The operator $\dfrac{D_0}{D\tau}$ represents the time rate of change as seen by an observer moving along with the mean flow. The third equation (1-13) therefore states that the entropy does not change with time for such an observer. Thus, if the entropy were uniform and steady far upstream, it would have to be constant everywhere. But equation (1-4) shows that, whenever the entropy is constant,

$$d\rho=\frac{1}{c^2}dp,$$

and the fourth equation (1-10) shows that for small ε,

$$c^2=c_0^2+O(\varepsilon).$$

Then, since c_0^2 is constant, integrating the previous equation from the background state implies that

$$\frac{p}{\rho_0 c_0^2}=\frac{\rho-\rho_0}{\rho_0}=\frac{\rho'}{\rho_0}, \quad \text{for } S=\text{Constant}. \qquad (1\text{-}14)$$

The quantity on the right is called the ***condensation***.

Since

$$\nabla \cdot \frac{D_0 \vec{u}}{D\tau} = \frac{D_0}{D\tau} \nabla \cdot \vec{u} + \frac{\partial U}{\partial y_2} \frac{\partial u_2}{\partial y_1}$$

taking the divergence of the first equation (1-13), operating with $\frac{D_0}{D\tau}$ on the second, and subtracting the result give

$$\nabla^2 p - \frac{1}{c_0^2} \frac{D_0^2}{D\tau^2} p + 2\rho_0 \frac{\partial u_2}{\partial y_1} \frac{dU}{dy_2} = \nabla \cdot \vec{f} - \rho_0 \frac{D_0 q}{D\tau} . \tag{1-15}$$

Because this equation has two dependent variables, it cannot by itself be solved to determine the disturbance field. However, in the special case where the mean velocity U is constant, the last term on the left side drops out and we obtain the equation

$$\nabla^2 p - \frac{1}{c_0^2} \frac{D_0^2}{D\tau^2} p = \nabla \cdot \vec{f} - \rho_0 \frac{D_0 q}{D\tau} , \tag{1-16}$$

which (together with suitable boundary conditions) can be solved to unambiguously determine the fluctuating pressure p. Once this pressure is found, the acoustic particle velocity $\vec{u}$ can be determined from the first equation (1-13). Equation (1-16) is an inhomogeneous wave equation for a uniformly moving medium. The reason for this terminology will be clear subsequently.

Equations (1-14) and (1-16) show that, if the entropy is everywhere constant, the density fluctuation also satisfies an inhomogeneous wave equation

$$\nabla^2 \rho - \frac{1}{c_0^2} \frac{D_0^2}{D\tau^2} \rho = \frac{1}{c_0^2} \left(\nabla \cdot \vec{f} - \rho_0 \frac{D_0 q}{D\tau} \right) \quad \text{for } S = \text{Constant}. \tag{1-17}$$

Finally, when U = 0, equation (1-16) reduces to the inhomogeneous wave equation for a stationary medium or simply the ***inhomogeneous wave equation***

$$\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2}{\partial \tau^2} p = \nabla \cdot \vec{f} - \rho_0 \frac{\partial q}{\partial \tau} , \tag{1-18}$$

which forms the basis of the field of classical acoustics.

We now return to the general equation (1-15). This equation closely resembles the wave equation (1-18) for a nonmoving medium with $\frac{\partial}{\partial \tau}$

replaced by $\frac{D_0}{D\tau}$. However, the additional term on the left side involves the velocity and must be eliminated in order to obtain a single differential equation for the pressure. To this end, we differentiate the y_2-component of the momentum equation in (1-13) with respect to y_1 to obtain

$$\rho_0 \frac{D_0}{D\tau}\frac{\partial u_2}{\partial y_1} = -\frac{\partial^2 p}{\partial y_2 \partial y_1} + \frac{\partial f_2}{\partial y_1}. \qquad (1\text{-}19)$$

Then operating on equation (1-15) with $\frac{D_0}{D\tau}$ and substituting equation (1-19) into the result yield

$$\frac{D_0}{D\tau}\left(\nabla^2 p - \frac{1}{c_0^2}\frac{D_0^2}{D\tau^2}p\right) - 2\frac{dU}{dy_2}\frac{\partial^2 p}{\partial y_2 \partial y_1} = \frac{D_0}{D\tau}\nabla\cdot\vec{f} - -2\frac{dU}{dy_2}\frac{\partial f_2}{\partial y_1} - \rho_0 \frac{D_0^2}{D\tau^2}q$$

(1-20)

Thus, in the general case of a transversely sheared unidirectional mean flow the wave equation is of higher order (in two of the variables) than it is for a uniformly moving medium.

1.3 ELEMENTARY SOLUTIONS OF ACOUSTIC EQUATIONS

In principle, all acoustic phenomena which occur in a transversely sheared flow can be analyzed simply by solving the wave equations derived in section 1.2. In this section we shall obtain a number of simple solutions to these equations which either illustrate certain physical principles or serve as tools to synthesize more complicated solutions. We shall first consider the case of a stationary medium.

1.3.1 Solutions of Stationary-Medium Wave Equation

The basic properties of the Fourier series and transforms which are used in this text are listed in appendix 1. A. The notation and sign conventions adopted therein are adhered to whenever possible.

Multiplying both sides of the stationary-medium wave equation

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial \tau^2} = \nabla\cdot\vec{f} - \rho_0 \frac{\partial q}{\partial \tau} \equiv -\gamma \qquad (1\text{-}21)$$

by $e^{i\omega t}$ and integrating by parts over the appropriate time interval reduce this equation to the ***inhomogeneous Helmholtz equation***

$$\left[\nabla^2 + \left(\frac{\omega}{c_0}\right)^2\right]P = -\Gamma, \qquad (1\text{-}22)$$

where P and Γ are the Fourier coefficients or Fourier transforms

(depending on whether the process is periodic, stationary, or vanishing at ∞) of p and γ, respectively. (We shall henceforth refer to quantities such as P and Γ simply as ***Fourier components***.)

Solutions to equation (1-21) can be obtained by inserting the solutions to equation (1-22) into the appropriate Fourier inversion formula. If the source terms and boundary conditions are simple harmonic functions of time, the solution p of equation (1-21) is also a simple harmonic function. That is,

$$p = Pe^{-i\omega\tau}.$$

1.3.1.1 Plane wave solutions. The simplest case occurs when the region under consideration is all of space and there are no sources present. Then equation (1-22) becomes

$$\left[\nabla^2 + \left(\frac{\omega}{c_0}\right)^2\right]P = 0. \qquad (1\text{-}23)$$

The three-dimensional Fourier transform of this equation is

$$\left[-k^2 + \left(\frac{\omega}{c_0}\right)^2\right]\widetilde{P} = \left(k + \frac{\omega}{c_0}\right)\left(\frac{\omega}{c_0} - k\right)\widetilde{P} = 0,$$

where

$$P = \int \widetilde{P}(\vec{k})e^{i\vec{k}\cdot\vec{y}}d\vec{k}. \qquad (1\text{-}24)$$

But since $x\delta(x) = 0$, this equation has the solution

$$\widetilde{P} = A(\vec{\kappa})\delta\left(k - \frac{\omega}{c_0}\right),$$

where A is an arbitrary function of the unit vector $\vec{\kappa} \equiv \frac{\vec{k}}{k}$ in the $\vec{k}$-direction. Hence, the solution to equation (1-23) is

$$\begin{aligned} P &= \int_{\vec{\kappa}}\int_0^\infty A(\vec{\kappa})e^{i\vec{k}\cdot\vec{y}}\delta\left(k - \frac{\omega}{c_0}\right)k^2\,dk\,d\vec{\kappa} \\ &= \left(\frac{\omega}{c_0}\right)^2 \int_{\vec{\kappa}} A(\vec{\kappa})e^{i(\omega/c_0)\vec{\kappa}\cdot\vec{y}}d\vec{\kappa} \end{aligned}, \qquad (1\text{-}25)$$

where $d\vec{\kappa}$ denotes the element of solid angle.

When

$$A(\vec{\kappa}) = A\frac{\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{\sin\theta}$$

where θ and ϕ are polar coordinates determined by

$$\vec{\kappa} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

and θ_0, ϕ_0 bear a similar relation to the fixed unit vector $\vec{\kappa}_0$, equation (1-25) becomes

$$P = \left(\frac{\omega}{c_0}\right)^2 A e^{i\vec{k}_0 \cdot \vec{y}},$$

where $k_0 = \dfrac{\omega}{c_0}$ and $\dfrac{\vec{k}_0}{k_0} = \vec{\kappa}_0$. Equation (1-25) shows that the general solution of equation (1-23) is simply a linear superposition of solutions of this type. Hence, the general solution of the homogeneous wave equation

$$\nabla^2 p = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial \tau^2} = 0 \qquad (1\text{-}28)$$

can be expressed as a superposition of solutions of the type

$$p = A e^{i(\vec{k}_0 \cdot \vec{y} - \omega\tau)} \quad \text{where} \quad k_0 = \frac{\omega}{c_0} \qquad (1\text{-}29)$$

called ***plane waves***.[5] The constant A is called the complex ***amplitude*** of the wave, $\Phi_0 \equiv \arg A = \tan^{-1} \dfrac{\operatorname{Im} A}{\operatorname{Re} A}$ is called the ***phase constant***, and

$$\Phi = \vec{k}_0 \cdot \vec{y} - \omega\tau + \Phi_0 \qquad (1\text{-}30)$$

is called the ***instantaneous phase*** or simply the ***phase***.

When the solution to equation (l-28) is given by equation (1-29), the pressure at each fixed point $\vec{y}$ executes a simple harmonic variation in time whose amplitude is $|A|$. The ***angular frequency*** of the motion is ω; its ***frequency*** f is $f = \dfrac{\omega}{2\pi}$ and its ***period*** T_p is $T_p = \dfrac{1}{f}$. The vector $\vec{k}_0$ is called the ***wave number***.

The pressure oscillations at every point have the same frequency and the same amplitude $|A|$. However, the pressure oscillations at different points will, in general, not be in phase. The difference in phase between any two points, say $\vec{y}_1$ and $\vec{y}_2$ is given by $\vec{k}_0 \cdot (\vec{y}_1 - \vec{y}_2)$ and hence remains constant in time. This also shows that the phase is constant on any plane perpendicular to the $\vec{k}_0$-direction. Since the trigonometric functions are periodic, with period 2π, the pressure fluctuation at any two points will be in phase whenever the distance $\left(\dfrac{\vec{k}_0}{k_0}\right) \cdot (\vec{y}_1 - \vec{y}_2)$ between the two points measured along the $\vec{k}_0$-direction is

$$\left(\frac{\vec{k}_0}{k_0}\right) \cdot (\vec{y}_1 - \vec{y}_2) = \frac{2\pi}{k_0} = \frac{2\pi c_0}{\omega} = \frac{c_0}{f} = T_p c_0 .$$

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[5] When complex solutions to the wave equation are given, generally the solution to the physical problem is understood to be the real part.

This distance, which we denote by λ, is called the ***wavelength***. Thus, at any time $t = t_0$, the pressure will vary along the $\vec{k}_0$-direction in the manner shown by the solid curve in figure 1-3 and will remain constant along any plane perpendicular to this direction. At a time 1/4 period later, the wave will

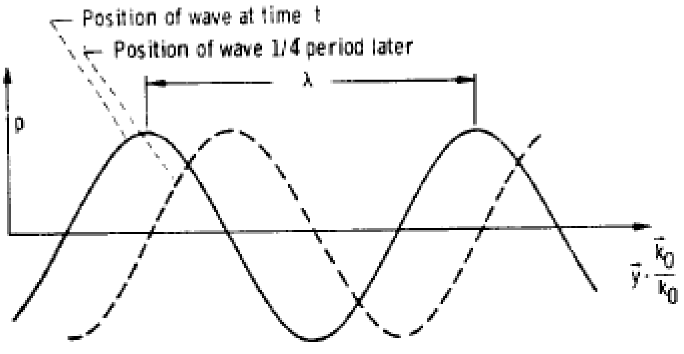

Figure 1-3. - Plane wave propagation 1/4 period after time t.

appear as the dotted curve. Hence, the individual pressure oscillations at each point are phased in such a way that they result in a wave of unchanged shape moving through the medium in the $\vec{k}_0$-direction. In other words, the pressure oscillations at each point are passed on to adjacent points with a phase relation that causes them to propagate as a wave with unchanging shape. Every surface of constant phase Φ (given by eq. (1-30)), called a **phase surface**, must be perpendicular to the $\vec{k}_0$-direction and move along with the wave, as shown schematically in figure 1-4.

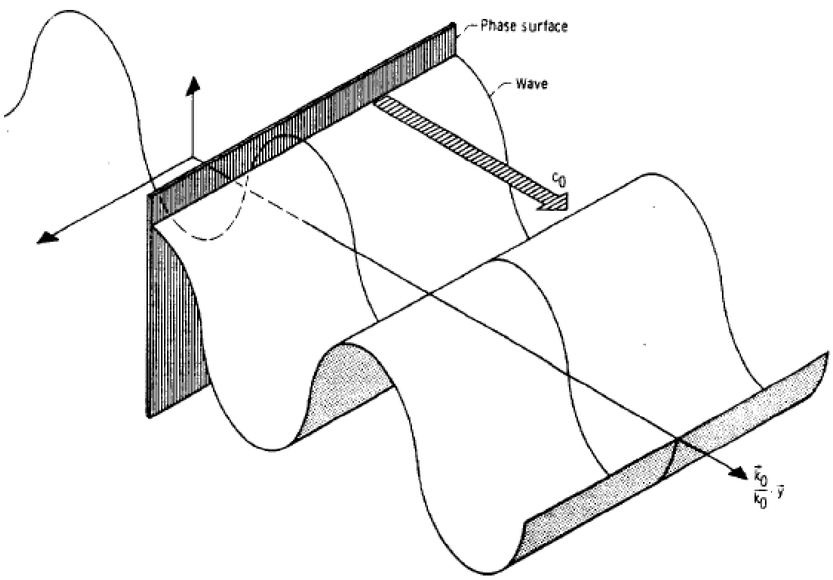

Figure 1-4. - Motion of phase surfaces for plane wave.

It can be seen from equation (1-30) that the common velocity of the phase surface and the disturbance is c_0. This velocity is called the ***speed of***

sound.[6] We have therefore shown that, at least in this special case, the initial assumption used in deriving the basic wave equations (i.e., that the propagation speed of a small disturbance is independent of the amplitude of that disturbance) is justified.

1.3.1.2 Solutions in arbitrary regions. - When the region in which the wave equation is to be solved is not all of space, the solution is usually not expressed as a superposition of plane waves but rather as the superposition of a number of eigenfunctions P_α of Helmholtz's equation, called ***modes***, which are appropriate to the region under consideration. Thus, the solution to the wave equation will appear as the sum or integral (or perhaps both) of a number of simple harmonic solutions $P_\alpha(\vec{y})r^{-i\omega\tau}$. Or upon expressing P_α in complex polar form, this becomes

$$A(\vec{y})e^{i\left|kS(\vec{y})-\omega\tau\right|},$$

where $k \equiv \omega / c_0$ and S and A are real.

We may regard the quantity $\Phi = k\left|S(\vec{y}) - c_0\tau\right|$ as being the analogue of the instantaneous phase which appeared in the plane wave solutions discussed in section 1.3.1.1. At any given instant of time, Φ will be constant on any surface $S(\vec{y}) = \text{Constant}$. The surfaces of constant phase are called ***wave fronts*** or ***wave surfaces***, and the function $S(\vec{y})$ is called the ***eikonal***. However, the amplitude of the wave $A(\vec{y})$ is not necessarily constant on the wave front as it is for plane waves.

Now the wave surface

$$k\left|S(\vec{y}) - c_0\tau\right| = \Phi = \text{Constant} = C_1$$

will, in general, move with time. Thus, the point $\vec{v}$ on $\Phi = C_1$ at time τ will move to the point $\vec{v} + \delta\vec{v}$ at time $\tau + \delta\tau$ and

$$\begin{aligned} k\left|S(\vec{y}) - c_0\tau\right| &= k\left|S(\vec{y} + \delta\vec{y}) - c_0(\tau + \delta\tau)\right| \\ &= k\left|S(\vec{y}) + \nabla S \cdot \delta\vec{y} - c_0(\tau + \delta\tau)\right| + O\left[(\delta\vec{y})^2\right]. \end{aligned}$$

This shows, to the first order in $\delta\tau$,

$$\nabla S \cdot \delta\vec{y} = c_0 \delta\tau .$$

Hence, in the limit as $\delta\tau \to 0$,

$$\nabla S \cdot \left(\frac{dy}{d\tau}\right)_{\Phi=\text{Constant}} = c_0 . \tag{1-31}$$

---

[6] For an ideal gas, this propagation speed c_0 is given in terms of the absolute temperature Θ_0 of the background state by

$$c_0 = \sqrt{\gamma \frac{p_0}{\rho_0}} = \sqrt{\gamma R \Theta_0}$$

which is equal to about 335 m/sec (1100 ft/sec) in air at standard conditions.

But since ∇S is always perpendicular to the wave fronts, $\nabla S/|\nabla S|$ is the unit normal to these surfaces (see fig. 1-5). And since $\left(\frac{d\vec{y}}{d\tau}\right)_{\Phi=\text{Constant}}$ is the time rate of change of position of a point which moves with the wave front $\Phi = C_1$.

$$V_p \equiv \frac{\nabla S}{|\nabla S|}\cdot\left(\frac{d\vec{y}}{d\tau}\right)_{\Phi=C_1}$$

is the velocity of the wave front $\Phi = C_1$ normal to itself. It is called the ***phase velocity***, and equation (1-31) shows that

$$V_p \equiv \frac{c_0}{|\nabla S|}. \tag{1-32}$$

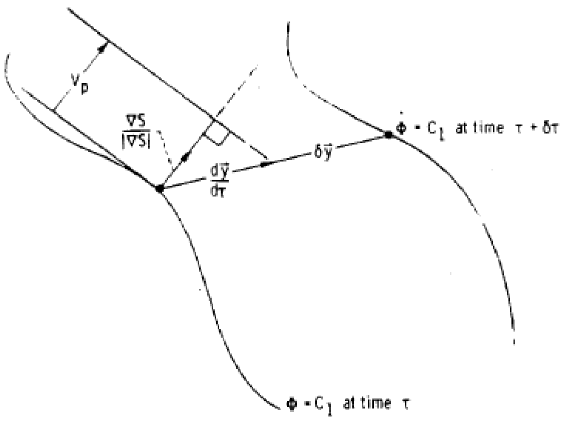

Figure 1-5. - Wave fronts.

1.3.1.3 Point source solutions. Returning to the general solution (1-25), we now take A to be independent of $\vec{\kappa}$. Then upon introducing the polar coordinates given by equation (1-26) with the polar axis now taken along the $\vec{y}$-direction, we obtain a solution

$$\begin{aligned} P &= \left(\frac{\omega}{c_0}\right)^2 A\int_0^{2\pi}\int_0^{\pi} e^{i(\omega/c_0)y\cos\theta}\sin\theta d\theta d\phi \\ &= 2\pi\left(\frac{\omega}{c_0}\right)\frac{A}{iy}e^{i(\omega/c_0)y} - 2\pi\left(\frac{\omega}{c_0}\right)\frac{A}{iy}e^{-i(\omega/c_0)y} \end{aligned}$$

to Helmholtz's equation (1-23) which depends only on the magnitude y of $|\vec{y}|$.

In fact, it is easy to see that, if $y \neq 0$, each of the terms

$$2\pi\left(\frac{\omega}{c_0}\right)\frac{A}{iy}e^{\pm i(\omega/c_0)y}$$

in this solution is itself a solution to equation (1-23). Hence, any superposition of solutions of the type

$$\frac{\Gamma_0}{4\pi y}e^{i\omega(\pm y/c_0-\tau)} \tag{1-33}$$

satisfies the wave equation (1-28). The wave fronts are given by $\Phi = \pm ky - \omega\tau$ and the eikonal is equal to $\pm y$ so that

$$|\nabla S| = 1\,.$$

But in view of equation (1-32), this shows that the phase velocity is again equal to the speed of sound c_0. Since the phase surfaces of the solution with the upper sign move in the direction of increasing *y*, this solution must represent an outward-propagating wave. The solution with the lower sign represents an inward-propagating wave.[7]

In any region including the origin $y = 0$, however, the equation

$$P^{\pm} = \frac{\Gamma_0}{4\pi y} e^{\pm i(\omega/c_0)y}$$

does not provide a solution to the Helmholtz equation (1-23) but rather satisfies the inhomogeneous Helmholtz equation

$$\nabla^2 P^{\pm} + \left(\frac{\omega}{c_0}\right)^2 P^{\pm} = -A\delta(\vec{y}) \tag{1-34}$$

with a delta function source term at the origin. In order to show this, we shall need to use the divergence theorem

$$\int_V \nabla \cdot \vec{A} d\vec{y} = \int_S \hat{n} \cdot \vec{A} dS\,, \tag{1-35}$$

where $\vec{A}$ is any vector and V is an arbitrary volume bounded by the surface *S* with outward-drawn normal $\hat{n}$. Thus, if V is taken to be a sphere of radius r_0 centered about the origin $\vec{y} = 0$ and if $d\Omega$ denotes an element of solid angle, this shows that

$$\begin{aligned}
&\int_V \left[\nabla^2 P^{\pm} + \left(\frac{\omega}{c_0}\right)^2 P^{\pm}\right] d\vec{y} \\
&= r_0^2 \int_{4\pi} \left(\frac{\partial P^{\pm}}{\partial y}\right)_{y=r_0} d\Omega + \left(\frac{\omega}{c_0}\right)^2 \int_{4\pi} \int_0^{r_0} P^{\pm} y^2 dy d\Omega \qquad . \\
&= \Gamma_0 \left(\pm i r_0 \frac{\omega}{c_0} - 1\right) e^{\pm i(\omega/c_0)r_0} \mp i\Gamma_0 \frac{\omega^2}{c_0} \frac{\partial}{\partial\omega} \int_0^{r_0} e^{\pm 8(\omega/c_0)y} dy \\
&= -\Gamma_0
\end{aligned}$$

But since

$$\int_V \delta(\vec{y}) dy = 1$$

---

[7] It will be seen subsequently that this type of behavior is quite typical of solutions for any bounded source region. Hence, solutions which behave like $(1/v)e^{ikv}$ for large y are called outgoing wave solutions, and solutions which behave like $(1/v)e^{-ikv}$ are called ingoing wave solutions.

and $\delta(\vec{y}) = 0$ in any region where $P^{\pm}$ satisfies the homogeneous Helmholtz equation, we conclude that $P^{\pm}$ satisfies equation (1-34). By shifting the location of the origin, we find that

$$P^{\pm} = \frac{\Gamma_0}{4\pi r} e^{\pm i(\omega/c_0)r}$$

with

$$r \equiv |\vec{x} - \vec{y}|$$

satisfies the Helmholtz equation

$$\nabla^2 P^{\pm} + \left(\frac{\omega}{c_0}\right)^2 P^{\pm} = \Gamma_0 \delta(\vec{x} - \vec{y}),$$

with a delta function source term at the arbitrary point $\vec{x}$.

Taking the inverse Fourier transforms shows that

$$P^{\pm} = \frac{1}{4\pi r} \int e^{-i\omega(\tau \mp r/c_0)} \Gamma_0 d\omega = \frac{1}{4\pi r} \gamma_0 \left(\tau \mp \frac{r}{c_0}\right) \quad \text{(1-36)}$$

(where Γ_0 is the Fourier transform of γ_0) satisfies the in homogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial \tau^2}\right) P^{\pm} = -\gamma_0(\tau) \delta(\vec{y} - \vec{x}) \quad \text{(1-37)}$$

with a point source of strength $\gamma_0(\tau)$ located at the point $\vec{x}$.

In order to interpret this result, notice that rP^{+} is constant everywhere along each line $c_0\tau - r = \text{Conatant}$ in the $r - \tau$ plane shown in figure 1-6.

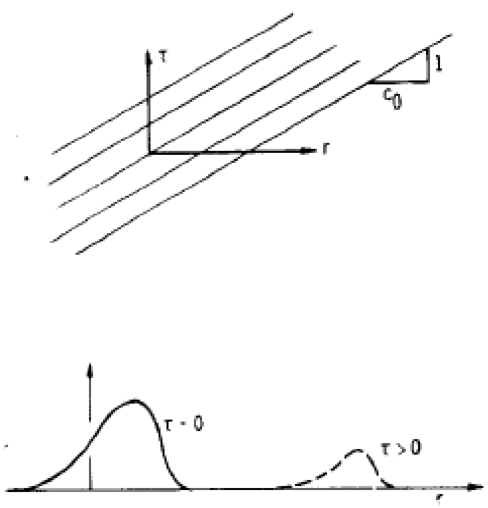

Figure 1-6. - Propagation of spherical waves.

It therefore represents an arbitrary pulse propagating outward in the radial direction with unchanged shape. The propagation speed is again equal to the speed of sound c_0. Hence, P^{+} represents a pressure pulse which propagates outward with unchanged shape in the radial direction with its

amplitude diminished by the factor $1/r$.

Upon choosing γ_0 to be the delta function $\delta(t-\tau)$, it follows from equations (1-36) and (1-37) that

$$G^0 \equiv \frac{1}{4\pi r}\delta(\tau - t + r/c_0) \qquad (1\text{-}38)$$

is an incoming wave which satisfies the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial \tau^2}\right)G^0 = -\delta(\tau - t)\delta(\vec{y} - \vec{x}) \qquad (1\text{-}39)$$

with an impulsive point source acting at the time t and located at the point $\vec{x}$. Since r is always positive, this solution together with all its derivatives must certainly vanish whenever $t < \tau$.

1.3.2 Solutions to Acoustic Equation for a Uniformly Moving Medium

Now suppose that the velocity U of the medium is constant so that the wave motion is governed by equation (1-16). The equation closely resembles the stationary-medium wave equation (1-18). This resemblance is not accidental, for suppose we carry out the analysis in a coordinate system moving at the constant velocity U. Then the medium ought to appear at rest, and therefore the equation for sound propagation in this coordinate system ought to be the stationary-medium wave equation. In fact, introducing the change in variable

$$\vec{y}' = \vec{y} - \hat{i}U\tau \text{, for } \tau' = \tau \qquad (1\text{-}40)$$

into equation (1-16) results in the **stationary-medium wave equation**

$$\left(\nabla'^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial \tau'^2}\right)P = \nabla' \cdot \vec{f} - \rho_0 \frac{\partial q}{\partial \tau'}, \qquad (1\text{-}41)$$

where ∇' denotes the operator

$$\hat{i}\frac{\partial}{\partial y'_1} + \hat{j}\frac{\partial}{\partial y'_2} + \hat{k}\frac{\partial}{\partial y'_3}.$$

Solutions to the **moving-medium wave equation** (1-17) can therefore frequently be obtained simply by transforming solutions to the stationary-medium wave equation (1-41) back to the laboratory frame. Thus, transforming the plane wave solution

$$P = e^{i(\vec{k}\cdot\vec{y}' - \omega'\tau')} \text{, for } k = \left|\vec{k}\right| = \frac{\omega'}{c_0}$$

the wave equation (1-41) (with the source term omitted) back to the fixed frame by equation (1-40) shows that

$$P = e^{i\vec{k}\cdot\vec{y} - (\omega' + \vec{k}\cdot\vec{U})\tau}$$

where $\vec{U} = U\hat{i}$. This solution represents a plane wave in the fixed laboratory

frame with a frequency

$$\omega \equiv= \omega' + \vec{k} \cdot \vec{U} = \omega'(1 + M \cos\theta)$$

where $M = U / c_0$ is the mean-flow Mach number and θ is the angle between the direction $\vec{k} / k$ of propagation and the mean flow direction (see fig. 1-7). The phase speed of the wave is

$$V_P = \frac{\omega}{k} = (1 + M \cos\theta) c_0 = c_0 + U \cos\theta$$

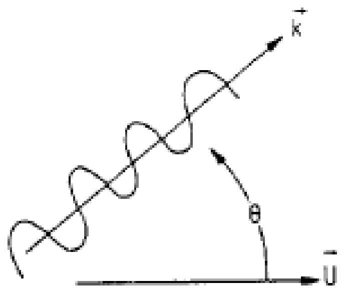

Figure 1-7. - Plane wave propagation in a constant-velocity medium.

This shows that the wave is traveling with a speed equal to c_0, the propagation speed relative to the medium, plus $U \cos\theta$, the component of the velocity of the medium in the direction of wave propagation. The frequency in the laboratory frame is increased if the medium has a component of its velocity in the direction of wave motion and is decreased if it has a component in the direction opposite to the wave motion. However, the wave has the same wavelength, $\lambda = 2\pi / k$, in both reference frames. This is simply a consequence of the fact that the moving wave pattern must appear the same to both a stationary and moving observer and only the frequency and apparent velocity of the wave can differ.

1.3.3 Solutions to Acoustic Equation with Velocity Gradients: Geometric Acoustics

Returning now to the general moving-medium wave equation (1-20), with source terms neglected, we find that the Fourier components of the pressure satisfy the transformed equation

$$i\left(k + iM \frac{\partial}{\partial y_1}\right)\left[\nabla^2 P + \left(k + iM \frac{\partial}{\partial y_1}\right)^2 P\right] - 2\frac{dM}{dy_2}\frac{\partial^2 P}{\partial y_2 \partial y_1} = 0$$

(1-42)

where $M = U / c_0$ is the mean-flow Mach number and $k \equiv \omega / c_0$. Then the

solution to equation (1-20) will be the sum or integral of terms of the form $Pe^{-i\omega\tau}$.

As in the case where the mean velocity is zero, we write *P* in the complex polar form

$$P = A(\vec{y})e^{ikS(\vec{y})}, \qquad (1\text{-}43)$$

so that the general term in the solution is of the form

$$A(\vec{y})e^{ik[S(\vec{y})-c_0\tau]}. \qquad (1\text{-}44)$$

Thus, the wave fronts (surfaces of constant phase) are given by $\Phi \equiv k[S(\vec{y} - c_0\tau] = \text{Constant}$; and the phase velocity is given by $V_P = \dfrac{c_0}{|\nabla S|}$.

In order to simplify the situation, we shall consider the case where the velocity varies slowly with y_2. Thus, we require that the length *L* over which *U* changes by a unit amount[8] be so large that

$$\varepsilon = \frac{1}{kL} << 1.$$

This means that $\dfrac{L}{\lambda} >> \dfrac{1}{2\pi}$ or $\lambda << L$. Hence, the velocity changes occur over a distance of many wavelengths.

We are interested in obtaining solutions to equation (1-42) which are analogous to the plane wave solutions discussed in the preceding sections. Since the mean velocity varies slowly on the scale of a wavelength, we anticipate that equation (1-42) will have solutions which behave locally as plane waves. Thus, suppose there exists a solution of equation (1-42) such that

$$\left.\begin{aligned} kS(\vec{y}) &= kLS_0(\vec{\eta}) \\ A(\vec{y}) &= A_0(\vec{\eta}) \end{aligned}\right\}, \qquad (1\text{-}45)$$

where $\vec{\eta} = \vec{y}L$, $S_0(0) = 0$ and the derivatives of S_0 and A_0 with respect to η_i are of order 1 (i.e., S_0 and A_0 change on the scale of $\vec{\eta}$). Then expanding S_0 and A_0 in a Taylor series about $\vec{\eta} = 0$ shows that, for $ky = O(1)$ or $y = O(\lambda)$,

$$A = A_0(0) + \vec{\eta} \cdot (\tilde{\nabla} A_0)_{\vec{\eta}=0} + O(\varepsilon^2),$$
$$ks = kL[\vec{\eta} \cdot (\tilde{\nabla} S_0)_{\vec{\eta}=0} + O(\varepsilon^2)],$$

where

$$\tilde{\nabla} = \hat{i}\frac{\partial}{\partial \eta_1} + \hat{j}\frac{\partial}{\partial \eta_2} + \hat{k}\frac{\partial}{\partial \eta_3}.$$

It follows that

[8] This is the length *L* for which $\dfrac{L}{U}\dfrac{dU}{dy_2} = O(1)$.

$$A \approx A_0(0)\,,$$
$$kS \approx kL\vec{\eta} \cdot (\tilde{\nabla} S_0)_{\vec{\eta}=0} = \vec{k} \cdot \vec{y}\,,$$

where we have put

$$\vec{k} = k(\tilde{\nabla} S_0)_{\vec{\eta}=0}\,.$$

Hence, for changes in η of the order of a wavelength, the solution (1-44) reduces approximately to the plane wave solution

$$A_0(0)e^{i(\vec{k}\cdot\vec{y}-\omega\tau)}\,.$$

In order to find an expression for this solution which is valid for all values of *y* (and not just for $y = O(\lambda)$), we nondimensionalize the length scales in equation (1-42) with respect to *L*. Introduce equation (1-43) for *P* with *A* and *S* given by equation (1-45), and neglect terms of order $\varepsilon = (kL)^{-1}$ in the resulting equation. Then upon reverting to dimensional quantities, we obtain for the real and imaginary parts of this equation, respectively,

$$\left(1 - M\frac{\partial S}{\partial y_1}\right)\left[2\nabla A \cdot \nabla S + A\nabla^2 S - 3M^2 A\frac{\partial^2 S}{\partial {y_1}^2} + \left(1 - M\frac{\partial S}{\partial y_1}\right)3M\frac{\partial A}{\partial y_1}\right],$$
$$- M\frac{\partial}{\partial y_1}A\left|\nabla S\right|^2 + 2\frac{\partial M}{\partial y_2}A\frac{\partial S}{\partial y_1}\frac{\partial S}{\partial y_2} = 0$$

and

$$\left[\left(1 - M\frac{\partial S}{\partial y_1}\right)^2 - \left|\nabla S\right|^2\right]\left(1 - M\frac{\partial S}{\partial y_1}\right)A = 0\,.$$

Since $A \neq 0$, the latter equation has two families of solutions. The interesting solution is

$$\left|\nabla S\right| = \pm\left(1 - M\frac{\partial S}{\partial y_1}\right) = \pm\left(1 - \frac{\vec{U}}{c_0}\cdot\nabla S\right), \qquad (1\text{-}46)$$

where $\vec{U} = \hat{i}U$ is the velocity vector. Since the unit normal to the phase surface $\hat{n}$ is given by

$$\hat{n} = \frac{\nabla S}{\left|\nabla S\right|}$$

and $U\cos\theta = \vec{U}\cdot\hat{n}$ is the component of mean velocity normal to the wave fronts (see fig. 1-8), equation (1-36) can be written as

$$\left|\nabla S\right| = \pm\left(1 - \frac{U\cos\theta}{c_0}\left|\nabla S\right|\right)$$

or

$$\left|\nabla S\right| = \frac{c_0}{U\cos\theta \pm c_0}\,.$$

Now suppose the flow is subsonic. Then since $\left|\nabla S\right| > 0$, only the plus sign

can hold and

$$|\nabla S| = \frac{c_0}{U\cos\theta + c_0}.$$

The phase velocity V_P is therefore given by

$$V_P = \frac{c_0}{|\nabla S|} = U\cos\theta + c_0.$$

This is identical to the expression for the phase speed in a uniformly moving medium given in section 1.3.2. In order to interpret this result, consider an initially plane wave moving to the right in a velocity field which is increasing in the upward direction, as shown in figure 1-9. The phase velocity will be larger on the upper part of the wave surface than on the bottom. Hence, the velocity of the wave surface normal to itself will be larger on the top than on the bottom. As a consequence, the wave front will bend in toward the lower velocity region as it moves. Similarly, if the wave is traveling to the left, it will bend upward toward the higher velocity region.

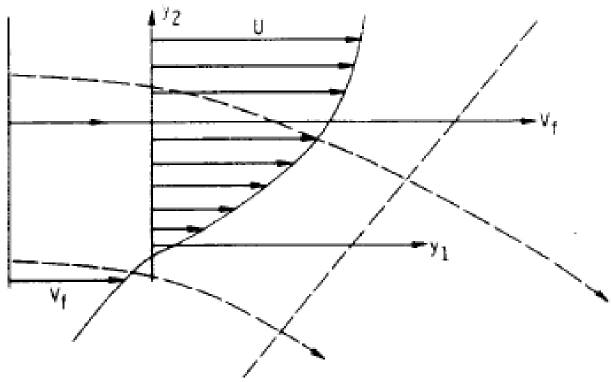

Figure 1-9. - Bending of phase surface by mean flow.

1.4 INTEGRAL FORMULAS FOR SOLUTIONS TO THE WAVE EQUATION

1.4.1 General Formulas

Before proceeding with the material of this section, it is helpful to recall three well-known integral formulas from vector analysis. Thus, let $v(\tau)$ denote an arbitrary region of space bounded (internally or externally) by the surface $S(\tau)$ (which is generally moving), and let $\vec{A}$ be an arbitrary vector defined on $v(\tau)$. Then the **divergence theorem** (1-35) states that

$$\int_{S(\tau)} \vec{A}\cdot\hat{n}dS(\vec{y}) = \int_{v(\tau)} \nabla\cdot\vec{A}d\vec{y}, \qquad (1\text{-}47)$$

provided the integrals exist. If $\vec{V}_S(\vec{y},\tau)$ denotes the velocity at any point $\vec{y}$ of the surface $S(\tau)$, the three-dimensional **Leibniz's rule** shows that

$$\frac{d}{\delta\tau}\int_{\nu(\tau)}\Psi d\vec{y}=\int_{\nu(\tau)}\frac{\partial\Psi}{\partial\tau}d\vec{y}+\int_{S(\tau)}\vec{V}_S\cdot\hat{n}\,\Psi dS(\vec{y}) \qquad (1\text{-}48)$$

for any function $\Psi(\vec{v},\tau)$ defined on $\nu(\tau)$. Finally, it is a direct consequence of the divergence theorem that **Green's theorem**

$$\int_{S(\tau)}\left(\Psi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\Psi}{\partial n}\right)dS(\vec{y})=\int_{\nu(\tau)}\left(\Psi\nabla^2\psi-\psi\nabla^2\Psi\right)dS(\vec{y})$$

(1-49)

holds for any two functions Ψ and ψ defined on ν. In this equation we have written $\frac{\partial\Psi}{\partial n}$ in place of $\hat{n}\cdot\nabla\Psi$.

In this section these formulas will be used to derive an integral formula which expresses the solution to the **inhomogeneous, uniformly moving medium, wave equation**

$$\nabla^2 P-\frac{1}{c_0^2}\frac{D_0^2}{D\tau^2}P=-\gamma(\vec{y},\tau) \qquad (1\text{-}50)$$

in terms of a solution $G(\vec{y},\tau|\vec{x},t)$ of the equation

$$\nabla^2 G-\frac{1}{c_0^2}\frac{D_0^2}{D\tau^2}G=-\delta(t-\tau)\delta(\vec{x}-\vec{y}) \qquad (1\text{-}51)$$

for an **impulsive point source**.[9] This result is used extensively in subsequent chapters to deduce the effects of solid boundaries on aerodynamic sound generation.

It was shown in section 1.3.1.3 for the special case of a stationary medium, that, equation (l-51) possesses a solution (given by eq. (1-38)) at all points of space which together with all its derivatives vanishes for $t-\tau$. In any region ν which does not include all of space, equation (1-51) possesses many such solutions. Hence, let *G* denote any solution of equation (1-51) satisfying the condition

$$G=\frac{D_0 G}{D\tau}=0\text{, for } t=\tau\,. \qquad (1\text{-}52)$$

Then applying Green's formula to *p* and G integrating the result with respect to τ from -*T* to +*T* (where *T* is some large interval of time) show that

$$\int_{-T}^{T}\int_{S(\tau)}\left(G\frac{\partial p}{\partial n}-p\frac{\partial G}{\partial n}\right)dSd\tau$$

$$=\int_{-T}^{T}\int_{\nu(\tau)}\left(G\nabla^2 p-p\nabla^2 G\right)d\vec{y}d\tau$$

$$=\frac{1}{c_0^2}\int_{-T}^{T}\int_{\nu(\tau)}\left(G\frac{D_0^2 p}{D\tau^2}-p\frac{D_0^2 G}{D\tau^2}\right)d\vec{y}d\tau-\int_{-T}^{T}\int_{\nu(\tau)}\left(G\gamma(\vec{y},\tau)-\delta(t-\tau)\delta(\vec{y}-\vec{x})p\right)d\vec{y}d\tau$$

---

[9] *G* is called a ***fundamental solution*** of the wave equation.

$$(1\text{-}53)$$

But since

$$\left(G\frac{D_0^2}{D\tau^2}p - p\frac{D_0^2}{D\tau^2}G \right) = \frac{\partial}{\partial\tau}\left(G\frac{D_0 p}{D\tau} - p\frac{D_0 G}{D\tau} \right) + U\frac{\partial}{\partial y_1}\left(G\frac{D_0 p}{D\tau} - p\frac{D_0 G}{D\tau} \right)$$

it follows from applying Leibniz's rule to the first term and the divergence theorem to the second that

$$\int_{v(\tau)}\left(G\frac{D_0^2 p}{D\tau^2} - p\frac{D_0^2 G}{D\tau^2} \right) d\vec{y} = \frac{d}{d\tau}\int_{v(\tau)}\left(G\frac{D_0 p}{D\tau} - p\frac{D_0 G}{D\tau} \right) d\vec{y} + \int_{v(\tau)}\left(U\hat{i} - \vec{V}_S \right)\cdot\hat{n}\left(G\frac{D_0 p}{D\tau} - p\frac{D_0 G}{D\tau} \right) dS(\vec{y})$$

Hence,

$$\int_{-T}^{T}\int_{v(\tau)}\left(G\frac{D_0^2 p}{D\tau^2} - p\frac{D_0^2 G}{D\tau^2} \right) d\vec{y}d\tau = \int_{v(\tau)}\left(G\frac{D_0 p}{D\tau} - p\frac{D_0 G}{D\tau} \right) d\vec{y}\Bigg|_{\tau=-T}^{\tau=T} - \int_{-T}^{T}\int_{S(\tau)} V'_n\left(G\frac{D_0 p}{D\tau} - p\frac{D_0 G}{D\tau} \right) dS(\vec{y})d\tau$$

where

$$V'_n \equiv (\vec{V}_S - \hat{i}U)\cdot\hat{n} \qquad (1\text{-}54)$$

is the velocity of the surface normal to itself relative to a reference frame moving with the velocity $\hat{i}U$. The causality condition (1-52) implies that the integrated (first) term vanishes at the upper limit ($\tau = T$). At the lower limit this term represents the effects of initial conditions in the remote past (ref. 2, p. 837). Since in most aerodynamic sound problems only the time-stationary[10] (and not the transient) sound field is of interest, this term will be omitted.[11] Hence,

$$\int_{-T}^{T}\int_{v(\tau)}\left(G\frac{D_0^2 p}{D\tau^2} - p\frac{D_0^2 G}{D\tau^2} \right) d\vec{y}d\tau = -\int_{-T}^{T}\int_{S(\tau)} V'_n\left(G\frac{D_0 p}{D\tau} - p\frac{D_0 G}{D\tau} \right) dS(\vec{y})d\tau \cdot$$

Substituting this result into equation (1-45) and carrying out the integrals over the delta functions show that

$$\int_{-T}^{T}d\tau\int_{v(\tau)}\gamma(\vec{y},\tau)G(\vec{y},\tau\,|\,\vec{x},t)d\vec{y}$$

$$+\int_{-T}^{T}d\tau\int_{S(\tau)}\left[G(\vec{y},\tau\,|\,\vec{x},t)\left(\frac{\partial}{\partial n} + \frac{V'_n}{c_0^2}\frac{D_0}{D\tau} \right)p(\vec{y},\tau) - p(\vec{y},\tau)\left(\frac{\partial}{\partial n} + \frac{V'_n}{c_0^2}\frac{D_0}{D\tau} \right)G(\vec{y},\tau\,|\,\vec{x},t) \right] dS(\vec{y})$$

$$= \begin{cases} p(\vec{x},t)\text{\_if\_x\_is\_in\_}v(t) \\ 0\text{\_if\_x\_is\_not\_in\_}v(t) \end{cases}$$

$$(1\text{-}55)$$

This equation provides an expression for the acoustic pressure at an arbitrary point $\vec{x}$ within a volume v in terms of the distribution γ of sources within v and the distribution of the pressure and its derivatives on the

[10] See appendix 1. A, section 1.A.3.

[11] It is assumed that the boundary condition is such that the effect of any initial state will decay with time. In any event, it is always possible to require that $p \quad \frac{D_0 p}{D\tau} = 0$ at $\tau = -T$.

boundary of ν. We make extensive use of it in chapters 3 and 4 to predict the emission of aerodynamic sound in the presence of solid boundaries.

The region $\nu(\tau)$ in equation (I-55) can be either exterior or interior to the closed surface (or surfaces) $S(\tau)$. However, for exterior regions the solution $P(\vec{y},\tau)$ of equation (1-50) must be such that the surface integral in equation (1-45) vanishes when carried out over any region enclosing $S(\tau)$ whose boundaries move out to infinity. This will usually occur whenever $P(\vec{y},\tau)$ behaves like an outgoing wave at large distances from the source. When applying equation(1-55), it is necessary to be sure that the direction of the outward drawn normal $\hat{n}$ to *S* is always taken to be from the region ν to the region on the other side of *S*.

The preceding argument applies just as well to the case where the surface $S(\tau)$ is absent. Hence, equation (1-55), with the surface integral omitted, holds even when the region ν is all of space. However, in this case, there is only one possible solution to equation (1-51) which satisfies condition (1-52) and vanishes at infinity. When U = 0, this is the function G^0 given by equation (1-38). Then, in this case, equation (1-55) becomes

$$p(\vec{x},t)=\int_{-T}^{T}\int\gamma(\vec{y},\tau)G^0(\vec{y},\tau\,|\,\vec{x},t)d\vec{y}d\tau\ . \qquad (1\text{-}56)$$

This equation can be used to compute the pressure at any point from the known source distribution γ whenever the region of interest is all of space.

More generally, if the surface *S* is stationary and the velocity *U* of the medium is zero or tangent to the surface (so that $\hat{n}\cdot\hat{i}=0$), the normal relative surface velocity V'_n becomes the normal surface velocity

$$V_n=\vec{V}_S\cdot\hat{n}\ , \qquad (1\text{-}57)$$

and equation (1-55) reduces to the usual integral formula for the **wave equation**

$$\int_{-T}^{T}d\tau\int_{\nu}\gamma G d\vec{y}+\int_{-T}^{T}d\tau\int_{S}\left(G\frac{\partial p}{dn}-p\frac{\partial G}{\partial n}\right)dS=\begin{cases}p(\vec{x},t)\text{\_if\_x\_is\_in\_}\nu\\ 0\text{\_if\_x\_is\_not\_in\_}\nu\end{cases}.$$

(1-58)

Of course, when U = 0, p and G satisfy the inhomogeneous stationary-medium wave equations

$$\nabla^2 p-\frac{1}{c_0^2}\frac{\partial^2 p}{\partial\tau^2}=-\gamma(\vec{y},\tau)\ , \qquad (1\text{-}59)$$

$$\nabla^2 G-\frac{1}{c_0^2}\frac{\partial^2 G}{\partial\tau^2}=-\delta(t-\tau)\delta(\vec{x}-\vec{y})\ . \qquad (1\text{-}60)$$

1.4.2 Boundary Conditions: Green's Function

1.5 SOURCE DISTRIBUTION IN FREE SPACE: MULTIPOLE EXPANSION

1.6 RADIATION FIELD

1.7 ENERGY RELATIONS

1.8 MOVING SOUND SOURCES