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DIFFERENTIAL EQUATIONS, DYNAMICAL SYSTEMS, AND AN INTRODUCTION TO CHAOS

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Table of Contents

1 First-Order Equations

1.1 The Simplest Example

The constant a in the equation $x' = ax$ can be considered a parameter. If a changes, the equation changes and so do the solutions. Can we describe qualitatively the way the solutions change? The sign of a is crucial here:

1. If $a > 0$, $\lim_{t\to\infty} ke^{at}$ equals ∞ when $k > 0$, and equals $-\infty$ when $k < 0$;
2. If $a = 0$, ke^{at} =constant;
3. If $a < 0$, $\lim_{t\to\infty} ke^{at} = 0$.

The qualitative behavior of solutions is vividly illustrated by sketching the graphs of solutions as in Figure 1.1. Note that the behavior of solutions is quite different when a is positive and negative. When $a > 0$, all nonzero solutions tend away from the equilibrium point at 0 as t increases, whereas when $a < 0$, solutions tend toward the equilibrium point. We say that the equilibrium point is a ***source*** when nearby solutions tend away from it. The equilibrium point is a ***sink*** when nearby solutions tend toward it.

We also describe solutions by drawing them on the ***phase line***. Because the solution $x(t)$ is a function of time, we may view $x(t)$ as a particle moving along the real line. At the equilibrium point, the particle remains at rest (indicated by a solid dot), while any other solution moves up or down the x-axis, as indicated by the arrows in Figure 1.1.

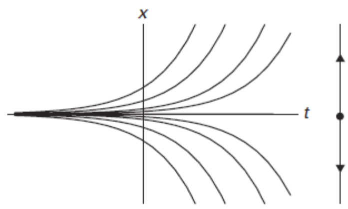

Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

The equation $x' = ax$ is ***stable*** in a certain sense if $a \neq 0$. More precisely, if a is replaced by another constant b whose sign is the same as a, then the qualitative behavior of the solutions does not change. But if $a = 0$, the slightest change in a leads to a radical change in the behavior of solutions. We therefore say that we have a ***bifurcation*** at $a = 0$ in the one-parameter family of equations $x' = ax$.

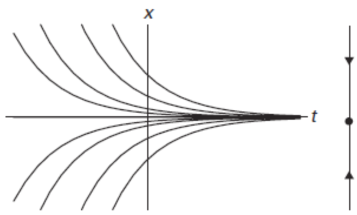

Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

1.3 Constant Harvesting and Bifurcations

Now let's modify the logistic model to take into account harvesting of the population. Suppose that the population obeys the logistic assumptions with the parameter $a = 1$, but is also harvested at the constant rate h. The differential equation becomes

$$x' = x(1-x) - h$$

where $h \geq 0$ is a new parameter.

Rather than solving this equation explicitly (which can be done — see Exercise 6 at the end of this chapter), we use the graphs of the functions

$$f_h(x) = x(1-x) - h$$

to “read off ” the qualitative behavior of solutions. In Figure 1.6 we display the graph of f_h in three different cases: $0 < h < 1/4$, $h = 1/4$, and $h > 1/4$. It is straightforward to check that f_h has two roots when $0 \le h < 1/4$, one root when $h = 1/4$, and no roots if $h > 1/4$, as illustrated in the graphs. As a consequence, the differential equation has two equilibrium points x_l and x_r with $0 \le x_l < x_r$ when $0 < h < 1/4$. It is also easy to check that $f_h'(x_l) > 0$, so that x_l is a **source**, and $f_h'(x_l) < 0$ so that x_r is a **sink**.

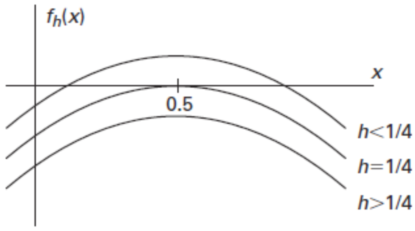

Figure 1.6 The graphs of the function $f_h(x) = x(1 - x) - h$.

As h passes through $h = 1/4$, we encounter another example of a **bifurcation**. The two equilibria x_l and x_r coalesce as h increases through 1/4 and then disappear when $h > 1/4$. Moreover, when $h > 1/4$, we have $f_h(x) < 0$ for all x. Mathematically, this means that all solutions of the differential equation decrease to $-\infty$ as time goes on.

We record this visually in the ***bifurcation diagram***. In this diagram we plot the parameter h horizontally. Over each h-value we plot the corresponding phase line. The curve in this picture represents the equilibrium points for each value of h. This gives another view of the sink and source merging into a single equilibrium point and then disappearing as h passes through 1/4 (see Figure 1.7).

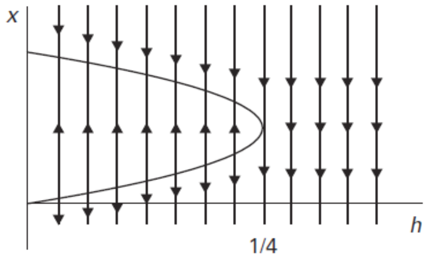

Figure 1.7 The bifurcation diagram for $f_h(x) = x(1 - x) - h$.

Ecologically, this bifurcation corresponds to a disaster for the species under study. For rates of harvesting 1/4 or lower, the population persists, provided the initial population is sufficiently large ($x(0) \ge x_l$). But a very small change in the rate of harvesting when $h = 1/4$ leads to a major change in the fate of the population: At any rate of harvesting $h > 1/4$, the species becomes extinct.

This phenomenon highlights the importance of detecting bifurcations in families of differential equations, a procedure that we will encounter many times in later chapters. We should also mention that, despite the simplicity of this population model, the prediction that small changes in harvesting rates can lead to disastrous changes in population has been observed many times in real situations on earth.

Example. As another example of a bifurcation, consider the family of differential equations

$$x' = g_a(x) = x^2 - ax = x(x - a)$$

which depends on a parameter a. The equilibrium points are given by $x = 0$ and $x = a$. We compute $g_a'(0) = -a$, so 0 is a **sink** if $a > 0$ and

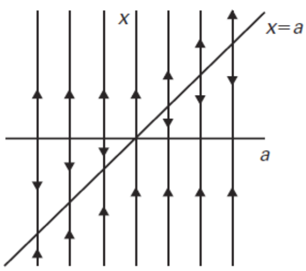

Figure 1.8 The bifurcation diagram for $x' = x^2 - ax$.

a **source** if $a < 0$. Similarly, $g'_a(a) = a$, so $x = a$ is a **sink** if $a < 0$ and a **source** if $a > 0$. We have a **bifurcation** at $a = 0$ since there is only one equilibrium point when $a = 0$. Moreover, the equilibrium point at 0 changes from a source to a sink as a increases through 0. Similarly, the equilibrium at $x = a$ changes from a sink to a source as a passes through 0. The bifurcation diagram for this family is depicted in Figure 1.8.

1.5 Computing the Poincare Map

Before computing the Poincare map for this equation, we introduce some important terminology. To emphasize the dependence of a solution on the initial value $x0$, we will denote the corresponding solution by $\varphi(t, x0)$. This function $\varphi : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ is called the ***flow*** associated to the differential equation. If we hold the variable $x0$ fixed, then the function

$$t \to \varphi(t, x0)$$

is just an alternative expression for the solution of the differential equation satisfying the initial condition $x0$. Sometimes we write this function as $\varphi_t(x_0)$.

Example. For our first example, $x' = ax$, the flow is given by

$$\varphi(t, x_0) = x_0 e^{at} .$$

For the logistic equation (without harvesting), the flow is

$$\varphi(t, x_0) \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}} .$$

Now we return to the logistic differential equation with periodic harvesting

$$x' = f(t, x) = ax(1 - x) - h(1 + \sin(2\pi t)) .$$

The solution satisfying the initial condition $x(0) = x0$ is given by $t \to \varphi(t, x0)$. While we do not have a formula for this expression, we do know that, by the **fundamental theorem of calculus**, this solution satisfies

$$\varphi(t, x_0) = x_0 + \int_0^t f(s, \varphi(s, x_0))ds$$

since

$$\frac{\partial \varphi}{\partial t}(t, x_0) = f(t, \varphi(t, x_0))$$

and $\varphi\ (0, x_0) = x_0$.

If we differentiate this solution with respect to $x0$, we obtain, using the chain rule:

$$\frac{\partial \varphi}{\partial x_0}(t, x_0) = 1 + \int_0^t \frac{\partial f}{\partial x_0}(s, \varphi(s, x_0)) \cdot \frac{\partial \varphi}{\partial x_0}(s, x_0) ds .$$

Now let

$$z(0) = \frac{\partial \varphi}{\partial x_0}(0, x_0) = 1 .$$

Differentiating z with respect to t, we find

$$z' = \frac{\partial f}{\partial x_0}(t, \varphi(t, x_0)) \cdot \frac{\partial \varphi}{\partial x_0}(t, x_0)$$
$$= \frac{\partial f}{\partial x_0}(t, \varphi(t, x_0)) \cdot z(t)$$

Again, we do not know $\varphi(t , x0)$ explicitly, but this equation does tell us that z(t) solves the differential equation

$$z' = \frac{\partial f}{\partial x_0}(t, \varphi(t, x_0))z(t)$$

with z(0)=1. Consequently, via **separation of variables**, we may compute that the solution of this equation is

$$z(t) = \exp \int_0^t \frac{\partial f}{\partial x_0}(s, \varphi(s, x_0))ds$$

and so we find

$$\frac{\partial \varphi}{\partial x_0}(1, x_0) = \exp \int_0^1 \frac{\partial f}{\partial x_0}(s, \varphi(s, x_0))ds .$$

Since $p(x0) = \varphi(1, x0)$, we have determined the derivative $p'\_(x0)$ of the Poincare map. Note that $p'(x0) > 0$. Therefore p is an increasing function.

Differentiating once more, we find

$$p''(x_0) = p'(x_0)\left(\int_0^1 \frac{\partial^2 f}{\partial x_0 \partial x_0}(s, \varphi(s, x_0)) \cdot \exp\left(\frac{\partial f}{\partial x_0}(u, \varphi(u, x_0))du \right) ds \right),$$

which looks pretty intimidating. However, since

$$f(t, x_0) = a x_0(1 - x_0) - h(1 + \sin(2\pi t)),$$

we have

$$\frac{\partial^2 f}{\partial x_0 \partial x_0} = -2a .$$

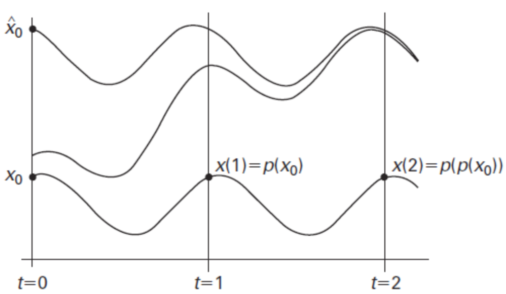

Figure 1.10 The Poincaré map for $x' = 5x(1 - x) - 0.8(1 + \sin(2\pi t))$.

Thus we know in addition that $p''(x0) < 0$. Consequently, the graph of the Poincare map is **concave down**. This implies that the graph of p can cross the diagonal line $y = x$ at most two times. That is, there can be at most two values of x for which $p(x) = x$. Therefore the Poincare map has at most two fixed points. These fixed points yield periodic solutions of

the original differential equation. These are solutions that satisfy $x(t+1) = x(t)$ for all t . Another way to say this is that the flow $\varphi(t, x0)$ is a periodic function in t with period 1 when the initial condition $x0$ is one of the fixed points. We saw these two solutions in the particular case when $h = 0.8$ in Figure 1.10. In Figure 1.11, we again see two solutions that appear to be periodic. Note that one of these solutions appears to attract all nearby solutions, while the other appears to repel them. We will return to these concepts often and make them more precise later in the book.

Recall that the differential equation also depends on the harvesting parameter h. For small values of h there will be two fixed points such as shown in Figure 1.11. Differentiating f with respect to h, we find

$$\frac{\partial f}{\partial h}(t, x_0) = 1(1 + \sin 2\pi t)$$

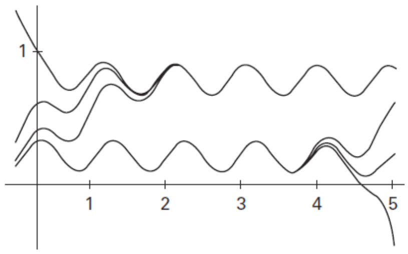

Figure 1.11 Several solutions of $x' = 5x(1 - x) - 0.8(1 + \sin(2\pi t))$.

Hence $\partial f / \partial h < 0$ (except when $t = 3/4$). This implies that the slopes of the slope field lines at each point $(t, x0)$ decrease as h increases. As a consequence, the values of the Poincare map also decrease as h increases. Hence there is a unique value $h*$ for which the Poincare map has exactly one fixed point. For $h > h*$, there are no fixed points for p and so $p(x0) < x0$ for all initial values. It then follows that the population again dies out.

2. Planar Linear Systems

2.4 Planar Linear Systems

Proposition. *The planar linear system* $X' = AX$ *has*

1. *A unique equilibrium point* (0, 0) *if* $\det A \neq 0$.
2. *A straight line of equilibrium points if* $\det A = 0$ *(and A is not the* 0 *matrix).*

2.5 Eigenvalues and Eigenvectors

Theorem. *Suppose that* $V0$ *is an eigenvector for the matrix A with associated eigenvalue λ. Then the function* $X(t) = e^{\lambda t} V_0$ *is a solution of the system* $X' = AX$.

2.6 Solving Linear Systems

Theorem. *Suppose A has a pair of real eigenvalues* $\lambda_1 \neq \lambda_2$ *and associated eigenvectors* $V1$ *and* $V2$. *Then the general solution of the linear system* $X' = AX$ *is given by*

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2$$

3. Phase Portraits for Planar Systems

3.1 Real Distinct Eigenvalues

Consider $X' = AX$ and suppose that A has two real eigenvalues $\lambda 1 < \lambda 2$. Assuming for the moment that $\lambda_i \neq 0$, there are three cases to consider:

1. $\lambda 1 < 0 < \lambda 2$;
2. $\lambda 1 < \lambda 2 < 0$;
3. $0 < \lambda 1 < \lambda 2$.

We give a specific example of each case; any system that falls into any one of these three categories may be handled in a similar manner, as we show later.

Example 1. (**Saddle**) First consider the simple system $X' = AX$ where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with$\lambda 1 < 0 < \lambda 2$. This can be solved immediately since the system decouples into two unrelated first-order equations:

$$x' = \lambda 1 x$$

$$y' = \lambda 2 y.$$

The characteristic equation is

$$(\lambda - \lambda 1)(\lambda - \lambda 2) = 0$$

so $\lambda 1$ and $\lambda 2$ are the eigenvalues. An eigenvector corresponding to $\lambda 1$ is (1, 0) and to $\lambda 2$ is (0, 1). Hence we find the general solution

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\lambda 1 < 0$, the straight-line solutions of the form $\alpha e^{\lambda_1 t}$ (1, 0) lie on the x-axis and tend to (0, 0) as $t \to$

∞. This axis is called the ***stable line***. Since $\lambda 2 > 0$, the solutions $\beta e^{\lambda_2 t}$ (0, 1) lie on the y-axis and tend away from (0, 0) as $t \to \infty$; this axis is the ***unstable line***. All other solutions (with $\alpha, \beta \neq 0$) tend to ∞ in the direction of the unstable line, as $t \to \infty$, since $X(t)$ comes closer and closer to (0, $\beta e^{\lambda_2 t}$) as t increases. In backward time, these solutions tend to ∞ in the direction of the stable line.

In Figure 3.1 we have plotted the ***phase portrait*** of this system. The phase portrait is a picture of a collection of representative solution curves of the system in R^2, which we call the ***phase plane***. The equilibrium point of a system of this type (eigenvalues satisfying $\lambda 1 < 0 < \lambda 2$) is called a ***saddle***.

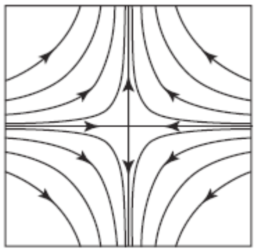

Figure 3.1 Saddle phase portrait for $x' = -x$, $y' = y$.

Example 2. (**Saddle**) We consider $X' = AX$ where

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

As we saw in Chapter 2, the eigenvalues of A are ±2. The eigenvector associated to $\lambda = 2$ is the vector (3, 1); the eigenvector associated to $\lambda = -2$ is (1,−1). Hence we have an unstable line that contains **straight-line solutions** of the form

$$X_1(t) = \alpha e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

each of which tends away from the origin as $t \to \infty$. The stable line contains the straight-line solutions

$$X_2(t) = \beta e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which tend toward the origin as $t \to \infty$. By the linearity principle, any other solution assumes the form

$$X(t) = \alpha e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for some α, β. Note that, if α _= 0, as $t \to \infty$, we have

$$X(t) \approx \alpha e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = X_1(t)$$

whereas, if β _= 0, as $t \to -\infty$,

$$X(t) \approx \beta e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = X_2(t).$$

Thus, as time increases, the typical solution approaches $X1(t)$ while, as time decreases, this solution tends toward $X2(t)$, just as in the previous case. Figure 3.2 displays this phase portrait.

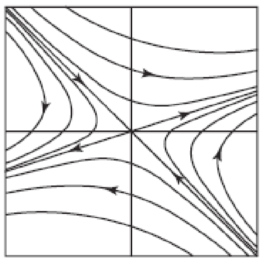

Figure 3.2 Saddle phase portrait for $x' = x + 3y$, $y' = x - y$.

In the general case where A has a positive and negative eigenvalue, we always find a similar stable and unstable line on which solutions tend toward or away from the origin. All other solutions approach the unstable line as t →∞, and tend toward the stable line as t →−∞.

Example 3. (**Sink**) Now consider the case $X' = AX$ where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

but $\lambda 1 < \lambda 2 < 0$. As above we find two straight-line solutions and then the general solution:

$$X(t) = \alpha e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Unlike the saddle case, now all solutions tend to (0, 0) as $t \to \infty$. The question is: How do they approach the origin? To answer this, we compute the slope dy/dx of a solution with $\beta \neq 0$. We write

$$x(t) = \alpha e^{\lambda_1 t}$$
$$y(t) = \beta e^{\lambda_2 t}$$

and compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t}.$$

Since $\lambda 2 - \lambda 1 > 0$, it follows that these slopes approach $\pm\infty$ (provided $\beta \neq 0$). Thus these solutions tend to the origin tangentially to the y-axis.

Since $\lambda 1 < \lambda 2 < 0$, we call $\lambda 1$ the ***stronger eigenvalue*** and $\lambda 2$ the ***weaker eigenvalue***. The reason for this in this particular case is that the x-coordinates of solutions tend to 0 much more quickly than the y-coordinates. This accounts for why solutions (except those on the line corresponding to the $\lambda 1$ eigenvector) tend to "hug" the straight-line solution corresponding to the weaker eigenvalue as they approach the origin. The phase portrait for this system is displayed in Figure 3.3a. In this case the equilibrium point is called a ***sink***.

More generally, if the system has eigenvalues $\lambda 1 < \lambda 2 < 0$ with eigenvectors ($u1$, $u2$) and ($v1$, $v2$), respectively, then the general solution is

$$\alpha e^{\lambda_1 t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The slope of this solution is given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2}{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1} \\ &= \left(\frac{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2}{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1} \right) \frac{e^{-\lambda_2 t}}{e^{-\lambda_2 t}} \\ &= \frac{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_2 + \lambda_2 \beta v_2}{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_1 + \lambda_2 \beta v_1} \end{aligned}$$

which tends to the slope $v2/v1$ of the $\lambda 2$ eigenvector, unless we have $\beta = 0$. If $\beta = 0$, our solution is the straight-line solution corresponding to the eigenvalue $\lambda 1$. Hence all solutions (except those on the straight line corresponding to the stronger eigenvalue) tend to the origin tangentially to the straight-line solution

corresponding to the weaker eigenvalue in this case as well.

Example 4. (**Source**) When the matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

satisfies $0 < \lambda 2 < \lambda 1$, our vector field may be regarded as the negative of the previous example. The general solution and phase portrait remain the same, except that all solutions now tend away from (0, 0) along the same paths. See Figure 3.3b. □

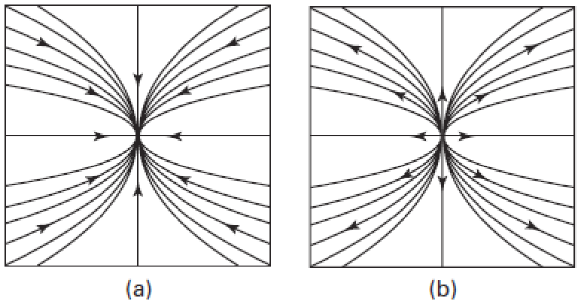

Figure 3.3 Phase portraits for (a) a sink and (b) a source.

3.2 Complex Eigenvalues

Example. (**Center**) Consider $X' = AX$ with

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

and $\beta \neq 0$. The characteristic polynomial is $\lambda^2 + \beta^2 = 0$, so the Eigenvalues are now the imaginary numbers $\pm i\beta$. Without worrying about the resulting complex vectors, we react just as before to find the eigenvector corresponding to $\lambda = i\beta$. We therefore solve

$$\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $i\beta x = \beta y$, since the second equation is redundant. Thus we find a complex eigenvector $(1, i)$, and so the function

$$X(t) = e^{i\beta t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

is a complex solution of $X' = AX$.

Now in general it is not polite to hand someone a complex solution to a real system of differential equations, but we can remedy this with the help of Euler's formula

$$e^{i\beta t} = \cos\beta t + i\sin\beta t\,.$$

Using this fact, we rewrite the solution as

$$X(t) = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ i(\cos\beta t + i\sin\beta t) \end{pmatrix} = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ -\sin\beta t + i\cos\beta t \end{pmatrix}.$$

Better yet, by breaking $X(t)$ into its real and imaginary parts, we have

$$X(t) = X_{\mathrm{Re}}(t) + iX_{\mathrm{Im}}(t)$$

where

$$X_{\mathrm{Re}}(t) = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, \quad X_{\mathrm{Im}}(t) = \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$$

But now we see that both $X_{\mathrm{Re}}(t)$ and $X_{\mathrm{Im}}(t)$ are (real!) solutions of the original system. To see this, we simply check

$$\begin{aligned} X'_{\mathrm{Re}}(t) + iX'_{\mathrm{Im}}(t) &= X'(t) \\ &= AX(t) \\ &= A(X_{\mathrm{Re}}(t) + iX_{\mathrm{Im}}(t)) \\ &= AX_{\mathrm{Re}}(t) + iAX_{\mathrm{Im}}(t). \end{aligned}$$

Equating the real and imaginary parts of this equation yields $X'_{\mathrm{Re}} = AX_{\mathrm{Re}}$ and $X'_{\mathrm{Im}} = AX_{\mathrm{Im}}$ which shows that both are indeed solutions. Moreover, since

$$X_{\mathrm{Re}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_{\mathrm{Im}}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the linear combination of these solutions

$$X(t) = c_1 X_{\mathrm{Re}}(t) + c_2 X_{\mathrm{Im}}(t)$$

where $c1$ and $c2$ are arbitrary constants provides a solution to any initial value problem.

We claim that this is the general solution of this equation. To prove this, we need to show that these are the only solutions of this equation. Suppose that this is not the case. Let

$$Y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

be another solution. Consider the complex function $f(t) = (u(t) + iv(t))e^{i\beta t}$. Differentiating this expression and using the fact that $Y(t)$ is a solution of the equation yields $f'(t) = 0$. Hence $u(t) + iv(t)$ is a complex constant times $e^{-i\beta t}$. From this it follows directly that $Y(t)$ is a linear combination of $X_{\mathrm{Re}}(t)$ and $X_{\mathrm{Im}}(t)$.

Note that each of these solutions is a periodic function with period $2\pi/\beta$. Indeed, the phase portrait shows that all solutions lie on circles centered at the origin. These circles are traversed in the clockwise direction if $\beta > 0$, counterclockwise if $\beta < 0$. See Figure 3.4. This type of system is called a ***center***.

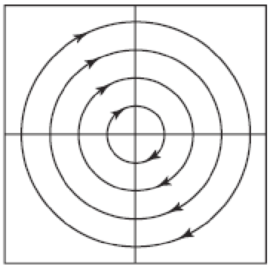

Figure 3.4 Phase portrait for a center.

Example. (**Spiral Sink**, **Spiral Source**) More generally, consider $X' = AX$ where

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

and $\alpha, \beta \neq 0$. The characteristic equation is now $\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = 0$, so the eigenvalues are $\lambda = \alpha \pm i\beta$. An eigenvector associated to $\alpha + i\beta$ is determined by the equation

$$(\alpha - (\alpha + i\beta))x + \beta y = 0 .$$

Thus $(1, i)$ is again an eigenvector. Hence we have complex solutions of the form

$$\begin{aligned} X(t) &= e^{(\alpha+i\beta)t}\begin{pmatrix}1\\ i\end{pmatrix} \\ &= e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} + ie^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix} \\ &= X_{\mathrm{Re}}(t) + iX_{\mathrm{Im}}(t). \end{aligned}$$

As above, both $X_{\mathrm{Re}}(t)$ and $X_{\mathrm{Im}}(t)$ yield real solutions of the system whose initial conditions are linearly independent. Thus we find the general solution

$$X(t) = c_1 e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} + c_2 e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}.$$

Without the term $e^{\alpha t}$, these solutions would wind periodically around circles centered at the origin. The $e^{\alpha t}$ term converts solutions into spirals that either spiral into the origin (when $\alpha < 0$) or away from the origin ($\alpha > 0$). In these cases the equilibrium point is called a ***spiral sink*** or ***spiral source***, respectively. See Figure 3.5.

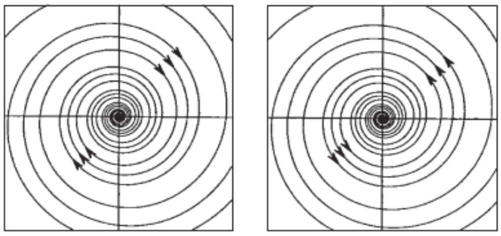

Figure 3.5 Phase portraits for a spiral sink and a spiral source.

3.3 Repeated Eigenvalues

The only remaining cases occur when A has repeated real eigenvalues. One simple case occurs when A is a diagonal matrix of the form

$$A = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}.$$

The eigenvalues of A are both equal to λ. In this case every nonzero vector is an eigenvector since

$AV = \lambda V$

for any $V \in R^2$.Hence solutions are of the form

$$X(t) = \alpha e^{\lambda t} V.$$

Each such solution lies on a straight line through (0, 0) and either tends to (0, 0) (if $\lambda < 0$) or away from (0, 0) (if $\lambda > 0$). So this is an easy case.

A more interesting case occurs when

$$A = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}.$$

Again both eigenvalues are equal to λ, but now there is only one linearly independent eigenvector given by (1, 0). Hence we have one straight-line solution

$$X_1(t) = \alpha e^{\lambda t}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

To find other solutions, note that the system can be written

$$x' = \lambda x + y$$
$$y' = \lambda y$$

Thus, if $y \neq 0$, we must have

$$y(t) = \beta e^{\lambda t}.$$

Therefore the differential equation for $x(t)$ reads

$$x' = \lambda x + \beta e^{\lambda t}.$$

This is a nonautonomous, first-order differential equation for $x(t)$. One might first expect solutions of the form $e^{\lambda t}$, but the nonautonomous term is also in this form. As you perhaps saw in calculus, the best option is to guess a solution of the form

$$x(t) = \alpha e^{\lambda t} + \mu t e^{\lambda t}$$

for some constants α and μ. This technique is often called "***the method of undetermined coefficients***." Inserting this guess into the differential equation shows that $\mu = \beta$ while α is arbitrary. Hence the solution of the system may be written

$$\varepsilon e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t}\begin{pmatrix} t \\ 1 \end{pmatrix}.$$

This is in fact the general solution (see Exercise 12).

Note that, if $\lambda < 0$, each term in this solution tends to 0 as $t \to \infty$. This is clear for the $\alpha e^{\lambda t}$ and $\beta e^{\lambda t}$ terms. For the term $\beta t e^{\lambda t}$ this is an immediate consequence of l'Hopital's rule. Hence all solutions tend to (0, 0) as $t \to \infty$. When $\lambda > 0$, all solutions tend away from (0, 0). See Figure 3.6. In fact, solutions tend toward or away from the origin in a direction tangent to the eigenvector (1, 0) (see Exercise 7).

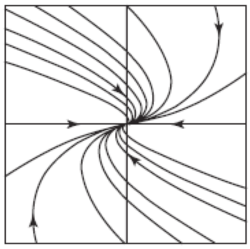

Figure 3.6 Phase portrait for a system with repeated negative eigenvalues.

3.4 Changing Coordinates

Example. Suppose

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$, which yields eigenvalues $\lambda = -1$ and $\lambda = -2$. And we have an eigenvectors (1, 1)for $\lambda = -1$ and (0, 1) for $\lambda = -2$.

We therefore have a pair of straight-line solutions, each tending to the origin as $t \to \infty$. The straight-line solution corresponding to the ***weaker*** eigenvalue lies along the line $y = x$; the straight-line solution corresponding to the ***stronger*** eigenvalue lies on the y-axis. All other solutions tend to the origin tangentially to the line $y = x$.

To put this system in canonical form, we choose T to be the matrix whose columns are these eigenvectors:

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that

$$T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Finally, we compute

$$T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix},$$

so $T^{-1}AT$ is in ***canonical form***. The general solution of the system $Y' = (T^{-1}AT)Y$ is

$$Y(t) = \alpha e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the general solution of $X' = AX$ is

$$\begin{aligned} TY(t) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left(\alpha e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \alpha e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}.$$

Thus the linear map T converts the phase portrait for the system

$$Y' = \begin{pmatrix} -1 & 0 \\ - & -2 \end{pmatrix}Y$$

to that of $X' = AX$ as shown in Figure 3.7.

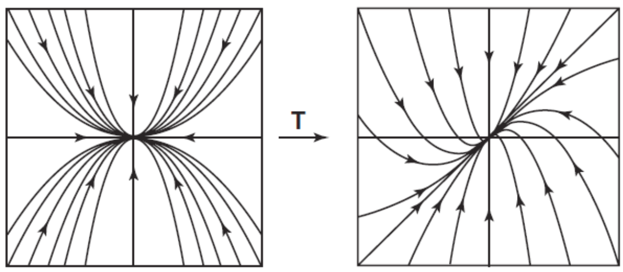

Figure 3.7 The change of variables T in the case of a (real) sink.

Example. (Another Harmonic Oscillator) Consider the second-order equation

$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}X = AX\ .$$

The characteristic equation is

$$\lambda^2 + 4 = 0$$

so that the eigenvalues are $\pm 2i$. A complex eigenvector associated to $\lambda = 2i$ is a solution of the system

$$\begin{aligned} -2ix + y &= 0 \\ -4x - 2iy &= 0 \end{aligned}$$

One such solution is the vector $(1, 2i)$. So we have a complex solution of the form

$$e^{2it}\begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Breaking this solution into its real and imaginary parts, we find the general solution

$$X(t) = c_1\begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

Thus the position of this oscillator is given by

$$x(t) = c_1 \cos 2t + c_2 \sin 2t \ ,$$

which is a periodic function of period π.

Now, let T be the matrix whose columns are the real and imaginary parts of the eigenvector $(1, 2i)$. That is

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then, we compute easily that

$$T^{-1}AT = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

which is in canonical form. The phase portraits of these systems are shown in Figure 3.8. Note that T maps the ***circular*** solutions of the system $Y' = (T^{-1}AT)Y$ to ***elliptic*** solutions of $X' = AX$.

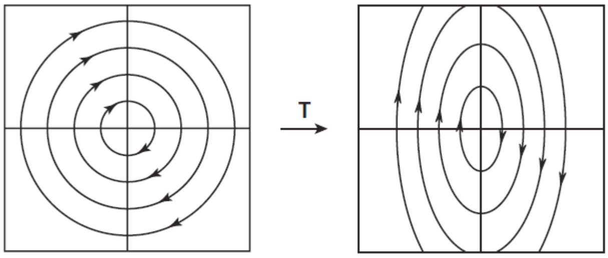

Figure 3.8 The change of variables T in the case of a center.

4. Classification of Planar Systems

4.1 The Trace-Determinant Plane

For a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we know that the eigenvalues are the roots of the characteristic equation, which can be written

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 .$$

The constant term in this equation is det A. The coefficient of λ also has a name: The quantity $a + d$ is called the ***trace*** of A and is denoted by tr A. Thus the eigenvalues satisfy

$$\lambda^2 - (trA)\lambda + \det A = 0$$

and are given by

$$\lambda_{\pm} = \frac{1}{2}\left(trA \pm \sqrt{(trA)^2 - 4\det A} \right).$$

Note that $\lambda_+ + \lambda_- = trA$ and $\lambda_+\lambda_- = \det A$, so the trace is the sum of the eigenvalues of A while the determinant is the product of the eigenvalues of A. We will also write T = tr A and D = det A. Knowing T and D tells us the eigenvalues of A and therefore virtually everything about the geometry of solutions of $X' = AX$. For example, the values of T and D tell us whether solutions spiral into or away from the origin, whether we have a center, and so forth.

We may display this classification visually by painting a picture in the ***trace-determinant plane***. In this picture a matrix with trace T and determinant D corresponds to the point with coordinates (T,D). The location of this point in the TD–plane then determines the geometry of the phase portrait as above. For example, the sign of $T^2 - 4D$ tells us that the eigenvalues are:

1. Complex with nonzero imaginary part if $T^2 - 4D < 0$;
2. Real and distinct if $T^2 - 4D > 0$;
3. Real and repeated if $T^2 - 4D = 0$.

Thus the location of (T,D) relative to the parabola $T^2 - 4D = 0$ in the TD–plane tells us all we need to know about the eigenvalues of A from an algebraic point of view.

In terms of phase portraits, however, we can say more. If $T^2 - 4D < 0$, then the real part of the eigenvalues is $T/2$, and so we have a

1. **Spiral sink** if $T < 0$;
2. **Spiral source** if $T > 0$;
3. **Center** if $T = 0$.

If $T^2 - 4D > 0$ we have a similar breakdown into cases. In this region, both eigenvalues are real. If $D < 0$, then we have a saddle. This follows since D is the product of the eigenvalues, one of which must be positive, the other negative. Equivalently, if $D < 0$, we compute

$$T^2 < T^2 - 4D$$

so that

$$\pm T < \sqrt{T^2 - 4D} \ .$$

Thus we have

$$T + \sqrt{T^2 - 4D} > 0$$
$$T - \sqrt{T^2 - 4D} < 0$$

so the eigenvalues are real and have different signs. If $D > 0$ and $T < 0$ then both

$$T \pm \sqrt{T^2 - 4D} < 0 \ ,$$

so we have a (real) ***sink***. Similarly, $T > 0$ and$D > 0$ leads to a (real) ***source***.
When $D = 0$ and $T \neq 0$, we have one zero eigenvalue, while both eigenvalues vanish if $D = T = 0$.

Plotting all of this verbal information in the *TD*–plane gives us a visual summary of all of the different types of linear systems. The equations above partition the *TD*–plane into various regions in which systems of a particular type reside. See Figure 4.1. This yields a geometric classification of 2×2 linear systems.

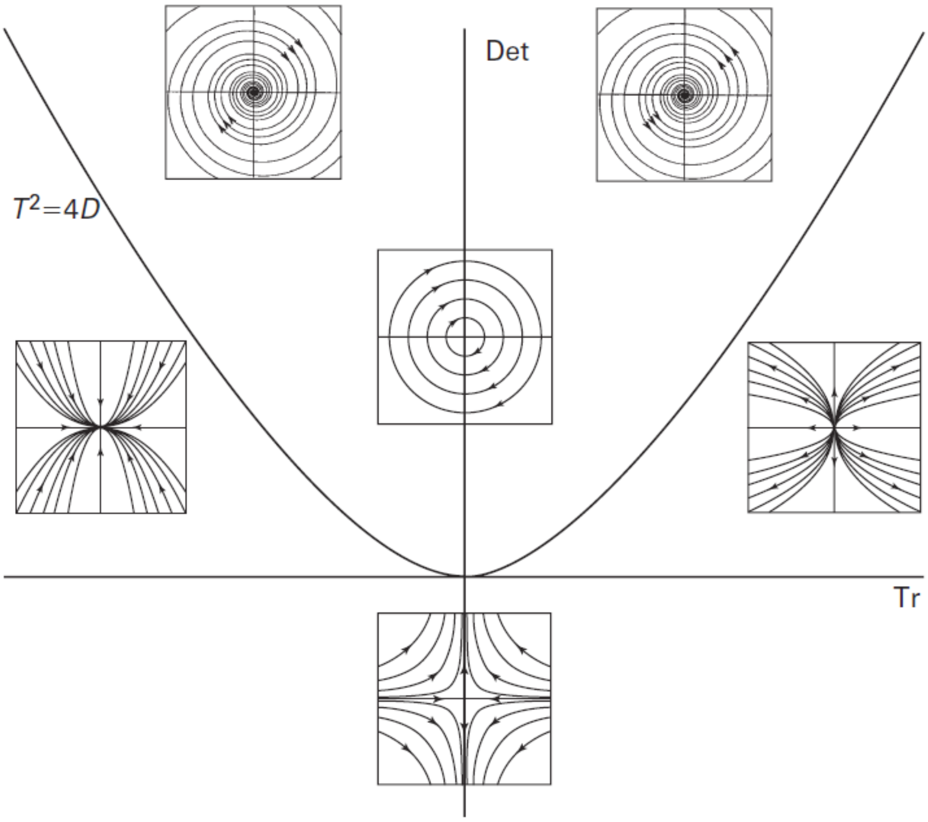

Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

A couple of remarks are in order. First, the trace-determinant plane is a ***two-dimensional representation*** of what is really a ***four-dimensional space***, since 2 × 2 matrices are determined by four

parameters, the entries of the matrix. Thus there are infinitely many different matrices corresponding to each point in the *TD*–plane. While all of these matrices share the same eigenvalue configuration, there may be subtle differences in the phase portraits, such as the direction of rotation for centers and spiral sinks and sources, or the possibility of one or two independent eigenvectors in the repeated eigenvalue case. We also think of the trace-determinant plane as the analog of the ***bifurcation diagram*** for planar linear systems. A one-parameter family of linear systems corresponds to a curve in the *TD*–plane. When this curve crosses the *T*-axis, the positive *D*-axis, or the parabola $T^2 - 4D = 0$, the phase portrait of the linear system undergoes a ***bifurcation***: A major change occurs in the geometry of the phase portrait.

Finally, note that we may obtain quite a bit of information about the system from *D* and *T* without ever computing the eigenvalues. For example, if $D < 0$, we know that we have a ***saddle*** at the origin. Similarly, if both *D* and *T* are positive, then we have a ***source*** at the origin.

4.2 Dynamical Classification

To emphasize the dependence of solutions on both time and the initial conditions *X*0, we let $\phi_t(X_0)$ denote the solution that satisfies the initial condition *X*0. That is, $\phi_0(X_0) = X_0$. The function $\phi(t, X_0) = \phi_t(X_0)$ is called the ***flow*** of the differential equation, whereas ϕ_t is called the ***time t map*** of the flow.

For example, let

$$X' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} X.$$

Then the ***time t map*** is given by

$$\phi_t(x_0, y_0) = \left(x_0 e^{2t}, y_0 e^{3t}\right).$$

Thus the flow is a function that depends on both time and the initial values.

We will consider two systems to be dynamically equivalent if there is a function *h* that takes one flow to the other. We require that this function be a ***homeomorphism***, that is, *h* is a **one-to-one**, **onto**, and **continuous** function whose **inverse** is also continuous.

Definition

Suppose $X' = AX$ and $X' = BX$ have flows ϕ^A and ϕ^B. These two systems are (topologically) *conjugate* if there exists a homeomorphism $h: R^2 \to R^2$ that satisfies

$$\phi^B(t, h(X_0)) = h(\phi^A(t, X_0)).$$

The homeomorphism *h* is called a *conjugacy*. Thus a conjugacy takes the solution curves of $X' = AX$ to those of $X' = BX$.

Example. For the one-dimensional linear differential equations

$$x' = \lambda_1 x \quad \text{and} \quad x' = \lambda_2 x$$

we have the flows

$$\phi^j(t, x_0) = x_0 e^{\lambda_j t}$$

for $j = 1, 2$. Suppose that $\lambda 1$ and $\lambda 2$ are nonzero and have the same sign. Then let

$$h(x) = \begin{cases} x^{\lambda_2/\lambda_1} \_if\_x \geq 0 \\ -|x|^{\lambda_2/\lambda_1} \_if\_x < 0 \end{cases}$$

where we recall that

$$x^{\lambda_2/\lambda_1} = \exp\left(\frac{\lambda_2}{\lambda_1}\log(x)\right).$$

Note that h is a ***homeomorphism*** of the real line. We claim that h is a ***conjugacy*** between $x' = \lambda_1 x$ and $x' = \lambda_2 x$. To see this, we check that when $x0 > 0$

$$\begin{aligned} h(\phi^1(t, x_0)) &= \left(x_0 e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} \\ &= x_0^{\lambda_2/\lambda_1} e^{\lambda_2 t} \\ &= \phi^2(t, h(x_0)) \end{aligned}$$

as required. A similar computation works when $x0 < 0$. □

Definition

A matrix A is ***hyperbolic*** if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is ***hyperbolic***.

Theorem. *Suppose that the* 2×2 *matrices A*1 *and A*2 *are* **hyperbolic**. *Then the linear systems* $X' = AX$ *are* ***conjugate*** *if and only if each matrix has the same number of eigenvalues with negative real part.* □

Thus two hyperbolic matrices yield conjugate linear systems if both sets of eigenvalues fall into the same category below:

1. One eigenvalue is positive and the other is negative;
2. Both eigenvalues have negative real parts;
3. Both eigenvalues have positive real parts.

Before proving this, note that this theorem implies that a system with a ***spiral sink*** *is* ***conjugate*** *to a system with a (real)* ***sink***. *Of course! Even though their phase portraits look very different, it is nevertheless the case that all solutions of both systems share the same fate: They tend to the origin as t* →∞.

5. Higher Dimensional Linear Algebra

5.2 Eigenvalues and Eigenvectors

Proposition. *Suppose* $\lambda_1, \ldots, \lambda_l$ *are real and* ***distinct*** *eigenvalues for A with associated eigenvectors* $V_1, \ldots, V_l$. *Then the* V_j *are* ***linearly independent***.

Corollary. *Suppose A is an n × n matrix with real,* ***distinct*** *eigenvalues. Then there is a matrix T such that*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

where all of the entries off the diagonal are 0.

Example. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Expanding det $(A - \lambda I)$ along the first column, we find that the characteristic equation of A is

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)\det\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \\ &= (1-\lambda)((3-\lambda)(-2-\lambda)+4) \\ &= (1-\lambda)(\lambda-2)(\lambda+1), \end{aligned}$$

so the eigenvalues are 2, 1, and −1. The eigenvector corresponding to $\lambda = 2$ is given by solving the equations $(A - 2I)X = 0$, which yields

$$\begin{cases} -x+2y-z=0 \\ y-2z=0 \\ 2y-4z=0 \end{cases}.$$

These equations reduce to

$$\begin{cases} x-3z=0 \\ y-2z=0 \end{cases}$$

Hence $V1 = (3, 2, 1)$ is an eigenvector associated to $\lambda = 2$. In similar fashion we find that (1, 0, 0) is an eigenvector associated to $\lambda = 1$, while (0, 1, 2) is an eigenvector associated to $\lambda = -1$. Then we set

$$T = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

A simple calculation shows that

$$AT = T\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since det $T = -3$, T is invertible and we have

$$T^{-1}AT = T\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

5.5 Repeated Eigenvalues

Proposition. *Let A be an n ×n matrix. Then there is a* ***change of coordinates*** *T for which*

$$T^{-1}AT = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}$$

where each of the Bj's is a square matrix (and all other entries are zero) of one of the following forms:

$$\text{(i)} \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} C_2 & I_2 & & & \\ & C_2 & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C_2 \end{pmatrix}$$

where

$$C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and where $\alpha, \beta, \lambda \in R$ *with* $\beta \neq 0$. *The special cases where Bj = (λ) or*

$$B_j = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

are, of course, allowed.

Proposition. *Suppose A is a* 3 × 3 *matrix for which λ is the* **only** *eigenvalue. Then we may find a change of coordinates T such that* $T^{-1}AT$ *assumes one of the following three forms:*

$$\text{(i)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Example 1.

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = (2 - \lambda)^3$$

V1=(1,-1,0), V3=(1,0,0), V2=(0,0,-1).

$$T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 2.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^3$$

V1=(-1,-1,-1), V2=(1,0,0), V3=(0,0,1)

$$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 3.

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$(\lambda^2 + 1)^2 = 0$$

V1=(1,1,-*i*,0,0), V2=(0.0.1.-*i*,1)

$$W_1 = (V + \overline{V_1})/2 = \operatorname{Re} V_1$$
$$W_2 = -i(V_1 - \overline{V_1})/2 = \operatorname{Im} V_1$$
$$W_3 = (V_2 + \overline{V_2})/2 = \operatorname{Re} V_2$$
$$W_4 = -i(V_2 - \overline{V_2})/2 = \operatorname{Im} V_2$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Example 4.

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

$$(2-\lambda)^2((2-\lambda)^2+1)=0$$

V=(0,-*i*,0,1), W1=(0,0,0,), W2=(0,-1,0,0), W3=(1,0,0,0), W4=(0,0,1,0)

$$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

5.6 Genericity

Recall that a set $U \subset R^n$ is ***open*** if whenever $X \in U$ there is an open ball about *X* contained in *U*; that is, for some *a* > 0 (depending on *X*) the open ball about *X* of radius *a*,

$$\{Y \in R^n : |Y - X| < a\},$$

is contained in *U*. Using geometrical language we say that if *X* belongs to an open set *U*, any point sufficiently near to *X* also belongs to *U*.

Another kind of subset of R^n is a ***dense*** set: $U \subset R^n$ is dense if there are points in *U* arbitrarily close to each point in R^n. More precisely, if $X \in R^n$, then for every $\varepsilon > 0$ there exists some $Y \subset U$ with $|X - Y| < \varepsilon$. Equivalently, *U* is dense in R^n if $V \cap U$ is nonempty for every nonempty open set $V \subset R^n$.

Theorem. *The set M of matrices in $L(R^n)$ that have n distinct eigenvalues is* ***open*** *and* ***dense*** *in $L(R^n)$*.

A property *P* of matrices is a ***generic property*** if the set of matrices having property *P* contains an **open** and **dense** set in $L(R^n)$. Thus a property is **generic** if it is shared by some open and dense set of matrices (and perhaps other matrices as well). Intuitively speaking, a generic property is one that “almost all” matrices have. Thus, having all distinct eigenvalues is a generic property of $n \times n$ matrices.

6 Higher Dimensional Linear Systems

6.1 Distinct Eigenvalues

Example. Consider

$$X' = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X .$$

In Section 5.2 in Chapter 5, we showed that this matrix has Eigenvalues 2, 1, and −1 with associated eigenvectors (3, 2, 1), (1, 0, 0), and (0, 1, 2), respectively. Therefore the matrix

$$T = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

converts $X' = AX$ to

$$Y' = (T^{-1}AT)Y = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Y ,$$

which we can solve immediately. Multiplying the solution by T then yields the general solution

$$X(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

of $X' = AX$. The straight line through the origin and (0, 1, 2) is the stable line, while the plane spanned by (3, 2, 1) and (1, 0, 0) is the unstable plane. A collection of solutions of this system as well as the

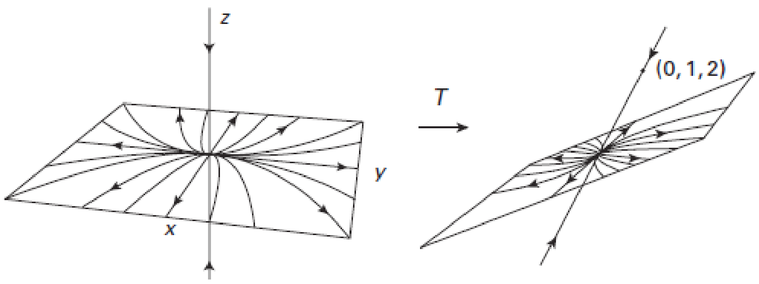

Figure 6.1 The stable and unstable subspaces of a saddle in dimension 3. On the left, the system is in canonical form.

system $Y' = (T^{-1}AT)Y$ is displayed in Figure 6.1.

Example. If the 3 × 3 matrix A has three real, distinct eigenvalues that are negative, then we may find a change of coordinates so that the system assumes the form

$$Y' = (T^{-1}AT)Y = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} Y$$

where $\lambda 3 < \lambda 2 < \lambda 1 < 0$. All solutions therefore tend to the origin and so we have a higher dimensional *sink*. See Figure 6.2. For an initial condition ($x0$, $y0$, $z0$) with all three coordinates nonzero, the corresponding solution tends to the origin tangentially to the x-axis (see Exercise 2 at the end of the chapter).

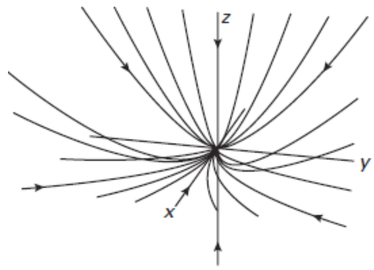

Figure 6.2 A sink in three dimensions.

Example. Consider the system

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} X$$

whose matrix is already in canonical form. The eigenvalues are $\pm i, -1$. The solution satisfying the initial condition ($x0$, $y0$, $z0$) is given by

$$Y(t) = x_0 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so this is the general solution. The phase portrait for this system is displayed in Figure 6.3. The stable line lies along the z-axis, whereas all solutions in the xy–plane travel around circles centered at the origin. In fact, each solution that does not lie on the stable line actually lies on a cylinder in R^3 given by $x^2 + y^2 = \text{constant}$. These solutions spiral toward the circular solution of radius $\sqrt{x_0^2 + y_0^2}$ in the xy–plane if $z_0 \neq 0$.

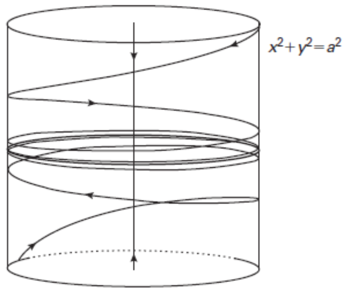

Figure 6.3 The phase portrait for a spiral center.

6.3 Repeated Eigenvalues

As we saw in the previous chapter, the solution of systems with repeated real eigenvalues reduces to solving systems whose matrices contain blocks of the form

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

Example. Let

$$X' = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} X \,.$$

The only eigenvalue for this system is λ, and its only eigenvector is (1, 0, 0). We may solve this system as we did in Chapter 3, by first noting that $x_3' = \lambda x_3$, so we must have

$$x_3(t) = c_3 e^{\lambda t} \,.$$

Now we must have

$$x_2' = \lambda x_2 + c_3 e^{\lambda t} \,.$$

As in Chapter 3, we guess a solution of the form

$$x_2(t) = c_2 e^{\lambda t} + \alpha t e^{\lambda t} \,.$$

Substituting this guess into the differential equation for x_2', we determine that $\alpha = c_3$ and find

$$x_2(t) = c_2 e^{\lambda t} + c_3 t e^{\lambda t} \,.$$

Finally, the equation

$$x_1' = \lambda x_1 + c_2 e^{\lambda t} + c_3 t e^{\lambda t}$$

suggests the guess

$$x_1(t) = c_1 e^{\lambda t} + \alpha t e^{\lambda t} + \beta t^2 e^{\lambda t} \,.$$

Solving as above, we find

$$x_1(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} + c_3 \frac{t^2}{2} e^{\lambda t} \,.$$

Altogether, we find

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix},$$

which is the general solution. Despite the presence of the polynomial terms in this solution, when $\lambda < 0$, the exponential term dominates and all solutions do tend to zero. Some representative solutions when $\lambda < 0$ are shown in Figure 6.9. Note that there is only one straight-line solution for this system; this solution lies on the x-axis. Also, the xy–plane is invariant and solutions there behave exactly as in the planar repeated eigenvalue case.

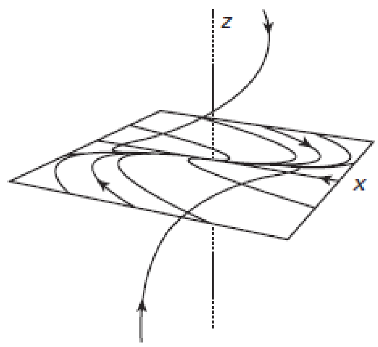

Figure 6.9 The phase portrait for repeated real eigenvalues.

6.4 The Exponential of a Matrix

Definition

Let A be an $n \times n$ matrix. We define the *exponential* of A to be the matrix given by

$$\exp(A)=\sum_{k=0}^{\infty}\frac{A^k}{k!}$$

Example. Let $A=\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ then $A^k=\begin{pmatrix}\lambda^k & 0\\ 0 & \mu^k\end{pmatrix}$, $\exp(A)=\begin{pmatrix}e^\lambda & 0\\ 0 & e^\mu\end{pmatrix}$.

Example. Let $A=\begin{pmatrix}0 & \beta\\ -\beta & 0\end{pmatrix}$ then $\exp(A)=\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix}$.

Example. Let $A=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ then $(tA)^k=\begin{pmatrix}(t\lambda)^k & kt^k\lambda^{k-1}\\ 0 & (t\lambda)^k\end{pmatrix}$, $\exp(tA)=\begin{pmatrix}e^{t\lambda} & te^{t\lambda}\\ 0 & e^{t\lambda}\end{pmatrix}$ ($\lambda\neq 0$)

Proposition. *Let A, B, and T be n × n matrices. Then:*

1. If $B=T^{-1}AT$, *then* $\exp(B)=T^{-1}\exp(A)T$.

2. If $AB=BA$, *then* $\exp(A+B)=\exp(A)\exp(B)$

3. $\exp(-A)=(\exp(A))^{-1}$

Lemma. *For any n × n matrices A and B, we have:*

$$\sum_{n=0}^{\infty}\left(\sum_{j+k=n}\frac{A^j}{j!}\frac{B^k}{k!}\right)=\left(\sum_{j=0}^{\infty}\frac{A^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right).$$

Proposition. *If* $V\in R^n$ *is an eigenvector of A associated to the eigenvalue λ, then V is also an eigenvector of* $\exp(A)$ *associated to* e^λ.

Proposition.

$$\frac{d}{dt}\exp(tA)=A\exp(tA)=\exp(tA)A.$$

In other words, the derivative of the matrix-valued function $t\to\exp(tA)$ is another matrix-valued function $A\exp(tA)$.

Theorem. *Let A be an n × n matrix. Then the solution of the initial value problem* $X'=AX$ *with* $X(0)=X_0$ *is* $X(t)=\exp(tA)X_0$. *Moreover, this is the only such solution.*

Example. Consider the system

$$X'=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}X.$$

By the theorem, the general solution is

$$X(t)=\exp(tA)X_0=\exp\begin{pmatrix}t\lambda & 1\\ 0 & \lambda\end{pmatrix}X_0.$$

But this is precisely the matrix whose exponential we computed earlier. We find

$$X(t)=\begin{pmatrix}e^{t\lambda} & te^{t\lambda}\\ 0 & e^{t\lambda}\end{pmatrix}X_0.$$

7 Nonlinear Systems

7.1 Dynamical Systems

We begin by collecting some of the terminology regarding dynamical systems that we have introduced at various points in the preceding chapters. A *dynamical system* is a way of describing the passage in time of all points of a given space *S*. The space *S* could be thought of, for example, as the space of states of some physical system. Mathematically, *S* might be a Euclidean space or an open subset of Euclidean space or some other space such as a surface in R3. When we consider dynamical systems that arise in mechanics, the space *S* will be the set of possible positions and velocities of the system. For the sake of simplicity, we will assume throughout that the space *S* is Euclidean space R^n, although in certain cases the important dynamical behavior will be confined to a particular subset of R^n.

Given an initial position $X \in R^n$, a dynamical system on R^n tells us where *X* is located 1 unit of time later, 2 units of time later, and so on. We denote these new positions of *X* by *X*1, *X*2, and so forth. At time zero, *X* is located at position *X*0. One unit before time zero, *X* was at *X*−1. In general, the "trajectory" of *X* is given by *Xt* . If we measure the positions *Xt* using only integer time values, we have an example of a *discrete* dynamical system, which we shall study in Chapter 15. If time is measured continuously with $t \in R$ t, we have a *continuous* dynamical system. If the system depends on time in a continuously differentiable manner, we have a *smooth* dynamical system. These are the three principal types of dynamical systems that arise in the study of systems of differential equations, and they will form the backbone of Chapters 8 through 14.

The function that takes *t* to *Xt* yields either a sequence of points or a curve in R^n that represents the life history of *X* as time runs from −∞ to ∞. Different branches of dynamical systems make different assumptions about how the function *Xt* depends on *t* . For example, ergodic theory deals with such functions under the assumption that they preserve a measure on R*n*. Topological dynamics deals with such functions under the assumption that *Xt* varies only continuously. In the case of differential equations, we will usually assume that the function *Xt* is continuously differentiable. The map *φt* : R*n* → R*n* that takes *X* into *Xt* is defined for each *t* and, from our interpretation of *Xt* as a state moving in time, it is reasonable to expect *φt* to have *φ*−*t* as its inverse. Also, *φ*0 should be the identity function $\phi_0(X) = X$ and $\phi_t(\phi_s(X)) = \phi_{t+s}(X)$ is also a natural condition. We formalize all of this in the following definition:

Definition

A *smooth dynamical system* on R^n is a continuously differentiable function $\phi : R \times R^n \to R^n$ where $\phi(t, X) = \phi_t(X)$ satisfies

1. $\phi_0 : R^n \to R^n$ is the identity function: $\phi_0(X_0) = X_0$;
2. The composition $\phi_t \circ \phi_s = \phi_{t+s}$ for each $t, s \in R$.

Example. For the first-order differential equation $x' = ax$, the function $\phi_t(x_0) = x_0 \exp(at)$ gives the solutions of this equation and also defines a smooth dynamical system on R.

Example. Let A be an $n \times n$ matrix. Then the function $\phi_t(X_0) = \exp(tA)X_0$ defines a smooth dynamical system on R^n. Clearly, $\phi_0 = \exp(0) = I$ and, as we saw in the previous chapter, we have $\phi_{t+s} = \exp((t+s)A) = (\exp(tA))(\exp(sA)) = \phi_t \circ \phi_s$.

7.2 The Existence and Uniqueness Theorem

The Existence and Uniqueness Theorem. *Consider the initial value problem*

$X' = F(X),\quad X(t_0) = X_0$

where $X_0 \in R^n$. *Suppose that* $F : R^n \to R^n$ *is* $C1$. *Then, first of all, there exists a solution of this initial value problem and, secondly, this is the only such solution. More precisely, there exists an* $a > 0$ *and a unique solution*

$X : (t_0 - a, t_0 + a) \to R^n$

of this differential equation satisfying the initial condition $X(t_0) = X_0$.

7.4 The Variational Equation

Theorem. (Smoothness of Flows). *Consider the system* $X' = F(X)$ *where* F *is* $C1$. *Then the flow* $\varphi(t, X)$ *of this system is a* $C1$ *function; that is,* $\partial\varphi/\partial t$ *and* $\partial\varphi/\partial X$ *exist and are continuous in* t *and* X.

8 Equilibria in Nonlinear Systems

8.1 Some Illustrative Examples

As a first example, consider the system:

$$\begin{cases} x' = x + y^2 \\ y' = -y \end{cases}.$$

There is a single equilibrium point at the origin. The linearized equation is

$$\begin{cases} x' = x \\ y' = -y \end{cases}.$$

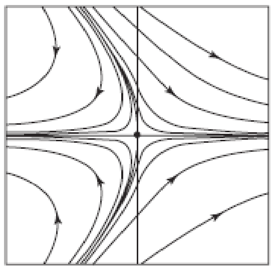

Figure 8.1 The phase plane for $x' = x + y^2$, $y' = -y$. Note the stable curve tangent to the y-axis.

We have a saddle at the origin with a stable line along the y-axis and an unstable line along the x-axis.

The general solution is

$$\begin{cases} x(t) = \left(x_0 + \frac{1}{3}y_0^2\right)e^t - \frac{1}{3}y_0^2 e^{-2t} \\ y(t) = y_0 e^{-t} \end{cases}$$

Example. In general, it is impossible to convert a nonlinear system to a linear one as in the previous example, since the nonlinear terms almost always make huge changes in the system far from the equilibrium point at the origin. For example, consider the nonlinear system

$$\begin{cases} x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + y^2 x) \\ y' = x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2 y) \end{cases}$$

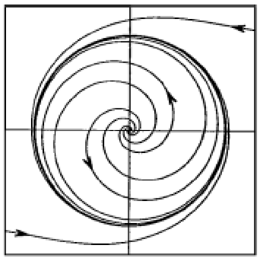

Figure 8.2 The phase plane for $r' = \frac{1}{2}(r - r^3), \theta' = 1$.

The linearized system is now

$$X' = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix} X,$$

which has eigenvalues 1/2+i, 1/2-i. All solutions of this system spiral away from the origin and toward ∞ in the counterclockwise direction. We have

$$\begin{cases} r' = r(1-r^2)/2 \\ \theta' = 1 \end{cases}.$$

From the equation $\theta' = 1$, we conclude that all nonzero solutions spiral around the origin in the counterclockwise direction. From the first equation, we see that solutions do not spiral toward ∞. Indeed, we have $r' = 0$ when $r = 1$, so all solutions that start on the unit circle stay there forever and move periodically around the circle. Since $r' > 0$ when $0 < r < 1$, we conclude that nonzero solutions inside the circle spiral away from the origin and toward the unit circle. Since $r' < 0$ when $r > 1$, solutions outside the circle spiral toward it. See Figure 8.2.

Example. Now consider one final example:

$$\begin{cases} x' = x^2 \\ y' = -y \end{cases}$$

The only equilibrium solution for this system is the origin. All other solutions (except those on the *y*-axis) move to the right and toward the *x*-axis. On the *y*-axis, solutions tend along this straight line to the origin. Hence the phase portrait is as shown in Figure 8.3.

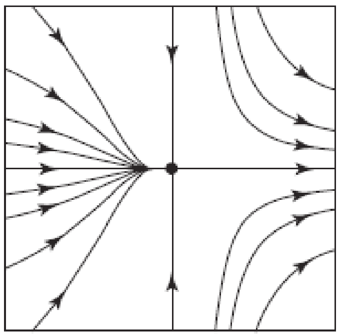

Figure 8.3 The phase plane for $x' = x^2$, $y' = -y$.

8.2 Nonlinear Sinks and Sources

Let $X' = F(X)$ and suppose that $F(X_0) = 0$. Let DF_{X_0} denote the Jacobian matrix of *F* evaluated at *X*0. Then, as in Chapter 7, the linear system of differential equations

$$Y' = DF_{X_0} Y$$

is called the *linearized system near X*0. Note that, if *X*0 = 0, the linearized system is obtained by simply dropping all of the nonlinear terms in *F*, just as we did in the previous section.

In analogy with our work with linear systems, we say that an equilibrium point *X*0 of a nonlinear system is *hyperbolic* if all of the eigenvalues of DF_{X_0} have nonzero real parts.

We now specialize the discussion to the case of an equilibrium of a planar system for which the linearized system has a sink at 0. Suppose our system is

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

with $f(x0, y0) = 0 = g(x0, y0)$. If we make the change of coordinates u =x-x0, $v = y - y0$ then the new system has an equilibrium point at (0, 0). Hence we may as well assume that $x0 = y0 = 0$ at the outset. We then make a further linear change of coordinates that puts the linearized system in canonical form. For simplicity, let us assume at first that the linearized system has distinct eigenvalues $-\lambda < -\mu < 0$. Thus after these changes of coordinates, our system become

$$\begin{cases} x' = -\lambda x + h_1(x, y) \\ y' = -\mu y + h_2(x, y) \end{cases}$$

where $hj = hj(x, y)$ contains all of the "higher order terms." That is, in terms of its Taylor expansion, each hj contains terms that are quadratic or higher order in x and/or y. Equivalently, we have

$$\lim_{(x,y)\to(0,0)} \frac{h_j(x,)}{r} = 0$$

where $r^2 = x^2 + y^2$.

The Linearization Theorem. *Suppose the n-dimensional system* $X' = F(X)$ *has an equilibrium point at* $X0$ *that is hyperbolic. Then the nonlinear flow is conjugate to the flow of the linearized system in a neighborhood of* $X0$.

8.3 Saddles

We turn now to the case of an equilibrium for which the linearized system has a saddle at the origin in R^2. As in the previous section, we may assume that this system is in the form

$$\begin{cases} x' = \lambda x + f_1(x, y) \\ y' = -\mu y + f_2(x, y) \end{cases}$$

where $-\mu < 0 < \lambda$ and $f_j(x, y)/r$ tends to 0 as $r \to 0$. As in the case of a linear system, we call this type of equilibrium point a *saddle*.

The Stable Curve Theorem. *Suppose the system*

$$\begin{cases} x' = \lambda x + f_1(x, y) \\ y' = -\mu y + f_2(x, y) \end{cases}$$

satisfies $-\mu < 0 < \lambda$ *and* $f_j(x, y)/r \to 0$ *as* $r \to 0$. *Then there is an* $\varepsilon > 0$ *and a curve* $x = hs(y)$ *that is defined for* $|y| < \varepsilon$ *and satisfies* $hs(0) = 0$.

Furthermore:

1. All solutions whose initial conditions lie on this curve remain on this curve for all $t \geq 0$ *and tend to the origin as* $t \to \infty$;

2. The curve $x = hs(y)$ *passes through the origin tangent to the y-axis;*

3. All other solutions whose initial conditions lie in the disk of radius ε centered at the origin leave this disk as time increases.

We conclude this section with a brief discussion of higher dimensional saddles. Suppose $X' = F(X)$ where $X \in R^n$. Suppose that $X0$ is an equilibrium solution for which the linearized system has k eigenvalues with negative real parts and $n - k$ eigenvalues with positive real parts. Then the local stable and unstable sets are not generally curves. Rather, they are "submanifolds" of dimension k and $n - k$, respectively. Without entering the realm of manifold theory, we simply note that this means there is a linear change of coordinates in which the local stable set is given near the origin by the graph of a $C\infty$ function g : $g : B_r \to R^{n-k}$ that satisfies $g(0) = 0$, and all partial derivatives of g vanish at the origin. Here Br is the disk of radius r centered at the origin in R^k. The local unstable set is a similar graph over an $n - k$-dimensional disk. Each of these graphs is tangent at the equilibrium point to the stable and unstable subspaces at $X0$. Hence they meet only at $X0$.

Example. Consider the system

$$\begin{cases} x' = -x \\ y' = -y \\ z' = z + x^2 + y^2 \end{cases}$$

The linearized system at the origin has eigenvalues 1 and −1 (repeated). The change of coordinates

$$\begin{cases} u = -x \\ v = y \\ w = z + \frac{1}{3}(x^2 + y^2) \end{cases}$$

converts the nonlinear system to the linear system

$$\begin{cases} u' = -u \\ v' = -v \\ w' = w \end{cases}$$

The plane $w = 0$ for the linear system is the stable plane. Under the change of coordinates this plane is transformed to the surface

$$z = -\frac{1}{3}(x^2 + y^2)$$

which is a paraboloid passing through the origin in R^3 and opening downward. All solutions tend to the origin on this surface; we call this the *stable surface* for the nonlinear system. See Figure 8.5.

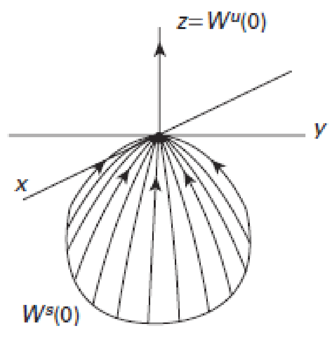

Figure 8.5 The phase portrait for $x' = -x$, $y' = -y$, $z' = z + x^2 + y^2$.

8.5 Bifurcations

Recall the elementary bifurcations we encountered in Chapter 1 for first-order equations $x' = f_a(x)$. If

$x0$ is an equilibrium point, then we have $fa(x0) = 0$. If $f_a'(x_0) \neq 0$, then small changes in a do not change the local structure near $x0$: that is, the differential equation

$$x' = f_{a+\varepsilon}(x)$$

has an equilibrium point $x0(\varepsilon)$ that varies continuously with ε for ε small. A glance at the (increasing or decreasing) graphs of $f_{a+\varepsilon}(x)$ near $x0$ shows why this is true. More rigorously, this is an immediate consequence of the implicit function theorem (see Exercise 3 at the end of this chapter). Thus bifurcations for first-order equations only occur in the nonhyperbolic case where $f_a'(x_0) = 0$.

Example. The first-order equation

$$x' = f_a(x) = x^2 + a$$

has a single equilibrium point at $x = 0$ when $a = 0$. Note $f_0'(0) = 0$, but $f_0''(0) \neq 0$. For $a > 0$ this equation has no equilibrium points since $fa(x) > 0$ for all x, but for $a < 0$ this equation has a pair of equilibria. Thus a bifurcation occurs as the parameter passes through $a = 0$.

This kind of bifurcation is called a *saddle-node bifurcation* (we will see the "saddle" in this bifurcation a little later). In a saddle-node bifurcation, there is an interval about the bifurcation value $a0$ and another interval I on the x-axis in which the differential equation has

1. Two equilibrium points in I if $a < a0$;
2. One equilibrium point in I if $a = a0$;
3. No equilibrium points in I if $a > a0$.

Of course, the bifurcation could take place "the other way," with no equilibria when $a > a0$. The example above is actually the typical type of bifurcation for first-order equations.

Theorem. (**Saddle-Node Bifurcation**) *Suppose* $x' = f_a(x)$ *is a first-order differential equation for which*

1. $f_{a0}(x_0) = 0$;
2. $f_{a0}'(x_0) = 0$;
3. $f_{a0}''(x_0) \neq 0$;
4. $\dfrac{\partial f_{a0}}{\partial a}(x_0) \neq 0$.

Then this differential equation undergoes a saddle-node bifurcation at $a = a0$.

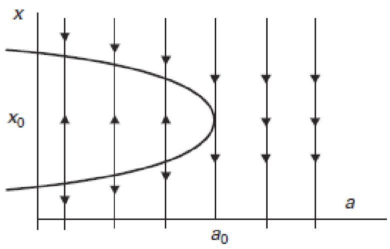

Figure 8.6 The bifurcation diagram for a saddle-node bifurcation.

Recall that the bifurcation diagram for $x' = f_a(x)$ is a plot of the various phase lines of the equation versus the parameter a. The bifurcation diagram for a typical saddle-node bifurcation is displayed in Figure 8.6. (The directions of the arrows and the curve of equilibria may change.)

Example. (Pitchfork Bifurcation) Consider

$x' = x^3 - ax$

There are three equilibria for this equation, at $x = 0$ and $x = \pm\sqrt{a}$

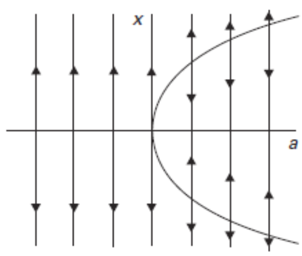

Figure 8.7 The bifurcation diagram for a pitchfork bifurcation.

when $a > 0$. When $a \le 0$, $x = 0$ is the only equilibrium point. The bifurcation diagram shown in Figure 8.7 explains why this bifurcation is so named.

Example. Consider the system

$$\begin{cases} x' = x^2 + a \\ y' = -y \end{cases}$$

When $a = 0$, this is one of the systems considered in Section 8.1. There is a unique equilibrium point at the origin, and the linearized system has a zero eigenvalue. When a passes through $a = 0$, a *saddle-node* bifurcation occurs. When $a > 0$, we have $x' > 0$ so all solutions move to the right; the equilibrium point disappears. When $a < 0$ we have a pair of equilibria, at the points $(\pm\sqrt{(-a)}, 0)$. The linearized equation is

$$X' = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix} X .$$

So we have a sink at $(-\sqrt{(-a)}, 0)$ and a saddle at $(\sqrt{(-a)}, 0)$. Note that solutions on the lines $x = \pm\sqrt{(-a)}$ remain for all time on these lines since $x' = 0$ on these lines. Solutions tend directly to the equilibria on these lines since $y' = -y$. This bifurcation is sketched in Figure 8.8.

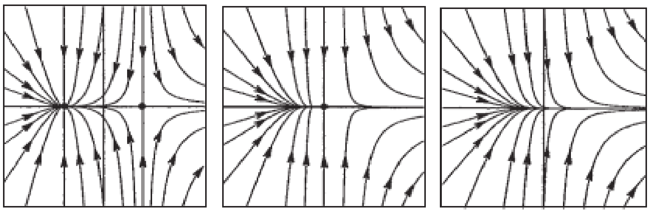

Figure 8.8 The saddle-node bifurcation when, from left to right, $a < 0$, $a = 0$, and $a > 0$.

Example. Consider the system given in polar coordinates by

$$\begin{cases} r' = r - r^3 \\ \theta' = \sin^2\theta + a \end{cases}$$

where a is again a parameter. See Figure 8.9.

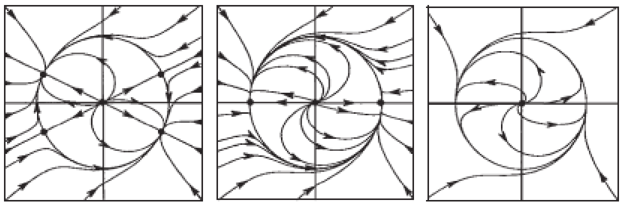

Figure 8.9 Global effects of saddle-node bifurcations when, from left to right, $a < 0$, $a = 0$, and $a > 0$.

Example. (**Hopf Bifurcation**) Consider the system

$$\begin{cases} x' = ax - y - x(x^2 + y^2) \\ y' = x + ay - y(x^2 + y^2) \end{cases}$$

There is an equilibrium point at the origin and the linearized system is

$$X' = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} X$$

The eigenvalues are $a \pm i$, so we expect a bifurcation when $a = 0$. To see what happens as a passes through 0, we change to polar coordinates. The system becomes

$$\begin{cases} r' = ar - r^3 \\ \theta' = 1 \end{cases}$$

Note that the origin is the only equilibrium point for this system, since $\theta' \neq 0$. For $a < 0$ the origin is a sink since $ar - r^3 < 0$ for all $r > 0$. Thus all solutions tend to the origin in this case. When $a > 0$, the equilibrium becomes a source. So what else happens? When $a > 0$ we have $r' = 0$ if $r = \sqrt{a}$. So the circle of radius $\sqrt{a}$ is a periodic solution with period 2π. We also have $r' > 0$ if $0 < r < \sqrt{a}$, while $r' < 0$ if $r > \sqrt{a}$. Thus, all nonzero solutions spiral toward this circular solution as $t \to \infty$.

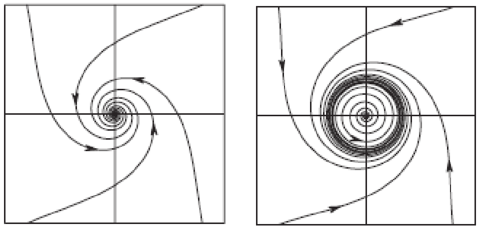

Figure 8.10 The Hopf bifurcation for $a < 0$ and $a > 0$.

This type of bifurcation is called a *Hopf bifurcation*. Thus at a Hopf bifurcation, no new equilibria arise. Instead, a periodic solution is born at the equilibrium point as a passes through the bifurcation value. See Figure 8.10.

9 Global Nonlinear Techniques

9.1 Nullclines

For a system in the form

$$\begin{cases} x_1' = f_1(x_1, \dots, x_n) \\ \dots \\ x_n' = f_n(x_1, \dots, x_n) \end{cases}$$

the xj -nullcline is the set of points where $x'j$ vanishes, so the xj -nullcline is the set of points determined by setting fj ($x1, \dots, xn$) = 0.

Example. For the system

$$\begin{cases} x' = y - x^2 \\ y' = x - 2 \end{cases}$$

the x-nullcline is the parabola $y = x^2$ and the y-nullcline is the vertical line x =2. These nullclines meet at (2, 4) so this is the only equilibrium point. The nullclines divide R^2 into four basic regions labeled A through D in Figure 9.1(a). By first choosing one point in each of these regions, and then determining the direction of the vector field at that point, we can decide the direction of the vector field at all points in the basic region. For example, the point (0, 1) lies in region A and the vector field is (1,−2) at this point, which points toward the southeast. Hence the vector field points southeast at all points in this region. Of course, the vector field may be nearly horizontal or nearly vertical in this region; when we say southeast we mean that the angle θ of the vector field lies in the sector $-\pi/2 < \theta < 0$. Continuing in this fashion we get the direction of the vector field in all four regions, as in Figure 9.1(b). This also determines the horizontal and vertical directions of the vector field on the nullclines.

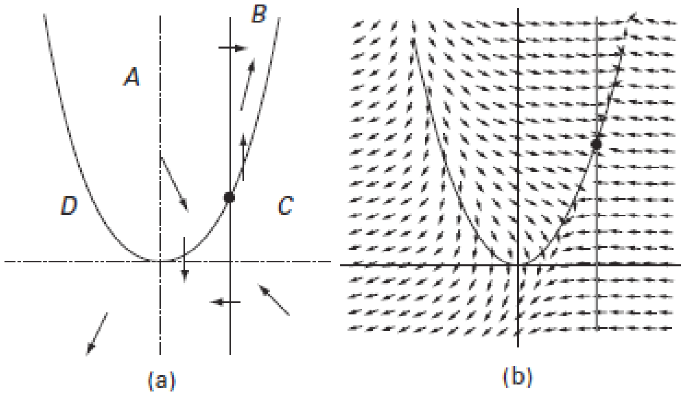

Figure 9.1 The (a) nullclines and (b) direction field.

Just from the direction field alone, it appears that the equilibrium point is a saddle. Indeed, this is the case because the linearized system at (2, 4) is

$$X' = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix} X ,$$

which has eigenvalues $-2 \pm \sqrt{5}$, one of which is positive, the other negative.

More importantly, we can fill in the approximate behavior of solutions everywhere in the plane. For example, note that the vector field points into the basic region marked B at all points along its boundary, and then it points northeasterly at all points inside B. Thus any solution in region B must stay in region B for all time and tend toward ∞ in the northeast direction. See Figure 9.2. Similarly, solutions in the basic

region D stay in that region and head toward ∞ in the southwest direction. Solutions starting in the basic regions A and C have a choice: They must eventually cross one of the nullclines and enter regions B and D (and therefore we know their ultimate behavior) or else they tend to the equilibrium point. However, there is only one curve of such solutions in each region, the stable curve at (2, 4). Thus we completely understand the phase portrait for this system, at least from a qualitative point of view. See Figure 9.3.

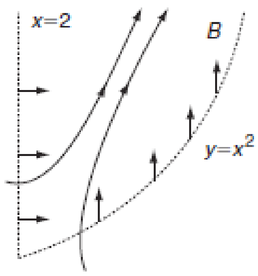

Figure 9.2 Solutions enter the basic region B and then tend to ∞.

Example. (**Heteroclinic Bifurcation**) Next consider the system that depends on a parameter a:

$$\begin{cases} x' = x^2 - 1 \\ y' = -xy + a(x^2 - 1) \end{cases}$$

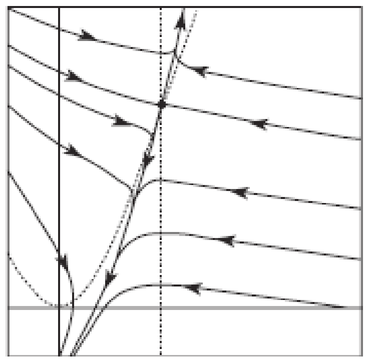
Figure 9.3 The nullclines and phase portrait for $x' = y - x^2$, $y' = x - 2$.

The x-nullclines are given by $x = \pm 1$ while the y-nullclines are $xy = a(x2-1)$. The equilibrium points are $(\pm 1, 0)$. Since $x' = 0$ on $x = \pm 1$, the vector field is actually tangent to these nullclines. Moreover, we have $y' = -y$ on $x = 1$ and $y' = y$ on $x = -1$. So solutions tend to (1, 0) along the vertical line $x = 1$ and tend away from $(-1, 0)$ along $x = -1$. This happens for all values of a.

Now, let's look at the case $a = 0$. Here the system simplifies to

$$\begin{cases} x' = x^2 - 1 \\ y' = -xy \end{cases}$$

so $y' = 0$ along the axes. In particular, the vector field is tangent to the x-axis and is given by $x' = x2 - 1$ on this line. So we have $x' > 0$ if $|x| > 1$ and $x' < 0$ if $|x| < 1$. Thus, at each equilibrium point, we have one straight-line solution tending to the equilibrium and one tending away. So it appears that each equilibrium is a saddle. This is indeed the case, as is easily checked by linearization.

There is a second y-nullcline along $x = 0$, but the vector field is not tangent to this nullcline. Computing the direction of the vector field in each of the basic regions determined by the nullclines yields Figure 9.4, from which we can deduce immediately the qualitative behavior of all solutions.

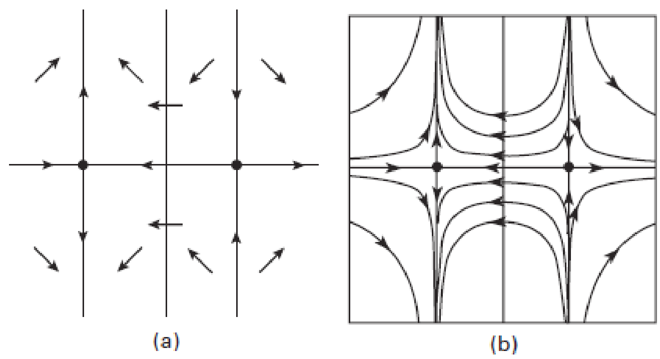

Figure 9.4 The (a) nullclines and (b) phase portrait for $x' = x^2 - 1$, $y' = -xy$.

Note that, when $a = 0$, one branch of the

unstable curve through (1, 0) matches up exactly with a branch of the stable curve at (−1, 0). All solutions on this curve simply travel from one saddle to the other. Such solutions are called *heteroclinic solutions* or *saddle connections*. Typically, for planar systems, stable and unstable curves rarely meet to form such heteroclinic "connections." When they do, however, one can expect a bifurcation.

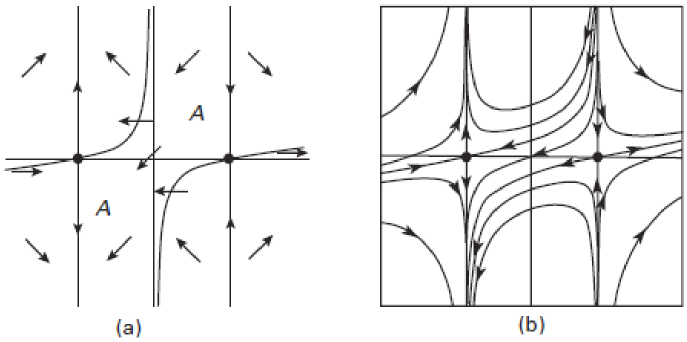

Figure 9.5 The (a) nullclines and (b) phase plane when $a > 0$ after the heteroclinic bifurcation.

Now consider the case where $a \neq 0$. The x-nullclines remain the same, at $x = \pm 1$. But the y-nullclines change drastically as shown in Figure 9.5. They are given by $y = a(x2 - 1)/x$. When $a > 0$, consider the basic region denoted by A. Here the vector field points southwesterly. In particular, the vector field points in this direction along the x-axis between $x = -1$ and $x = 1$. This breaks the heteroclinic connection: The right portion of the stable curve associated to (−1, 0) must now come from $y = \infty$ in the upper half plane, while the left portion of the unstable curve associated to (1, 0) now descends to $y = -\infty$ in the lower half plane. This opens an "avenue" for certain solutions to travel from $y = +\infty$ to $y = -\infty$between the two lines $x = \pm 1$. Whereas when $a = 0$ all solutions remain for all time confined to either the upper or lower half-plane, the *heteroclinic bifurcation* at $a = 0$ opens the door for certain solutions to make this transit.

A similar situation occurs when $a < 0$ (see Exercise 2 at the end of this chapter).

9.2 Stability of Equilibria

Theorem. (**Liapunov Stability**) *Let* X^* *be an equilibrium point for* $X' = F(X)$. *Let* $L^* O \to R$ *be a differentiable function defined on an open set* O *containing* X^*. *Suppose further that*

(a) $L(X^*) = 0$ *and* $L(X) > 0$ *if* $X \neq X^*$*;*

(b) $\dot{L} \leq 0$ *in* $O - X^*$.

Then X^* *is stable. Furthermore, if* L *also satisfies*

(c) $\dot{L} < 0$ *in* $O - X^*$,

then X^* *is asymptotically stable.*

A function L satisfying (a) and (b) is called a *Liapunov function* for X^*. If (c) also holds, we call L a *strict* Liapunov function.

Example. Consider the system of differential equations in R^3 given by

$$\begin{cases} x' = (\varepsilon x + 2y)(z+1) \\ y' = (-x + \varepsilon y)(z+1) \\ z' = -z^3 \end{cases}$$

where ε is a parameter. The origin is the only equilibrium point for this system. The linearization of the

system at (0, 0, 0) is

$$Y' = \begin{pmatrix} \varepsilon & 2 & 0 \\ -1 & \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} Y$$

The eigenvalues are 0 and $\varepsilon \pm \sqrt{2}i$. Hence, from the linearization, we can only conclude that the origin is unstable if $\varepsilon > 0$. When $\varepsilon \le 0$, all we can conclude is that the origin is not hyperbolic.

When $\varepsilon \le 0$ we search for a Liapunov function for (0, 0, 0) of the form

$$L(x, y, z) = ax^2 + by^2 + cz^2 ,$$

with $a, b, c > 0$. For such an L, we have

$$\dot{L} = 2(xx' + byy' + czz') ,$$

so that

$$\begin{aligned} \dot{L}/2 &= ax(\varepsilon x + 2y)(z+1) + by(-x + \varepsilon y)(z+1) - cz^4 \\ &= \varepsilon(ax^2 + by^2)(z+1) + (2a - b)(xy)(z+1) - cz^4 . \end{aligned}$$

For stability, we want $\dot{L} \le 0$; this can be arranged, for example, by setting $a = 1$, $b = 2$, and $c = 1$. If $\varepsilon = 0$, we then have $\dot{L} = -z^4 \le 0$, so the origin is stable. It can be shown (see Exercise 4) that the origin is not asymptotically stable in this case.

If $\varepsilon < 0$, then we find

$$\dot{L} = \varepsilon(x^2 + 2y^2)(z+1) - z^4$$

so that $\dot{L} < 0$ in the region O given by $z > -1$ (minus the origin). We conclude that the origin is asymptotically stable in this case, and, indeed, from Exercise 4, that all solutions that start in the region O tend to the origin.

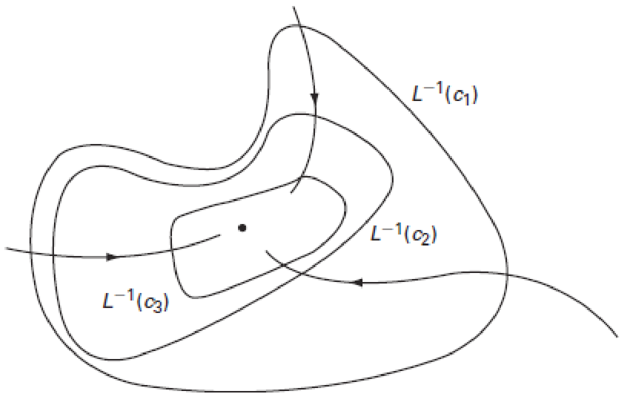

Figure 9.7 Solutions decrease through the level sets $L^{-1}(c_j)$ of a strict Liapunov function.

Figure 9.7 makes the theorem intuitively obvious. The condition $\dot{L} < 0$ means that when a solution crosses a "level surface" $L^{-1}(c)$, it moves inside the set where $L \le c$ and can never come out again. Unfortunately, it is sometimes difficult to justify the diagram shown in this figure; why should the sets $L^{-1}(c)$ shrink down to X^*? Of course, in many cases, Figure 9.7 is indeed correct, as, for example, if L is a quadratic function such as $ax2 + by2$ with $a,b > 0$. But what if the level surfaces look like those shown in Figure 9.8? It is hard to imagine such an L that fulfills all the requirements of a

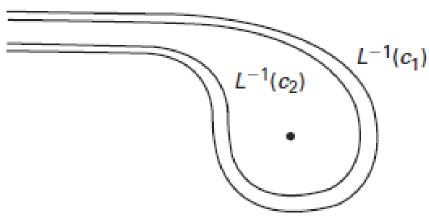

Figure 9.8 Level sets of a Liapunov function may look like this.

Liapunov function; but rather than trying to rule out that possibility, it is simpler to give the analytic proof as above.

Example. Now consider the system

$$\begin{cases} x' = -x^3 \\ y' = -y(x^2 + z^2 + 1) \\ z' = -\sin z \end{cases}$$

The origin is again an equilibrium point. It is not the only one, however, since (0, 0, $n\pi$) is also an equilibrium point for each $n \in Z$. Hence the origin cannot be globally asymptotically stable. Moreover, the planes $z = n\pi$ for $n \in Z$ are *invariant* in the sense that any solution that starts on one of these planes remains there for all time. This occurs since $z' = 0$ when $z = n\pi$. In particular, any solution that begins in the region $|z| < \pi$ must remain trapped in this region for all time.

Linearization at the origin yields the system

$$Y' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} Y$$

which tells us nothing about the stability of this equilibrium point.

However, consider the function

$$L(x, y, z) = x^2 + y^2 + z^2 .$$

Clearly, $L > 0$ except at the origin. We compute

$$\dot{L} = 2x^4 - 2y^2(x^2 + z^2 + 1) - 2z \sin z .$$

Then $\dot{L} < 0$ at all points in the set $|z| < \pi$ (except the origin) since $z \sin z > 0$ when $z \neq 0$. Hence the origin is asymptotically stable.

Moreover, we can conclude that the basin of attraction of the origin is the entire region $|z| < \pi$. From the proof of the Liapunov stability theorem, it follows immediately that any solution that starts inside a sphere of radius $r < \pi$ must tend to the origin. Outside of the sphere of radius π and between the planes $z = \pm\pi$, the function L is still strictly decreasing. Since solutions are trapped between these two planes, it follows that they too must tend to the origin.

Theorem. (**Lasalle's Invariance Principle**) *Let X^* be an equilibrium point for $X' = F(X)$ and let L :* $U \to \mathbb{R}$ *be a Liapunov function for X^*, where U is an open set containing X^*. Let $P \subset U$ be a neighborhood of X^* that is closed and bounded. Suppose that P is positively invariant, and that there is no entire solution in $P - X^*$ on which L is constant. Then X^* is asymptotically stable, and P is contained in the basin of attraction of X^*.*

9.3 Gradient Systems

A *gradient system* on R^n is a system of differential equations of the form

$$X' = -gradV(X)$$

where $V : R^n \to R$ is a $C\infty$ function, and

$$gradV = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n} \right).$$

(The negative sign in this system is traditional.) The vector field grad V is called the *gradient* of V. Note that −grad $V(X)$ = grad $(-V(X))$.

Gradient systems have special properties that make their flows rather simple. The following equality is fundamental:

$$DV_X(Y) = gradV(X) \cdot Y \text{.}$$

This says that the derivative of V at X evaluated at $Y = (y_1, \dots, y_n) \in R^n$ is given by the dot product of the vectors grad $V(X)$ and Y. This follows immediately from the formula

$$DV_X(Y) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(X) y_j$$

Let $X(t)$ be a solution of the gradient system with $X(0) = X0$, and let $\dot{V} : R^n \to R$ be the derivative of V along this solution. That is,

$$\dot{V}(X) = \frac{d}{dt} V(X(t)) \text{.}$$

Proposition. *The function V is a Liapunov function for the system* $X' = -gradV(X)$. *Moreover,* $\dot{V}(X) = 0$ *if and only if X is an equilibrium point.*

Theorem. (**Properties of Gradient Systems**) *For the system* $X' = -gradV(X)$ *:*

1. If c is a regular value of V, then the vector field is perpendicular to the level set $V^{-1}(c)$.

2. The critical points of V are the equilibrium points of the system.

3. If a critical point is an isolated minimum of V, then this point is an asymptotically stable equilibrium point.

Example. Let $V : R^2 \to R$ be the function $V(x, y) = x^2(x-1)^2 + y^2$. Then the gradient system

$$X' = F(X) = -gradV(X)$$

is given by

$$\begin{cases} x' = -2x(x-1)(2x-1) \\ y' = -2y \end{cases}$$

There are three equilibrium points: (0, 0), (1/2, 0), and (1, 0). The linearizations at these three points yield the following matrices:

$$DF(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad DF(1/2,0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \quad DF(1,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence (0, 0) and (1, 0) are sinks, while (1/2, 0) is a saddle. Both the x- and y-axes are invariant, as are the lines $x = 1/2$ and $x = 1$. Since $y' = -2y$ on these vertical lines, it follows that the stable curve at (1/2, 0) is the line $x = 1/2$, while the unstable curve at (1/2, 0) is the interval (0, 1) on the x-axis.

The level sets of V and the phase portrait are shown in Figure 9.10. Note that it appears that all solutions tend to one of the three equilibria. This is no accident, for we have:

Proposition. *Let Z be an α-limit point or an ω-limit point of a solution of a gradient flow. Then Z is an equilibrium point.*

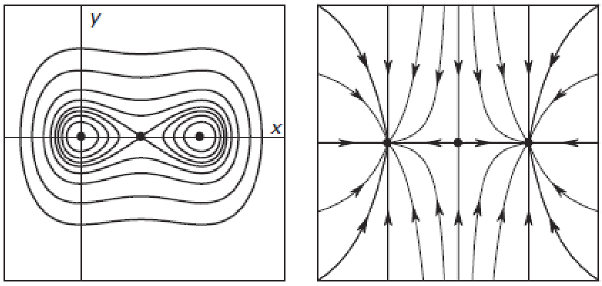

Figure 9.10 The level sets and phase portrait for the gradient system determined by $V(x, y) = x^2 (x-1)^2 + y^2$.

There is one final property that gradient systems share. Note that, in the previous example, all of the eigenvalues of the linearizations at the equilibria have real eigenvalues. Again, this is no accident, for the linearization of a gradient system at an equilibrium point X^* is a matrix $[a_{ij}]$ where

$$a_{ij} = -\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)(X^*).$$

Since mixed partial derivatives are equal, we have

$$\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)(X^*) = \left(\frac{\partial^2 V}{\partial x_j \partial x_i}\right)(X^*)$$

and so $a_{ij} = a_{ji}$. It follows that the matrix corresponding to the linearized system is a *symmetric matrix*. It is known that such matrices have only real eigenvalues. For example, in the 2 × 2 case, a symmetric matrix assumes the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

and the eigenvalues are easily seen to be

$$\frac{a+c}{2} \pm \frac{\sqrt{(a-c)^2 + 4b^2}}{2}$$

both of which are real numbers. A more general case is relegated to Exercise 15. We therefore have:

Proposition. *For a gradient system* $X' = -\mathrm{grad}V(X)$, *the linearized system at any equilibrium point has only real eigenvalues.*

9.4 Hamiltonian Systems

We shall restrict attention in this section to Hamiltonian systems in R^2. A *Hamiltonian system* on R^2 is a system of the form

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x, y) \\ y' = -\dfrac{\partial H}{\partial x}(x, y) \end{cases}$$

where $H : R^2 \to R$ is a $C\infty$ function called the *Hamiltonian function*.

Example. (**Undamped Harmonic Oscillator**) Recall that this system is given by

$$\begin{cases} x' = y \\ y' = -kx \end{cases}$$

where $k > 0$. A Hamiltonian function for this system is

$$H(x, y) = \frac{1}{2}y^2 + \frac{k}{2}x^2.$$

Example. (**Ideal Pendulum**) The equation for this system, as we saw in Section 9.2, is

$$\begin{cases} \theta' = v \\ v' = -\sin\theta \end{cases}$$

The total energy function

$$E(\theta, v) = \frac{1}{2}v^2 + 1 - \cos\theta$$

serves as a Hamiltonian function in this case. Note that we say a Hamiltonian function, since we can always add a constant to any Hamiltonian function without changing the equations.

What makes Hamiltonian systems so important is the fact that the Hamiltonian function is a *first integral* or *constant of the motion*. That is, H is constant along every solution of the system, or, in the language of the previous sections, $\dot{H} = 0$. This follows immediately from

$$\begin{aligned} \dot{H} &= \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y' \\ &= \frac{\partial H}{\partial x}\frac{\partial}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0 \end{aligned}$$

Proposition. *For a Hamiltonian system on R^2, H is constant along every solution curve.*

Example. Consider the system

$$\begin{cases} x' = y \\ y' = -x^3 + x \end{cases}$$

A Hamiltonian function is

$$H(x, y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{4}.$$

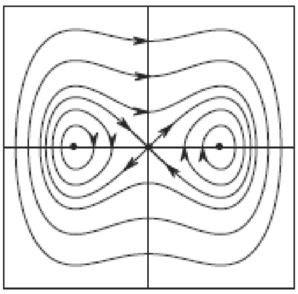

Figure 9.11 The phase portrait for $x' = y$, $y' = -x^3 + x$.

The constant value 1/4 is irrelevant here; we choose it so that H has minimum value 0, which occurs at (±1, 0), as is easily checked. The only other equilibrium point lies at the origin. The linearized system is

$$X' = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix} X$$

At (0, 0), this system has eigenvalues ±1, so we have a saddle. At (±1, 0), the eigenvalues are $\pm\sqrt{2}i$, so we have a center, at least for the

linearized system. Plotting the level curves of H and adding the directions at nonequilibrium points yields the phase portrait depicted in Figure 9.11. Note that the equilibrium points at $(\pm 1, 0)$ remain centers for the nonlinear system. Also note that the stable and unstable curves at the origin match up exactly. That is, we have solutions that tend to (0, 0) in both forward and backward time Such solutions are known as *homoclinic solutions* of *homoclinic orbits*.

Proposition. *Suppose* $(x0, y0)$ *is an equilibrium point for a planar Hamiltonian system. Then the eigenvalues of the linearized system are either* $\pm\lambda$ *or* $\pm i\lambda$ *where* $\lambda \in R$.

10 Closed Orbits and Limit Sets

10.1 Limit Sets

We begin by describing the limiting behavior of solutions of systems of differential equations. Recall that $Y \in R^n$ is an ω-limit point for the solution through X if there is a sequence $tn \to \infty$ such that $\lim_{n\to\infty} \phi_n(X) = Y$. That is, the solution curve through X accumulates on the point Y as time moves forward. The set of all ω-limit points of the solution through X is the ω-limit set of X and is denoted by $\omega(X)$. The α-limit points and the α-limit set $\alpha(X)$ are defined by replacing $tn \to\infty$ with $tn \to -\infty$ in the above definition. By a *limit set* we mean a set of the form $\omega(X)$ or $\alpha(X)$.

Here are some examples of limit sets. If $X*$ is an asymptotically stable equilibrium, it is the ω-limit set of every point in its basin of attraction. Any equilibrium is its own α- and ω-limit set. A periodic solution is the α-limit and ω-limit set of every point on it. Such a solution may also be the ω-limit set of many other points.

Example. Consider the planar system given in polar coordinates by

$$\begin{cases} r' = \frac{1}{2}(r - r^3) \\ \theta' = 1 \end{cases}$$

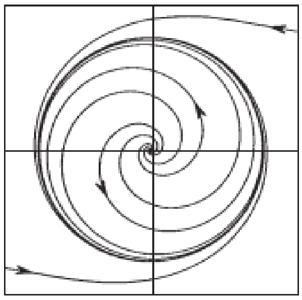

Figure 10.1 The phase plane for $r' = \frac{1}{2}(r-r^3)$, $\theta' = 1$.

As we saw in Section 8.1, all nonzero solutions of this equation tend to the periodic solution that resides on the unit circle in the plane. See Figure 10.1. Consequently, the ω-limit set of any nonzero point is this closed orbit.

Example. Consider the system

$$\begin{cases} x' = \sin x(-0.1\cos x - \cos y) \\ y' = \sin y(\cos x - 0.1\cos y) \end{cases}$$

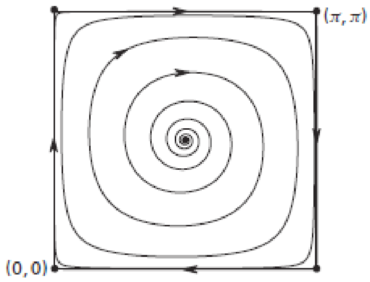

Figure 10.2 The ω-limit set of any solution emanating from the source at $(\pi/2, \pi/2)$ is the square bounded by the four equilibria and the heteroclinic solutions.

There are equilibria which are saddles at the corners of the square (0, 0), (0, π), (π, π), and (π, 0), as well as at many other points. There are heteroclinic solutions connecting these equilibria in the order listed. See Figure 10.2. There is also a spiral source at ($\pi/2$, $\pi/2$). All solutions emanating from this source accumulate on the four heteroclinic solutions connecting the equilibria (see Exercise 4 at the end of this chapter). Hence the ω-limit set of any point on these solutions is the square bounded by $x = 0, \pi$ and $y = 0, \pi$.

Proposition.

1. If X and Z lie on the same solution curve, then $\omega(X) = \omega(Z)$ *and* $\alpha(X) = \alpha(Z)$*;*

2. If D is a closed, positively invariant set and $Z \in D$*, then* $\omega(Z) \subset D$*, and similarly for negatively invariant sets and α-limits;*

3. A closed invariant set, in particular, a limit set, contains the α-limit and ω-limit sets of every point in it.

10.3 The Poincare Map

Proposition. *Let* $X' = F(X)$ *be a planar system and suppose that* $X0$ *lies on a closed orbit* γ *. Let P be a Poincare map defined on a neighborhood of* $X0$ *in some local section. If* $|P'(X0)| < 1$*, then* γ *is asymptotically stable.*

10.5 The Poincare-Bendixson Theorem

Theorem. (**Poincaré-Bendixson**) *Suppose that* Ω *is a nonempty, closed and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then* Ω *is a closed orbit.*

Example. Another example of an ω-limit set that is neither a closed orbit nor an equilibrium is provided by a *homoclinic solution.* Consider the system

$$\begin{cases} x' = -y - \left(\dfrac{x^4}{4} - \dfrac{x^2}{2} + \dfrac{y^2}{2}\right)(x^3 - x) \\ y' = xH3s - x - \left(\dfrac{x^4}{4} - \dfrac{x^2}{2} + \dfrac{y^2}{2}\right)y \end{cases}$$

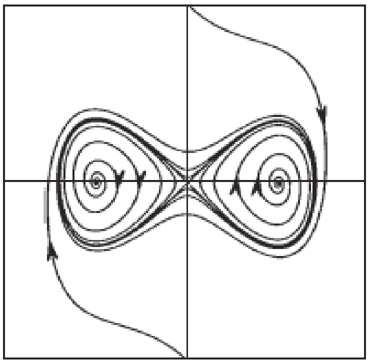

Figure 10.9 A pair of homoclinic solutions in the ω-limit set.

A computation shows that there are three equilibria: at (0, 0), (−1, 0), and (1, 0). The origin is a saddle, while the other two equilibria are sources. The phase portrait of this system is shown in Figure 10.9. Note that solutions far from the origin tend to accumulate on the origin and a pair of homoclinic solutions, each of which leaves and then returns to the origin. Solutions emanating from either source have ω-limit set that consists of just one homoclinic solution and (0, 0). See Exercise 6 for proofs of these facts.

12 Applications in Circuit Theory

12.2 The Lienard Equation

In this section we begin the study of the phase portrait of the Lienard system from the circuit of the previous section, namely:

$$\begin{cases} \dfrac{dx}{dt} = y - f(x) \\ \dfrac{dy}{dt} = -x \end{cases}$$

In the special case where $f(x) = x^3 - x$, this system is called the *van der Pol equation*.

First consider the simplest case where f is linear. Suppose $f(x) = kx$, where $k > 0$. Then the Lienard system takes the form $Y' = AY$ where

$$A = \begin{pmatrix} -k & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues of A are given by $\lambda_{\pm} = (-k \pm (k^2 - 4)^{1/2})/2$. Since $\lambda_{\pm}$ is either negative or else has a negative real part, the equilibrium point at the origin is a sink. It is a spiral sink if $k < 2$. For any $k > 0$, all solutions of the system tend to the origin; physically, this is the dissipative effect of the resistor.

Note that we have

$$y'' = -x' = -y + kx = -y - ky',$$

so that the system is equivalent to the second-order equation

$$y'' + ky' + y = 0,$$

which is often encountered in elementary differential equations courses.

Next we consider the case of a general characteristic f. There is a unique equilibrium point for the Lienard system that is given by $(0, f(0))$. Linearization yields the matrix

$$\begin{pmatrix} -f'(0) & 1 \\ -1 & 0 \end{pmatrix}$$

whose eigenvalues are given by

$$\lambda_{\pm} = \frac{1}{2}\left(-f'(0) \pm \sqrt{(f'(0))^2 - 4}\right).$$

We conclude that this equilibrium point is a sink if $f'(0) > 0$ and a source if $f'(0) < 0$. In particular, for the van der Pol equation where $f(x) = x3 - x$, the unique equilibrium point is a source.

To analyze the system further, we define the function $W : R^2 \to R^2$ by $W(x, y) = (1/2)(x^2 + y^2)$. Then we have

$$\dot{W} = x(y - f(x) + y(-x) = -xf(x).$$

In particular, if f satisfies $f(x) > 0$ if $x > 0$, $f(x) < 0$ if $x < 0$, and $f(0) = 0$, then W is a strict Liapunov function on all of R^2. It follows that, in this case, all solutions tend to the unique equilibrium point lying at the origin.

In circuit theory, a resistor is called *passive* if its characteristic is contained in the set consisting of (0, 0) and the interior of the first and third quadrant. Therefore in the case of a passive resistor, $-xf(x)$ is negative except when $x = 0$, and so all solutions tend to the origin. Thus the word *passive* correctly describes the dynamics of such a circuit.

12.3 The van der Pol Equation

In this section we continue the study of the Lienard equation in the special case where $f(x) = x^3 - x$. This is the *van der Pol equation*:

$$\begin{cases} \dfrac{dx}{dt} = y - x^3 + x \\ \dfrac{dy}{dt} = -x \end{cases}$$

Let φ_t denote the flow of this system. In this case we can give a fairly complete phase portrait analysis.

Theorem. *There is one nontrivial periodic solution of the van der Pol equation and every other solution (except the equilibrium point at the origin) tends to this periodic solution. "The system oscillates."*

We know from the previous section that this system has a unique equilibrium point at the origin, and that this equilibrium is a source, since $f'(0) < 0$. The next step is to show that every nonequilibrium solution "rotates" in a certain sense around the equilibrium in a clockwise direction. To see this, note that the x-nullcline is given by $y = x^3 - x$ and the y-nullcline is the y-axis.

We subdivide each of these nullclines into two pieces given by

$v^+ = \{(x, y) \mid y > 0, x = 0\}$

$v^- = \{(x, y) \mid y < 0, x = 0\}$

$g^+ = \{(x, y) \mid x > 0, y = x^3 - x\}$

$g^- = \{(x, y) \mid x < 0, y = x^3 - x\}$.

These curves are disjoint; together with the origin they form the boundaries of the four basic regions A, B, C, and D depicted in Figure 12.3. From the configuration of the vector field in the basic regions, it appears that all nonequilibrium solutions wind around the origin in the clockwise direction. This is indeed the case.

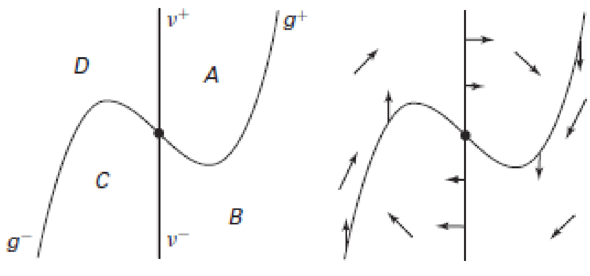

Figure 12.3 The basic regions and nullclines for the van der Pol system.

Proposition. *Solution curves starting on v^+ cross successively through g^+, v^-, and g^- before returning to v^+.*

As a consequence of this result, we may define a Poincare map P on the half-line v^+. Given $(0, y_0) \in v^+$, we define $P(y_0)$ to be the y coordinate of the first return of $\varphi_t(0, y_0)$ to v^+ with $t > 0$. See Figure 12.4. As in Section 10.3, P is a one to one C^∞ function. The Poincare map is also onto. To see this,

simply follow solutions starting on $v+$ backward in time until they reintersect $v+$, as they must by the proposition. Let $P^n = P \circ P^{n-1}$ denote the n-fold composition of P with itself.

Our goal now is to prove the following theorem:

Theorem. *The Poincaré map has a unique fixed point in* $v+$. *Furthermore, the sequence* $P^n(y_0)$ *tends to this fixed point as* $n \to \infty$ *for any nonzero* $y_0 \in v^+$.

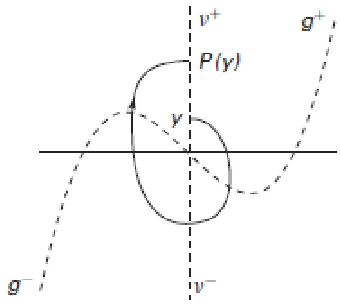

Figure 12.4 The Poincaré map on v^+.

12.4 A Hopf Bifurcation

We now describe a more general class of circuit equations where the resistor characteristic depends on a parameter μ and is denoted by $f\mu$. (Perhaps μ is the temperature of the resistor.) The physical behavior of the circuit is then described by the system of differential equations on R^2:

$$\begin{cases} \dfrac{dx}{dt} = y - f_\mu(x) \\ \dfrac{dy}{dt} = -x \end{cases}$$

Consider as an example the special case where $f\mu$ is described by

$f_\mu(x) = x^3 - \mu x$

and the parameter μ lies in the interval $[-1, 1]$. When $\mu = 1$ we have the van der Pol system from the previous section. As before, the only equilibrium point lies at the origin. The linearized system is

$$Y' = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix} Y$$

and the eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 - 4}\right).$$

Thus the origin is a spiral sink for $-1 \le \mu < 0$ and a spiral source for $0 < \mu \le 1$. Indeed, when $-1 \le \mu \le 0$, the resistor is passive as the graph of $f\mu$ lies in the first and third quadrants. Therefore all solutions tend to the origin in this case. This holds even in the case where $\mu = 0$ and the linearization yields a center. Physically the circuit is dead in that, after a period of transition, all

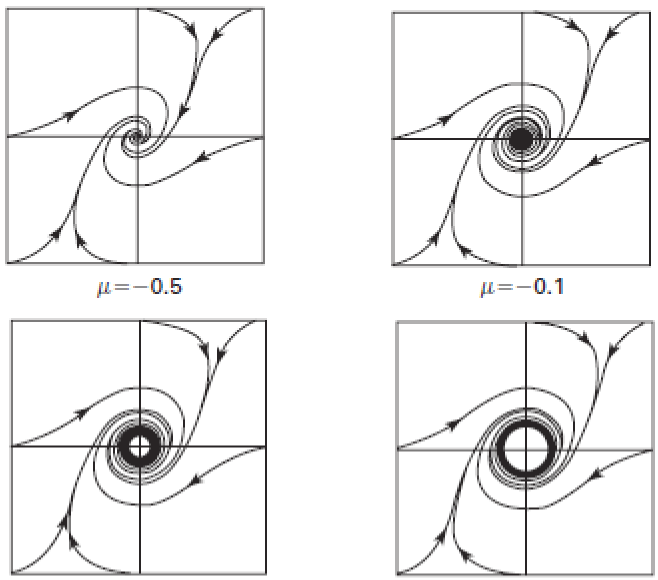

Figure 12.9 The Hopf bifurcation in the system $x' = y - x^3 + \mu x$, $y' = -x$.

currents and voltages stay at 0 (or as close to 0 as we want).

However, as μ becomes positive, the circuit becomes alive. It begins to oscillate. This follows from the fact that the analysis of Section 12.3 applies to this system for all μ in the interval (0, 1]. We therefore see the birth of a (unique) periodic solution $\gamma\mu$ as μ increases through 0 (see Exercise 4 at the end of this chapter). Just as above, this solution attracts all other nonzero solutions. As in Section 8.5, this is an example of a *Hopf bifurcation*. Further elaboration of the ideas in Section 12.3 can be used to show that $\gamma\mu \to 0$ as $\mu \to 0$ with $\mu > 0$. Figure 12.9 shows some phase portraits associated to this bifurcation.

13 Applications in Mechanics

13.1 Newton's Second Law

The connection between the physical concept of a force field and the mathematical concept of a differential equation is *Newton's second law*: $F = ma$. This law asserts that a particle in a force field moves in such a way that the force vector at the location X of the particle, at any instant, equals the acceleration vector of the particle times the mass m. That is, Newton's law gives the second-order differential equation

$$mX'' = F(X).$$

As a system, this equation becomes

$$\begin{cases} X' = V \\ V' = \dfrac{1}{m} F(X) \end{cases}$$

where $V = V(t)$ is the velocity of the particle. This is a system of equations on $R^n \times R^n$. This type of system is often called a *mechanical system with n degrees of freedom*.

A solution $X \subset R^n$ of the second-order equation is said to lie in *configuration space*. The solution of the system $(X(t), V(t)) \subset R^n \times R^n$ lies in the *phase space* or *state space* of the system.

Example. Recall the simple undamped harmonic oscillator from Chapter 2. In this case the mass moves in one dimension and its position at time t is given by a function $x(t)$, where $x : \mathbb{R} \to \mathbb{R}$. As we saw, the differential equation governing this motion is

$mx'' = -kx$

for some constant $k > 0$. That is, the force field at the point $x \in R$ is given by $-kx$.

13.2 Conservative Systems

Theorem. (**Conservation of Energy**) *Let $(X(t), V(t))$ be a solution curve of a conservative system. Then the total energy E is constant along this solution curve.*

13.3 Central Force Fields

Proposition. *Let F be a conservative force field. Then the following statements are equivalent:*

1. *F is central;*
2. *$F(X) = f(|X|)X$;*
3. *$F(X) = -\text{grad } U(X)$ and $U(X) = g(|X|)$.*

Proposition. *A particle moving in a central force field in* $\mathbb{R}^3$ *always moves in a fixed plane containing the origin*.

Corollary. (**Conservation of Angular Momentum**) *Angular momentum is constant along any solution curve in a central force field.*

13.7 Blowing Up the Singularity

The singularity at the origin in the Newtonian central force problem is the first time we have encountered

such a situation. Usually our vector fields have been well defined on all of R^n. In mechanics, such singularities can sometimes be removed by a combination of judicious changes of variables and time scalings. In the Newtonian central force system, this may be achieved using a change of variables introduced by McGehee [32].

We first introduce scaled variables

$$u_r = r^{1/2} v_r,$$
$$u_\theta = r^{1/2} v_\theta$$

In these variables the system becomes

$$r' = r^{-1/2} u_r,$$
$$\theta' = r^{-3/2} u_\theta,$$
$$u_r' = r^{-3/2}\left(\frac{1}{2}u_r^2 + u_\theta^2 - 1\right),$$
$$u_\theta' = r^{-3/2}\left(-\frac{1}{2}u_r u_\theta\right).$$

We still have a singularity at the origin, but note that the last three equations are all multiplied by $r^{-3/2}$. We can remove these terms by simply multiplying the vector field by $r^{3/2}$. In doing so, solution curves of the system remain the same but are parameterized differently.

More precisely, we introduce a new time variable τ via the rule

$$\frac{dt}{d\tau} = r^{3/2}.$$

By the chain rule we have

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$$

and similarly for the other variables. In this new timescale the system becomes

$$\dot{r} = r u_r$$
$$\dot{\theta} = u_\theta$$
$$\dot{u}_r = \frac{1}{2}u_r^2 + u_\theta^2 - 1$$
$$\dot{u}_\theta = -\frac{1}{2}u_r u_\theta$$

where the dot now indicates differentiation with respect to τ. Note that, when r is small, $dt/d\tau$ is close to zero, so "time" τ moves much more slowly than time t near the origin.

This system no longer has a singularity at the origin. We have "blown up" the singularity and replaced it with a new set given by $r = 0$ with θ, u_r, and u_θ being arbitrary. On this set the system is now perfectly well defined. Indeed, the set $r = 0$ is an invariant set for the flow since $\dot{r} = 0$ when $r = 0$. We have thus introduced a fictitious flow on $r = 0$. While solutions on $r = 0$ mean nothing in terms of the

real system, by continuity of solutions, they can tell us a lot about how solutions behave near the singularity.

We need not concern ourselves with all of $r=0$ since the total energy relation in the new variables becomes

$$hr = \frac{1}{2}(u_r^2 + u_\theta^2) - 1 .$$

On the set $r=0$, only the subset Λ defined by

$$u_r^2 + u_\theta^2 = 2 , \quad \theta \text{ is arbitrary}$$

matters. The set Λ is called the *collision surface* for the system; how solutions behave on Λ dictates how solutions move near the singularity since any solution that approaches $r=0$ necessarily comes close to Λ in our new coordinates. Note that Λ is a two-dimensional torus: It is formed by a circle in the θ direction and a circle in the $u_r u_\theta$–plane.

On Λ the system reduces to

$$\dot\theta = u_\theta$$

$$\dot u_r = \frac{1}{2}u_\theta^2$$

$$\dot u_\theta = -\frac{1}{2}u_r u_\theta$$

where we have used the energy relation to simplify $\dot u_r$. This system is easy to analyze. We have $\dot u_r > 0$ provided $u_\theta \neq 0$. Hence the u_r coordinate must increase along any solution in Λ with $u_\theta \neq 0$.

On the other hand, when $u_\theta = 0$, the system has equilibrium points. There are two circles of equilibria, one given by $u\theta=0, ur=\sqrt{2}$, and θ arbitrary, the other by $u\theta=0, ur=-\sqrt{2}$, and θ arbitrary. Let $C\pm$ denote these two circles with $ur=\pm\sqrt{2}$ on $C\pm$. All other solutions must travel from $C-$ to $C+$ since $v\theta$ increases along solutions.

To fully understand the flow on Λ, we introduce the angular variable ψ in each $u_r u_\theta$–plane via

$$u_r = \sqrt{2}\sin\psi ,$$
$$u_\theta = \sqrt{2}\cos\psi .$$

The torus is now parameterized by θ and ψ. In $\theta\psi$ coordinates, the system becomes

$$\dot\theta = \sqrt{2}\cos\psi ,$$

$$\dot\psi = \frac{1}{\sqrt{2}}\cos\psi .$$

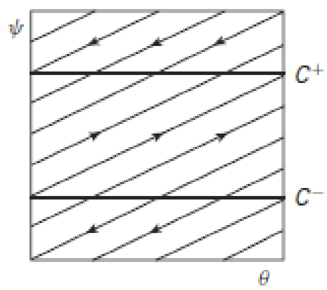

Figure 13.4 Solutions on Λ in $\theta\psi$ coordinates. Recall that θ and ψ are both defined mod 2π, so opposite sides of this square are identified to form a torus.

The circles $C\pm$ are now given by $\psi=\pm\pi/2$. Eliminating time from this equation, we find

$$\frac{d\psi}{d\theta} = \frac{1}{2} ,$$

Thus all nonequilibrium solutions have constant slope 1/2 when viewed in $\theta\psi$ coordinates. See Figure 13.4.

Now recall the collision-ejection solutions described in Section 13.4. Each of these solutions leaves the origin and then returns along a ray $\theta = \theta^*$ in configuration space. The solution departs with $v_r > 0$ (and so $u_r > 0$) and returns with $v_r < 0$ ($u_r < 0$). In our new four-dimensional coordinate system, it follows that this solution forms an unstable curve associated to the equilibrium point $(0, \theta^*, \sqrt{2}, 0)$ and a stable curve associated to $(0, \theta^*, -\sqrt{2}, 0)$. See Figure 13.5. What happens to nearby noncollision solutions? Well, they come close to the "lower" equilibrium point with $\theta = \theta^*, u_r = -\sqrt{2}$, then follow one of two branches of the unstable curve through this point up to the "upper" equilibrium point $\theta = \theta^*, u_r = +\sqrt{2}$, and then depart near the unstable curve leaving this equilibrium point. Interpreting this motion in configuration space, we see that each near-collision solution approaches the origin and then retreats after θ either increases or decreases by 2π units. Of course, we know this already, since these solutions whip around the origin in tight ellipses.

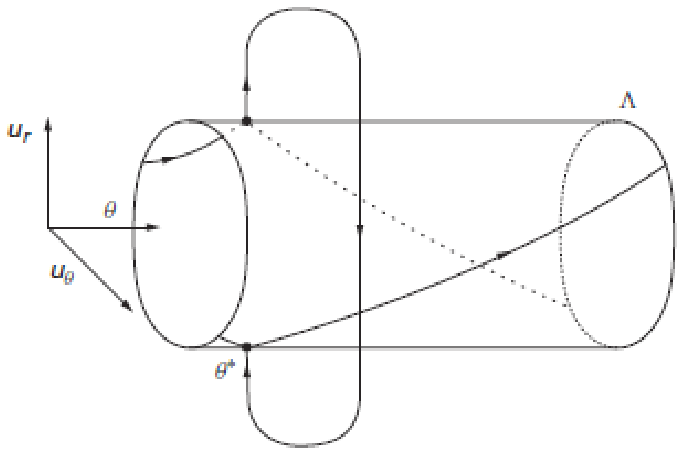

Figure 13.5 A collision-ejection solution in the region $r > 0$ leaving and returning to Λ and a connecting orbit on the collision surface.

14 The Lorenz System

14.1 Introduction to the Lorenz System

The resulting motion led to a three-dimensional system of differential equations that involved three parameters: the Prandtl number σ, the Rayleigh number r , and another parameter b that is related to the physical size of the system. When all of these simplifications were made, the system of differential equations involved only two nonlinear terms and was given by

$$\begin{cases} x' = \sigma(y - x) \\ y' = rx - y - xz \\ z' = xy - bz \end{cases}$$

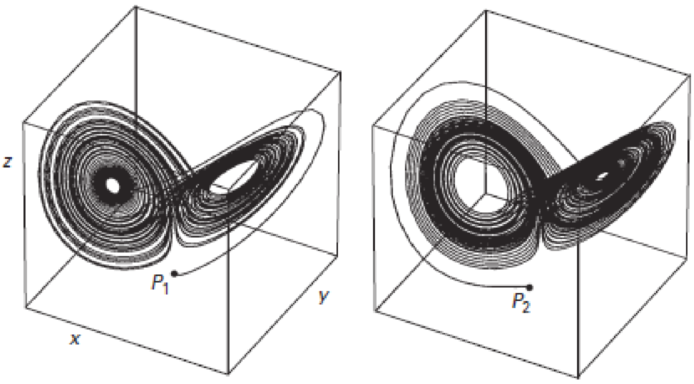

Figure 14.1 The Lorenz attractor. Two solutions with initial conditions P_1 = (0, 2, 0) and P_2 = (0, −2, 0).

In this system all three parameters are assumed to be positive and, moreover, $\sigma > b + 1$. We denote this system by $X' = L(X)$. In Figure 14.1, we have displayed the solution curves through two different initial conditions $P1$ =(0, 2, 0) and $P2$ = (0,−2, 0) when the parameters are $\sigma = 10$, $b = 8/3$, and $r = 28$. These are the original parameters that led to Lorenz's discovery. Note how both solutions start out very differently, but eventually have more or less the same fate: They both seem to wind around a pair of points, alternating at times which point they encircle. This is the first important fact about the Lorenz system: All nonequilibrium solutions tend eventually to the same complicated set, the so-called *Lorenz attractor*.

14.2 Elementary Properties of the Lorenz System

As usual, to analyze this system, we begin by finding the equilibria. Some easy algebra yields three equilibrium points, the origin, and

$$Q_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1).$$

The latter two equilibria only exist when $r > 1$, so already we see that we have a bifurcation when $r = 1$. Linearizing, we find the system

$$Y' = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix} Y.$$

At the origin, the eigenvalues of this matrix are $-b$ and $\lambda_{\pm} = \frac{1}{2}\left(-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}\right)$. Note that both $\lambda\pm$ are negative when $0 \le r < 1$. Hence the origin is a sink in this case.

The Lorenz vector field $L(X)$ possesses a symmetry. If we let $S(x, y, z) = (-x, -y, z)$, then we have

$S(L(X)) = L(S(X))$. That is, reflection through the z-axis preserves the vector field. In particular, if $(x(t), y(t), z(t))$ is a solution of the Lorenz equations, then so is $(-x(t), -y(t), z(t))$.

When $x = y = 0$, we have $x' = y' = 0$, so the z-axis is invariant. On this axis, we have simply $z' = -bz$, so all solutions tend to the origin on this axis. In fact, the solution through any point in R^3 tends to the origin when $r < 1$, for we have:

Proposition. *Suppose* $r < 1$. *Then all solutions of the Lorenz system tend to the equilibrium point at the origin.*

When r increases through 1, two things happen. First, the eigenvalue $\lambda+$ at the origin becomes positive, so the origin is now a saddle with a two-dimensional stable surface and an unstable curve. Second, the two equilibria $Q\pm$ are born at the origin when $r = 1$ and move away as r increases.

Proposition. *The equilibrium points* $Q\pm$ *are sinks provided*

$$1 < r < r^* = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right).$$

We remark that a Hopf bifurcation is known to occur at r^*, but proving this is beyond the scope of this book.

When $r > 1$ it is no longer true that all solutions tend to the origin. However, we can say that solutions that start far from the origin do at least move closer in. To be precise, let

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2.$$

Note that $V(x, y, z) = v > 0$ defines an ellipsoid in R^3 centered at $(0, 0, 2r)$. We will show:

Proposition. *There exists* v^* *such that any solution that starts outside the ellipsoid* $V = v^*$ *eventually enters this ellipsoid and then remains trapped therein for all future time.*

Proposition. *The volume of _ is zero.*

14.3 The Lorenz Attractor

Definition Let $X' = F(X)$ be a system of differential equations in R^n with flow φt. A set Λ is called an *attractor* if

1. Λ is compact and invariant;
2. There is an open set U containing Λ such that for each $X \in U$, $\phi_t(X) \in U$ for all $t \geq 0$ and $\cap_{t \geq 0} \phi_t(U) = \Lambda$;
3. (Transitivity) Given any points $Y_1, Y_2 \in \Lambda$ and any open neighborhoods Uj about Yj in U, there is a solution curve that begins in $U1$ and later passes through $U2$.

As a remark, there is no universally accepted definition of an attractor in mathematics; some people choose to say that a set Λ that meets only conditions 1 and 2 is an attractor, while if Λ also meets condition 3, it would be called a transitive attractor. For planar systems, condition 3 is usually easily verified; in higher dimensions, however, this can be much more difficult, as we shall see.

For the rest of this chapter, we restrict attention to the very special case of the Lorenz system where

the parameters are given by σ = 10, b = 8/3, and r = 28. Historically, these are the values Lorenz used when he first encountered chaotic phenomena in this system. Thus, the specific Lorenz system we consider is

$$X' = L(X) = \begin{pmatrix} 10(y-x) \\ 28x - y - xz \\ xy - (8/3)z \end{pmatrix}.$$

As in the previous section, we have three equilibria: the origin and $Q\pm = (\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$. At the origin we find eigenvalues $\lambda 1 = -8/3$ and $\lambda_{\pm} = -\frac{11}{2} \pm \frac{\sqrt{1201}}{2}$. For later use, note that these eigenvalues satisfy $\lambda_{-} < -\lambda_{+} < \lambda_{1} < 0 < \lambda_{+}/$

The linearized system at the origin is then

$$Y' = \begin{pmatrix} \lambda_{-} & 0 & 0 \\ 0 & \lambda_{+} & 0 \\ 0 & 0 & \lambda_{1} \end{pmatrix} Y.$$

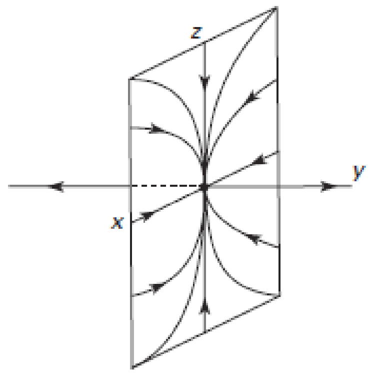

Figure 14.4 Linearization at the origin for the Lorenz system.

The phase portrait of the linearized system is shown in Figure 14.4. Note that all solutions in the stable plane of this system tend to the origin tangentially to the z-axis.

At $Q\pm$ a computation shows that there is a single negative real eigenvalue and a pair of complex conjugate eigenvalues with positive real parts. Note that the symmetry in the system forces the rotations about $Q+$ and $Q-$ to have opposite orientations.

In Figure 14.5, we have displayed a numerical computation of a portion of the left- and right-hand branches of the unstable curve at the origin. Note that the right-hand portion of this curve comes close to $Q-$ and then spirals away. The left portion behaves symmetrically under reflection through the z-axis. In Figure 14.6, we have displayed a significantly larger portion of these unstable curves. Note that they appear to circulate around the two equilibria, sometimes spiraling around $Q+$, sometimes about $Q-$. In particular, these curves continually reintersect the portion of the plane z = 27 containing $Q\pm$ in which the vector field points downward. This suggests that we may construct a Poincaré map on a portion of this plane. As we have seen before, computing a Poincare map is often impossible, and this case is no different. So we will content ourselves

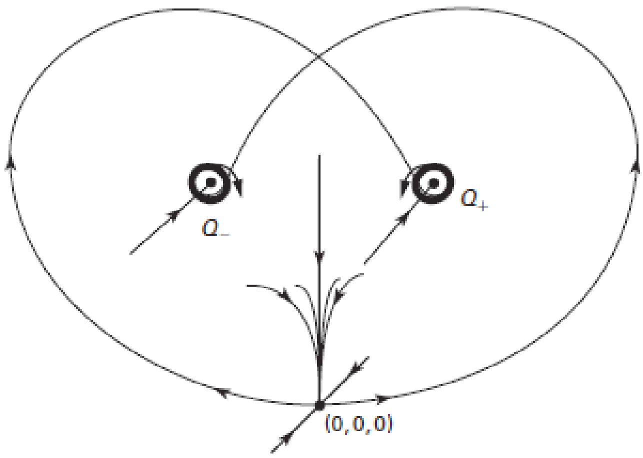

Figure 14.5 The unstable curve at the origin.

with building a simplified model that exhibits much of the behavior we find in the Lorenz system. As we shall see in the following section, this model provides a computable means to assess the chaotic behavior of the system.

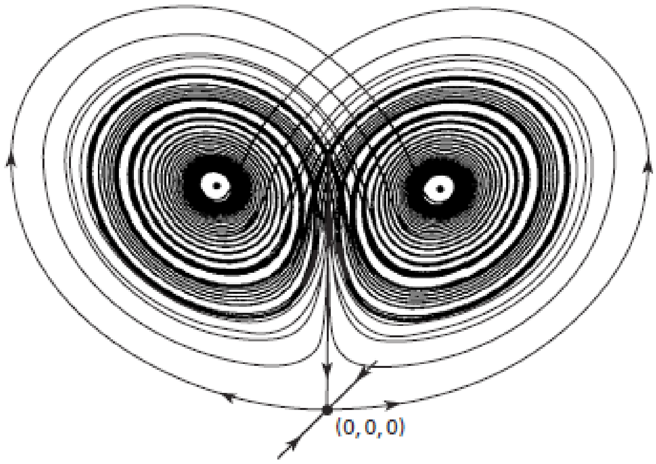

Figure 14.6 More of the unstable curve at the origin.

16 Homoclinic Phenomena

16.1 The Shil'nikov System

For this example, we do not specify the full system of differential equations. Rather, we first set up a linear system of differential equations in a certain cylindrical neighborhood of the origin. This system has a two-dimensional stable surface in which solutions spiral toward the origin and a one-dimensional unstable curve. We then make the simple but crucial dynamical assumption that one of the two branches of the unstable curve is a homoclinic solution and thus eventually enters the stable surface. We do not write down a specific differential equation having this behavior. Although it is possible to do so, having the equations is not particularly useful for understanding the global dynamics of the system. In fact, the phenomena we study here depend only on the qualitative properties of the linear system described previously a key inequality involving the eigenvalues of this linear system, and the homoclinic assumption.

The first portion of the system is defined in the cylindrical region S of R^3 given by $x^2 + y^2 \le 1$ and $|z| \le 1$. In this region consider the linear system

$$X' = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} X .$$

The associated eigenvalues are $-1 \pm i$ and 2. Using the results of Chapter 6, the flow φt of this system is easily derived:

$$\begin{cases} x(t) = x_0 e^{-t} \cos t + y_0 e^{-t} \sin t, \\ y(t) = -x_0 e^{-t} \sin t + y_0 e^{-t} \cos t, \\ z(t) = z_0 e^{2t} . \end{cases}$$

Using polar coordinates in the xy–plane, solutions in S are given more succinctly by

$$\begin{cases} r(t) = r_0 e^{-t} , \\ \theta(t) = \theta_0 - t, \\ z(t) = z_0 e^{2t} . \end{cases}$$

This system has a two-dimensional stable plane (the xy–plane) and a pair of unstable curves $\zeta\pm$ lying on the positive and negative z-axis, respectively. We remark that there is nothing special about our choice of eigenvalues for this system. Everything below works fine for eigenvalues $\alpha \pm i\beta$ and λ where $\alpha < 0$, $\beta \neq 0$, and $\lambda > 0$ subject only to the important condition that $\lambda > -\alpha$.

The boundary of S consists of three pieces: the upper and lower disks $D\pm$ given by $z = \pm1$, $r \le 1$, and the cylindrical boundary C given by $r = 1$, $|z| \le 1$. The stable plane meets C along the circle $z = 0$ and divides C into two pieces, the upper and lower halves given by $C+$ and $C-$, on which $z > 0$ and $z < 0$, respectively. We may parameterize $D\pm$ by r and θ and C by θ and z. We will concentrate in this section on $C+$.

Any solution of this system that starts in $C+$ must eventually exit from S through $D+$. Hence we can define a map $\psi 1 : C+ \to D+$ given by following solution curves that start in $C+$ until they first meet $D+$. Given $(\theta_0, z_0) \in C^+$, let $\tau = \tau(\theta 0, z0)$ denote the time it takes for the solution through $(\theta 0, z0)$ to make the transit to $D+$. We compute immediately using $z(t) = z_0 e^{2t}$ that $\tau = -\log(\sqrt{z_0})$. Therefore

$$\psi_1\begin{pmatrix} 1 \\ \theta_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} r_1 \\ \theta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{z_0} \\ \theta_0 + \log(\sqrt{z_0}) \\ 1 \end{pmatrix}.$$

For simplicity, we will regard $\psi 1$ as a map from the $(\theta 0, z0)$ cylinder to the $(r1, \theta 1)$ plane. Note that a vertical line given by $\theta 0 = \theta^*$ in $C+$ is mapped by $\psi 1$ to the spiral

$$z_0 \to (\sqrt{z_0}, \theta^* + \log(\sqrt{z_0})),$$

which spirals down to the point $r = 0$ in $D\pm$, since $\log\sqrt{z0} \to -\infty$ as $z0 \to 0$.

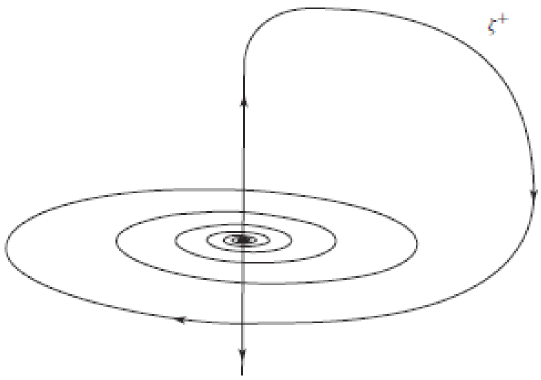

Figure 16.1 The homoclinic orbit ζ^+.

To define the second piece of the system, we assume that the branch $\zeta+$ of the unstable curve leaving the origin through $D+$ is a homoclinic solution. That is, $\zeta+$ eventually returns to the stable plane. See Figure 16.1. We assume that $\zeta+$ first meets the cylinder C at the point $r = 1$, $\theta = 0$, $z = 0$. More precisely, we assume that there is a time $t1$ such that $\varphi t1\ (0, \theta, 1) = (1, 0, 0)$ in r, θ, z coordinates.

Therefore we may define a second map $\psi 2$ by following solutions beginning near $r = 0$ in $D+$ until they reach C. We will assume that $\psi 2$ is, in fact, defined on all of $D+$. In Cartesian coordinates on $D+$, we assume that $\psi 2$ takes $(x, y) \in D^+$ to $(\theta_1, z_1) \in C$ via the rule

$$\psi_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \theta_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} y/2 \\ x/2 \end{pmatrix}.$$

In polar coordinates, $\psi 2$ is given by

$$\theta 1 = (r \sin \theta)/2$$

$$z1 = (r \cos \theta)/2.$$

Of course, this is a major assumption, since writing down such a map for a particular nonlinear system would be virtually impossible.

Now the composition $\Phi = \psi_2 \circ \psi_1$ defines a Poincare map on $C+$. The map $\psi 1$ is defined on $C+$ and takes values in $D+$, and then $\psi 2$ takes values in C. We have $\Phi : C^+ \to C$ where

$$\Phi\begin{pmatrix} \theta_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} (1/2)\sqrt{z_0}\,\sin(\theta_0 + \log(\sqrt{z_0})) \\ (1/2)\sqrt{z_0}\,\cos(\theta_0 + \log(\sqrt{z_0})) \end{pmatrix}.$$

See Figure 16.2.

As in the Lorenz system, we have now reduced the study of the flow of this three-dimensional system to the study of a planar discrete dynamical system. As we shall see in the next section, this type of mapping has incredibly rich dynamics that may be (partially) analyzed using symbolic dynamics. For a little taste of what is to come, we content ourselves here with just finding the fixed points of Φ. To do this we need to solve

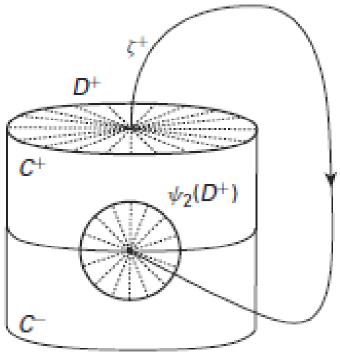

Figure 16.2 The map ψ_2: $D^+ \to C$.

$$\begin{cases} \theta_0 = (1/2)\sqrt{z_0}\,\sin(\theta_0 + \log(\sqrt{z_0})) \\ z_0 = (1/2)\sqrt{z_0}\,\cos(\theta_0 + \log(\sqrt{z_0})) \end{cases}.$$

These equations look pretty formidable. However, if we square both equations and add them, we find

$$\theta_0^2 + z_0^2 = \frac{z_0^2}{4}$$

so that

$$\theta_0 = \pm\frac{1}{2}\sqrt{z_0 - 4z_0^2}\ ,$$

which is well defined provided that $0 \le z0 \le 1/4$. Substituting this expression into the second equation above, we find that we need to solve

$$\cos\left(\pm\frac{1}{2}\sqrt{z_0 - 4z_0^2} + \log(\sqrt{z_0})\right) = 2\sqrt{z_0}\ .$$

Now the term $\sqrt{z_0 - 4z_0^2}$ tends to zero as $z0 \to 0$, but $\log(\sqrt{z0}) \to -\infty$. Therefore the graph of the left-hand side of this equation oscillates infinitely many times between ± 1 as $z0 \to 0$. Hence there must be infinitely many places where this graph meets that of $2\sqrt{z0}$, and so there are infinitely many solutions of this equation. This, in turn, yields infinitely many fixed points for Φ. Each of these fixed points then corresponds to a periodic solution of the system that starts in $C+$, winds a number of times around the z-axis near the origin, and then travels around close to the homoclinic orbit until closing up when it returns to $C+$. See Figure 16.3.

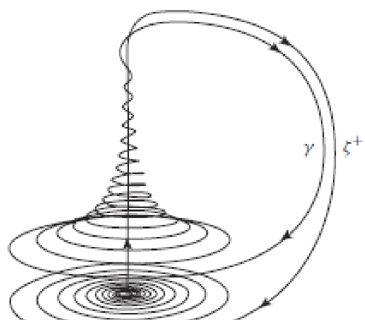

Figure 16.3 A periodic solution γ near the homoclinic solution ζ^+.

We now describe the geometry of this map; in the next section we use these ideas to investigate the dynamics of a simplified version of this map. First note that the circles $z0 = \alpha$ in $C+$ are mapped by $\psi 1$ to circles $r = \sqrt{\alpha}$ centered at $r = 0$ in $D+$ since

$$\psi_1\begin{pmatrix}\theta_0\\ \alpha\end{pmatrix}=\begin{pmatrix}r_1\\ \theta_1\end{pmatrix}=\begin{pmatrix}\sqrt{\alpha}\\ \theta_0+\log(\sqrt{\alpha})\end{pmatrix}.$$

Then $\psi 2$ maps these circles to circles of radius $\sqrt{\alpha}/2$ centered at $\theta 1 = z1 = 0$ in C. (To be precise, these are circles in the θz–plane; in the cylinder, these circles are "bent.") In particular, we see that "one-half" of the domain $C+$ is mapped into the lower part of the cylinder $C-$ and therefore no longer comes into play.

Let H denote the half-disk $\Phi(C^+)\cap\{z\geq 0\}$. Half-disk H has center at $\theta 1 = z1 = 0$ and radius 1/2. The preimage of H in $C+$ consists of all points $(\theta 0, z0)$ whose images satisfy $z1 \geq 0$, so that we must have

$$z_1=\frac{1}{2}\sqrt{z_0}\cos(\theta_0+\log(\sqrt{z_0}))\geq 0\,.$$

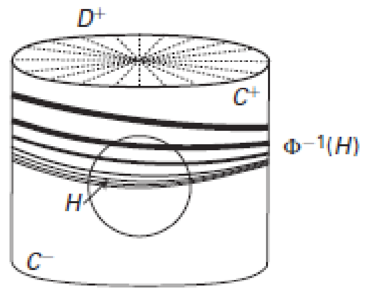

Figure 16.4 The half-disk H and its preimage in C^+.

It follows that the preimage of H is given by

$$\Phi^{-1}(H)=\{(\theta_0,z_0):-\pi/2\leq\theta_0+\log(\sqrt{z_0})\leq\pi/2\}$$

where $0 < z0 \leq 1$. This is a region bounded by the two curves $\theta 0+\log(\sqrt{z0}) =\pm\pi/2$, each of which spirals downward in $C+$ toward the circle $z = 0$. See Figure 16.4. This follows since, as $z0 \to 0$, we must have $\theta 0 \to \infty$. More generally, consider the curves l_α given by

$$\theta_0+\log(\sqrt{z_0})=\alpha$$

for $-\pi/2 \leq \alpha \leq \pi/2$. These curves fill the preimage $\Phi^{-1}(H)$ and each spirals around C just as the boundary curves do. Now we have

$$\Phi(l_\alpha)=\frac{\sqrt{z_0}}{2}\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix},$$

so Φ maps each l_α to a ray that emanates from $\theta = z = 0$ in $C+$ and is parameterized by $\sqrt{z0}$. In particular, Φ maps each of the boundary curves $l_{\pm\pi/2}$ to z=0 in C.

Since the curves $l_{\pm\pi/2}$ spiral down toward the circle $z = 0$ in C, it follows that $\Phi^{-1}(H)$ meets H in infinitely many strips, which are nearly horizontal close to $z = 0$. See Figure 16.4. We denote these strips by H_k for k sufficiently large. More precisely, let H_k denote the component of $\Phi^{-1}(H)\cap H$ for which we have

$$2k\pi-\frac{1}{2}\leq\theta_0\leq 2k\pi+\frac{1}{2}\,.$$

The top boundary of H_k is given by a portion of the spiral $l_{\pi/2}$ and the bottom boundary by a piece of $l_{-\pi/2}$. Using the fact that

$$-\frac{\pi}{2}\leq\theta_0+\log(\sqrt{z_0})\leq\frac{\pi}{2},$$

we find that, if $(\theta_0,z_0)\in H_k$, then

$$-(4k+1)\pi - 1 \le -\pi - 2\theta_0 \le 2\log\sqrt{z_0} \le \pi - 2\theta_0 \le -(4k-1)\pi + 1$$

from which we conclude that

$$\exp(-(4k+1)\pi - 1) \le z_0 \le \exp(-(4k-1)\pi + 1) .$$

Now consider the image of H_k under Φ. The upper and lower boundaries of H_k are mapped to $z = 0$. The curves $l_\alpha \cap H_k$ are mapped to arcs in rays emanating from $\theta = z = 0$. These rays are given as above by

$$\frac{\sqrt{z_0}}{2}\begin{pmatrix}\sin\alpha \\ \cos\alpha\end{pmatrix}.$$

In particular, the curve l_0 is mapped to the vertical line $\theta 1 = 0$, $z1 = \sqrt{z0}/2$. Using the above estimate of the size of $z0$ in H_k, one checks easily that the image of l_0 lies completely above H_k when $k \ge 2$. Therefore the image of $\Phi(H_k)$ is a "horseshoe-shaped" region that crosses H_k twice as shown in Figure 16.5. In particular, if k is large, the curves $l_\alpha \cap H_k$ meet the horseshoe $\Phi(H_k)$ in nearly horizontal subarcs.

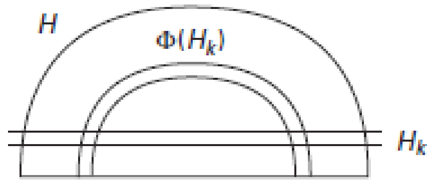

Figure 16.5 The image of H_k is a horseshoe that crosses H_k twice in C^+.

Such a map is called a *horseshoe map*; in the next section we discuss the prototype of such a function.