CHAPTER2 Aerodynamic Sound

2.1 INTRODUCTION

In an unsteady flow, pressure fluctuations must occur in order to balance the fluctuations in momentum. But since all real fluids possess elasticity (i.e., they are compressible), the pressure fluctuations can be communicated to the surrounding fluid and propagate outward from the flow. It is these pressure waves in the surrounding fluid which we recognize as sound.

At fairly low Mach numbers the pressure fluctuations in the vicinity of the flow are substantially uninfluenced by compressibility and can be determined from the velocity field by solving a **Poisson's equation**[1]

$$\nabla^2 p = \gamma$$

in which the source term γ is a known function of the flow velocity. However, the **Biot-Savart law** shows that we can consider the velocity field to be in turn driven by a prescribed vorticity field. And since **Kelvin's theorem of conservation of circulation** shows that the vorticity in an inviscid fluid is simply carried along with the flow, an initially localized region of vorticity will remain that way for sometime to come. Thus, many flows can be envisioned as relatively localized regions of vorticity which drive not only the pressure fluctuations in their immediate vicinity but also those which occur at large distances.

The pressure fluctuations at large distances are weak and satisfy the **acoustic wave equation**. Thus, in this region, which we shall often call the ***acoustic field***, the effects of compressibility and the finite propagation speed of acoustic waves are important.[2]

Although the localized pressure fluctuations have been extensively studied, the theory of aerodynamic sound is principally concerned with the study of the pressure fluctuations in the acoustic field.[3] This subject probably began with Gutin's theory (ref. 1) of the noise produced by the rotating

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1 These pressure fluctuations are sometimes called ***pseudosound***.

2 If the Mach number is sufficiently low, there will be an intermediate region where the pressure fluctuations have some of the properties of both the localized pressure fluctuations and those in the sound field. Thus, in this intermediate region the pressure fluctuations are as weak as in the sound field, but the distances involved are small enough so that the effects of finite propagation speed and hence of compressibility can be neglected.

3 The difference in character between the pressure fluctuations in the acoustic field and those in the vicinity of the flow is evidenced by their relation to the flow velocity. Thus, the localized pressure fluctuations are of the order $\rho u'^2$, where u' is a characteristic velocity. But it was shown in chapter 1 that the pressure fluctuations in the sound field are of order $\rho c_0 u'$.

pressure field of propellers, developed in 1937. However, it was not until 1952, when Lighthill (refs. *2* and 3) introduced his acoustic analogy to deal with the problem of jet noise, that a general theory began to emerge. Lighthill's ideas were extended by Curie (ref. 4), Powell (ref. 5), and Ffowcs Williams and Hall (ref. 6) to include the effects of solid boundaries. These extensions include the theory developed by Gutin and, in fact, provide a complete theory of aerodynamically generated sound which can be used to predict blading noise as well as jet noise.

The fundamental equation which forms the basis of the acoustic analogy approach is derived in the next section. The methods of classical acoustics given in chapter 1 are then used to obtain solutions to this equation for the case where no solid boundaries are present. (The treatment of solid boundaries is deferred to chapters 3 and 4.) These solutions are applied to high-speed subsonic jets, and fairly detailed results are obtained. Supersonic and low-speed subsonic jets are treated in a somewhat more qualitative fashion.

In Lighthill's acoustic analogy, certain terms associated with the propagation of sound are treated as source terms. In practice, this places certain limitations on the accuracy of the theory. Alternative approaches developed to overcome these limitations are presented in chapter 6.

2.2 LIGHTHILL'S ACOUSTIC ANALOGY

In this section we develop the acoustic analogy approach introduced by Lighthill in two classical papers published in 1952 and 1954 (refs. 2 and 3). This approach was initially evolved to calculate acoustic radiation from relatively small regions of turbulent flow embedded in an infinite homogeneous fluid in which the speed of sound c_0 and the density ρ_0 are constants.

In this case the density fluctuations, $\rho' \equiv \rho - \rho_0$, at large distances from the turbulent region ought to behave like acoustic waves and hence satisfy the **homogeneous wave equation**[4]

$$\frac{1}{c_0^2}\frac{\partial^2 \rho'}{\partial \tau^2} - \nabla^2 \rho' = 0 .$$

Lighthill arranged the exact **equations of continuity and momentum** in such a way that they reduce to this equation outside the region of flow.

2.2.1 Derivation of Lighthill's Equation

[4] The notation introduced at the beginning of section 1, 2 will be used in this section.

In order to derive Lighthill's result, notice that upon using the summation convention the **continuity** and **momentum** equations can be written as

$$\begin{cases} \dfrac{\partial \rho}{\partial \tau} + \dfrac{\partial}{\partial y_j} \rho v_j = 0 \\ \rho \left(\dfrac{\partial v_i}{\partial \tau} + v_j \dfrac{\partial}{\partial y_j} v_i \right) = -\dfrac{\partial p}{\partial y_i} + \dfrac{\partial e_{ij}}{\partial y_j} \end{cases} \qquad (2\text{-}1)$$

where e_{ij} is the (i,j)-th component of the viscous stress tensor. For a **Stokesian gas** it can be expressed in terms of the velocity gradients by

$$e_{ij} = \mu \left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial y_k} \right), \qquad (2\text{-}2)$$

where μ is the viscosity of the fluid.

Multiplying the continuity equation (2-1) by v_i, adding the result to the momentum equation, and combining terms show that

$$\frac{\partial}{\partial \tau} \rho v_i = -\frac{\partial}{\partial y_j} (\rho v_i v_j + \delta_{ij} p - e_{ij}) .$$

But after adding and subtracting the term[5] $c_0^2 \dfrac{\partial \rho}{\partial y_i}$, this equation can be written as

$$\frac{\partial \rho v_i}{\partial \tau} + c_0^2 \frac{\partial \rho}{\partial y_i} = -\frac{\partial T_{ij}}{\partial y_j}, \qquad (2\text{-}3)$$

where

$$T_{ij} = \rho v_i v_j + \delta_{ij} [(p - p_0) - c_0^2 (\rho - \rho_0)] - e_{ij} \qquad (2\text{-}4)$$

is ***Lighthill's turbulence stress tensor***. Finally, differentiating equation (2-l) with respect to τ, taking the divergence of equation (2-3), and then subtracting the results yield ***Lighthill's equation***

$$\frac{\partial^2 \rho'}{\partial \tau^2} - c_0^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} . \qquad (2\text{-}5)$$

2.2.2 Interpretation of Lighthill's Equation

Equation (2-5) clearly has the same form as the **wave equation** governing the propagation of sound emitted by a quadrupole source[6] $\dfrac{\partial^2 T_{ij}}{\partial y_i \partial y_j}$ in a

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[5] The subscript 0 is used here to denote constant reference values, winch will usually be taken to be the corresponding properties at large distances from the flow.

[6] It is shown in the next section that this source term should vanish outside the region of turbulent flow and hence (as indicated in the beginning of this section) eq. (2-5) does indeed reduce to a homogeneous wave equation in this region.

nonmoving medium (see section 1.5.2). It therefore shows that there is an exact analogy between the density fluctuations in any real flow in arbitrary motion and those in an ideal acoustic medium at rest (with sound speed c_0) due to a distribution of quadrupoles of strength T_{ij}.

The crucial step in Lighthill's analysis is to regard this source term as known *a priori*. (Notice that the nonlinear terms are all contained in the source term). However, we never have complete prior knowledge of this term since it involves the fluctuating density, which is precisely the variable for which equation (2-5) is to be solved. In fact, since Lighthill's equation is an exact consequence of the laws of conservation of mass and momentum, it must be satisfied by all real flows: most of which are certainly not sound like. Thus, in most cases, a knowledge of T_{ij} is equivalent to solving the complete nonlinear equations governing the flow problem, which is virtually impossible for most flows of interest.

Even for those flows which are sound like, the source term $\dfrac{\partial^2 T_{ij}}{\partial y_i \partial y_j}$, aside from representing the sound emission, includes such real fluid effects as the convection and refraction of the sound by the mean flow, the scattering of the sound by turbulence and entropy spottiness, the back reaction of the sound field on the flow itself, and the viscous dissipation of the sound by the flow. The prediction of any of these effects requires that the sound field (which is not known until eq. (2-5) is already solved) be included in the source term.

In spite of these drawbacks the acoustic analogy approach serves as a foundation for most aerodynamic sound analyses. This is probably due to the fact that this approach allows us to use the powerful methods of classical acoustics to treat aerodynamic sound problems. In chapter 6 we discuss procedures which have been developed to alleviate the difficulties associated with this approach.

By incorporating suitable boundary conditions, we can apply Lighthill's acoustic analogy to flow in the presence of solid boundaries. As a first step, however, we shall consider the case where the effect of solid boundaries on the sound field is negligible. Then the only important applications of the results will be to jet noise. (In fact, Lighthill actually developed his theory specifically to deal with this problem.) In chapter 3 we show how solid boundaries can be included in the analysis and apply the theory to a number of special cases.

2.2.3 Approximation of Lighthill's Stress Tensor

Lighthill's equation can only serve as the starting point for the solution of aerodynamic sound problems if it is possible to regard its right side as a known source term. We shall now show that there are at least some flows for which this is a reasonable assumption.

To this end, consider a subsonic turbulent airflow (or for that matter any unsteady high Reynolds number subsonic flow) of relatively small spatial extent (such as the flow in a jet) embedded in a uniform stationary atmosphere. The subscript 0 will now be used to denote the constant values of the thermodynamic properties in this atmosphere. Within the flow we anticipate that the viscous stress e_{ij}, which appears in T_{ij}, will always be negligible compared with the far larger Reynolds stress term $\rho v_i v_j$. In fact, it is well known from the study of turbulence that the ratio of these terms is of the order of magnitude of the Reynolds number $\dfrac{\rho UL}{\mu}$, which in virtually all applications of aerodynamic noise theory is quite large.

In the region outside the flow (or at least at sufficiently large distances from this flow) the acoustic approximation should apply, and hence the velocity v_i should be small. Then the quadratic Reynolds stress term $\rho v_i v_j$ will be negligible. In addition, the effects of viscosity and heat conduction can be expected to act in this region in the same way as they do for any sound field. This means (as shown by Kirchoff, see ref. 8) that they only cause a slow damping due to the conversion of acoustic energy into heat and have a significant effect only over very large distances. Thus, it should be possible to neglect e_i entirely.

Now assuming that the flow emanates from a region of uniform temperature, the effects of heat conduction ought to be of the same order of magnitude as the viscous effects (provided the Prandtl number is of order 1 as it is for most fluids). Hence, heat conduction should also be negligible within the flow. Then the entropy changes will be governed by the inviscid energy equation (1-3). And, since it is assumed that the flow emanates from a region of uniform temperature, this equation shows that the entropy should be relatively constant. But it is shown in section 1.2 that

$$p - p_0 = c_0^2(\rho - \rho_0) \qquad (2\text{-}6)$$

in any isentropic flow in which (as is usually the case in subsonic flows) $\dfrac{p - p_0}{p_0}$ and $\dfrac{\rho - \rho_0}{\rho_0}$ are sufficiently small.

We have therefore shown that T_{ij} is approximately equal to $\rho v_i v_j$

inside the flow and approximately equal to zero outside this region. Hence, upon assuming that the density fluctuations are negligible within the flow, we can approximate Lighthill's stress tensor by[7]

$$T_{ij} \approx \rho_0 v_i v_j . \tag{2-7}$$

But within the flow it is reasonable to suppose that the **Reynolds stress** $\rho_0 v_i v_j$ can be determined, say from measurements or estimates of the turbulence, without any prior knowledge of the sound field. Then the right side of Lighthill's equation (2-5) can indeed be treated as a source term.

2.3 SOLUTION TO LIGHTHILL'S EQUATION WHEN NO SOLID BOUNDARIES ARE PRESENT

It is shown in section 2.2 that the problem of predicting the sound emission from a region of unsteady flow embedded in a uniform atmosphere can be reduced to the classical problem of predicting the sound field from a known quadrupole source of limited spatial extent. If any solid boundaries which may be present do not influence the sound field to any appreciable extent, the solution to this problem can be expressed in terms of the **free-space Green's function**. Indeed after comparing equation (2-5) with equation (1-59), we see from equation (1-82) that this solution is given by[8]

$$\rho(\vec{x},t) - \rho_0 = \frac{1}{4\pi c_0^2} \int \frac{1}{r} \left[\frac{\partial^2 T_{ij}}{\partial y_i \partial y_j}(\vec{y},\tau) \right]_{\tau = t-(r/c_0)} d\vec{y} \,, \tag{2-8}$$

where

$$r \equiv |\vec{x} - \vec{y}| .$$

In order to transform this equation into a more suitable form, it is convenient to introduce the differential operator $\frac{\delta}{\delta y_i}$, which denotes partial differentiation with respect to y_i with not only *t* but also *r* held fixed to obtain

$$\rho(\vec{x},t) - \rho_0 = \frac{1}{4\pi c_0^2} \int \frac{\delta^2}{\delta y_i \delta y_j} \frac{T_{ij}(\vec{y}, t - r/c_0)}{r} d\vec{y} \,. \tag{2-9}$$

Then since the operator $\frac{\delta}{\delta y_i}$ denotes partial differentiation with respect to

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[7] Of course, it is being assumed that no combustion occurs in the flow. This could result in large fluctuations in entropy and hence in $(p - p_0) - c_0^2(\rho - \rho_0)$. This term would then have to be included in T_{ij}.

[8] As indicated in chapter 1, the omission of the limits on a volume integral denotes an integration over all space.

y_i with $\vec{x}$ and t held fixed and $\frac{\partial}{\partial x_i}$ denotes partial differentiation with respect to x_i with $\vec{y}$ and t held fixed, the chain rule for partial differentiation shows that for any function $F(\vec{y},\vec{r},t)$

$$\frac{\delta F}{\delta y_i} = \frac{\partial F}{\partial y_i} + \frac{\partial F}{\partial x_i}$$

and hence that

$$\frac{\delta^2 F}{\delta y_i \delta y_j} = \frac{\partial^2 F}{\partial y_i \partial y_j} + \frac{\partial^2 F}{\partial y_i \partial x_j} + \frac{\partial^2 F}{\partial y_j \partial x_i} + \frac{\partial^2 F}{\partial x_i \partial x_j}$$

Using this result in equation (2-9) shows that

$$\rho(\vec{x},t) - \rho_0 = \frac{1}{4\pi c_0^2} \int \frac{\partial^2}{\partial y_i \partial y_j} \left[\frac{T_{ij}}{r}\right] d\vec{y} + \frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_j} \int \frac{\partial}{\partial y_j} \left[\frac{T_{ij}}{r}\right] d\vec{y}$$
$$+ \frac{1}{4\pi c_0^2} \frac{\partial}{\partial x_i} \int \frac{\partial}{\partial y_j} \left[\frac{T_{ij}}{r}\right] d\vec{y} + \frac{1}{4\pi c_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int \left[\frac{T_{ij}}{r}\right] d\vec{y}$$

(2-10)

provided the integrals exist. In this equation the notation $\left[\frac{T_{ij}}{r}\right]$ is used to denote $\frac{T_{ij}(\vec{y}, t - r/c_0)}{r}$. Notice that the integrand in each of the first three integrals is the divergence of a vector. But if S_R denotes a sphere of radius R, the divergence theorem shows that

$$\int \nabla \cdot \vec{A} d\vec{y} = \lim_{R\to\infty} \int_{S_R} \vec{A} \cdot d\vec{S}$$

for any vector $\vec{A}$ for which the integrals exist. Hence, upon assuming[9] that T_{ij} is smooth and decays faster than y^{-1} for large y, we can conclude that these integrals vanish and that equation (2-10) becomes

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[9] We show in section 2. 2 that outside a localized region of turbulent flow where the viscous and heat conduction effects are negligible, T_{ij} behaves like $\rho v_i v_j$. But in the outer region, v_i will not decay any slower than the rate y^{-1} at which the acoustic particle velocity decays (eqs. (1-93) and (1-94)). Hence, T_{ij} must decay at least as fast as y^{-2}. But we cannot be sure that the last integral in eq. (:2-10) will converge unless T_{ij} is known to decay faster than y^{-2}. However, the incompressible … velocities, which dominate (at sufficiently low Mach numbers) in the region of a localized flow, decay as y^{-3} for large values of y. Thus, if we could begin by completely neglecting the contribution of the acoustic velocities, T_{ij} would decay as y^{-6} and the last integral in eq. (2-10) would certainly converge. By using the method of matched asymptotic expansion, it can be shown (ref. 9) that this approximation is valid whenever the wavelength of the sound is large compared with the size of the source region.

$$\rho(\vec{x},t)-\rho_0=\frac{1}{4\pi c_0^2}\frac{\partial^2}{\partial x_i \partial x_j}\int \frac{T_{ij}}{r}(\vec{y},t-r/c_0)d\vec{y}\ . \tag{2-11}$$

In aerodynamic sound problems we are usually interested in the sound at large distances from the source where, as we have seen, the expression for the sound field becomes particularly simple. Thus, first consider the case where the observation point $\vec{x}$ is many wavelengths away from any point in the source region. (This distance need not be large relative to the dimensions of the source region.) Then upon using the manipulations described in section 1.5.2 the second partial derivative of the integrand in equation (2-11) becomes

$$\frac{\partial^2}{\partial x_i \partial x_j}\frac{T_{ij}(\vec{y},t-r/c_0)}{r}=\frac{r_i r_j}{c_0^2 r^3}\frac{\partial^2 T_{ij}(\vec{y},t-r/c_0)}{\partial t^2}+O(r^{-2})$$

where

$$\vec{r}=\vec{x}-\vec{y}\ .$$

Hence, for large r,

$$\rho(\vec{x},t)-\rho_0\approx\frac{1}{4\pi c_0^2}\int \frac{r_i r_j}{r^3 c_0^2}\frac{\partial^2 T_{ij}}{\partial t^2}(\vec{y},t-r/c_0)d\vec{y}\ .$$

If the distance between any source point and the observation point is also large compared with the dimensions of the source region (i.e., if the observation point is in the radiation field), we can (upon assuming that the origin of the coordinate system is in the source region) replace $\frac{r_i r_j}{r^3}$ by $\frac{x_i x_j}{x^3}$ to obtain

$$\rho(\vec{x},t)-\rho_0\approx\frac{1}{4\pi c_0^2}\frac{x_i x_j}{x^3}\int \frac{1}{c_0^2}\frac{\partial^2 T_{ij}}{\partial t^2}(\vec{y},t-r/c_0)d\vec{y} \tag{2-12}$$

provided the integral converges.[10] This equation allows us to calculate the density fluctuations in the radiation field once the source term is known.

2.4 APPLICATION OF LIGHTHILL'S THEORY TO TURBULENT FLOWS

2.4. 1 Derivation of Basic Equations

The most important application of the solution (2-12) is the prediction of sound from turbulent jets.[11] But for turbulent flows it is reasonable to assume

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[10] The convergence of this integral now requires that T_{ij} decay faster than y^{-6} for large *y*.

[11] It can also be used to predict the sound from periodic jets. See section 2.5.3.

that the stress tensor T_{ij} is a stationary random function of time. Then equation (2-12) shows that the density fluctuation in the radiation field must also be a function of this type. For such sound fields (see section I.7.3.2.1) both the average intensity and its spectrum can readily be determined from the normalized pressure autocorrelation function

$$\Gamma(\vec{x},\tau) \equiv \frac{\overline{[p(\vec{x},t+\tau)-p_0][p(\vec{x},t)-p_0]}}{\rho_0 cl0}.$$

And since equation (2-6) must certainly hold in the radiation field, it follows from equation (2-12) that this function is related to the source term by

$$\Gamma(\vec{x},\tau) = \frac{1}{16\pi^2 c_0^5 \rho_0} \frac{x_i x_j x_k x_l}{x^6} \iint \overline{\frac{\partial^2 T_{ij}}{\partial t^2}(\vec{y}',t')\frac{\partial^2 T_{kl}}{\partial t^2}(\vec{y}'',t'')} d\vec{y}' d\vec{y}'',$$

(2-13)

where

$$\left.\begin{aligned} t' &= t - \frac{|\vec{x}-\vec{y}'|}{c_0} \\ t'' &= t + \tau - \frac{|\vec{x}-\vec{y}''|}{c_0} \end{aligned}\right\}. \qquad (2\text{-}14)$$

It is shown in the appendix that the integrand in equation (2-13) can be put in the form

$$\overline{\frac{\partial^2 T_{ij}}{\partial t^2}(\vec{y}',t')\frac{\partial^2 T_{kl}}{\partial t^2}(\vec{y}'',t'')} = \frac{\partial^4}{\partial \tau^4}\overline{T_{ij}(\vec{y}',t')T_{kl}(\vec{y}'',t'')}. \qquad (2\text{-}15)$$

But since (as shown in appendix 1.A.3) the cross correlation of a stationary function is independent of time translations, it follows from equation (2-14) that

$$\overline{T_{ij}(\vec{y}',t')T_{kl}(\vec{y}'',t'')} = \overline{T_{ij}(\vec{y}',t)T_{kl}\left(\vec{y}'',t+\tau+\frac{|\vec{x}-\vec{y}'|-|\vec{x}-\vec{y}''|}{c_0}\right)}.$$

(2-16)

And since $|\vec{x}-\vec{y}'|$ behaves like

$$|\vec{x}-\vec{y}'| = x - \frac{\vec{x}}{x}\cdot\vec{y}' + O(x^{-1})$$

for large x it follows that

$$\frac{|\vec{x}-\vec{y}'|-|\vec{x}-\vec{y}''|}{c_0} \approx \frac{\vec{x}}{x}\cdot\frac{\vec{y}''-\vec{y}'}{c_0}. \qquad (2\text{-}17)$$

Finally, inserting equations (2-15) to (2-17) into equation (2-18) shows that

$$\Gamma(\vec{x},\tau) = \frac{1}{16\pi^2 c_0^5 \rho_0} \frac{x_i x_j x_k x_l}{x^6} \frac{\partial^4}{\partial \tau^4} \iint \overline{T_{ij}(\vec{y}',t)T_{kl}(\vec{y}'',\tau_0)} d\vec{y}' d\vec{y}'',$$

(2-18)

where

$$\tau_0 \equiv t + \tau + \frac{\vec{x}}{xc_0} \cdot (\vec{y}'' - \vec{y}') .$$

It is now convenient to introduce the separation vector $\vec{\eta} \equiv \vec{y}'' - \vec{y}'$ as a new variable of integration in equation (2-18) and to define a two-point time-delayed fourth-order correlation tensor by

$$R_{ijkl}(\vec{y}', \vec{\eta}, \tau) \equiv \frac{\overline{T_{ij}(\vec{y}', t) T_{kl}(\vec{y}'', t+\tau)} - \phi_{ijkl}(\vec{y}', \vec{\eta})}{\rho_0{}^2} ,$$

(2-19)

where ϕ_{ijkl} is an arbitrary time-independent tensor which will eventually be chosen to simplify the equations. Then, since the Jacobian of the transform $\vec{y}'$, $\vec{y}'' - \vec{y}'$, $\vec{\eta}$ is unity, inserting these quantities into equation (2-19) shows that

$$\Gamma(\vec{x}, \tau) = \frac{\rho_0 x_i x_j x_k x_l}{16\pi^2 c_0^5 x^6} \frac{\partial^4}{\partial \tau^4} \iint R_{ijkl}\left(\vec{y}', \vec{\eta}, \tau + \frac{\vec{\eta}}{c_0} \cdot \frac{\vec{x}}{x}\right) d\vec{y}' \, d\vec{\eta} .$$

(2-20)

This equation relates the pressure autocorrelation in the sound field to the source correlation tensor R_{ijkl} . Taking its Fourier transform and using equation (1-125) and table 1-1 in appendix 1.A show that the intensity spectrum in the radiation field is given by

$$\bar{I}_\omega(\vec{x}) = \frac{\omega^4 \rho_0}{32\pi^3 c_0^5} \frac{x_i x_j x_k x_l}{x^6} \int_{-\infty}^{\infty} \int \int e^{i\omega[\tau - \frac{\vec{x}}{x} \cdot \frac{\vec{\eta}}{c_0}]} R_{ijkl}(\vec{y}', \vec{\eta}, \tau) d\vec{y}' \, d\vec{\eta} \, d\tau .$$

(2-21)

This equation can, in principle, be used to calculate the spectrum of the sound field emitted from a turbulent flow whenever solid boundaries do not play a direct role in the process. However, most turbulent flows which are not in the immediate vicinity of solid boundaries (e.g., jets, wakes, etc.) have nearly parallel mean flows. In the next section we deduce certain properties of the correlation tensor which will be helpful in understanding the sound fields produced by such flows.

2.4.2 Parallel or Nearly Parallel Mean Flows

Whenever the mean flow is nearly parallel, it is of interest to consider the case where the velocity $\vec{v}(\vec{y}, t)$ is the sum of a parallel mean flow $\hat{i} U(y_2)$ as shown in figure 2-1 and a fluctuating part $\vec{u}(\vec{y}, t)$ with zero mean so

that[12]

$$v_i = \delta_{1i} U + u_i \,. \tag{2-22}$$

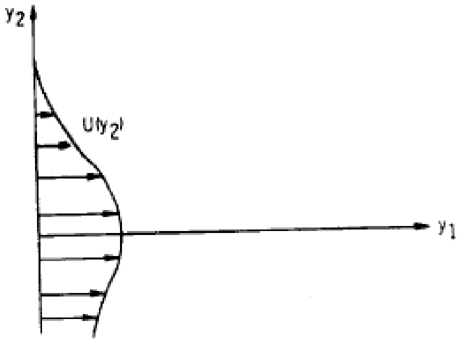

Figure 2-1. - Unidirectional transversely sheared mean flow.

2.4.2.1 Special form of Reynolds stress approximation to correlation tensor.

Before turning to more general considerations, we shall attempt to gain some insight into the connection between the turbulence velocity correlations and the correlation tensor R_{ijkl} by approximating T_{ij} by the Reynolds stress. Thus, substituting equation (2-22) into the Reynolds stress approximation (2-7) and choosing ϕ_{ijkl} in equation (2-19) to be

$$U'^2\,\delta_{1i}\delta_{1j}\,\overline{u''_k\,u''_l} + U''^2\,\delta_{1k}\delta_{1l}\,\overline{u'_i\,u'_j} + U'^2\,U''^2\,\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} + \overline{u'_i\,u'_j}\;\overline{u''_k\,u''_l}$$

show, after carrying out a very tedious calculation, that

$$R_{ijkl}(\vec{y}',\vec{\eta},\tau)\int= (\overline{u'_i\,u'_j\,u''_k\,u''_l} - \overline{u'_i\,u'_j}\;\overline{u''_k\,u''_l}) + 4U'\delta_{1i}\overline{u'_j\,u''_k\,u''_l} + 4U'U''\delta_{1i}\delta_{1k}\overline{u'_j\,u''_l} \tag{2-23}$$

where the double primes indicate that the quantities are to be evaluated at $\vec{y}''$ and $t+\tau$, while the primed quantities are to be evaluated at $\vec{y}'$ and t. The notation $\int=$ indicates that the quantities on both sides of the equal signs are not necessarily equal but merely make equal contributions to equations (2-20) and (2-21). In order to obtain this relation, we changed the names of dummy indices in the summations and used the equation

$$U'\delta_{1i}\overline{u'_j\,u''_k\,u''_l}\int= U''\delta_{ik}\overline{u'_i\,u'_j\,u''_l}$$

obtained by changing the variables of integration from $y',\vec{\eta}$ to $-\vec{\eta}$ and

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[12] This type of model for the turbulence correlation tensor appears to have been introduced by Ribner (refs. 10 and 11).

$\vec{y}'+\vec{\eta}$ and then using the invariance of the turbulence correlations under time translations.[13]

2.4.2.2 Introduction of moving coordinates.

Let l denote a typical correlation length of the turbulence. Then l is roughly the smallest length for which

$$\frac{R_{ijkl}(\vec{y}',\vec{\eta},\tau)}{R_{ijkl}(\vec{y}',0,\tau)} \approx 0\,, \quad \text{whenever } |\vec{\eta}| > l\,.$$

If R_{ijkl} changed so slowly with time that it was practically constant for time changes of the order of l/c_0 (the change in retarded time across a turbulent eddy) or, what is the same thing, if τ_η (the characteristic decay time of a turbulent eddy) satisfied the inequality

$$\tau_\eta >> \frac{l}{c_0} \qquad (2\text{-}24)$$

it would be possible to replace $R_{ijkl}(\vec{y}',\vec{\eta}',\tau+\vec{x}\cdot\vec{\eta}/xc_0)$ by $R_{ijkl}(\vec{y}',\vec{\eta},\tau)$, since $(\vec{\eta}/c_0)\cdot\vec{x}/x = O(l/c_0)$ in the region where the integrand in equation (2-20) is of significant magnitude. Indeed, if it were not for the mean flow, a plot of constant correlation contours might appear as shown in figure 2-2 and

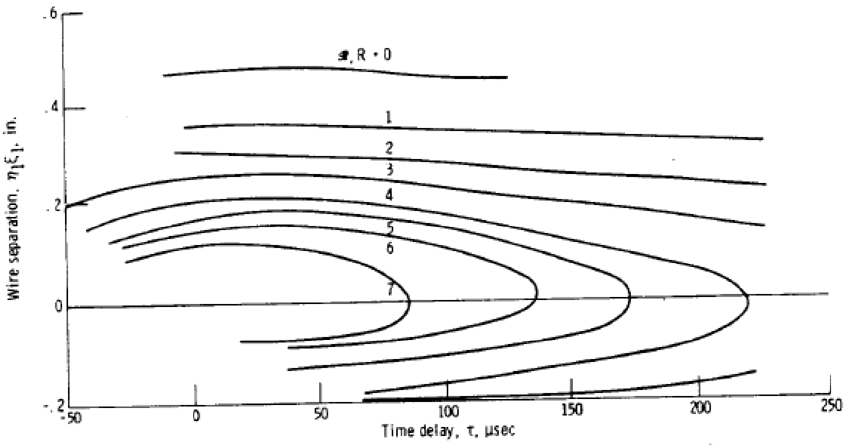

Figure 2-2. - Isocorrelation contours in moving frame (measurements in mixing region $1\frac{1}{2}$ diameters downstream). (From ref. 13.)

the inequality (2-24) would then be satisfied. However, for moving eddies, especially at higher velocities, the turbulent fluctuations (seen by a fixed observer) will appear to be much more rapid because of the convection of the random spatial pattern of the turbulence by the mean flow. This rapid convection of the eddy pattern therefore causes the turbulence fluctuations with time seen by an observer rowing with the mean flow to be much slower than those seen by a fixed observer. Hence, the eddy pattern appears to be

[13] The calculations are carried out in more detail in ref. 12.

nearly frozen.[14] As a result the constant correlation contours in an actual flow will resemble those shown in figure 2-3. In fact, this figure is a plot of

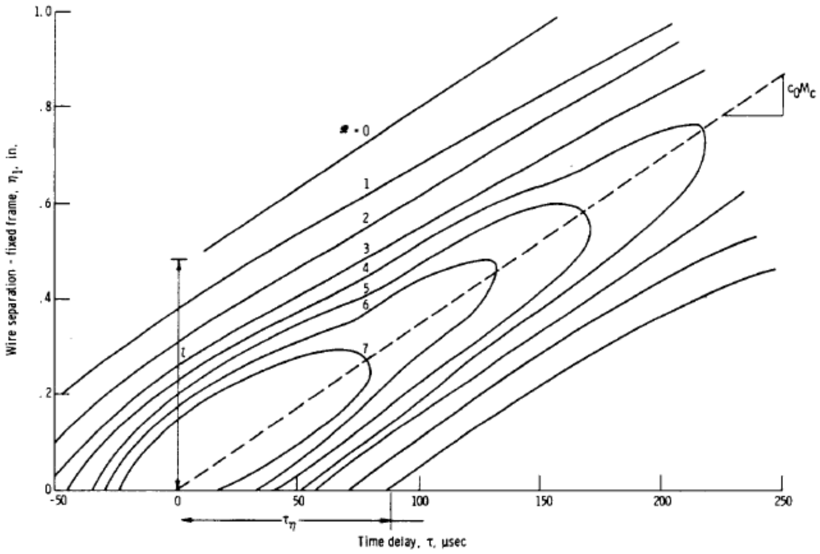

Figure 2-3. - Isocorrelation contours for fixed observer (measurements in center of mixing region $1\frac{1}{2}$ diameters downstream). (From ref. 13.)

actual measurements of the second-order time-delayed correlation $u_1(\vec{y}',t)u_1(\vec{y}'+\hat{i}\,\eta_1,t+\tau)$ carried out in the mixing region of a jet by Davies, Fisher, and Barratt (ref. 13). The inequality (2-24) will therefore not generally be satisfied in most real flows. But in any coordinate system which, roughly speaking, "moves with the eddies" the constant correlation contours should again resemble those shown in figure 2-2. (In fact this figure was obtained from fig. 2-3 by introducing just such a coordinate system.)

Thus, suppose that the correlation tensor $R_{ijkl}(\vec{\eta},\tau)$ is expressed in terms of the variables τ and

$$\vec{\xi} = \vec{\eta} - \hat{i}\,c_0 M_0 \tau \,, \qquad (2\text{-}25)$$

where $\hat{i}$ is a unit vector in the mean flow direction (i.e., y_1-direction) and $c_0 M_c$ is the slope of the dashed line in figure 2-3. Then ξ_1 will remain constant along any line having this slope. Hence, a change in ξ_1 with τ held fixed corresponds to a movement in the direction perpendicular to these lines. The constant correlation contours in the $\xi_1 - \tau$ plane must therefore resemble those shown in figure 2-2. And, as a consequence, the decay time τ_ξ of the "***moving-axis correlation tensor***" $\hat{R}_{ijkl}$ defined by

$$\hat{R}_{ijkl}(\vec{y}',\vec{\xi},\tau) = R_{ijkl}(\vec{y}',\vec{\eta},\tau) \qquad (2\text{-}26)$$

is more likely to satisfy the inequality

[14] This result is frequently referred to as ***Taylor's hypothesis***.

$$\tau_{\xi} >> \frac{l}{c_0} \qquad (2\text{-}27)$$

than is the fixed-frame decay time τ_{η}.

Substituting equation (2-26) together with the change of variable (2-25) into equation (2-21) shows that

$$\bar{I}_{\omega}(\vec{x}) = \frac{\omega^4 \rho_0}{32\pi^3 c_0^5} \frac{x_i x_j x_k x_l}{x^6} \int \int \int \exp\left\{ i\omega \left[(1 - M_c \cos\theta)\tau \cdot \frac{\vec{x}}{x} \cdot \frac{\vec{\xi}}{c_0} \right] \right\} \hat{R}_{ijkl}(\vec{y}', \vec{\xi}, \tau) d\vec{y}' d\vec{\xi} d\tau$$

(2-28)

where

$$\cos\theta = \frac{x_1}{x}$$

the angle between the direction of mean flow and the line between the observation and source points shown in figure 2-4. The essential simplicity of

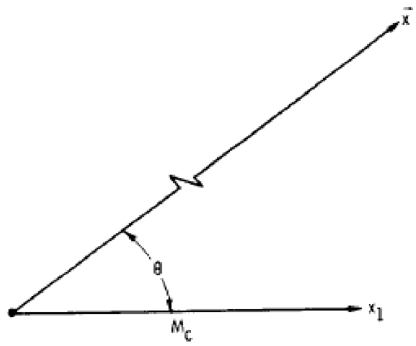

Figure 2-4. - Orientation of observation point relative to flow direction.

this equation becomes especially apparent when the four-dimensional power spectral density tensor

$$H_{ijkl}(\vec{y}', \vec{k}, \omega) \equiv \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int e^{i(\omega\tau - \vec{k}\cdot\vec{\xi})} \hat{R}_{ijkl}(\vec{y}', \vec{\xi}, \tau) d\vec{\xi} d\tau$$

is introduced to obtain

$$\bar{I}_{\omega}(\vec{x}) = \frac{\pi\omega^4 \rho_0}{2c_0^5} \frac{x_i x_j x_k x_l}{x^6} \int H_{ijkl}\left[\vec{y}', \frac{\omega}{c_0} \frac{\vec{x}}{x}, \omega(1 - M_c \cos\theta) \right] d\vec{y}'.$$

(2-29)

Instead of carrying out a similar operation on equation (2-20) for the pressure autocorrelation function, it is simpler to take the inverse transform of equation (2-28) to obtain

$$r(\vec{x},t)=\frac{\rho_0}{16\pi^2 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\int\left(\frac{1}{1-M_c\cos\theta}\right)\frac{\partial^4}{\partial t^4}\int \hat{R}_{ijkl}\left(\vec{y}',\vec{\xi},\frac{1-(\vec{x}/x)\cdot(\vec{\xi}/c_0)}{1-M_c\cos\theta}\right)d\vec{\xi}d\vec{y}'$$

$$=\frac{\rho_0}{16\pi^2 c_0^5}\frac{x_i x_j x_k x_l}{x^6}\int\left(\frac{1}{(1-M_c\cos\theta)^5}\right)\left\{\frac{\partial^4}{\partial t^4}\int \hat{R}_{ijkl}\left(\vec{y}',\vec{\xi},\tau+\frac{\vec{x}}{x}\cdot\frac{\vec{\xi}}{c_0(1-M_c\cos\theta)}\right)'\right\}_{\tau=t/(1-M_c\cos\theta)} d\vec{\xi}d\vec{y}$$

(2-30)

Aside from the possible advantage of being able to neglect the retarded time, this equation possesses the additional advantage over equation (2-20) of being less sensitive to small errors in the correlation function. In order to see this, notice that the largest changes of the correlation function with respect to time occur as a result of the convection of the frozen eddy pattern by the mean flow. Hence, the largest part of the time derivatives of R_{ijkl} and therefore of the integrand in equation (2-20) will be due to the convection. But the uniform subsonic convection of a frozen eddy pattern cannot contribute to the sound field. Hence, only a small part of the integrand does not integrate to zero. This difficulty does not occur with equation (2-30) since the changes with respect to time now occur on the time scale of the sound-producing turbulence fluctuations. The integrand in this equation should therefore be much less sensitive to small errors made either in the measurement Or in the analytical approximation of the turbulence correlation. This is extremely important since this quantity is quite difficult to determine accurately.

As pointed out by Ffowcs Williams (ref. 14), equation (2-29) shows in a particularly explicit way which components of the turbulence generate the sound field. Thus, it shows that for turbulence measured in the moving frame the wave number vector of the sound field $\frac{\vec{x}}{x}\frac{\omega}{c_0}$ is the same as that of the turbulence which generates it. However, the frequency of the turbulence is equal to the Doppler factor $(1-M_c\cos\theta)$ times the frequency of the sound it generates. A plot of a typical moving-frame turbulence power spectral density function (ref. 14) in wave number-frequency space is shown in figure 2-5. It reflects the fact that in the moving frame the turbulent energy is concentrated around the low frequencies. But equation (2-29) implies that all the sound-emitting elements must lie along the line shown in the figure. Hence, the part of the turbulence spectrum containing the maximum energy is by no means always the part which emits the most sound. At subsonic convection speeds these parts coincide more closely for forward emission ($|\theta|<\pi/2$) and high Mach numbers than they do for backward emission and

low Math numbers. Accordingly, more sound is emitted in the forward

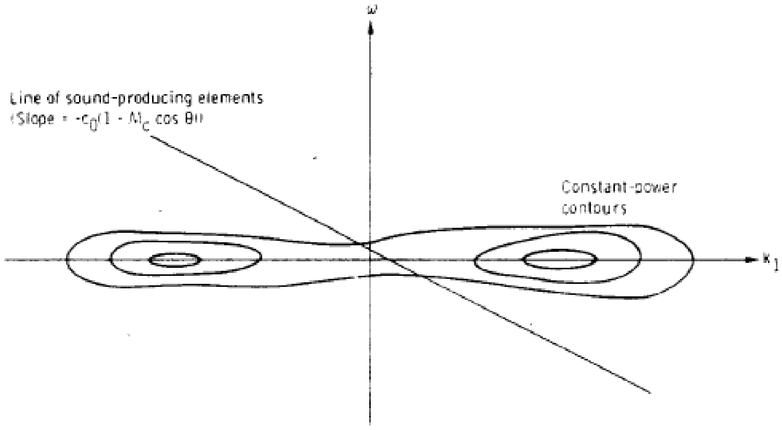

Figure 2-5. - Moving-frame turbulence power spectral density function.

direction than in the backward direction; and the higher the Mach number, the greater the forward emission.

2.4.2.3 Neglect of retarded time in subsonic flows.

(to be continued)