8-5. THE MOTION OF A TOP

In this section, we shall consider the motion of an **axially symmetric body**, such as a **top**, which has a fixed point on its axis of symmetry and is acted upon by a uniform force field. The top was chosen because it is a relatively simple example of a body whose forced motion is markedly affected by gyroscopic moments associated with the spin about its axis of symmetry. The results, however, have application in the analysis of other systems, such as gyroscopes and spinning projectiles.

General Equations. Consider the rotational motion of the **top** shown in Fig. 8-13. It is assumed to spin without friction such that the point O on the axis of symmetry is fixed. The only external moment about O is that due to the constant gravitational force mg acting through its center of mass at C.

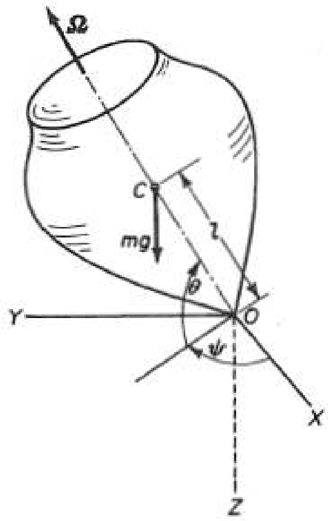

Fig. 8-13. A symmetrical top with a fixed point at *O*.

Let us analyze the motion of the top by using Lagrange's equations and choosing the Eulerian angles as coordinates. We note that this is an example of forced motion; hence ***H*** is not fixed in space. So let us use the original Euler angle definition of Sec. 7-12 in which the ψ vector is assumed to point vertically downward in the direction of the gravitational force.

If we choose the fixed point O as the reference point, the ***total kinetic energy*** may be written in terms of the Euler angle rates. Noting that $I_{xx} = I_a$ and $I_{yy} = I_{zz} = I_t$ for this case of symmetry about the x axis, we find from Eq. (8-87) that

$$T = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi} \sin\theta)^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) \tag{8-140}$$

where the moments, of inertia are taken about principal axes at O. Assuming a horizontal reference level through O, the gravitational potential energy is

$$V = mgl \sin\theta \tag{8-141}$$

where l is the distance of the mass center from . the fixed point. Now we can write the Lagrangian function:

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} I_a (\dot{\phi} - \dot{\psi}\sin\theta)^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) - mgl\sin\theta \end{aligned} \tag{8-142}$$

The generalized momenta are of the same form as we obtained in Eq. (8-101) for the unforced case. We see that

$$\begin{cases} p_\psi = \dfrac{\partial L}{\partial \dot\psi} = -I_a \Omega \sin\theta + I_t \dot\psi \cos^2\theta \\ p_\theta = \dfrac{\partial L}{\partial \dot\theta} = I_t \dot\theta \\ p_\psi = \dfrac{\partial L}{\partial \dot\phi} = I_a \Omega \end{cases} \tag{8-143}$$

where we recall that the total spin Ω is given by

$$\Omega = \dot\phi - \dot\psi \sin\theta .$$

The standard form of Lagrange's equation, namely,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

is now applied together with Eq. (8-143) to obtain

$$\begin{cases} \dfrac{dp_\psi}{dt} = 0 \\ \dfrac{dp_\phi}{dt} = 0 \end{cases} , \tag{8-144}$$

from which we see that both p_ψ and p_ϕ are constant. Hence we find that Ω is constant for this case where there is no applied moment about the symmetry axis. Also, the precession rate $\dot\psi$ can be obtained from Eq. (8-143) with the following result:

$$\dot\psi = \frac{p_\psi + I_a \Omega \sin\theta}{I_t \cos^2\theta} . \tag{8-145}$$

Now let us use the principle of conservation of energy to obtain an integral of the θ equation of motion. From Eqs. (8-140) and (8-141), we see that the total energy is

$$E = T + V = \frac{1}{2} I_a \Omega^2 + \frac{1}{2} I_t (\dot\theta^2 + \dot\psi^2 \cos^2\theta) + mgl \sin\theta \tag{8-146}$$

where we recall that $\Omega = \dot\phi - \dot\psi \sin\theta$ is constant. It follows that the total energy minus the kinetic energy associated with the total spin Ω is also a constant. Calling this quantity *E',* we can write

$$E' = E - \frac{1}{2} I_a \Omega^2 = \frac{1}{2} I_t (\dot\theta^2 + \dot\psi^2 \cos^2\theta) + mgl \sin\theta . \tag{8-147}$$

Substituting for $\dot\psi$ from Eq. (8-145) and solving for $\dot\theta^2$, we obtain

$$\dot\theta^2 = \frac{2E'}{I_t} - \left(\frac{p_\psi + I_a \Omega \sin\theta}{I_t \cos\theta}\right)^2 - \frac{2mgl}{I_t} \sin\theta . \tag{8-148}$$

Note that θ is the only variable on the right-hand side of this equation. Thus we see from Eqs. (8-145) and (8-148) that the *precession rate* $\dot\psi$ and the *nutation rate* $\dot\theta$ can be written as functions of θ alone for any given case.

In order to simplify the statement of Eq. (8-148), let us make the substitution

$$u = \sin\theta\,, \tag{8-149}$$

from which it follows that

$$\dot{u} = \dot{\theta}\cos\theta\,. \tag{8-150}$$

Also, let us define the constant parameters *a, b,* c, and *e* as follows:

$$a = \frac{p_\psi}{I_t}\,,\quad b = \frac{I_a\Omega}{I_t}\,,\quad c = \frac{2mgl}{I_t}\,,\quad e = \frac{2E'}{I_t}\,. \tag{8-151}$$

Now multiply Eq. (8-148) by $\cos^2\theta$ and make the foregoing substitutions. The resulting equation is

$$\dot{u}^2 = (1-u^2)(e-cu)-(a+bu)^2\,. \tag{8-152}$$

If we use the notation

$$f(u) = (1-u^2)(e-cu)-(a+bu)^2\,, \tag{8-153}$$

then we can write Eq. (8-152) in the form

$$\dot{u} = f(u)\,. \tag{8-154}$$

Now let us consider the function $f(u)$ in greater detail. Figure 8-14 shows a plot of $f(u)$ versus *u* for a typical case. We can assume that the parameter *c* is positive since we can always consider the distance *l* from the support point to the center of mass to be positive. Furthermore, we see from Eq. (8-153) that the cubic term predominates for large absolute magnitudes of *u*. Hence we find that $f(u)$ must be negative for large negative values of *u* and must be positive for large positive values of *u*. Now $f(u)$ is a continuous function of *u* so it must be zero for at least one real value of *u*. That $f(u)$ is actually zero for three real values of *u* will now be shown.

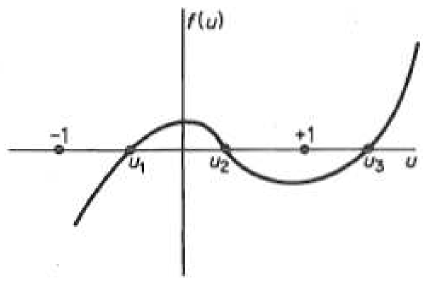

Fig. 8-14. A graph showing the three roots of $f(u)=0$ for a typical case of top motion.

It is evident that $\dot{u}^2$ is zero or positive for all physically realizable situations, so $f(u)$ must be zero or positive at some point in the interval $-1 \le u \le 1$ which is the range in which *u* must lie in the actual case. But if we set $u = \pm 1$ in Eq. (8-153) we find that $f(u)$ is zero or negative at these two points since $(a+bu)^2$ must be zero or positive. Therefore there are two roots in the interval $-1 \le u \le 1$ and the third root u_3 must lie in the range $u_3 \ge 1$. Summarizing, we can write

$$-1 \le u_1 \le u_2 \le 1 \le u_3\,.$$

Looking again at Eq. (8-148) or (8-152), we find that the motion in θ stops only when $u = u_1$ or $u = u_2$. It is apparent, then, that *u* must oscillate between these values and θ will undergo a corresponding oscillation. To analyze the motion of θ in time, let us write Eq. (8-154) in the form

$$\dot{u}^2 = f(u) = c(u-u_1)(u-u_2)(u-u_3)\,. \tag{8-155}$$

Now we define

$$w = \sqrt{\frac{u - u_1}{u_2 - u_1}}\,, \quad k = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}\,, \quad p = \frac{1}{2}\sqrt{c(u_3 - u_1)}\,, \tag{8-156}$$

and we see that

$$\dot{w} = \frac{\dot{u}}{2\sqrt{(u - u_1)(u_2 - u_1)}}\,. \tag{8-157}$$

Making these substitutions, we can write Eq. (8-155) I the following form:

$$\dot{w} = p^2(1 - w^2)(1 - k^2 w^2)\,. \tag{8-158}$$

If we measure the time *t* from the instant when θ is at its minimum value and $u = u_1$, we find that

$$pt = \int_0^w \frac{dw}{\sqrt{(1 - w^2)(1 - k^2 w^2)}} = F(\sin^{-1} w, k)\,. \tag{8-159}$$

where we recognize the integral as an ***elliptic integral of the first kind***. Conversely, we can solve for *w,* obtaining

$$w = \mathrm{sn}\; pt\,, \tag{8-160}$$

where, we recall, the elliptic function, sn, was introduced previously in Sec. 3-9. Now we use the definition of *w* in Eq. (8-156) to solve for *u*

$$u = u_1 + (u_2 - u_1)\,\mathrm{sn}^2\; pt\,. \tag{8-161}$$

Because the sn function is squared, the period of the nutation in θ is just half the period of *w*. Thus the period of θ or *u* is

$$T = \frac{2K(k)}{p} = \frac{4K(k)}{\sqrt{c(u_3 - u_1)}}\,, \tag{8-162}$$

where *K(k)* is the ***complete elliptic integral of the first kind***.

Referring to Eqs. (8-145) and (8-151), we see that the precession rate $\dot{\psi}$ can be expressed as a function of *u* as follows:

$$\dot{\psi} = \frac{a + bu}{1 - u^2}\,. \tag{8-163}$$

Having solved for *u* as a function of time, we can also obtain $\dot{\psi}$ as a function of time. Also, from the definition of the total spin Ω given in Eq. (8-100), we see that

$$\dot{\phi} = \Omega + \dot{\psi} u\,, \tag{8-164}$$

where Ω is constant. Hence $\dot{\phi}$ is a known function of time. Note that both $\dot{\psi}$ and $\dot{\phi}$ have the same period as θ.

Path of the Symmetry Axis. Let us consider the path of a point *P* located on the axis of symmetry at a unit distance from the fixed point *O*. Taking a horizontal reference through *O*, we find that $u = \sin\theta$ represents the height of *P*. The values u_1 and u_2 at which $\dot{u}$ is zero correspond to positions of

minimum and maximum height and are known as ***turning points***. If we represent the motion of the symmetry axis by the path of P on the surface of a unit sphere, we find that P will remain between two horizontal "latitude" circles given by $\theta = \theta_1$ and $\theta = \theta_2$, corresponding to the extreme values of u.

The path of P in any given case can be classified as one of *three general types*. In order to simplify this classification, let us define

$$u_0 = -\frac{a}{b} = -\frac{p_\psi}{p_\phi}. \tag{8-165}$$

From Eq. (8-163) we see that $\dot{\psi} = 0$ when $u = u_0$ (except if $u_0 = \pm 1$, in which case $\dot{\psi}$ is indeterminate). Hence the motion of the symmetry axis at the instant when $u = u_0$ is such that the path of P is tangent to a vertical circle of "longitude."

(a) Let us consider first the case where $u_0 > u_2$. In other words, u_0 lies outside the possible range of u, and therefore the precessional rate $\dot{\psi}$ does not equal zero at any time during the motion. This case is shown in Fig. 8-15(a), where the arrows indicate the direction of the velocity of P for positive Ω.

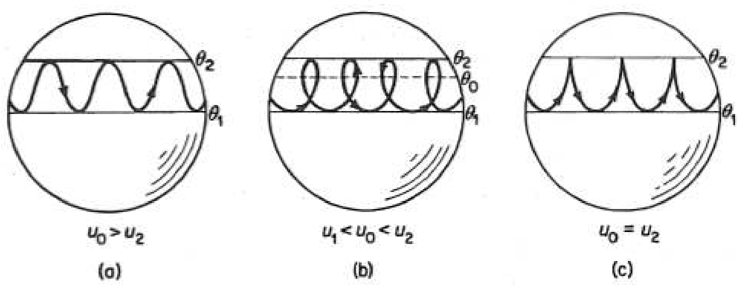

Fig. 8-15. Possible paths of the axis of symmetry.

(b) Next consider the case in which $u_1 < u_0 < u_2$. As shown in Fig. 8-15(b), we see that $\dot{\psi}$ is zero twice during each nutation cycle at $\theta = \theta_0$, and, for $\theta > \theta_0$, it actually reverses its sign compared to the average value of $\dot{\psi}$. In general, for a certain top with a given total spin Ω, the path traced by P is dependent upon the initial values of $\theta, \dot{\theta}$, and $\dot{\psi}$. It can be seen, for example, that the formation of loops as in Fig. 8-15(b) can be accomplished by giving the axis of symmetry an initial precession rate which is opposite in direction to its average value.

Cuspidal Motion. **(c)** The case in which $u_0 = u_2$ is known as ***cuspidal motion*** and is illustrated in Fig. 8-15(c). Note that u and $-\dot{\psi}$ are both zero when $u = u_0$, resulting in a ***cusp*** which points upward. This type of motion

will occur, for example, if the axis of a spinning top is released with zero initial velocity. The axis begins to fall vertically, causing gyroscopic inertial moments which result in a combination of nutation and precession such that the axis is motionless at a cusp at regular intervals. Incidentally, we see from energy considerations that the axis can be stationary only when P is at its highest point, corresponding to $u = u_2$. In other words, the potential energy must be at a maximum when the kinetic energy is minimum.

To find the limits of the nutation for cuspidal motion, let us set $\dot{u}^2$ or $f(u)$ equal to zero. Consider the following **two cases**: (1) $\dot{u} = 0$ because $\dot{\theta} = 0$; (2) $\dot{u} = 0$ because $\cos\theta = 0$.

Case 1: *$u = u_0$ when $\dot{\theta} = 0$.* Using the definition of u_0 given in Eq. (8-165), we see from Eq. (8-152) that

$$e = cu_0 , \tag{8-166}$$

since $\dot{u} = 0$ at the cusp. We can then write an expression for $f(u)$ in the form

$$f(u) = c(u_0 - u)(1-u) - b^2(u_0 - u)^2 = 0 . \tag{8-167}$$

We have already shown that one limit of the nutational motion occurs at $u_2 = u_0$. Dividing out this root, we obtain

$$c(1-u^2) - b^2(u_0 - u) = 0 ,$$

or

$$u^2 - 2\lambda u + (2\lambda u_0 - 1) = 0 , \tag{8-168}$$

where λ is a positive constant given by

$$\lambda = \frac{b^2}{2c} = \frac{I_a{}^2\Omega^2}{4I_t mgl} . \tag{8-169}$$

Solving for the roots of Eq. (8-168) and including the one found previously, we can summarize the roots of $f(u) = 0$ for this case as follows:

$$\begin{cases} u_1 = \lambda - \sqrt{\lambda^2 - 2\lambda u_0 + 1} \\ u_2 = u_0 \\ u_3 = \lambda + \sqrt{\lambda^2 - 2\lambda u_0 + 1} \end{cases} , \tag{8-170}$$

where we take the positive square root. For $-1 < u_0 \le 1$, an evaluation of Eq. (8-170) will show that

$$-1 \le u_1 \le u_0 \le 1 \le u_3$$

regardless of the value of λ and in agreement with our earlier results.

The limits of the motion in u or θ are found by evaluating u_0 from Eq. (8-166), λ from Eq. (8-169), and then using Eq. (8-170) to obtain u_1. Note again that $ul3$ has no physical meaning.

Case 2: *$\dot{\theta} \ne 0$ when $u = u_0 = \pm 1$.* Consider first that $u_0 = 1$. This case

applies when the symmetry axis passes through the upper vertical position with a non-zero angular velocity. We see from Eq. (8-165) that

$$a = -b \tag{8-171}$$

and, since $\dot{\theta}^2 > 0$ at this moment, we note from Eqs. (8-148) and (8-151) that

$$e > c\,. \tag{8-172}$$

So we can express $f(u)$ in the form

$$f(u) = (1-u^2)(e-cu) - b^2(1-u)^2 = 0 \tag{8-173}$$

and solve for the turning points. Dividing out the known facto (1-u) corresponding to $u_2 = 1$, there remains

$$(1+u)(e-cu) - b^2(1-u) = 0\,,$$

which can be written in the form

$$u^2 - \left(\frac{b^2+e}{c} - 1\right)u + \frac{b^2-e}{c} = 0\,. \tag{8-174}$$

One of the roots of this equation corresponds to a turning point at u_1. The other root u_3 is greater than 1 and has no physical meaning.

A similar procedure can be followed for the case where $\dot{\theta} \neq 0$ at $u = u_0 = -1$. In this instance, the symmetry axis passes through the bottom point on the unit sphere with a non-zero velocity. We see from Eq. (8-165) that

$$a = b\,. \tag{8-175}$$

Also, since $\dot{\theta}^2 > 0$ at $u = -1$, we see that

$$e + c > 0\,. \tag{8-176}$$

So we can write

$$f(u) = (1-u^2)(e-cu) - b^2(1+u)^2 = 0\,. \tag{8-177}$$

Dividing out the known factor (1 + *u)* corresponding to the root $u_1 = -1$, we obtain

$$u^2 - \left(\frac{b^2+e}{c} + 1\right)u - \frac{b^2-e}{c} = 0\,. \tag{8-178}$$

The roots of this equation are the upper turning point u_2 and the nonphysical root u_3 which again is larger than unity.

Stability of Motion near the Vertical. An example of top motion of particular interest occurs if the axis points vertically upward when $\dot{\theta}$ is zero. This falls under Case 1 of cuspidal motion, and $u_0 = 1$ in this instance. From Eq. (8-170), we see that the roots are

$$u_1, u_2, u_3 = (2\lambda - 1), 1, 1 \tag{8-179}$$

where $(2\lambda - 1)$ is designated as u_1 or u_3 according as λ . is less than or

greater than 1. Now let us consider the motion for the following values of λ : **(a)** $\lambda < 1$, **(b)** $\lambda = 1$ and **(c)** $\lambda > 1$.

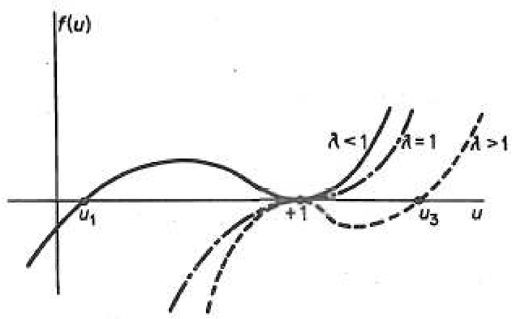

Fig. 8-16. Graphs of $f(u)$ indicating the stability of a vertical top.

(a) ***Case 1***: $\lambda < 1$ *or* $\Omega^2 < 4I_t mgl / I_a^2$. For this case the cusp occurs at a position of ***unstable equilibrium*** at u = 1. This can be seen from Fig. 8-16 by noting that a small disturbance at this position will cause a **nutational motion** during which θ decreases until the minimum value corresponding to $u_1 = 2\lambda - 1$ is reached; whereupon a reversal of this motion returns the axis to the vertical. The double root at $u = u_0 = 1$ is illustrated by the fact that the slope $f'(u)$ is zero at this point, as may be seen from Eq. (8-167).

It is interesting to note that the period of a nutation cycle is **infinite**. To show this, we refer to Eq. (8-156) and we find that the modulus k = 1; hence the *complete elliptic integral* *K(k)* is infinite. It follows from Eq. (8-162) that the period is also infinite.

(b) ***Case 2***: $\lambda = 1$ *or* $\Omega^2 = 4I_t mgl / I_a^2$. Here Here is the borderline case in which there is neutral stability at the vertical. All three roots are equal to unity and thus there is an inflection point of $f(u)$ at u = 1.

(c) ***Case 3***: $\lambda > 1$ *or* $\Omega^2 > 4I_t mgl / I_a^2$. The spin is sufficiently large in this case that the vertical position is ***stable***, that is, an infinitesimal disturbance will not cause a finite deviation of the symmetry axis from the vertical. Again we find that $f'(1) = 0$, but in this case the third root $u_3 = (2\lambda - 1) > 1$.

A top which is spinning as in Case 3 is known as a ***sleeping top***. This name arises because a smooth, axially symmetric top with its axis vertical and $\lambda > 1$ might appear at first glance to be not moving at all, and hence "sleeping."

In an actual case there are small frictional moments which slowly decrease the spin until λ becomes less than unity. At this point a wobble or *nutation* appears and gradually increases until the body of the top hits the horizontal surface. Incidentally, the ability of an actual top with a large spin to right itself and reach the sleeping condition is due to the action of frictional

forces on its rounded point. These forces have been omitted in this analysis.

Nutation Frequency and Amplitude for the Case of Large Spin. Now let us use Eq. (8-170), which was obtained in the analysis of *cuspidal motion*, to find the approximate amplitude and frequency of the nutational motion of a ***fast top***, that is, one for which $\lambda >> 1$. We can write the expression for u_1 in the form

$$u_1 = \lambda - \lambda\sqrt{1 - \frac{2u_0}{\lambda} + \frac{1}{\lambda^2}} ,$$

or, using the binomial expansion and neglecting terms of order higher than $1/\lambda$, we obtain

$$u_1 \cong \lambda - \lambda\left(1 - \frac{u_0}{\lambda} + \frac{1}{2\lambda^2} - \frac{{u_0}^2}{2\lambda^2}\right) = u_0 - \frac{1}{2\lambda}(1 - {u_0}^2) . \quad (8\text{-}180)$$

The ***amplitude of the nutation*** is

$$u_0 - u_1 \cong \frac{1}{2\lambda}(1 - {u_0}^2) = \frac{2I_t mgl}{I_a^2 \Omega^2}(1 - {u_0}^2) . \quad (8\text{-}181)$$

Note that the nutation amplitude varies inversely with the square of the total spin rate Ω. It becomes zero for $u_0 = 1$, that is, for a vertical top, the spin being much larger than that required for stability.

We see from Eq. (8-170), then, that the roots u_1 and $u_2 = u_0$ have a small separation; but the third root u_3 is approximated by

$$u_3 \cong 2\lambda , \quad (8\text{-}182)$$

and thus is widely separated from the others.

To find the nutation frequency of a *fast top*, let us refer to Eq. (8-161) which gave the general solution for *u* in terms of elliptic functions. However, because u_3 is widely separated from u_1 and u_2, we see from Eq. (8-156) that the modulus *k* is very small; hence we can approximate the *elliptic function* sn *pt* by the *trigonometric function* sin *pt.* Thus we obtain

$$u = u_1 + (u_0 - u_1)\sin^2 pt , \quad (8\text{-}183)$$

where the time *t* is measured from the moment when $u = u_1$. Using Eq. (8-181) and trigonometric identities, we can rewrite this result in the form:

$$u = u_0 - \frac{I_t mgl}{I_a^2 \Omega^2}(1 - {u_0}^2)(1 + \cos 2pt) . \quad (8\text{-}184)$$

From Eqs. (8-151), (8-156), (8-169), and (8-182), we' see that the ***circular frequency of the nutation*** is

$$2p = \sqrt{c(u_3 - u_1)} \cong \sqrt{2c\lambda} = \frac{I_a \Omega}{I_t} . \quad (8\text{-}185)$$

An expression for the *precession rate* $\dot{\psi}$ can now be obtained, for we

note from Eqs. (8-163) and (8-165) that

$$\dot{\psi} = -b\frac{u_0 - u}{1-u^2}.$$

Substituting for b and $(u_0 - u)$ from Eqs. (8-151) and (8-184), we obtain

$$\dot{\psi} = -\frac{mgl}{I_a\Omega}\frac{1-u_0^{\ 2}}{1-u^2}(1+\cos 2pt). \qquad (8\text{-}186)$$

But we see from Eq. (8-184) that

$$1-u^2 \cong (1-u_0^{\ 2}) + 2u_0(1-u_0^{\ 2})\frac{I_t mgl}{I_a^2\Omega^2}(1+\cos 2pt)$$

or

$$\frac{1-u_0^{\ 2}}{1-u^2} \cong 1 - \frac{u_0}{2\lambda}(1+\cos 2pt) \cong 1, \qquad (8\text{-}187)$$

where we neglect terms of order $1/\lambda$ or higher. So we obtain for this case of a fast top that the ***precession rate*** is

$$\dot{\psi} = -\frac{mgl}{I_a\Omega}(1+\cos 2pt); \qquad (8\text{-}188)$$

the approximation being valid even for cuspidal motion near the vertical, so long as $\lambda >> 1$.

Precession with No Nutation. Precession with no nutation is indicated by having $u_1 = u_2$, that is, by the presence of a *double root* of $f(u)$ at a point other than at $u = \pm 1$. The conditions for a double root are that $f(u)$ and $f'(u)$ be zero for the same value of *u*, as shown in Fig. 8-17. Thus, using Eq.

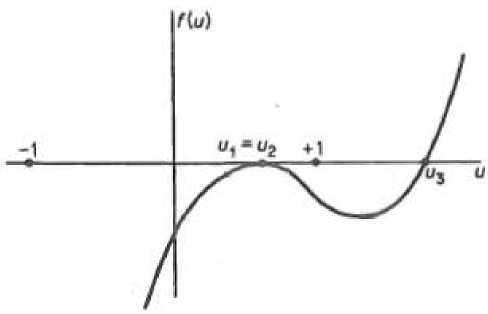

Fig. 8-17. The form of $f(u)$ for the case of steady precession.

(8-153) we can write

$$f(u) = (1-u^2)(e-cu) - (a+bu)^2 = 0. \qquad (8\text{-}189)$$

Also,

$$f'(u) = -2u(e-cu) - c(1-u^2) - 2b(a+bu) = 0. \qquad (8\text{-}190)$$

From Eqs. (8-189) and (8-190), we have

$$e - cu = \frac{(a+bu)^2}{1-u^2} = \frac{-c(1-u^2) - 2b(a+bu)}{2u},$$

from which we obtain the following quadratic equation in $(a+bu)$:

$$2u(a+bu)^2 + 2b(1-u^2)(a+bu) + c(1-u^2)^2 = 0 . \qquad (8\text{-}191)$$

Solving for $(a+bu)$, we obtain

$$a+bu = \frac{1-u^2}{2u}\left\{-b \pm \sqrt{b^2 - 2cu}\right\}. \qquad (8\text{-}192)$$

But we recall from Eq. (8-163) that

$$a+bu = \dot{\psi}(1-u^2) = \dot{\psi}\cos^2\theta , \qquad (8\text{-}193)$$

so we find that the precession rate is

$$\dot{\psi} = \frac{-b}{2u}\left\{1 \pm \sqrt{1-\frac{2cu}{b^2}}\right\}, \qquad (8\text{-}194)$$

or, in terms of θ,

$$\dot{\psi} = \frac{-b}{2u\sin\theta}\left\{1 \pm \sqrt{1-\frac{2c\sin\theta}{b^2}}\right\}. \qquad (8\text{-}195)$$

It is interesting to note that **two steady precession rates** are possible, provided that the values of θ and $b = I_a\Omega / I_t$ are such that the square root in Eq. (8-195) is real. This last requirement is similar to requiring sufficient spin for stability in a vertical top. Thus we see that the condition on the total spin Ω in order that steady precession be possible at a given value of θ is that

$$\Omega^2 > \frac{4I_t mgl}{I_a^2}\sin\theta . \qquad (8\text{-}196)$$

Consider now the case in which $\Omega^2 >> 4I_t mgl\sin\theta / I_a^2$; that is, the spin is large enough so that the second term in the square root of Eq. (8-195) is small compared to unity. Then we can approximate (8-195) by

$$\dot{\psi} = \frac{-b}{2u\sin\theta}\left\{1 \pm \left(1-\frac{c\sin\theta}{b^2}\right)\right\},$$

from which we obtain the following possible rates of uniform precession:

$$\begin{cases} (\dot{\psi})_1 = \dfrac{-c}{2b} = -\dfrac{mgl}{I_a\Omega} \\ (\dot{\psi})_2 = \dfrac{-b}{\sin\theta} = -\dfrac{I_a\Omega}{I_t\sin\theta} \end{cases} . \qquad (8\text{-}197)$$

Note that the slow precession rate $(\dot{\psi})_1$ is independent of θ. This is the precession rate which is usually observed in a fast top or gyroscope and is also equal to the mean value of $\dot{\psi}$ found in Eq. (8-188). On the other hand, the fast precession rate $(\dot{\psi})_2$ is independent of the acceleration of gravity and, in fact, is identical with the free precession rate obtained previously in Eq. (8-104).

***Example* 8-7.** A top with a total spin Ω and velocity v is sliding on a smooth horizontal floor with its symmetry axis vertical (Fig. 8-18). Suddenly, at t = 0, the point strikes a crack at O and is prevented from moving further although the angular motion is unhindered. If θ is the angle between the axis of symmetry and the floor, find

a. The angular velocity $\dot{\theta}(0+)$ just after the vertex is stopped.

b. The linear impulse exerted on the top at t = 0.

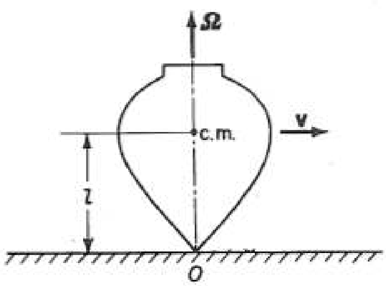

Fig. 8-18. A sliding top on a horizontal floor.

c. $\theta_{\min}$ in the ensuing motion, assuming that $\Omega = 20v/l = 20\sqrt{g/l}$ and $I_a = (1/4)ml^2 = I_t/5$, where the moments of inertia are taken about axes through the vertex.

Our approach will be to note that the angular momentum about the vertex O is conserved during the initial instant because the reaction forces of the floor pass through this point. Hence the angular impulse applied at t = 0 is zero. Equating expressions for the horizontal component of angular momentum before and after the impact at t= 0, we obtain

$$H_h = mvl = -I_t\dot{\theta}(0+)\,, \tag{8-198}$$

where this component is directed into the page. Thus we find that

$$\dot{\theta}(0+) = -\frac{mvl}{I_t}\,. \tag{8-199}$$

The linear impulse is found by calculating the change in the linear momentum during impact. Noting that

$$v(0+) = -\dot{\theta}(0+)l\,,$$

we obtain

$$\tilde{F} = mv(0+) - mv = -mv\left(1 - \frac{ml^2}{I_t}\right), \tag{8-200}$$

where the impulse is positive when directed to the right.

In order to calculate the minimum value of θ, we note first that $\dot{\theta} \neq 0$ at $u = \sin\theta = 1$. Therefore we have the ***cuspidal motion*** which was described previously under Case 2 on page 407. Evaluating the constant

parameters *b, c*, and *e* from Eq. (8-151), we have

$$\begin{cases} b = \dfrac{I_a \Omega}{I_t} = 4\sqrt{\dfrac{g}{l}} \\ c = \dfrac{2mgl}{I_t} = \dfrac{8g}{5l} \\ e = \dfrac{2E'}{I_t} = \left(\dfrac{mvl}{I_t}\right)^2 + \dfrac{2mgl}{I_t} = \dfrac{56g}{25l} \end{cases} . \tag{8-201}$$

The turning point is found by substituting these values into Eq. (8-174) with the result:

$$u^2 - 10.4u + 8.6 = 0 .$$

The roots of this equation are u_1 and u_3, the third root of $f(u) = 0$ being $u_2 = u_0 = 1$. We obtain

$$\begin{cases} u_1 = 0.9058 \\ u_3 = 9.494 \end{cases} .$$

Thus we find that

$$\theta_{\min} = \sin^{-1}(0.9058) = 64.9° .$$

***Example* 8-8.** A uniform circular disk of mass *m*, radius *a*, rolls on a horizontal surface in such a manner that its plane is inclined with the vertical at a constant angle θ and its center of mass describes a circular path of radius *R* - Fig. 8-19(a). Solve for the precession rate $\dot{\psi}$.

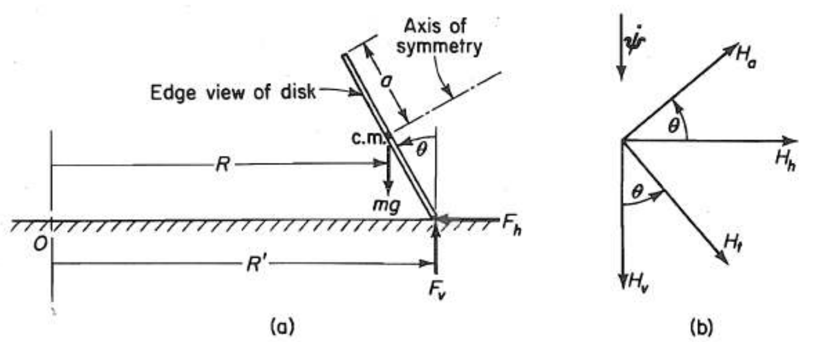

Fig. 8-19. (a) The forces acting on a disk rolling in a circular path. (b) The components of angular momentum in the vertical plane.

First Method: Although the disk is an axially symmetric body subject to gravitational moments, the geometry and the constraints are different from those encountered in the analysis of a top. So let us start again with the basic rotational equation

$$\boldsymbol{M} = \dot{\boldsymbol{H}} ,$$

where we shall choose the fixed point *O* as a reference.

Let us use Eulerian angles to describe the orientation of the disk. The precession rate $\dot{\psi}$ can be obtained in terms of the total spin Ω by

equating two expressions for the translational velocity of the center of mass.

$$v = R\dot{\psi} = a\Omega\,,$$

from which we have

$$\dot{\psi} = \frac{a}{R}\Omega\,. \tag{8-202}$$

It can be seen that $\dot{\psi}$ must be constant. A changing $\dot{\psi}$ would imply a changing kinetic energy. But, since θ is constant and no slipping occurs, there are no working forces acting on the disk; hence, by the principle of work and kinetic -energy, $\dot{\psi}$ cannot change. From Eq. (8-202), we see that Ω is also constant.

Consider now the axial and transverse components of the angular momentum *about the center of mass.* From Eqs. (8-106) and (8-202), we can write

$$\begin{cases} H_a = I_a\Omega = (R/a)I_a\dot{\psi} \\ H_t = I_t\dot{\psi}\cos\theta \end{cases}.$$

To find the total angular momentum about *O*, we must add the angular momentum due to the translational velocity of the center of mass about *O.* Taking horizontal and vertical components of this portion, we obtain

$$\begin{cases} H_a = mR^2\dot{\psi} \\ H_t = mRa\dot{\psi}\cos\theta \end{cases}$$

As the disk travels in its circular path, the system of four component vectors rotates with angular velocity $\dot{\psi}$, as shown in Fig. 8-19(b). Thus we see that the total angular momentum vector $\boldsymbol{H}$ is of constant magnitude but ***precesses*** about a vertical axis at the same rate as the disk.

Now we can evaluate $\dot{\boldsymbol{H}}$ by noting that

$$\dot{\boldsymbol{H}} = \dot{\boldsymbol{\psi}} \times \boldsymbol{H} \tag{8-203}$$

for this case in which the magnitude *H* is constant. We find that

$$\dot{\boldsymbol{H}} = \dot{\psi}(H_h + H_a\cos\theta + H_t\sin\theta)\boldsymbol{e}_\theta \tag{8-204}$$

or, substituting the preceding expressions for H_h, H_a and H_t, we obtain

$$\dot{\boldsymbol{H}} = \dot{\psi}^2\cos\theta\left(mRa + \frac{R}{a}I_a + I_t\sin\theta\right)\boldsymbol{e}_\theta\,, \tag{8-205}$$

where $\boldsymbol{e}_\theta$ is a unit vector directed out of the page. The applied moment about *O* is independent of F_h and is given by

$$\boldsymbol{M} = mga\sin\theta\boldsymbol{e}_\theta\,, \tag{8-206}$$

where we note that $F_v = mg$ since the mass center has no vertical motion. Equating the expressions on the right hand sides of Eqs. (8-205) and (8-206), we obtain

$$\dot{\psi}^2 \cos\theta\left(mRa + \frac{R}{a} I_a + I_t \sin\theta\right) = mga \sin\theta ,$$

or

$$\dot{\psi}^2 = \frac{mga \tan\theta}{mRa + \frac{R}{a} I_a + I_t \sin\theta} . \qquad (8\text{-}207)$$

F or a thin uniform disk, we recall that

$$I_a = \frac{1}{2} ma^2 , \quad I_t = \frac{1}{4} ma^2 .$$

Substituting these values into Eq. (8-207), we obtain

$$\dot{\psi}^2 = \frac{4g \tan\theta}{6R + a \sin\theta} . \qquad (8\text{-}208)$$

Second Method: Now let us consider this problem from the ***Lagrangian viewpoint***. Using Eq. (8-98) for the rotational kinetic energy and adding the translational kinetic energy, we obtain the total kinetic energy

$$T = \frac{1}{2} m(R^2 \dot{\psi}^2 + a^2 \dot{\theta}^2) + \frac{1}{2} I_a (\dot{\phi} - \dot{\psi} \sin\theta)^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2 \cos^2\theta) .$$

(8-209)

In order to use a standard form of Lagrange's equation, the coordinates must be ***independent***. But we saw in Eq. (8-202) that the condition of rolling imposes the constraint that

$$\Omega = \dot{\phi} - \dot{\psi} \sin\theta = \frac{R}{a} \dot{\psi} ,$$

So the coordinates ψ, θ, ϕ are not independent as they stand. If, however, we write the constraint equation in the form

$$\dot{\phi} - \dot{\psi} \sin\theta = \frac{R' - a \sin\theta}{a} \dot{\psi} , \qquad (8\text{-}210)$$

and substitute into the kinetic energy expression of Eq. (8-209), we obtain the result

$$T = \frac{1}{2}\left(m + \frac{I_a}{a^2}\right)(R' - a \sin\theta)^2 \dot{\psi}^2 + \frac{1}{2}(I'_t \dot{\theta}^2 + I_t \dot{\psi}^2 \cos^2\theta) , \qquad (8\text{-}211)$$

where we have let $I'_t = I_t + ma^2$ and

$$R = R' - a \sin\theta , \qquad (8\text{-}212)$$

in accordance with Fig. 8-19(a). Now ϕ has been eliminated as a generalized coordinate and the remaining coordinates ψ and θ are independent in the kinematic sense. Also, we have introduced a new constant, *R'*, because *R* is no longer constant if θ is considered to be variable.

It can be seen that the system is conservative and the potential energy is

given by

$$V = mga\cos\theta\ . \tag{8-213}$$

The generalized forces are

$$\begin{cases} M_\psi = -\dfrac{\partial V}{\partial \psi} = 0 \\ M_\theta = -\dfrac{\partial V}{\partial \theta} = mga\sin\theta \end{cases}. \tag{8-214}$$

Let us use Lagrange's equation in the form of Eq. (6-73), namely,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i\ .$$

First we obtain the generalized momenta:

$$\begin{cases} p_\psi = \dfrac{\partial T}{\partial \dot{\psi}} = \left[\left(m + \dfrac{I_a}{a^2}\right)(R' - a\sin\theta)^2 + I_t\cos^2\theta\right]\dot{\psi} \\ p_\theta = \dfrac{\partial T}{\partial \dot{\theta}} = I'_t\,\dot{\theta} \end{cases}. \tag{8-215}$$

Also, we evaluate the following terms:

$$\begin{cases} \dfrac{\partial T}{\partial \psi} = 0 \\ \dfrac{\partial T}{\partial \theta} = -\left[\left(m + \dfrac{I_a}{a^2}\right)(R' - a\sin\theta)a + I_t\sin\theta\right]\cos\theta\dot{\psi}^2 \end{cases}. \tag{8-216}$$

From Eqs. (8-214), (8-215), and (8-216), we can now write the ψ equation in the form

$$\frac{dp_\psi}{dt} = 0\ , \tag{8-217}$$

indicating that p_ψ is constant. For this particular case in which θ is constant, it follows from Eq. (8-215) that the precession rate $\dot{\psi}$ is constant.

In a similar fashion we find that the θ equation is

$$I'_t\,\ddot{\theta} + \left[\left(m + \frac{I_a}{a^2}\right)(R' - a\sin\theta)a + I_t\sin\theta\right]\cos\theta\dot{\psi}^2 = mga\sin\theta\ . \tag{8-218}$$

Setting $\ddot{\theta} = 0$ and solving for $\dot{\psi}^2$, we obtain

$$\dot{\psi}^2 = \frac{mga\tan\theta}{mRa + \dfrac{R}{a}I_a + I_t\sin\theta}$$

in agreement with our earlier result.