SUPERSONIC AERODYNAMICS
- PRINCIPLES AND APPLICATIONS[1]

by

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1. VELOCITY OF SOUND, PROPAGATION OF PRESSURE

The first theoretical computation of sound velocity was given by Isaac Newton in his *Principles of Natural Philosophy*. He found that the velocity of propagation of a pressure pulse is directly proportional to the square root of the **elastic force** resisting the compression of air and inversely proportional to the square root of the density of the medium. Carrying out the calculation, he obtained a value of 979 ft per sec for sound velocity in air at sea level under standard conditions and found that this value is about 15 per cent lower than the experimental value of 1142 ft per sec deduced from gunshot observations. He explained the discrepancy by the presence of suspended **solid particles** and **water vapor** in the atmosphere. Later, Laplace found that Newton's method of calculation of the elastic force is equivalent to the assumption of **isothermal compression**, whereas the true process is very nearly **adiabatic**. Newton, of course, could not foresee the thermodynamic relations that were unknown at his time; nevertheless, it is interesting to see that even such a genius could **succumb** to the temptation of explaining an essential discrepancy between theory and experiment by wishful thinking.

$$\boxed{\sqrt{\frac{p}{\rho}} \leftrightarrow \sqrt{\frac{\gamma p}{\rho}}}$$

Until recently, the study of the motion of bodies having velocity faster than sound belonged to the realm of **ballistics**. As a matter of fact, when the science of modern aerodynamics was developed a few decades ago, most theories were based on the assumption that the air could be considered as an incompressible fluid. It was shown that the error in the computation of air forces produced by the motion of an airplane is about one-half times the square of the ratio of flight velocity to sound velocity. This ratio is known as **Mach number**, in honor of the Viennese physicist and philosopher, to whom we owe many findings on high-speed flow and beautiful optical methods of observation. If the flight velocity is 150 m.p.h., the error is about

$$\boxed{\frac{1}{2}\sqrt{\frac{V}{c}}}$$

[1] The Tenth Wright Brothers Lecture. Presented before the Institute of the Aeronautical Sciences in the U.S. Chamber of Commerce Auditorium, Washington, D.C.

$\frac{1}{2}\left(\frac{1}{5}\right)^2 = 2$ per cent. As the flight velocity of airplanes increased, it became necessary to consider the so-called "**compressibility effects**".

I believe we have now arrived at the stage where knowledge of supersonic aerodynamics should be considered by the aeronautical engineer as a necessary prerequisite to his art. This branch of aerodynamics should cease to be a collection of mathematical formulas and half-digested, isolated, experimental results. The aeronautical engineer should start to get the same feeling for the facts of supersonic flight as he acquired in the domain of subsonic velocities by a long process of theoretical study, experimental research, and flight experience.

2. THE THREE RULES OF SUPERSONIC AERODYNAMICS

The following rules are based on the assumption that the disturbance caused in the air by a moving body can be considered slight. The influence of finite disturbances will be discussed in Section 10.

(a) The Rule of Forbidden Signals

Since a slight pressure change is **propagated at sound velocity**, it is evident that the effect of pressure changes produced in the air by a body moving at a speed faster than sound cannot reach points ahead of the body. It may be said that the body is **unable to send signals ahead**. It is seen that there is a fundamental difference between subsonic and supersonic motion. Consider the case of subsonic stationary motion - for example, the uniform level flight of an airplane. Then a pressure signal travels ahead at sound velocity minus flight velocity relative to the airplane, whereas a signal travels backward at a speed equal to the sum of the flight and sound velocity. So the distribution of the effects is no longer symmetric; nevertheless, every point in space is reached by a signal, provided the flight started from an infinitely remote point. (For this consideration we neglect viscosity - i.e., the absorption of energy in the air.) As can easily be seen, this is not the case in supersonic flight, and one obtains the second rule, which refers to the **zone of action** and the **zone of silence**.

(b) Zane of Action and Zone of Silence

Consider the simplest case of a point source (Figure 1). Figure 1a shows the spherical surfaces that the pressure effect reaches in equal time intervals in the case of a point source at rest. Figure 1b shows the same surfaces relative to the point

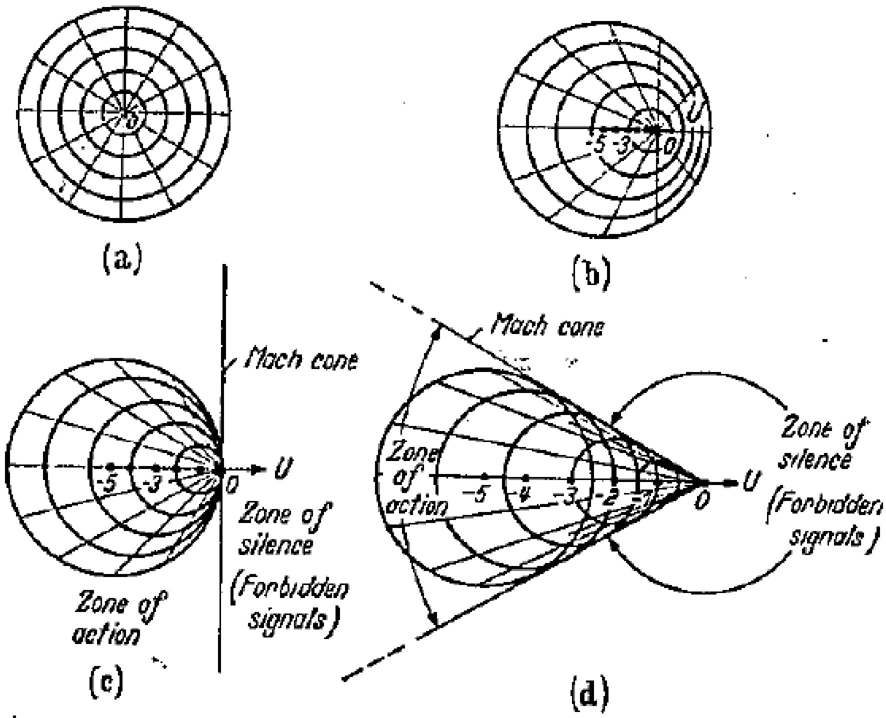

Figure 1. Point source moving in compressible fluid. (a) Stationary source. (b) Source moving at half the speed of sound. (c) Source moving at the speed of sound. (d) Source moving at twice the speed of sound.

source moving with a speed less than that of sound. Figure 1c represents the case of a point source moving with sonic velocity, and Figure ld, the case of a source moving faster than sound. It is seen that in the last case all action is restricted to the interior of a cone that includes all spheres. The outside of this cone can be called the **zone of silence**. It is easily seen that the trigonometrical sine of the half vertex angle of the cone is equal to the reciprocal of the Mach Number. This angle is called the **Mach angle**. The cone that separates the **zone of action** from the zone of silence is called the **Mach cone**.

According to this rule, a stationary point source in a supersonic stream produces action only on points that lie on or inside the Mach cone extending downstream from the point source. Conversely, pressure and velocity at an arbitrary point of the stream can be influenced only by disturbances acting at points that lie on or inside a cone of the same vertex angle extending upstream from the point considered.

(c) The Rule of Concentrated Action

This rule expresses another characteristic difference between subsonic and supersonic motion. It concerns the distribution of the pressure effect in space relative

to the moving body, The points plotted in Figures 1a to 1d show the location of mass points that are supposed to be emitted from the source and move at sound velocity in all directions. They illustrate qualitatively the distribution of the density of action in the various cases. In the subsonic case one finds that the pressure effect not only decreases with increasing distance from the source but is also dispersed in all directions. In the case of a body moving at supersonic velocity, the bulk of the effect is concentrated in the neighborhood of the Mach cone that forms the outer limit of the zone of action. This phenomenon is also easily seen in the examples to be considered in the following sections.

3. THE MECHANISM OF DRAG

There are two concepts of the drag of a body moving in a fluid medium. First, one may consider the **pressure** and **friction** forces acting on the surface of the body and determine their resultant. Second, one may consider the body and a suitably bounded **region** of the surrounding air as one mechanical system and compute or measure the normal and tangential stresses acting on the boundaries of this system, called control surfaces, plus the flow of momentum due to the fluid entering or leaving the system through the same boundaries. The engineer may look at the first concept as a. more direct one. Often, the second concept is more practical, however, as shown, for example, by its successful application as an experimental method. It is commonly known that the best determination of the profile drag of an airfoil section is achieved by the so-called **wake method**, which is precisely an application of the second concept. Also, in the age of propulsion by reaction, every engineer should be acquainted with the definition of forces acting on a moving body by action and reaction.

According to the so-called **paradox of d'Alembert**, the motion of a body in a nonviscous incompressible fluid does not involve drag if the motion does not produce eddies (vorticity) and the flow does not separate from the body. This statement is based on the existence of a vortex-free motion around the body, in which all disturbances completely disappear at infinity. The resistance in an actual fluid is then due to friction forces and flow separation. The loss of momentum equivalent to the drag can be found in the wake that follows the body. This statement concerning the relation between wake and drag is also correct for the motion of the body in a compressible fluid if the motion is subsonic. Moreover, the drag is zero - i.e., the d'Alembert theorem holds in a nonviscous compressible fluid, provided a

vortex-free continuous motion exists around the body. Whereas, however, in the case of an incompressible fluid, the equations of flow always admit such a solution, in the case of a compressible fluid this is true only below a certain critical Mach Number that is less than unity. Between this Mach Number and, Mach Number 1, the wake is due not only to friction and flow separation but, as will be shown in a later section, also to the existence of shock waves. Hence, although the mechanism of the wake formation may be more complex in this case, nevertheless the total loss of momentum equivalent to the drag always appears in the wake as long as the body moves at **subsonic** velocity.

If the body moves at **supersonic** velocity, a new type of drag appears. For convenience, consider the air nonviscous and assume that the motion of the body produces disturbances that can be considered small. At a certain distance from the moving body this second assumption will, in general, be satisfied. Consider, now, the body and the surrounding air inside of a cylindrical control surface as one mechanical system. Then one finds that, because of the concentrated action that characterizes the pressure propagation from a source moving at supersonic velocity, the total flux of momentum of the air masses entering and leaving the cylindrical boundary remains finite even when the boundary is removed to an arbitrarily large distance.

Figure 2 refers to the case of a **two-dimensional symmetric airfoil**, with sharp leading edge, moving through air initially at rest. Let us consider the flow through a plane parallel to the plane of symmetry at a certain distance from the body. The diagram shows the distribution of **induced velocities** and the **horizontal component of momentum transfer** along this plane for three cases. It is evident that the reaction of outgoing flow having a horizontal component opposite to the flight direction and incoming flow with a horizontal component in the flight direction is equivalent to a **propulsive thrust** acting on the body. Conversely, outgoing flow with a component in the flight direction and incoming flow directed opposite to the flight direction give rise to a drag. In the two subsonic cases ($M \sim 0$ and $M = 0.707$), thrust and drag contributions are balanced, and the total horizontal momentum transfer is equal to zero. The influence of increasing Mach Number is essentially increased magnitude of the induced velocities and the increasing concentration of the disturbance in the region extending laterally outward from the body. The increasing concentration of action is also illustrated by the pressure distribution on the control surface. In the supersonic case ($M = 1.414$), the disturbance is restricted

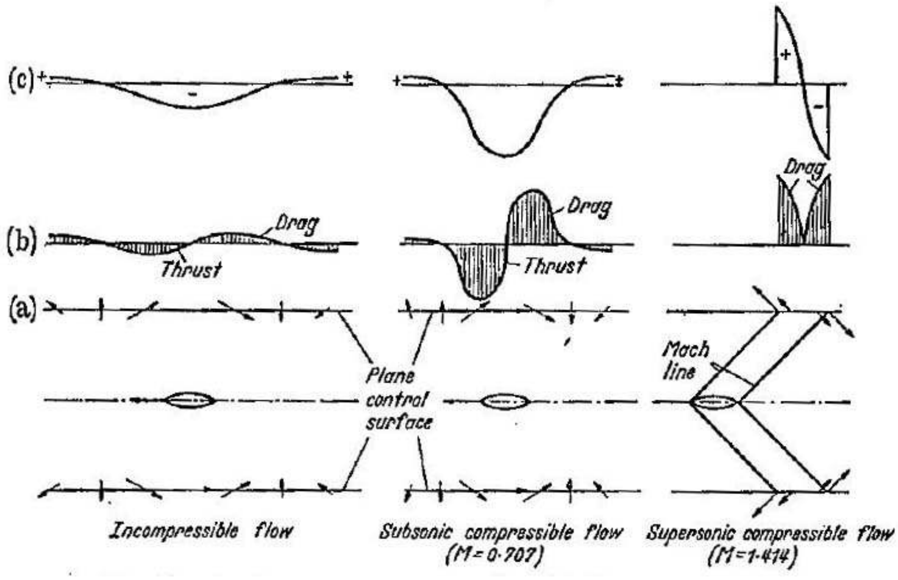

Figure 2. Two-dimensional symmetric airfoil in flight. (a) Velocities through control surface. (b) Horizontal momentum transfer through control surface. (c) Pressure at control surface.

to two strips bounded by two Mach lines. These lines are the intersections of planes that are envelopes of the Mach cones starting from points of /0.

the leading and trailing edges of the airfoil. The component of the outward flow is in the flight direction; that of the inward flow is opposite to the flight direction. Hence, both represent drag on the body. This type of drag is called "**wave drag**". The calculation of the wave drag is the first of the important problems to be solved by supersonic aerodynamics. It appears that one can obtain a fair approximation to the actual drag of a body moving at supersonic speed by addition of the calculated wave drag coefficient and a drag coefficient corresponding to friction and separation, extrapolated from subsonic data.

4. THE LINEARIZED THEORY OF DRAG

The theory of wave drag requires the solution of the following problem: A body is placed in an initially uniform and parallel airstream moving with supersonic velocity. What are the changes in the flow due to the presence of the body?

The exact solution of this problem requires the use of mathematical methods that in principle should be sufficient but in general involve much labor. Hence, approximate methods are of great value. The most important simplification consists of the linearization of the equations of flow by the assumption of small perturbations, more precisely of velocity perturbations that are small in comparison to both flight velocity and sound velocity. This theory gives a fair approximation for the drag of

slender or flat bodies with pointed noses or sharp leading edges. Fortunately, most parts of future supersonic airplanes or missiles will necessarily have such geometrical shapes.

The linearized theory can be carried out with relatively simple analytical means, since the **linearized equations** of the flow are essentially identical with the equations for **wave motion** of small amplitude. Thus, many known methods of the wave theory can be applied to such simplified supersonic aerodynamics. This is especially true in the case of slender bodies of revolution (for example, a fuselage of an airplane or the main body of a missile) and of flat bodies, like airplane wings. In these cases a further step of simplification can be made concerning the **boundary conditions** of the flow problem - i.e., the requirement that the air follow the surface of the body. This condition determines, in the case of axially symmetric flow, the direction of the velocity vector at the surface and, in the ease of a flat body, the direction of the component of the velocity vector that lies in the plane normal to the center surface. It is possible to solve the linearized differential equations and satisfy these boundary conditions exactly, but, in general, **numerical** or **graphical** methods must be applied. Hence, it is desirable to simplify the problem further by replacing the exact boundary conditions by conditions that, by a limiting process, are referred to the axis of the body of revolution or the plane plan form of the wing instead of the actual surface. The following results are based on this approximation. Strictly speaking, this approximation is the only one that is consistent with the assumptions of the linearized theory. If the boundary conditions are satisfied at the actual surface, terms of higher order are generally taken into account which are neglected in the differential equations.

(a) Two-Dimensional Flow

This case is that of a **wing** of infinite span. A **symmetric** wing section is assumed; wings producing lift will be discussed later.

In this case the linearized theory yields a simple result. The supersonic flow of velocity U produces at each surface element of the wing a pressure of the magnitude $\frac{2\delta}{\sqrt{M^2-1}}\cdot\frac{\rho U^2}{2}$, where ρ is the density of the air, δ is the local angle of attack, and M is the Mach Number of the flow (i.e., of flight). The remarkable simplicity of this result arises from the fact that the pressure acting at a surface element is independent of the shape of the rest of the section and depends only on the inclination of the element itself. It is known that in the case of subsonic motion there

is an interaction between all elements of the surface.

According to this simple result, the drag of a section of unit length in span direction can be expressed as the product of the pressure $\frac{\rho U^2}{2}$, the chord c, and a drag coefficient of the magnitude $C_D = \frac{4\bar{\delta}^2}{\sqrt{M^2 - 1}}$, where $\bar{\delta}^2$ is the mean square of the angles of inclination occurring at the surface of the section. For a double wedge section, $\bar{\delta}^2$ is equal to the square of the thickness-chord ratio. Thus, for example, for a 6 per cent double wedge section and for Mach Number $\sqrt{2}$, the wave drag coefficient is 0.0144, about twice the coefficient of the profile drag of a good subsonic section at low Mach Number.

In the case of two-dimensional flow, pressure effects are restricted to two strips whose angles of inclination with respect to the flow direction are equal to the Mach angle. With the approximation used, compressions and expansions are propagated with unchanged intensity along the Mach lines. As was mentioned previously, the transfer of momentum is also restricted to these strips. By calculating the action and reaction of the air passing through lateral surfaces, the preceding result concerning drag can easily be confirmed.

(b) Body of Revolution

One of the best-known methods for construction of an incompressible flow around a body of revolution uses the concepts of **sources** and **sinks**. These concepts can be carried over in the approximate theory of compressible flow, both subsonic and supersonic. The formula for the velocity potential of the flow produced in an incompressible fluid by a source located on the x-axis at the point $x = \xi$ is

$$\phi = \frac{Q}{4\pi} \frac{1}{\sqrt{(x-\xi)^2 + r^2}}, \tag{4.1}$$

in which x and r are cylindrical coordinates. Q is called the **intensity** of the source and is the total volume of fluid emitted by the source in unit time. This formula can be generalized by writing

$$\phi = \frac{A}{\sqrt{(x-\xi)^2 - (M^2 - 1)r^2}}. \tag{4.2}$$

The function ϕ, given by equation (4.2), is a solution of the linearized flow equations for arbitrary Mach Number M of the main flow. Figure 3 shows schematically a source defined by equation (4.2) for the three cases $M \to 0$, $M < 1$, and M >l. In accordance with the rules of the forbidden signals and zone of action

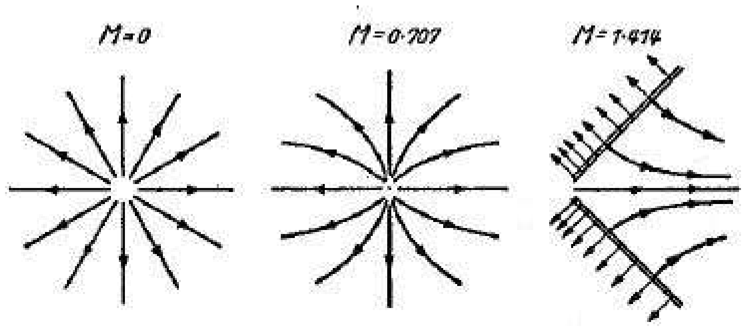

Figure 3. Streamlines of source flows used in the theory of compressible fluids.

and zone of silence, in the supersonic case the flow is restricted to the inside of the Mach cone. In fact, equation (4.2) gives real values for ϕ only in the interior of the Mach cone. The flow velocity is infinite along the surface of the cone itself. For this reason, no concentrated sources and sinks can be used in the supersonic theory. One has to construct line sources with continuously distributed intensity. Accordingly the application of this theory is restricted to body shapes with **pointed nose** and **pointed tail**. If the simplified boundary conditions are used, sharp corners in the meridian section must also be excluded.

With the simplified boundary conditions the theory yields the following results:

(1) The intensity of the source distribution along the axis of the body, $f(\xi)$, defined as the volume of fluid per unit: time per unit length of axis emitted at the location ξ on the axis, is given by

$$f(\xi) = U\frac{dS}{dx}, \qquad (4.3)$$

where U is the free stream velocity and S is the cross-sectional area of the body.

(2) The potential of the source distribution representing the body is

$$\phi = \frac{1}{2\pi}\int_0^l \frac{f(\xi)d\xi}{\sqrt{(x-\xi)^2-(M^2-1)r^2}}, \qquad (4.4)$$

in which x and ξ are length coordinates along the axis and l is the length of the body.

(3) The wave drag of the body is given by the formula

$$D_w = -\frac{\rho}{4\pi}\int_0^l\int_0^l dxd\xi f'(x)f'(\xi)\log|x-\xi|, \qquad (4.5)$$

where $f'(x) = \dfrac{df}{dx}$ and ρ is the density of air in the undisturbed flow.

It will be noted that there is a remarkable **analogy** between the **wave drag** of a

slender body of revolution and the **induced drag** of a loaded line. In fact, if the function $f(x)$ denotes the spanwise distribution of the circulation on a loaded line, the induced drag is given by the equation:

$$D_i = -\frac{\rho}{4\pi}\int_0^l \int_0^l dx d\xi f'(x) f'(\xi) \log|x-\xi|, \tag{4.6}$$

which is identical with equation (4.5). This analogy is useful to the aeronautical engineer who is well acquainted with the concept and theory of the induced drag.

The distribution of induced velocities and horizontal momentum transfer produced by a slender body in supersonic flow is shown in Figure 4. It is noted that in this case the disturbance caused by the body extends into the interior of the Mach cone starting from the rear end of the body. We recall that in the two-dimensional case the effect is confined to a strip.

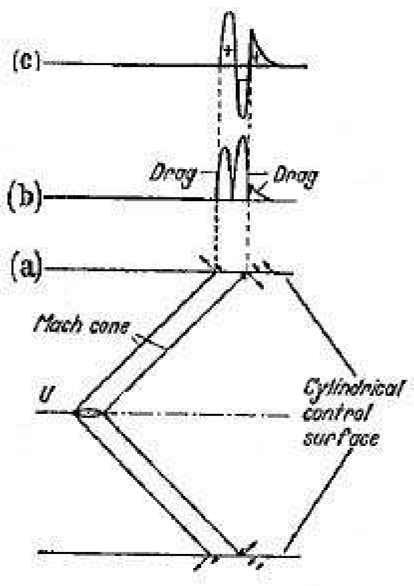

Figure 4. Body of revolution in supersonic flight. (a) Velocities at the control surface. (b) Transfer of horizontal momentum. (c) Pressure distribution at the control surface.

(c) Airplane Wing with Arbitrary Plan Farm and Thin Symmetrical Sections

The theory for this problem can be built up by using the concept of sources and sinks distributed continuously over the center-plane of the wing. One finds that in this case the surface density of the source of distribution is proportional to the angle of inclination of the wing surface measured in a vertical plane erected through the direction of flight. The pressure distribution over the wing and the total value of the wave drag can be calculated by the summation of the actions of all the sources.

R. T. Jones found useful methods for carrying out such summations, especially

for wings with sweepback and taper. He uses appropriately chosen oblique coordinates for this purpose.

Another promising method is given by the concept of Fourier analysis.

In this respect, a consideration that can be called the "**acoustic analogy**" is extremely useful. It was mentioned that within the linear approximation, the mathematical problem of the flow produced by the supersonic motion of a slender or flat body is identical to the problem of two-dimensional acoustic waves. In fact, if

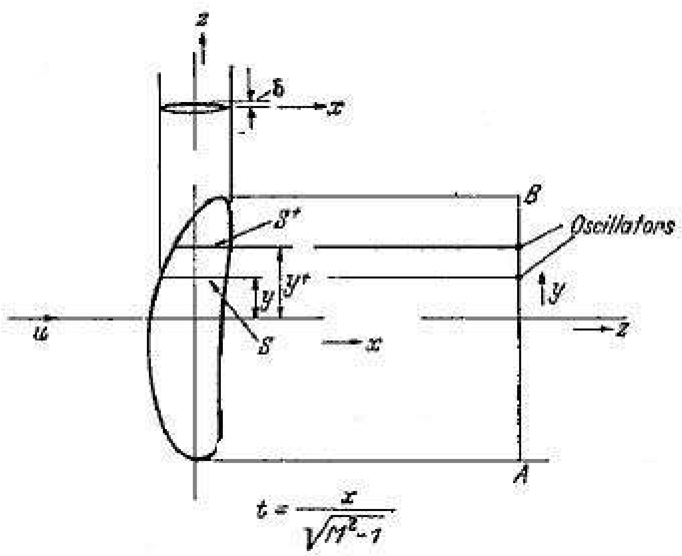

Figure 5. Schematic representation of a flat body with symmetric sections. Acoustic analog

one considers the length $\frac{x}{\sqrt{M^2-1}} = t$ as a. time coordinate, the three-dimensional flow produced by the presence of a thin airplane wing shown in Figure 5 can be described by the time history of a two-dimensional flow produced in the y,z-plane by appropriately chosen acoustic oscillators located along the line AB - i.e., the projection of the plan form of the wing on the y,z-plane. The effect of an arbitrary section can be replaced by a pulse communicated to the air by an oscillator; the time law of the pulse intensity is determined by the shape of the section. For example, in the case of a cylindrical wing normal to flow direction, all oscillators simultaneously carry out identical pulses, whereas sweepback means a phase delay in the action of the oscillators. Now it is known that a pulse can be replaced by an infinite number of elementary sinusoidal oscillations.

This concept leads to the application of **Fourier integrals** in the linearized supersonic wing theory and to the following main results:

(1) The "acoustic impulse" - i.e., the vertical velocity generated by the oscillator in the *y*,*z*-plane at any instant *t* - is proportional to the vertical velocity in the three-dimensional flow produced by the presence of a section, and therefore within the approximation used, proportional to the inclination of the surface of the section at the corresponding point *x*. Therefore, the first task is to express the distribution of the angle of inclination δ along an arbitrary section by a Fourier integral. This is done by the expression of δ in the form

$$\delta = a\int d\nu[f_1(\nu)\sin\nu t + f_2(\nu)\cos\nu t]. \qquad (4.7)$$

In this equation, *a* is a suitably chosen length parameter –for example, the half chord of one master section. The parameter ν, which is used as variable of integration, is inversely proportional to the wave length in the three-dimensional flow problem and proportional to the frequency of the oscillators in the acoustic analogy. The functions $f_1(\nu)$ and $f_2(\nu)$ give the amplitudes of the sine and cosine components of the Fourier analysis. With the exception of an infinite normal wing with constant section, f_1 and f_2 are also functions of the coordinate *y* (spanwise location); outside the span, $f_1 = f_2 = 0$.

(2) The **wave drag** of the wing appears in the acoustic analogy as the energy transmitted to infinity during the entire period of the process. Since one has to consider an infinite number of oscillators distributed along the span, or, more correctly, a chord oscillating in the two-dimensional *y*,*z*-plane, one has to compute the mutual interaction of the oscillators or the elements of the chord.

Consider two wing sections, *S* and *S** (Figure 5) at a distance |*y* - *y**|, measured spanwise, and replace these sections by oscillators. The amplitudes of the sine and cosine components as functions of the frequency are given by $f_1(\nu)$, $f_2(\nu)$ and $f_1{}^*(\nu)$, $f_2{}^*(\nu)$, respectively. Then it can be shown that the contribution of the two sections to the energy transmitted to infinity is proportional to the quantity

$$\int (f_1 f_1{}^* + f_2 f_2{}^*)\nu\tilde{J}_0(\nu\,|\,y - y^*|)d\nu\,,$$

where ν is the frequency parameter and $\tilde{J}_0$ denotes the Bessel function of zero order. This rather simple result indicates a straightforward way for computation of the wave drag of wings with various plan forms. The equation for the total wave drag can be written in the form:

$$D = \pi\rho U^2 a^2 \int\int dy dy^* \int \nu d\nu (f_1 f_1 + f_2 f_2{}^*)\tilde{J}_0(\nu\,|\,y - y^*|). \qquad (4.8)$$

5. SOME RESULTS OF: WAVE DRAG CALCULATIONS

(a) The Wave Drag of Normal Straight Wings with Constant Airfoil Section

The results for infinite aspect ratio were given in the foregoing section.

It is evident from the rule of the forbidden signals that the influence of the tips of a wing with finite aspect ratio is restricted to the inside of the Mach cones starting from the tip fronts.

The theory shows that the drag coefficient C_D can be expressed in the form:

$$C_D = C_{D_0}\phi(A.R.\sqrt{M^2-1})$$

where C_{D_0} is: the wave drag coefficient of the two-dimensional airfoil of the same section and A.R. is the geometrical aspect ratio. The parameter $A.R.\sqrt{M^2-1}$ can be considered as the effective aspect ratio parameter. It is equal to the ratio of the span to the length c^* shown in Figure 6, which is actually the portion of the trailing edge within the Mach cone starting from the tip of the leading edge. The function ϕ is equal to unity when $A.R.\sqrt{M^2-1} \geq 1$ and decreases with the aspect ratio when $A.R.\sqrt{M^2-1} < 1$. Hence, the aspect ratio influences the average drag coefficient only if the entire trailing edge is within both Mach cones, which start from the two tips of the leading edge.

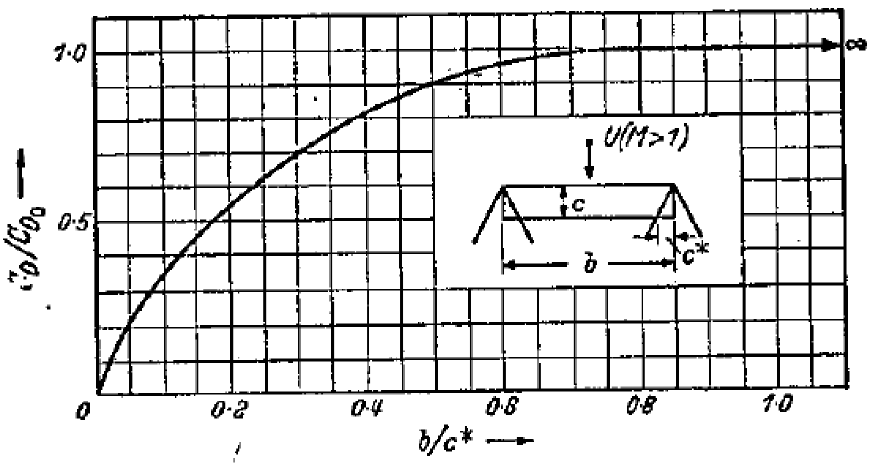

Figure 6. Wave drag coefficient of normal straight wings as a function of the aspect ratio.

Figure 6 shows the function ϕ for the special case of a wing with double wedge section. The distribution of the sectional wave drag coefficient C_d is shown in Figure 7 for $A.R.\sqrt{M^2-1}=2$. The sectional wave drag coefficient C_d is

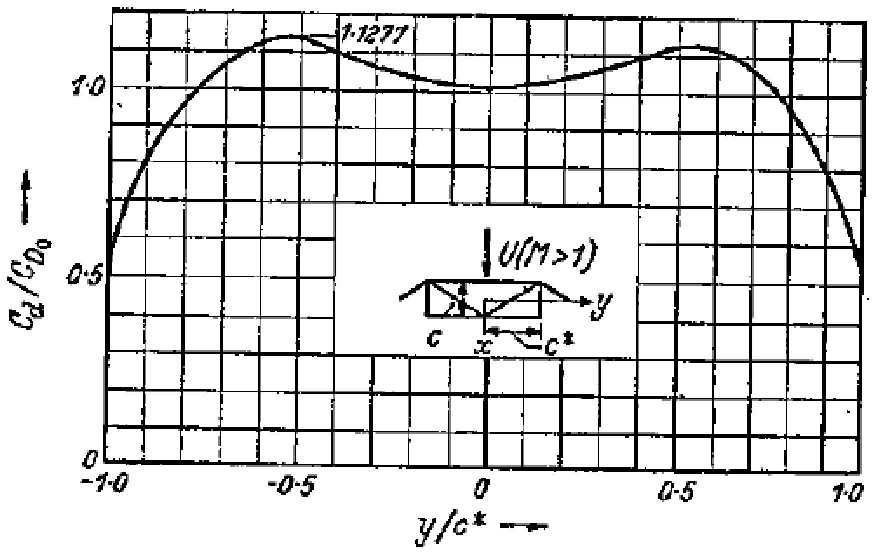

Figure 7. Spanwise distribution of the sectional wave drag coefficient of a normal wing ($A.R.\sqrt{M^2-1}=2$).

connected with C_D by the relation

$$C_D = \frac{1}{b}\int_0^b C_d\, dy\ ,$$

where *b* is the span and *y* is measured along the span. At either tip C_d is only one-half of the two-dimensional value. For a rectangular wing of geometrical aspect ratio 2, this figure corresponds to $M=\sqrt{2}$. For a wing of geometrical aspect ratio 8, it corresponds to $M=\sqrt{1.0625}$. For $A.R.\sqrt{M^2-1}>2$, the drag distribution along each portion of the tip length equal to *c** is identical to that shown in the diagram, whereas for the portion of the wing between the tips $C_D / C_{D_0} = 1$. If properly used, this simple diagram furnishes the drag distribution for wings of any aspect ratio at any Mach Number larger than unity.

(b) Infinite Uniform Wing with Sweepback

The pressures acting on an infinite uniform wing with sweepback depend only on the component of the main flow normal to the wing axis. This is made evident by the consideration that in a. nonviscous fluid the velocity component parallel to the wing axis is undisturbed by the presence of the wing and therefore cannot have any influence on pressure and drag.

One concludes that the wave drag of a uniform wing with infinite span is zero if the sweepback angle is so large that the velocity normal to the wing axis becomes subsonic.

This occurs when $\tan\gamma > \sqrt{M^2 - 1}$, where γ is the angle of sweepback. The ratio $\beta = \dfrac{\tan\gamma}{\sqrt{M^2 - 1}}$ is one of the fundamental parameters of supersonic aerodynamics. It may be called the **effective sweepback parameter**. When $\beta > 1$, the velocity normal to the wing is subsonic; when $\beta < 1$, it is supersonic.

It follows from elementary considerations that, for $\beta < 1$, the drag coefficient of an infinite wing with sweepback is given by

$$C_D = C_{D_0} \cos\gamma \frac{\sqrt{M^2 - 1}}{\sqrt{M^2 \cos^2\gamma - 1}} = \frac{C_{D_0}}{\sqrt{1 - \beta^2}}, \qquad (5.1)$$

where C_{D_0} is the wave drag coefficient of a normal wing with the same section and same chord measured in flight direction.

For $\beta > 1$, $C_D = 0$. Figure 8 shows the drag coefficient for two wings with infinite span: one without sweepback ($\gamma = 0$) and one with 45° sweepback. It is seen that the occurrence of wave drag is postponed from M =1 to $M = \sqrt{2} = 1.414$. However, for $M > \sqrt{2}$, the drag of the wing with sweepback is always larger than that of the normal wing.

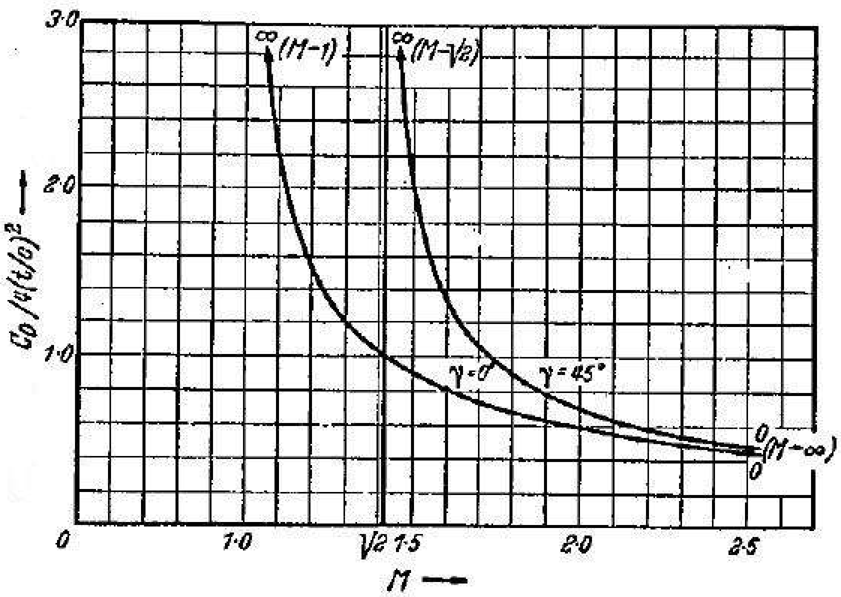

Figure 8. Wave drag coefficients of two infinite wings (zero and 45° sweepback).

(c) Half Infinite Wing with Sweepback and Wing with Finite Aspect Ratio

The equation for the wave drag of a half infinite uniform wing with double

wedge section and sweepback in the case of $\tan\gamma > \sqrt{M^2 - 1}$ is the following:

$$D = \rho U^2 c^2 \left(\frac{t}{c}\right)^2 \frac{2\log 2}{\pi} \frac{\sin\gamma\cos^2\gamma}{[1 - M^2\cos^2\gamma]^{3/2}} . \qquad (5.2)$$

In this equation γ is the sweepback angle, c is the chord measured in flight direction, and t is the thickness of the section. It is seen that the wing has finite wave drag, although the span is infinite. The drag coefficient, referred to an area equal to c^2 as function of Mach Number, is shown in Figure 9 for $\gamma = 45°$. When $M > \sqrt{2}$, the drag is, of course, infinite.

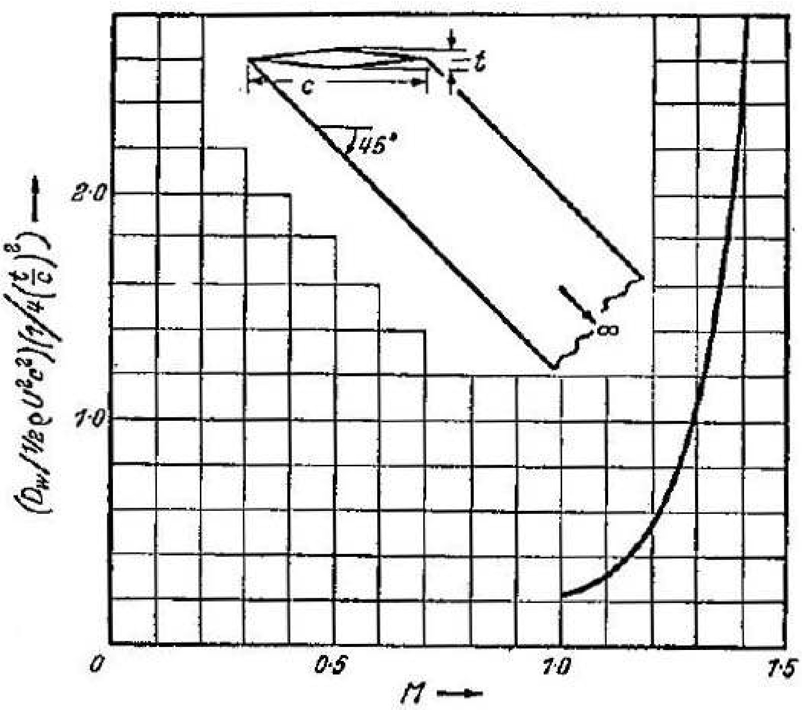

Figure 9. Drag coefficient of a half infinite wing with sweepback (referred to an area equal to the square of the chord).

In the case of sweepback and finite aspect ratio, the drag coefficient will be a function of the two parameters $A.R.\sqrt{M^2 - 1}$ and β. For example, for the double wedge wing

$$C_D = \frac{4(t/c)^2}{\sqrt{M^2 - 1}} \phi(A.R.\sqrt{M^2 - 1}, \beta) .$$

The parameter $A.R.\sqrt{M^2 - 1}$ is the ratio of the span to the length $c^* = c\tan\alpha$ (α = Mach angle). The geometrical meaning of c^* is analogous to that defined

before: the radius of the base of the tip of the Mach cone measured from the tip of the trailing edge.

The behavior of the drag coefficient is different for $\beta > 1$ and $\beta < 1$ - i.e., according to whether the velocity component normal to the wing axis is subsonic or supersonic. In Figure 10 the drag coefficients of three wings with different sweepback are compared with that of a normal wing. The Mach Number chosen for this comparison is equal to $M = \sqrt{2}$; the three sweepback angles are equal to 26.6°, 45°, and 63.4°. If the sweepback is equal to 45°, the velocity component normal to the wing axis is equal to sound velocity. The drag coefficients are represented versus geometrical aspect ratio. It is seen that the behavior of the drag curve for 26.6° sweepback ($\beta = 1/2$) is similar to that found for the normal wing, with the

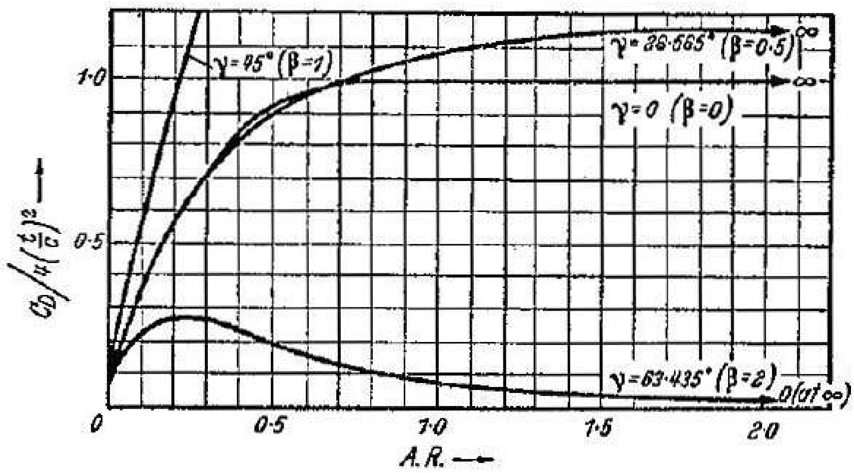

Figure 10. Wave drag coefficient as functions of the aspect ratio for four wings with different sweepbacks. (M=1.414)

exception of a larger asymptotic value for the two-dimensional case and an extension of the tip effect to larger aspect ratios. The wing with 45° sweepback ($\beta = 1$) is, of course, unfavorable for this Mach Number. The drag coefficient increases to infinity for the two-dimensional case. A wing with 63.4° sweepback ($\beta = 2$) is extremely favorable as to drag in comparison with the normal wing. It is seen, for example, that, for *A.R.* =1, $C_D / C_{D_0} = 0.092$ - i.e., the sweptback wing has only 9 per cent of the drag of a normal wing of the same geometrical aspect ratio. It was mentioned that the theoretical wave drag coefficient of a normal double-edge wing, at the Mach Number considered, is equal to 0.0144; the wave drag coefficient of the same wing with 63.4° sweepback would be only 0.00132. For 26.6 and 45° sweepback, $C_D / C_{D_0} = 1.15$ and 2.38, respectively; hence, the corresponding drag

coefficients would be equal to 0.0165 and 0.0343.

Figure 11 is instructive as to the mechanism of the drag in the case of subsonic normal velocity component. The diagram represents the distribution of the section drag coefficient compared with the drag coefficient of a two-dimensional normal wing at the same Mach Number. The abscissa is the ratio *y*/*c**. The calculation was carried out for $A.R.\sqrt{M^2-1} = b/c^* = 2$ and $\beta = 2$. It is seen that the effects of the upstream and downstream wing tips are clearly separated. The effect of the

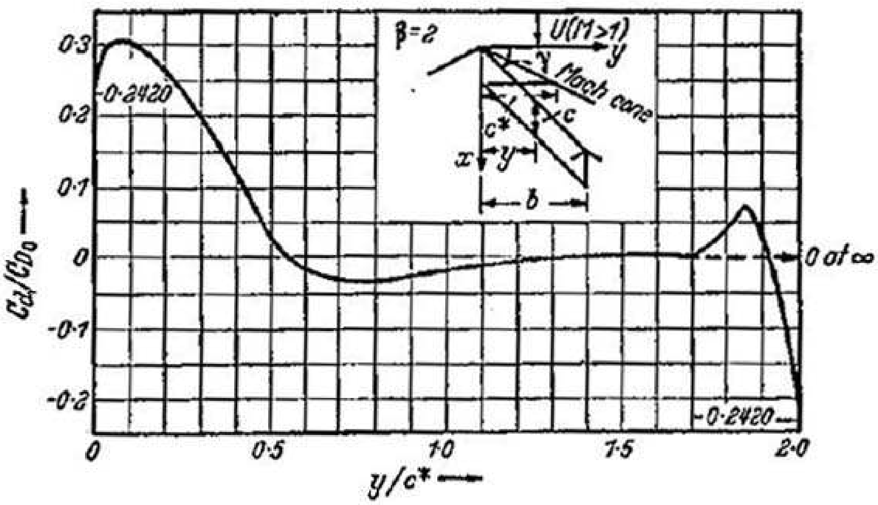

Figure 11. Spanwise distribution of the sectional wave drag coefficient for wings with sweepback.

upstream tip is significant over a portion of the span slightly greater than *c**. The amount of drag corresponding to this effect is about equal to the drag of the half infinite wing with the same sweepback. The effect of the downstream wing tip extends to a relatively small portion of the span equal to $c^*/(1+\beta)$. The drag resulting from this tip effect is zero.

These considerations show a relatively easy way to calculate with good approximation the drag of a sweptback wing with arbitrary aspect ratio, provided $A.R.\sqrt{M^2-1} > 1$. Since in this case the total drag is about the same as the tip drag of a half infinite drag, one obtains from equation (5.2)

$$C_D = 4\left(\frac{t}{c}\right)^2 \frac{1}{\sqrt{M^2-1}} \frac{\log 2}{\pi} \frac{\beta}{(\beta^2-1)^{3/2} A.R.}. \qquad (5.2a)$$

This approximate value is shown by the dotted line in Figure 12 for the value of the sweepback parameter $\beta = 2$. The full line gives the exact value of the ratio C_D / C_{D_0} for the same case. The approximation furnished by equation (5.2a) is

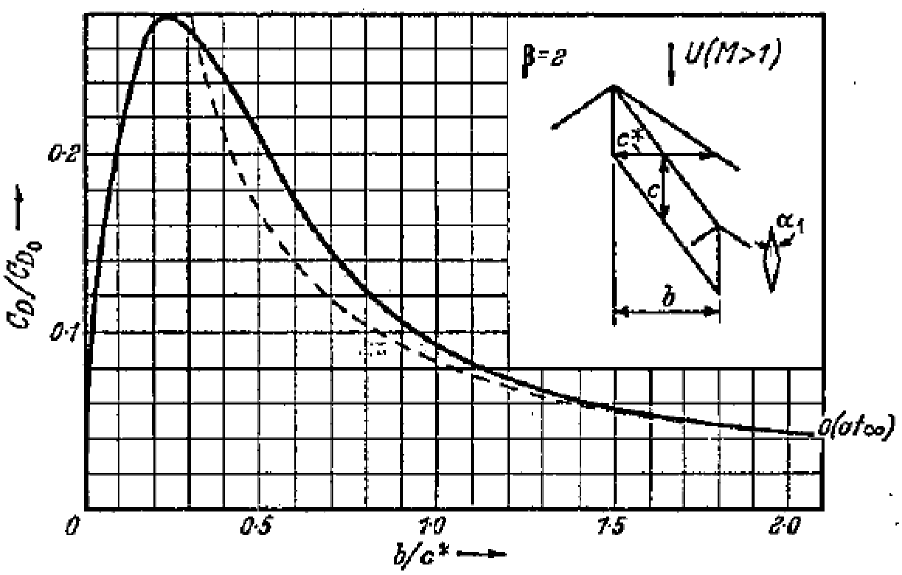

Figure 12. Wave drag coefficient as a function of the aspect ratio for wings with sweepback ($\beta = 2$). Dotted line corresponds to approximation by equation (5.2a).

satisfactory almost down to $A.R.\sqrt{M^2-1} \cong 0.3$. If the aspect ratio is further reduced C_D / C_{D_0} reaches a peak and after that decreases to zero. It is evident that, in the immediate neighborhood of the point $A.R.\sqrt{M^2-1} = 0$, C_D / C_{D_0} becomes independent of β - i.e., wings of any aspect ratio behave in the same way.

The following consideration is of interest: Let two identical wings with the same sweepback be located on a common axis. If the two wings have sufficiently large aspect ratio and are sufficiently far apart, each one has its own wave drag of equal amount. If the two wings approach each other, there is an interference that reduces the drag of the rear wing. When the two wings are matched together tip to tip, the drag of the rear wing entirely disappears. A somewhat analogous interference effect is well known in the theory of the induced drag in subsonic flow.

(d) Arrowhead Wings

The equation for the wave drag of all infinite arrowhead wing in the case of $\tan\gamma > \sqrt{M^2-1}$ is the following:

$$D = \rho U^2 c^2 \left(\frac{t}{c}\right)^2 \left(\frac{2\log 2}{\pi}\right) \left[\frac{(1+2\sin^2\gamma - M^2\cos^2\gamma)\cos^2\gamma}{\sin\gamma(1-M^2\cos^2\gamma)^{3/2}}\right]. \quad (5.3)$$

Figure 13 shows the value of $\dfrac{C_D}{4(t/c)^2}$ as a function of Mach Number for an arrowhead wing with 45° sweepback and aspect ratio A.R. = 8. It is seen that the drag coefficient has a finite peak for $M = \sqrt{2}$, i.e., at the Mach Number that corresponds to $\beta = 1$. For $\beta < 1$ - i.e., when the sweepback is not sufficiently large to cause subsonic flow normal to the wing axis the conditions can be described as follows: For $M >$ 1.61, the drag coefficient of the arrowhead wing is equal to that of a single two-dimensional wing with the same span and sweepback. In this case $C_D = \dfrac{4(t/c)^2}{\sqrt{M^2 - 2}}$ [compare equation (5.1)]. Between M =1.5 and 1.61, this approximation is still satisfactory. Between M =1.41 and 1.5, the approximate formula leads to too large values, tending to infinity for $M = \sqrt{2}$.

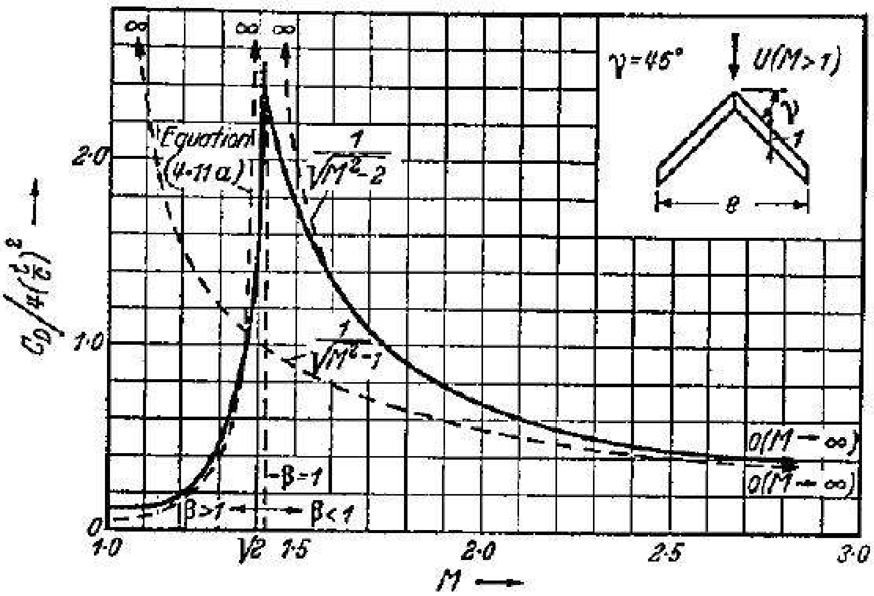

Figure 13. Wave drag coefficient of an arrowhead wing with large aspect ratio (A.R.= 8) as a function of Mach Number.

For $\beta > 1$ - i.e., M $< \sqrt{2}$ - one obtains a fair approximation (with the exception of the immediate neighborhood of M =1 and $M = \sqrt{2}$) if one divides the drag given by equation (5.3) for the arrowhead wing with infinite aspect ratio by the actual wing surface. This° leads, with $\gamma = 45°$, to the approximate formula

$$C_D = 4\left(\frac{t}{c}\right)^2 \left[\frac{\log 2(4 - M^2)}{\pi \cdot A.R. \cdot (2 - M^2)^{3/2}}\right]. \qquad (5.3a)$$

The values corresponding to equation (5.3a) are shown in Figure 13 by a dotted line. For comparison, the values of $\dfrac{C_D}{4(t/c)^2}$ for a normal wing with infinite aspect ratio are also shown. The minimum value of the drag occurs at about $M = 1.08$. For 6 per cent thickness ratio, $C_{D_{\min}} = 0.00082$. It is seen that this value is extremely low, so that, practically, such a wing combination has negligible wave drag.

Figure 14 shows the corresponding results for A.R. = 2. The drag coefficient at large Mach Numbers - namely, for $M > 2.236$ - is again equal to that of a two-dimensional wing with the same sweepback. The peak at $M = \sqrt{2}$ ($\beta = 1$) is considerably lower than for A.R.=8. However, the minimum, which occurs at about the same Mach Number as for A.R. = 8, is much larger. It is equal to about $C_D = 4\left(\dfrac{t}{c}\right)^2 \times 0.47$, or for 6 per cent thickness ratio, $C_{D_{\min}} = 0.00678$. To be sure, the reduction as compared with the normal wing is still great.

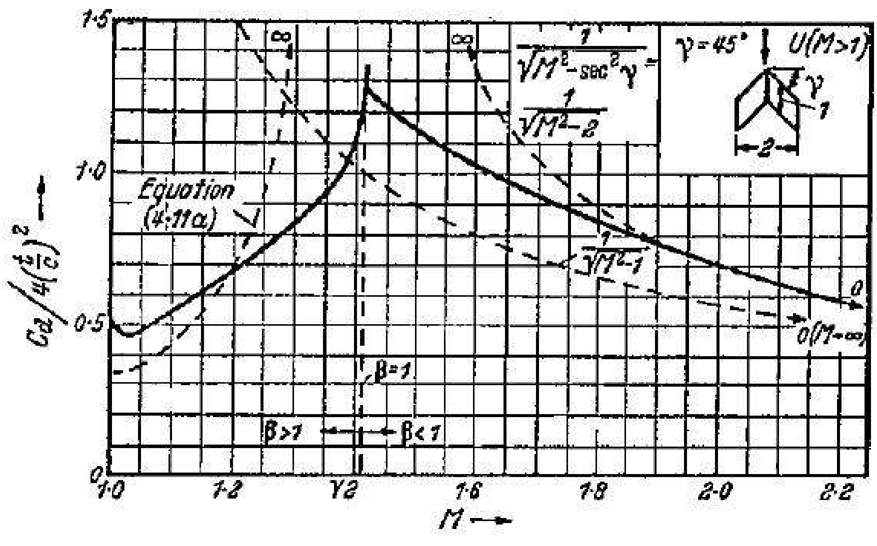

Figure 14. Wave drag coefficient of an arrowhead wing with small aspect ratio (A.R. = 2) as a function of Mach Number.

It is interesting to note that the drag of an arrowhead wing remains unchanged when the flight direction is reversed. However, the spanwise drag distribution is different. For example, in the case of sweptback wings of sufficiently large aspect ratio, the total drag acts on the center section, whereas **in the case of sweep forward,**

a considerable portion of the drag acts on the tips.

The invariance of drag with respect to reversal of flight direction is a special case of a general result of the linearized wave drag theory. The **wave drag is independent of flight direction** in any case in which the source distribution representing the flow is the same. Since, within the approximation used in this theory, the source distribution is inverted but not changed when the body is turned around relative to the flight direction, the theorem that the drag is independent of flow direction applies to a body of arbitrary shape. The body can be a flat one, like an airplane wing, or it can be a body of revolution. However, one should keep in mind that this is only true **within the limits of validity of the linearized theory** with approximate boundary conditions.

6. THE MECHANISM OF LIFT

The theory of lift of an airplane wing moving at subsonic velocities is based on the concept of **circulation**. The origin of the circulation can be described in the following way. Consider an airplane wing initially at rest and suddenly given a forward velocity. The equations of motion admit a solution that represents a flow without circulation and therefore without lift. However, this flow has infinite velocity around the sharp trailing edge of the airfoil section. Since some viscosity always exists, the flow separates with consequent formation of a vortex, called the "**starting vortex**". The reaction to the starting vortex produces circulation around the airfoil; the final magnitude of the circulation is determined by the condition of smooth flow at the trailing edge, called the **Kutta-Joukowski condition**. The condition of smooth flow is equivalent to the statement that the lift per unit area is zero at the trailing edge; the pressures acting on the edge at the upper and lower surface of the airfoil are equal.

It is easily seen that this process is, in general, impossible in the case of supersonic motion. Consider, for example, the two-dimensional case - i.e., the case of an airplane wing of infinite span - the trailing edge being normal to the flow direction. It is evident that, according to the rule of forbidden signals, no process at the trailing edge can have an effect upstream. Correspondingly, the lift density at the trailing edge may well have a finite value. This also follows from **Ackeret's rule** that the pressure produced at every element of the surface depends on the local angle of attack. Therefore, for example in the case of a flat plate with an angle of incidence *i*, one obtains a uniform pressure of the amount $\frac{2i}{\sqrt{M^2-1}} \cdot \frac{\rho U^2}{2}$ acting

on the lower surface and a uniform suction of the same magnitude on the upper surface (Figure 15). This causes a discontinuity of pressure at the trailing edge, the effect of which, however, is restricted to the flow downstream. Hence, a flow around the airfoil which would equalize the pressures at the trailing edge cannot be created.

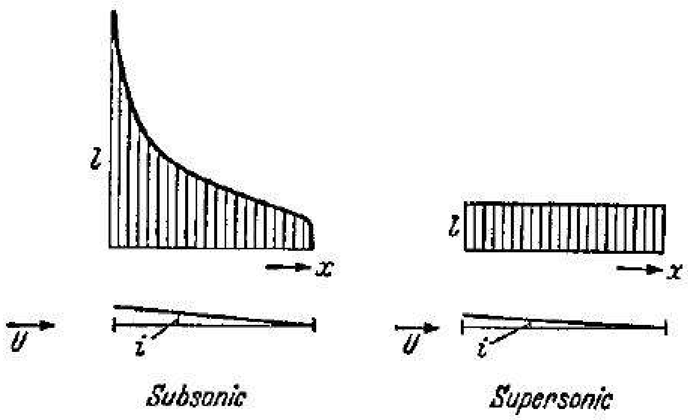

Figure 15. Lift distribution over the chord of a flat plate.

It is seen that the lift distribution near the leading edge is also different in both cases. In the **subsonic** case, the theory gives **infinite lift density** for a plate with mathematically sharp leading edge. This is consistent with the fact that the air is supposed to flow around the leading edge with infinite velocity. One might object to the inconsistency that the theory admits infinite velocity at the leading edge and forbids the same at the trailing edge. The reason is that, although the fluid separates at the sharp leading edge, it will again become attached to the upper surface, at least below the stalling angle. In the case of suitably rounded leading edge, the flow will follow the surface continuously. The flow around the edge produces negative pressure, which in the simplified theory of the flat plate or infinitely thin wing is taken into account by the assumption of a concentrated force at the nose. This force balances the horizontal component of the resulting pressure forces on the plate and makes the drag of a two-dimensional wing in a perfect fluid equal to zero, as required by **d'Alembert's theorem**.

In the **supersonic** case - more exactly, if the flow velocity normal to the leading edge is supersonic - no such flow occurs around the leading edge. The pressure difference between upper and lower surfaces will be finite, and we do not have the benefit of a suction force acting at the leading edge. The resultant of the pressure forces is normal to the plate, and its horizontal component represents an actual drag.

The theory of lift of a wing with finite span moving at subsonic speed uses particular solutions of the linearized flow equations which represent elementary **horseshoe vortices**. A horseshoe vortex consists of the so-called **bound vortex**, which carries the lifting force, and two free vortices. These latter create the induced velocities (Figure 16). It is known that the kinetic energy of‘ the two free vortices, which remains in the air behind the moving wing, represents the work done against the induced drag - i.e., the work necessary to create lift.

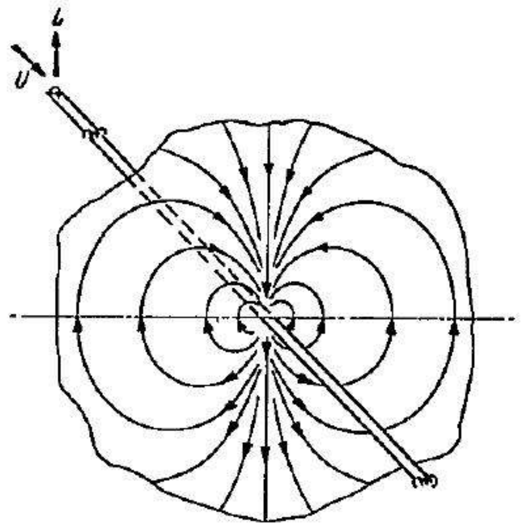

Figure 16 Horseshoe vortex in subsonic flow.

In the supersonic case it is possible to build up a flow by means of analogous particular solutions of the wave equation. Each such solution represents a concentrated lift force. The elementary flow pattern in this case is restricted to the surface and interior of the Mach cone whose vertex lies at the point of application of the lift force (Figure 17). The lines indicating the direction of flow in a plane normal to the main flow direction are also shown in the figure. It is seen that, at a large distance behind the wing, the flow in the neighborhood of the axis is identical to that produced by a subsonic horseshoe vortex. One concludes from this consideration that the induced drag exists in the supersonic case as well. Since the wave drag is also due to induced velocities, in the supersonic case, the specific expression “**induced vortex drag**” appears more appropriate. One finds, in fact, that the relation between induced vortex drag and lift distribution is the same for subsonic and supersonic flow, at least within the approximation corresponding to the linearized theory.

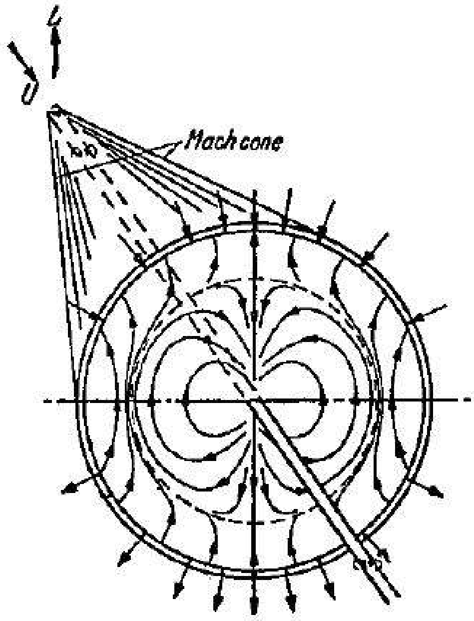

Figure 17 Horseshoe vortex in supersonic flow.

However, in the supersonic case the production of lift requires, in addition to the induced drag, **a certain amount of wave drag corresponding to the energy** radiated to infinity along the Mach cone. This drag is found to be **proportional to the square of the lift** produced - i.e., it follows a similar law as the induced drag. The drag acting on the flat plate, which was discussed previously, is evidently a simple example of the wave drag produced by lift.

It is interesting to compare the pressure produced by a wing moving at subsonic and supersonic velocity over the ground (Figure 18). Consider first, for the sake of simplicity, the wing of infinite span. In the case of $M \to 0$ - i.e., at a velocity that is small in comparison with sonic velocity - the weight of an airplane wing flying over the ground is carried by pressures created on the ground and distributed over a large area. The center of the pressures lies vertically under the center of gravity of the airplane, so that the weight of the airplane and the resultant of the pressures acting on the ground are in equilibrium. For increasing Mach Numbers, one finds that the area over which the bulk of the pressure is distributed becomes narrower and narrower. In the **supersonic** case, the pressure effect below the wing is restricted to a strip inclined at the Mach angle and intersecting the ground at a certain distance behind the plane. If the plane flies at high altitude, this distance is evidently extremely large. It is seen that in this case the equilibrium of the forces is restored by

the additional pressure effects shown in Figure 18. In fact, if the weight of the plane per unit span is denoted by W, the resultant of the pressures acting on the ground is equal to W and, together with the weight, constitutes a couple. Consider now a horizontal control plane above the airplane. It is seen that a total negative pressure force equal to $W/2$ is acting on the intersection of the horizontal plane and the wavestrip extending from the wing upwards, and an equal amount of positive pressure force is acting on the intersection of the plane and the wavestrip reflected from the ground. It is easily seen that the level arms of the two couples are in the ratio 1 to 2, so that the whole system is in equilibrium.

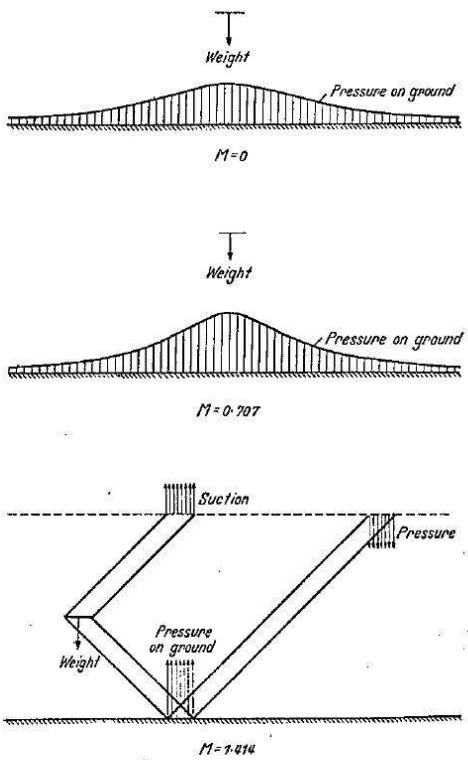

Figure 18. Pressure distribution on the ground produced by an airfoil with infinite span.

Similar considerations can readily be made for the case of wings with finite span (Figure 19).

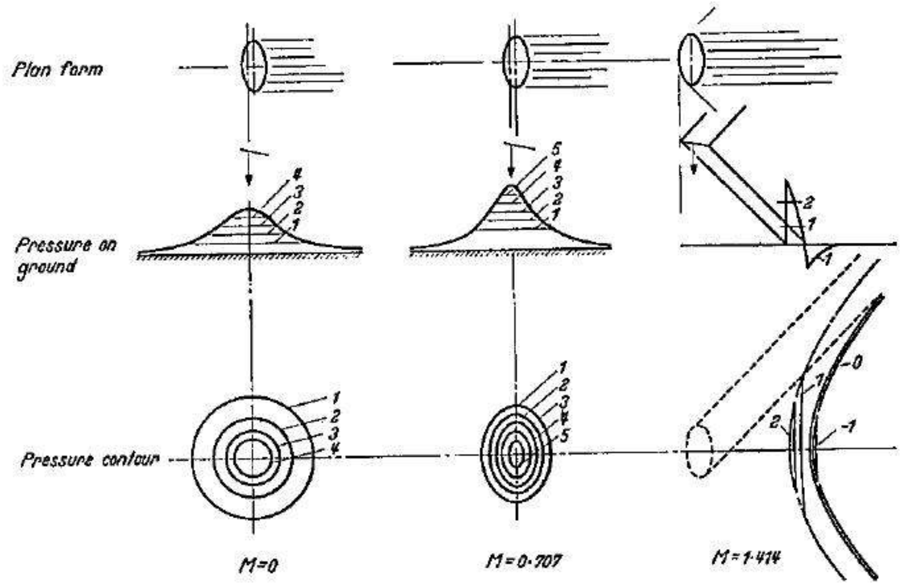

Figure 19. Pressure distribution on the ground produced by an airfoil with finite aspect ratio.

7. THE LINEARIZED THEORY OF LIFTING SURFACES

(a) Two-Dimensional Case

From the fundamental fact that the supersonic flow of the velocity U produces a normal pressure equal to

$$\frac{2\delta}{\sqrt{M^2-1}}\frac{\rho U^2}{2},$$

where δ denotes the local angle of inclination of the surface, one easily obtains the equations for the lift and drag coefficients of an arbitrary thin section as follows:

$$C_L = \frac{2(\bar{\delta}_u + \bar{\delta}_l)}{\sqrt{M^2-1}}, \tag{7.1}$$

$$C_D = \frac{4\bar{\delta}^2}{\sqrt{M^2-1}}. \tag{7.2}$$

In equation (7.1), $\bar{\delta}_u$ and $\bar{\delta}_l$ denote the angles of inclination at an arbitrary point of the upper and lower surfaces, respectively. For a symmetric section with the angle of incidence i, equations (7.1) and (7.2) yield the results:

$$C_L = \frac{4i}{\sqrt{M^2-1}}, \tag{7.3}$$

$$C_D = \frac{4}{\sqrt{M^2-1}}(i^2 + \bar{\delta}^2). \tag{7.4}$$

For a double wedge section of thickness ratio t/c, one obtains

$$C_D = \frac{4}{\sqrt{M^2-1}}(i^2 + (t/c)^2).$$

It is seen that, if the drag due to friction and flow separation is neglected, the maximum of the lift drag ratio occurs when $i = t/c$, and its value is equal to half the reciprocal of the thickness ratio.

(b) Lifting Surface with Finite Span

Both methods presented in Section 4 can be applied to the calculation of the flow produced by a lifting surface, provided the lift distribution over the plan form of the wing is given. Since, in the approximation used, the flow caused by the finite thickness of a wing can be separated from the flow caused by lift forces, the lifting surfaces can be considered to be without thickness.

Instead of the sources and sinks used in the theory of drag, it is necessary to use, in the theory of the lifting surface, the elemental solution sketched in Figures 16 and 17. The potential function corresponding to this flow pattern can-be written in cylindrical coordinates (x, r, and θ)

$$\phi = \frac{\cos\theta}{r} \frac{x}{\sqrt{x^2 - r^2(M^2 - 1)}} . \tag{7.5}$$

This expression corresponds to the effect of a concentrated lifting force of the amount $2\pi\rho U$. By computing the vertical velocity produced by such flow patterns continuously distributed over the plan form of the wing, it becomes possible to compute the **camber distribution** that would produce the **given lift distribution**. Also, since the flow is completely known, the wing drag can be computed. Finally, the induced drag is calculated in the classical way and the problem is so far solved.

The method of the **Fourier integrals** will greatly simplify these calculations. The results analogous to those presented in Section 4 are the following:

(1) One starts with the Fourier analysis of the lift distribution and expresses the lift per unit area distributed over an arbitrary section in the form

$$l = 2\rho U^2 \frac{a}{\sqrt{M^2 - 1}} \int_0^\infty d\nu (g_1 \cos \nu t - g_2 \sin \nu t) . \tag{7.6}$$

(2) Then the contribution of two arbitrary sections S and S* to the total wave drag of the wing is given by the equation:

$$D = \frac{\pi\rho U^2 a^2}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy dy^* \int_0^\infty d\nu (g_1 g_1{}^* + g_2 g_2{}^*) \frac{\tilde{J}_1(\nu | y - y^* |)}{| y - y^* |} \tag{7.7}$$

where $\tilde{J}_1$ denotes the Bessel function of the first kind and the order one.

(3) In this case, also, the **acoustic analogy** can be easily constructed. A lift distributed over a section corresponds to a pulse transmitting vertical impulses to the air over a limited period of time. Every section represents an oscillator. If we imagine that the impulse is transmitted by the motion of a physically existing surface, this, of course, has to disappear after the pulse is terminated.

It can easily be seen that in this case the energy transmitted to the fluid consists of two parts. A certain amount of energy will be radiated to infinity; another part remains in the fluid in the domain in which the impulse has been transmitted. The first amount of energy corresponds to the **wave drag**; the second amount, to the **induced vortex drag**.

(4) The wing camber distribution that produces a given lift distribution can be calculated by the Fourier method without intrinsic difficulty.

However, the direct problem, which in many cases may be more interesting to the designer - namely, the determination of the lift distribution for a given wing shape (i.e., given plan form, camber, and angle of attack)-is much more difficult.

The exact solution of the analogous problem causes great difficulties in the subsonic case as well. However, in the subsonic case the **concept of the lifting line** gives a fair approximation, at least for wings with large aspect ratio and without large sweepback. Unfortunately, the concept of lifting line cannot be used in the supersonic case, since it yields infinitely large velocities at the lifting line and an infinite amount of wave drag. Because of this fact, a satisfactory general solution will require much more analytical and numerical work than in the subsonic ease.

The conclusion of Section 5 concerning the **absence of wave drag** for wings of infinite span and sufficiently large sweepback applies also to the theory of lifting surfaces. In fact, one sees immediately that, if the sweepback angle is larger than 90° minus α, where α is the Mach angle, the flow conditions must be the same as over a wing moving normal to its axis with subsonic velocity.

Hence, one has to conclude that the statement concerning the absence of a Kutta-Joukowski condition in supersonic flow cannot be generally true. One has to introduce the **concept of supersonic and subsonic trailing edges**.

For example, it is evident that in the case of the wing with large sweepback, signals emitted from the trailing edge may cover a certain part of the wing plan form and reach even a portion of the leading edge. Consequently, the statement that trailing edge conditions cannot influence the flow over the wing is evidently not correct in such a case.

The fundamental ideas entering the problem can best be illustrated by considering a wing with elliptic plan form (Figure 20). The tangents to the ellipse parallel to the flow direction and to the direction of the Mach lines subdivide the boundary of the wing plan form into four types:

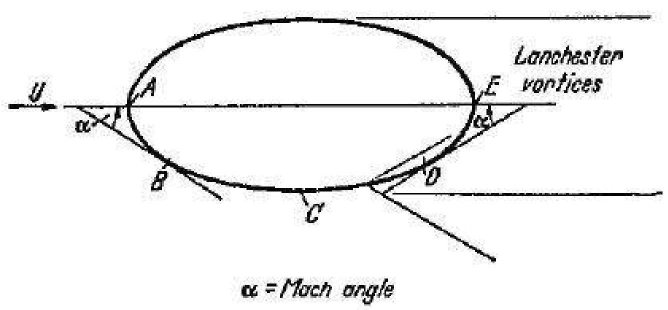

Figure 20. Schematic diagram for a wing with subsonic and supersonic leading and trailing edges.

(1) The leading edge from *A* to *B* will be called a **supersonic leading edge**. The local flow conditions are the same as at the leading edge of a supersonic normal wing with infinite span. One has at the leading edge a finite lift per unit surface, determined by the normal component of the flow velocity and the local angle of attack.

(2) The boundary *B* to *C* will be called a **subsonic leading edge**. As in the case of a sharp-edged flat plate in subsonic flow, the lift density at the leading edge is infinite but integrable. If an infinite lift density exists, a horizontal suction force will also appear at the leading edge.

(3) The portion of the boundary between *C* and *D* has the **character of a trailing edge in subsonic flow**. It is seen that signals emitted from points between *C* and *D* cover a portion of the plan form, and they can produce an additional flow that secures smooth flow at this portion of the trailing edge, fulfilling the Kutta-Joukowski condition. In other words, if there are portions of the trailing edge at which the component, of the main flow velocity normal to the trailing edge is subsonic, the solution of the flow problem is only unique provided a condition similar to the Kutta-Joukowski condition is imposed. This can be done by requiring that the lift density be zero along the line *C-D*.

(4) Finally, the boundary *D-E* is a **supersonic trailing edge**. In general, the lift density will be finite along *D-E*.

Along the whole trailing edge - i.e., between the points at which tangents parallel to the flow direction touch the plan form - **Lanchester-type vortices** originate and extend to infinity in the wake of the wing.

8. ESCAPE FROM WAVE DRAG. INTERFERENCE, SWEEPBACK, DELTA WING.

It was shown in the foregoing sections that, in general, supersonic motion of bodies causes a wave drag that does not exist in subsonic motion. In the case of **slender bodies** and **thin symmetric airfoils**, the wave drag is proportional to the square of the thickness ratio; in the case of **lifting surfaces**, an additional wave drag occurs which is proportional to the square of the lift produced. These results give a general hint in favor of small thickness ratios and small lift coefficients. However, this general advice is limited by requirements of weight and strength of construction. Therefore, the airplane designer will welcome ideas leading to reduction of the wave drag without going to extreme slenderness and small lift coefficients.

The physical significance of wave drag consists of a continuous **transport of**

momentum from the moving body to the air to infinity if there are no boundaries. One can also say that **energy** is continuously transmitted to the air, radiated from the moving body to infinity. Work done by a propulsive thrust is necessary to furnish this energy. Therefore, if there is a method of preventing the energy from leaving the moving system, there will be no wave drag, and propulsive thrust will be spared. This offers the possibility, at least theoretically, of escaping totally or partially from wave drag by appropriate interference between components of the moving system. It is possible, for example, to combine two airfoil sections with flat external surfaces parallel to the flow direction, so that their internal curved surfaces face each other. By properly designing the two inner surfaces, one can prevent the reflection of pressure waves on the inner surfaces, and the air leaves the system without change in velocity or energy. Of course, **this system does not produce lift**.

It is also possible to design a **biplane** - as was first indicated by A. Busemann - in such a way that the expansion waves emanating from the upper surface are compensated by the pressure produced at the lower surface of the other. This measure may substantially reduce the total wave drag. A biplane, of course, has disadvantages from several points of view. Nevertheless, the problem of interference action should be studied further with a view toward reduction of wave drag.

Another general idea has already been mentioned in connection with the supersonic airfoil theory. It was shown that, in the case of a **sweptback wing with infinite span**, the wave drag disappears when the sweepback is sufficiently large for the flow velocity normal to the wing axis to become subsonic. We have shown that in the case of wings with finite span, the wave drag is also essentially reduced by sufficiently large sweepback. Of course, when the velocity normal to the leading edge approaches sonic velocity, one will have a similar increase in drag, as in the case of straight normal wings in the transonic speed range, due to shock and separation. Furthermore, the sweptback wing design has its peculiar problems and difficulties because of the plan form.

One serious problem is the appropriate choice of **airfoil sections**. If a wing has a subsonic leading edge, a supersonic section will produce large peaks of negative pressure at the leading edge, eventually causing separation. Therefore, the designer, in this case, should use sections suitable for transonic flow, similar to the laminar flow sections used in airplanes designed for high subsonic speeds. At the center section of an arrowhead wing, or at the sections of a sweptback wing near the fuselage, supersonic conditions prevail, and one must therefore probably change the

basic shape of the section along the span.

The **aerodynamic design of control surfaces** constitutes a branch of supersonic aerodynamics which is very much in need of fundamental investigation. The suggestion of **boundary-layer control** as a direct method of airplane control by changing lift distribution over the wing deserves careful study. It may have extraordinary possibilities and help to eliminate the difficulties connected with the change from subsonic to supersonic flow regimes.

One variation of the sweepback scheme is the wing with triangular plan form, the so-called **delta wing**. This plan form also has considerable theoretical interest, since, for delta wings with certain simple angle of attack distributions, the direct problem of the wing theory can be solved in a relatively easy way. In the foregoing sections two general methods of computing flow produced by slender bodies and airfoils were presented: the **methods of sources** and of **Fourier integrals**. A third elegant method is that of the so-called "**conical flows**". A flow is called conical if its three velocity components are constant along straight lines that diverge from one point. The simplest example is a flow along a cone of circular cross section. Conical flow is a generalization of this flow pattern. It is convenient to introduce spherical polar coordinates: the radius vector *r*, the meridian angle ϕ, and an azimuth parameter ξ (connected with the azimuth angle ω by the relation $\cosh\xi = \frac{\cot\omega}{\sqrt{M^2-1}}$). Since the three velocity components *u*, *v*, *w* of a conical flow are independent of *r*, ϕ and ξ are chosen as independent coordinates. Then it can be shown that *u*, *v*, and *w* satisfy the Laplace equation; for example,

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \phi^2} = 0 .$$

Hence, the mathematical problem of finding conical flows to satisfy given boundary conditions is relatively easy. For example, the elegant methods of the conformal transformation are available for this purpose.

The flow produced by a thin delta wing is a conical flow within the approximation of the linearized wing theory, provided the local angle of attack is only a function of the azimuth angle. The simplest example of this kind is the delta wing with constant local angle of attack - i.e., a flat plate with triangular plan form. In this case, closed mathematical expressions can he obtained for the lift and drag distribution of an infinite triangular wing. Then, according to the rule of forbidden

signals and the general properties of plan forms with subsonic and supersonic trailing edges, the solution for the infinite wing is valid without change also for a finite wing, provided it has supersonic trailing edges.

Consider, for example, a triangular wing with sufficiently large sweepback so that the leading edge is subsonic. The trailing edge is assumed to be normal to the flight direction. A simple analysis shows that the lift density is infinite but integrable along the leading edges and finite and different from zero at the trailing edge, as follows from general considerations previously given. It is found that the spanwise lift distribution is elliptic. The lift per unit span is equal to

$$\frac{\rho U^2}{2}\frac{is}{E}\sqrt{1-\frac{4r^2}{s^2}}\,.$$

The total lift is equal to

$$\frac{\rho U^2}{2} is^2 \frac{\pi}{2E}\,.$$

In these equations E is a numerical factor given by the elliptic integral of the second kind:

$$E=\int_0^{\pi/2}\sqrt{1-\left[1-\frac{\tan^2\omega_0}{\tan^2\alpha}\right]\sin^2\beta d\beta}\,.$$

The total drag is equal to

$$D=Li-\frac{Li}{2E}\sqrt{1-\frac{\tan^2\omega_0}{\tan^2\alpha}}\,, \tag{8.1}$$

where α is the Mach angle and i is the angle of incidence. The quantities r, s and ω_0 are defined in Figure 21.

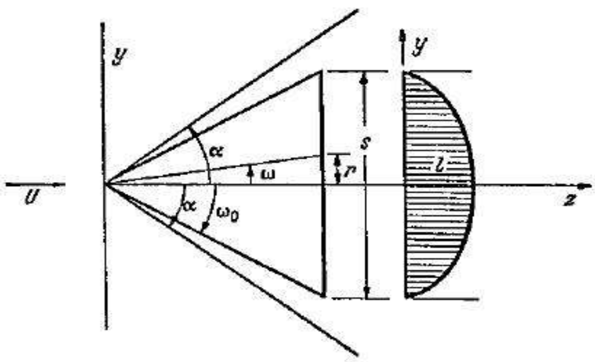

Figure 21. Delta wing.

The first term represents the sum of the horizontal components of the pressures acting on the upper and lower surfaces. The second, negative, term corresponds to the suction acting at the leading edge. One can also easily separate the wave drag from the induced drag. The amount of the **induced drag** is given by the simple relation

$$D_i = \frac{Li}{2E}. \tag{8.2}$$

The rest, represented by the equation

$$D_w = Li\left\{1 - \frac{1+\sqrt{1-\dfrac{\tan^2 \omega_0}{\tan^2 \alpha}}}{2E}\right\} \tag{8.3}$$

gives the wave drag.

One can show by means of calculation that, with sufficiently large sweepback, favorable values of lift/drag ratios can be reached with satisfactory lift coefficients.

The **drag produced by thickness of the wing** is not included in this calculation. The final answer on the practical merit of the delta wing will be given by further comprehensive theoretical and experimental investigations.

The method of conical flows, which led to such simple results in the case of the triangular wing, is also useful for the solution of the lift problem for a large class of wings with various sweepback and taper combinations. It can also be used in the theory of drag of wings with given sectional shape, especially if the section is composed of straight lines.

9. FRICTION AND BOUNDARY LAYER

In the foregoing sections two assumptions were made: absence of viscosity and small magnitude of the perturbations caused by the presence of a. body in a supersonic stream. This and the following sections contain a few remarks on the influence of **viscosity** and **finite perturbation**.

There is not much information available on the magnitude of the friction between a solid and-air moving at supersonic speed relative to the solid surface. However, pressure drop measurements in pipes indicate that the friction coefficient obtained for subsonic flow can be applied to the supersonic case as well, at least in the case of turbulent flow. Ballistic experience confirms this result. It appears that the frictional resistance of projectiles is of the order of magnitude to be expected by extrapolation of the friction coefficients from subsonic to supersonic speeds as

functions of the Reynolds Number only.

The **boundary-layer theory** was extended to the supersonic velocity range by various authors in the case of laminar flow. This is, of course, no longer solely a problem of aerodynamics; it is rather one of **aerothermodynamics** (a term introduced by G. A. Crocco), since the heat produced by viscosity and the heat exchange across the boundary layer are essential factors which cannot be neglected.

One interesting result is that if the wall is thermally insulated, the temperature of the air immediately adjacent to the wall reaches the value corresponding to an adiabatic compression of the gas to stagnation pressure, although no pressure rise occurs. If the temperature of a conducting wall surface is lower than this value, heat is transferred to the wall. Hence, even if there is a considerable temperature difference between a moving hot body and cool outside air at a certain flight Mach Number, the **cooling is reversed to heating**. This is due to the heat produced by internal friction in the boundary layer. The **reversal** occurs at

$$M = \sqrt{\frac{2}{\gamma - 1}\frac{T_w - T_s}{T_s}} \ ,$$

where T_w and T_s are the wall and stream temperatures, respectively. When H. S. Tsien and the present author considered this problem in 1938, it appeared to have merely academic interest. However, it is evident that today it is a practical question, for example, with reference to V-2-type rockets. The most recent progress in the integration of the aerothermodynamic equations relative to laminar boundary layers in compressible fluids has been made by L. Crocco.

The question of stability of the laminar boundary layer in an incompressible fluid has been cleared up in final form by the mathematical work of C. C. Lin and the experimental research of H. L. Dryden. The **stability problem of a laminar compressible boundary layer** was recently investigated by C. C. Lin and L. Lees. In general, if there is a heat flow through the wall, the compressibility appears to have a stabilizing effect, whereas, if the wall is thermally insulated, the effect is in the opposite direction. However, one must bear in mind that, as will be pointed out in Section 12, the boundary-layer theory applies only to the case where the outside flow determines the boundary layer completely, and no interaction from the boundary layer to the outside flow can be expected. Hence, the problem of stability in a broader sense should include the mutual **interaction between boundary layer and outside flow**, especially between boundary layer and shock wave. At high altitude - i.e., in a medium of low density - the heat radiated must also be taken into account. The cooling of the wall by **radiation** can increase the stability of the

laminar boundary layer to a large extent.

As to the theory of the **fully developed turbulent boundary layer** and **turbulent separation**, these problems have not even been solved for the case of incompressible fluids. The problem of separation of supersonic streams is closely connected with the problem of formation of shock waves. This question will be discussed in Section 12. It has a fundamental bearing on the problem of transonic flow.

10. THE EXACT THEORY OF SUPERSONIC FLOW

If one wants to proceed to a more exact analysis of the forces acting on bodies in a supersonic stream beyond the linear approximation based on the assumption of small perturbations, the simple rules of Section 2 and the theories of drag and lift presented in Sections 3 to 7 must be modified in various essential aspects.

(a) In the linear approximation, the velocity of sound is assumed to be constant throughout the whole field of the fluid motion, and equal to the velocity of sound in the undisturbed flow. Now it is known that the velocity of sound is proportional to the square root of the temperature of the gas; therefore it will vary from point to point if the pressure change is not exactly isothermal. If the variation of the velocity of sound and the influence of the difference between local and main flow velocity on the wave propagation are taken into account, the equations of motion are no longer linear and the method of superposition of particular solutions is no longer available to the mathematician. Correspondingly, one can no longer talk of Mach cones traversing the whole flow. The concept of the Mach cone is restricted to **infinitesimal local Mach cones**, which determine the pressure propagation in the infinitesimal neighborhood of one point. The axis of such an infinitesimal Mach cone is parallel to the direction of the local velocity, and its vertex angle corresponds to the local Mach Number. The method of superposition of particular solutions valid for the whole field must be replaced by a method of step-by-step construction. In the case of two-dimensional flow and three-dimensional flow with axial symmetry, such step-by-step integrations have been carried out using numerical or graphical methods. The lines of intersection of the elementary Mach cones with the plane of the two-dimensional flow, or with a meridian plane of the axially symmetric flow, constitute networks of curves called the **characteristic curves**. The tangents of the flow lines are the bisectors of the characteristics, the angle between a flow line and each of the two characteristic curves intersecting it being equal to the local Mach angle.

The **method of characteristics** makes it possible to obtain essential refinements of the drag computations of the linearized theory. This is especially valuable for a check of the degree of approximation which can be attained by the latter.

(b) A second point that has to be emphasized is the fact that the velocity of sound is the velocity of propagation of infinitesimally small pressure variations, whereas a **finite pressure rise propagates at an essentially higher speed**. The velocity of propagation is a function of the magnitude of the pressure rise. Therefore if the pressure rise produced by the moving body has finite magnitude, the rule of forbidden signals as stated in Section 2 is not exactly true. In general, the surface that separates the zone of silence from the zone of action is no longer a straight cone starting from the source of perturbance; in most cases it is a curved surface at which pressure, density, and velocity undergo sudden finite changes. The sudden change of these characteristic quantities is commonly called a **shock**, and the surface at which the change occurs is called a **shock wave**. This terminology reminds us that the discontinuity is caused by wave propagation - namely, propagation of a wave front with finite amplitude at a speed that is greater than the velocity of sound.

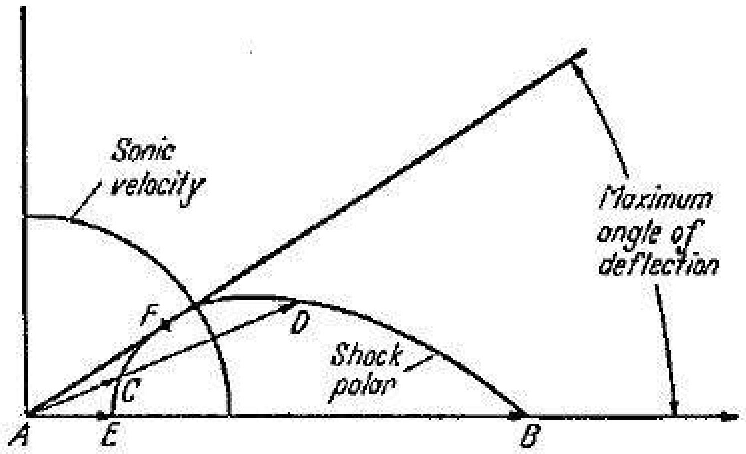

Figure 22 Shock polar.

The possibility of discontinuous change is not revealed by the linear theory, which deals only with infinitesimally small perturbations. It is indicated in certain cases by infinite values for velocity or pressure gradients.

When the velocity is perpendicular to the discontinuity surface, the shock is called a **normal shock**. In a normal shock only the magnitude of the velocity changes. The velocity upstream of the discontinuity surface must, of course, be supersonic, and it always becomes subsonic at the downstream surface. If the discontinuity surface is not normal to the velocity, the tangential component of the

velocity remains unaltered by the transition through the surface. The velocity component normal to the surface changes from supersonic to subsonic magnitude. In the so-called **hodograph diagram**, the **shock polar** represents all the velocity vectors in which a given velocity vector, say the vector AB in the diagram of Figure 22, can be discontinuously changed without violating the three pertinent dynamic and thermodynamic theorems - namely, those of the conservation of matter, momentum, and energy. The diagram shows that, if the initial velocity is given, the magnitude of the angle of deflection is limited. This explains why, for example, in the case of a body with pointed nose, a shock wave can

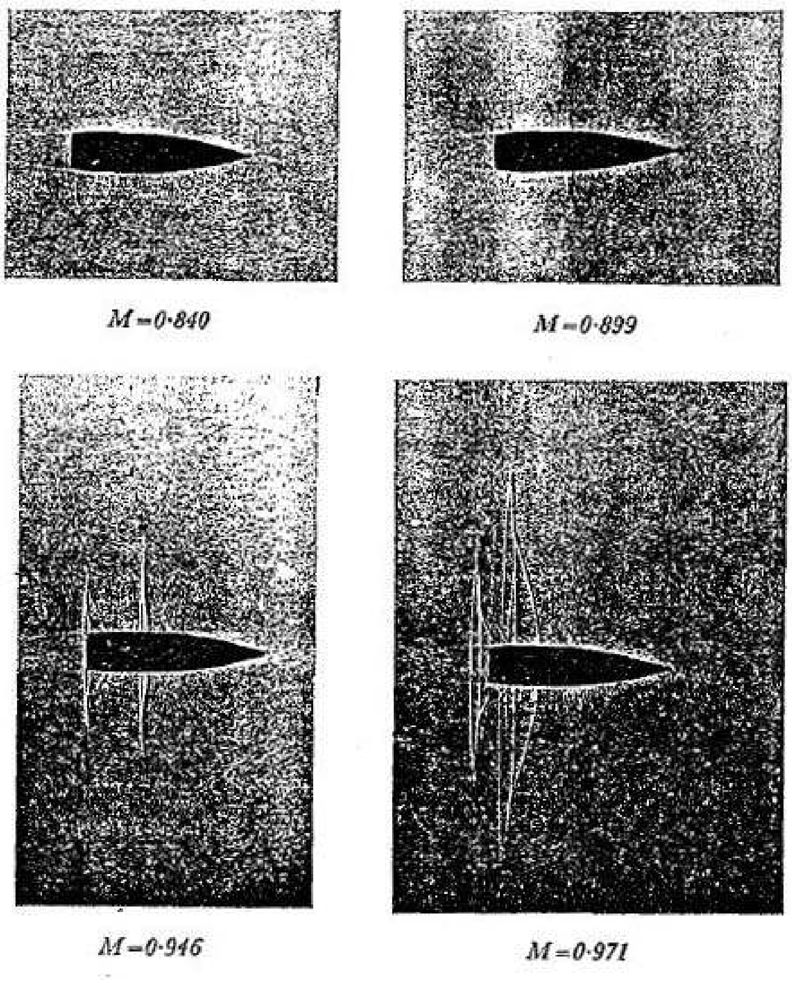

Figure 23. Shadow photograph (on this and following pages) of 155-mm: projectile in flee flight through sonic velocity. (Released by courtesy of Ballistic Research Laboratory, Aberdeen, Md.)

be attached to the conical nose if the vertex angle is not too large. For larger angles one obtains a so-called **detached shock wave**. The detached shock wave also appears if the body has a. blunt nose or edge. Figure 22 shows that in the case of

attached shock waves there are theoretically two possibilities of sudden change of direction - namely, from the vector AB to AC or AD. For reasons that are **theoretically not yet clear**, at a pointed wedge or cone the change from AB to AD usually occurs - i.e., the change that involves smaller variation in the magnitude of the velocity. The angle of maximum deflection depends on the Mach Number and approaches zero when $M \to 1$; therefore, when the speed of a moving body passes through sonic velocity, a detached shock wave always occurs first. Theoretically, the detached shock appears at infinite distance ahead of the body when $M = 1$. Shadow photographs of a projectile passing through sonic velocity are shown in Figures 23. With increasing Mach Number, the distance between nose and shock wave gradually decreases, as shown in Figure 24. In this particular case (20° half vertex angle), the shock wave is first attached at M = 1.18.

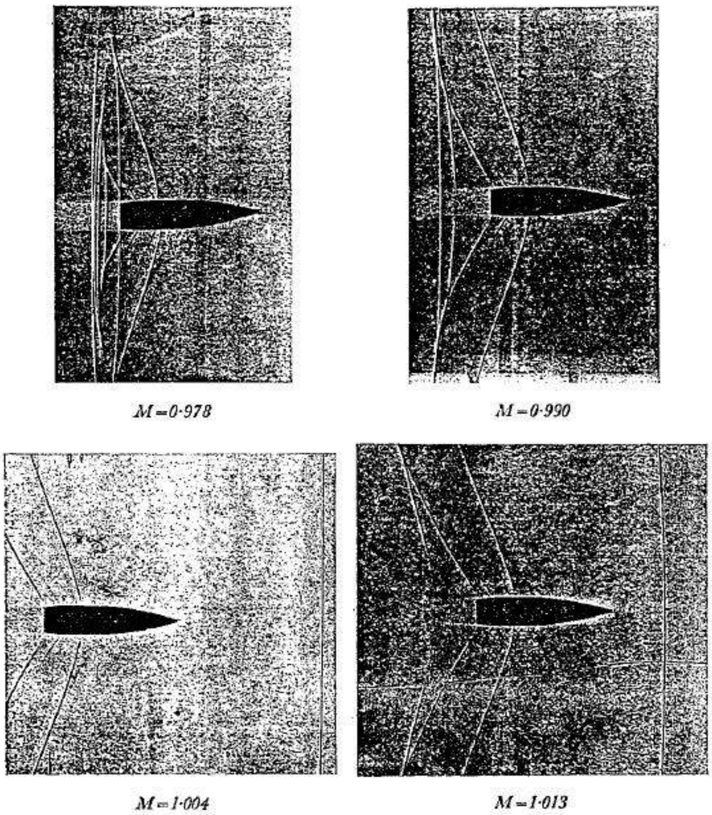

Figure 23. Continued

Consider now a detached shock wave ahead of a wedge. Then point B in Figure 22 corresponds to infinity, and E corresponds to the intersection of the shock wave with the plane of symmetry. As the Mach Number increases, the portion of the shock wave corresponding to the are EF becomes smaller and smaller and vanishes when the shock wave becomes attached. This is in accordance with the fact mentioned

above, that for higher Mach Numbers the portion of the shock wave in the neighborhood of the edge corresponds to the point *D* rather than to the point *C*.

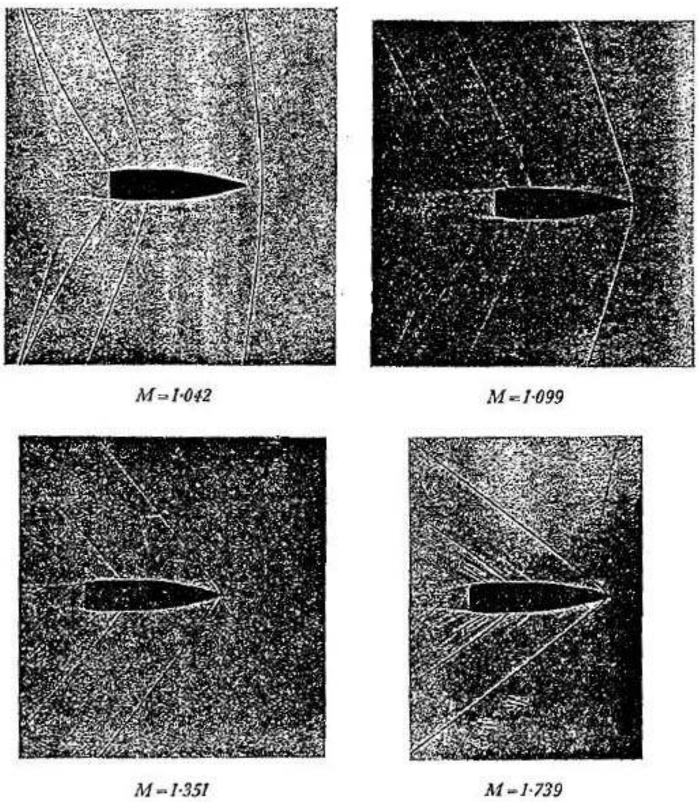

Figure 23. Concluded.

The process of sudden pressure rise is not reversible, since it involves an increase of the entropy of the gas. In other words, no sudden pressure drop can be maintained in a stationary flow. If one wants to express this fact in another way, one can say that **sudden finite velocity decrease is possible**, but **sudden finite velocity increase is impossible**, in a compressible fluid. This behavior of the compressible fluid in supersonic flow is illustrated by the flow pattern shown in Figure 25. The fluid passing along the concave corner undergoes an instantaneous change in velocity and pressure, whereas the fluid passing around the convex corner changes its velocity and pressure continuously. The lines of constant pressure constitute a fan-like formation starting from the corner. The remarkable fact is that in the supersonic flow, the fluid may go around the corner without infinite velocity or separation of flow, whereas it is known that in the subsonic case either the velocity

becomes infinite or the flow separates.

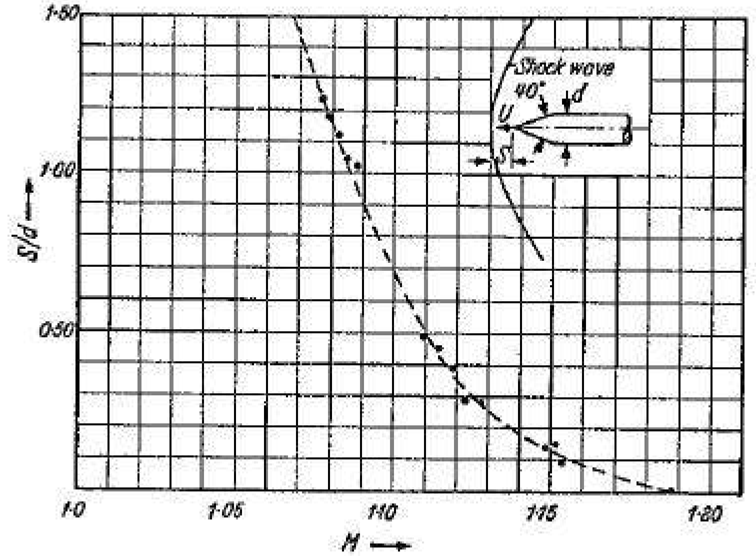

Figure 24. The ratio between the distance of the detached shock wave and the projectile diameter as function of Mach Number.

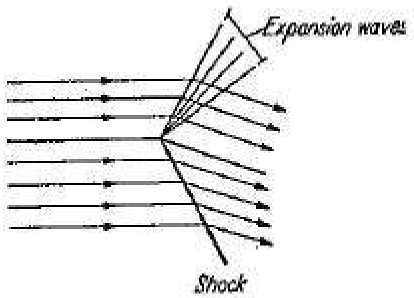

Figure 25 Compression and expansion waves.

The discussion of the phenomena given in these lines refers to a nonviscous fluid. In a **viscous fluid**, because of viscosity and heat transfer, the pressure and velocity changes will always be continuous. However, it can be shown that the region in which the **major part of the change takes place is of the order of magnitude of the mean free path** of the gas molecules, and therefore, in general, very small, unless the density of the gas is extremely low. The thickness and the physical nature of the transition region are also influenced by the internal thermal properties of the gas - namely, by the distribution of thermal energy to the various degrees of freedom in a molecule. This effect is called the **relaxation effect** and is especially important in the case of certain gases with sluggish internal vibrations. The investigation of this last problem requires the methods of quantum mechanics.

(c) The principal importance of the phenomenon of the shock wave, from the point of view of the practical aerodynamicist, is the **drag** caused by its presence, which was not revealed by the linear theory. I believe that the French mathematician, Hadamard, first showed that the flow of a gas passing through a curved discontinuity surface is not vortex-free, even in the case of uniform parallel flow ahead of the shock wave. Consequently, if a moving body produces a shock wave, the body is followed by a wake equivalent to a loss of momentum. This loss of momentum represents a certain amount of drag similar to the **wake drag** caused by flow separation.

One can look at the phenomenon also from the point of view of **entropy**. The shock wave involves an increase of entropy, which means that the kinetic energy that has been transformed into heat content cannot be retransformed entirely into kinetic energy. Consequently, the air at a large distance behind the moving body has a higher heat content than the air at a large distance in front of the body. To produce this heat, work has to be done – i.e., propulsive force is necessary to maintain the stationary motion of the body.

The **drag corresponding to the wake of the shock wave** and the **wave drag** cannot always be easily separated in the exact theory of the drag of bodies moving with supersonic speed. Consider, for example, a two-dimensional airfoil, and assume an attached shock wave at the sharp leading edge. If one draws the Mach lines starting from the surface of the airfoil, one notices that they intersect the shock wave. The Mach lines starting from the surface represent expansion waves of the type indicated previously in connection with the flow of a compressible fluid around a corner. They are sometimes called **Prandtl-Meyer waves**, since the mathematical solution of the expansion process was first given by these authors. As the expansion waves intersect the compression shock wave, they reduce its intensity and may also create infinitesimal compression waves reflected from the shock wave. By these processes the compression wave gradually loses in intensity, and at infinity a simple Mach line remains, analogous to the Mach lines encountered in the linear theory. However, one sees from this qualitative discussion of the exact process, that the expansion waves starting from the surface of the airfoil do not extend to infinity, and therefore they cannot contribute to the wave drag, which was defined as the equivalent of the momentum transferred to infinity. The wave drag is transformed into wake drag - i.e., into a drag due to the shadow of the shock wave.

However, the comparison of the calculated wave drag and the drag actually

measured shows that, in spite of this difference in the physical nature of the drag mechanism, the concept of the wave drag leads to a fair approximation. The reason is that it represents correctly the conditions at a large but finite distance from the body. The case is somewhat **analogous to that of the induced drag in the subsonic three-dimensional wing theory**. The plane vortex sheet behind the wing assumed by the linearized subsonic theory cannot extend to infinity. Nevertheless, the computation of the induced drag based on this assumption gives a good approximation.

A further approximation beyond the linearized theory can be obtained - at least for the case of two-dimensional or axially symmetric flow - by using the **exact flow equations** instead of the linearized ones but **neglecting the reflection** of the wavelines at the shock wave. This method was applied by several authors to the calculation of pressure distributions on airfoils. One interesting result of this analysis-not revealed by the linear theory-is that the **angle of deflection of the flow along an airfoil surface is limited**. The fan-like expansion waves represented in Figure 25 cannot be continued indefinitely, for at a certain angle of deflection zero pressure is reached. The same phenomenon occurs on the cambered surface of an airfoil. If the limit is reached, flow separation must occur. This phenomenon has important bearing on the **maximum lift** of wings and especially on the limitation of **control surface effectiveness** in supersonic flight.

In the case of a **detached shock wave**, a great portion of the flow in the neighborhood of the body may be subsonic. In this case the main part of the drag corresponds to the wake of the shock wave. The same is true in the case of transonic flow with shock wave of finite length. Suppose the velocity of the body is less than sound velocity but the relative flow around the body is partially supersonic. If a shock wave develops in such a case, it can only have a finite length, inasmuch as no shock wave can exist in the subsonic region. Since the flow is subsonic at a large distance from the body, certainly no wave drag can exist. The drag due to shock wave and separation, being in general much larger than the frictional drag, causes a considerable rise in the value of the coefficient of the total drag. The value of the Mach Number at which this rise of drag coefficient occurs is called the **critical Mach Number**. The transonic problem is discussed in some detail in Sections 11 and 12.

(d) The formation of shock waves is of fundamental importance in problems involving internal flows with supersonic velocity. The aeronautical engineer necessarily faces such problems in connection with the **intake** and **ducting** of the air required by propulsive devices. One characteristic problem is that of the **supersonic diffuser**. The problem consists of decelerating air approaching a duct at supersonic speed to a speed that can conveniently be used in machines or combustion chambers. It appears that in this process shock waves cannot be completely avoided. The **efficiency** of the process can be essentially improved, however, by proper control of the location and magnitude of the shocks. The main design principle is avoiding normal shocks of great intensity. The aim of the designer is either to create a sequence of oblique shocks, such that the change of the normal velocity component is small at every step, or to produce weak normal shocks at sections where the velocity is only slightly greater than sound velocity. The scope of this paper does not allow a detailed discussion of this problem. Reference is made to work done by Oswatitsch, Crocco, and others.

11. THE TRANSONIC PROBLEM

The main effects of compressibility encountered in practical subsonic airplane design are: **increase of drag**; **breakdown of lift**; and, as a consequence of the second phenomenon, **loss of control and maneuverability** of wing-tail combinations.

These effects are due to the **breakdown of the continuous vortex-free flow**, rather than to the direct effect of compressibility of the air. The effects that occur before the breakdown have been analyzed by several methods.

The first approximate theories of the compressibility effects were proposed by Prandtl and Glauert, using the assumption of small perturbations - i.e., the same assumption that was made in the supersonic case leading to the results given in this paper. In the **subsonic** case it is possible to deduce a simple rule for the compressibility correction known as the **Prandtl-Glauert correction**. This amounts to the statement that, in the case of thin airfoil sections, the pressures acting on a surface element are to be multiplied by the factor $\frac{1}{\sqrt{1-M^2}}$, where M is the Mach Number of flight. It soon appeared that this correction was not sufficient when the Mach Number was increased beyond 0.5 or 0.6. A more adequate correction was obtained by H. S. Tsien and the author. This correction does not use the assumption of small perturbations. It is based on a different linearization of the flow equations

originally proposed by P. Molenbrock and A. Chaplygin and applied by several authors, especially to the problem of gas jets. By a modification of the original method of Chaplygin, H. S. Tsien and the author made the method applicable to problems of high-speed flow and especially useful for the calculation of forces acting on airfoils. The mathematical simplification is attained by replacing a portion of the adiabatic curve in the pressure vs. volume diagram of the gas by a straight line. The Kármán-Tsien correction gave satisfactory results in a speed range in which the Prandtl-Glauert correction did not seem satisfactory. However, neither of the two methods can give a correct answer if the local velocity at some point on the airfoil reaches sound velocity. For this case, the theory must be reconsidered. It is the belief of the author that it will be difficult to find a simple unified method that remains valid in the speed range in which the flow is partially supersonic. It may be possible to find useful approximations for the pressure distribution by combination of typical subsonic and supersonic flow patterns. Extrapolations without safe theoretical foundation, however, although in accord with the measurements in a few cases, are bound to be inadequate if the experimental material is further extended.

Consider the problem of a symmetric airfoil in uniform parallel flow of a nonviscous compressible fluid, and assume that the velocity of the undisturbed flow is gradually increased. Up to a certain Mach Number there will be only one unique solution of the flow equations, the velocity at every point being subsonic. At a certain Mach Number, sonic velocity is reached at some point, and it can be shown that this necessarily happens on the surface of the airfoil. Contrary to some previous opinions, it appears that the occurrence of sonic velocities does not necessarily mean that a continuous flow is not possible and that the airfoil has a resulting drag. The solution, in fact, can be composed of a subsonic and a supersonic domain without discontinuity. The supersonic domain is adjacent to the airfoil surface. If the Mach Number is further increased, at least at one point an **infinite acceleration** will occur, and beyond this limiting Mach Number **no continuous solution is possible**. Beyond this limit the airfoil must have a drag even in a nonviscous fluid. This drag is caused by the presence of shock waves, by separation, or by both. From the theoretical point of view, this limiting Mach Number should be considered as the **critical Mach Number**. There are several doubtful aspects of the question, however:

(1) It has not been shown that the continuous solution composed of a subsonic and supersonic domain is the **unique solution**.

(2) It has not been shown that a **discontinuous solution** - i.e., a flow with shock wave - cannot exist below the theoretical critical Mach Number. Certainly no such

solution is possible before the velocity of sound is reached locally. However, the behavior of the fluid between these two limits has not yet been sufficiently explored theoretically, and the experimental evidence is also somewhat spotty. At the present state of knowledge it seems to be appropriate to call (a) the Mach Number at which the local velocity reaches sound velocity, the **lower critical Mach Number**; and (b) the Mach Number beyond which no continuous flow is possible, the **upper critical Mach Number**. The condition (a) gives a necessary, the condition (b) a sufficient, condition for the occurrence of a shock wave.

(3) It can certainly be stated that the actual breakdown causing a rapid rise in drag and a drop in lift must occur between the lower and upper critical limits. In no case has it been shown that the upper limit was attained experimentally. A major handicap in such investigations is the difficulty of finding the upper limit theoretically. Moreover, according to a classification by H. S. Tsien, the following factors may, in the case of such large accelerations as necessarily occur near the upper critical limit, give appreciable alterations in the play of the dynamic forces: (a) viscous stresses due to ordinary internal fluid friction; (b) viscous stresses due to rapid compression and expansion; (e) relaxation lag in internal molecular vibrations; and (d) heat conduction.

The author believes that especially instability of the continuous flow can lead to a premature appearance of shock waves.

The experimental evidence also shows that the boundary layer has considerable influence on the formation of shock waves. This problem is known as that of the interaction between boundary layer and shock waves. It appears necessary to devote at least a short discussion to this question.

12. INTERACTION BETWEEN BOUNDARY LAYER AND SHOCK WAVE

The concept of the boundary layer is based on the assumption that the flow conditions in the boundary layer, its growth or decrease, are fully determined by the velocity and pressure distribution of the nonviscous outside flow. In other words, one assumes that the flow outside of the boundary layer influences the development of the boundary layer but that there is no influence of the boundary layer on the main flow. In the case of an incompressible fluid or a compressible fluid moving at low velocity, it can be shown that this statement is really true **unless the flow separates**. The separation can be caused by a sharp corner or by a large magnitude of the so-called **adverse pressure gradient**. Unless there is separation, the pressure across the boundary layer is sensibly constant and changes in thickness of the

boundary layer do not influence the main flow appreciably.

The **fundamental assumption of the boundary layer theory cannot be applied to a flow in the transonic speed range** for two reasons:

(a) In the neighborhood of sonic velocity a slight cross-section change causes large changes in pressure and velocity. It is known that, in a flow through a supersonic nozzle, sonic velocity occurs at the minimum cross section, so that both slightly smaller or slightly larger velocities require a larger cross section. This fact indicates that a flow near sonic velocity must be **extremely sensitive** to changes in cross section. Therefore, if one considers the streamline tubes passing just outside of the boundary layer, one must conclude that increase or decrease of the boundary layer thickness must have an essential influence on the flow in the neighborhood by narrowing or widening the cross sections of the neighboring streamline tubes. Consequently, near the velocity of sound there is **interaction between the main flow and the boundary layer** in both directions.

(b) Since the flow in the boundary layer, at least near the wall, is certainly subsonic, no shock wave can extend through the whole boundary layer and end at the wall. Now if the shock wave ends at some point inside of the boundary layer, the conditions that prevail must contradict the assumptions made in the boundary layer theory. The pressure discontinuity at the shock wave must create a steep pressure rise in the subsonic part of the boundary layer. The pressure across the boundary layer will no longer be constant, and therefore the ordinary boundary layer theory can hardly be applied. The large pressure rise may cause separation, and the separation will, in general, react on the direction and magnitude of the shock wave.

The importance of the interaction between the boundary layer and transonic flow, especially the formation of shock waves, was recognized at about the same time by scientific workers at the N.A.C.A., at the California Institute of Technology, and by Professor Ackeret in Zurich. The research on this subject is still far from being finished. Several important findings can, however, be mentioned. It has been recognized that the formation of shock waves on a cambered airfoil surface depends, to a large extent, on the type of flow in the boundary layer - i.e., on whether it is laminar or turbulent. In the case of **laminar** boundary layer, a wave combination is observed which, by its configuration, resembles the Greek letter λ. It appears that the increase in thickness of the laminar boundary layer produces in the main flow a system of oblique waves or weak shocks, apparently followed by one strong shock. The photographic evidence and the evidence obtained by measurements of the wall

pressure distribution seemed contradictory at first. In spite of the large intensity of the shock wave apparent from the photographs, almost no pressure difference was measured at the wall. This discrepancy was cleared up by improvement of the technique and a more exact analysis of the photographs. H. Liepmann, of the Guggenheim Laboratory in Pasadena, found that the shock is followed by a fanlike system of expansion waves. As the shock is inclined in the upstream direction and the expansion waves are inclined in the downstream direction, this phenomenon can be considered as **reflection of a compression wave at a free boundary**, which in this case is the subsonic part of the laminar boundary layer. It is known that a compression wave is reflected at a fixed wall as compression and at a free boundary as expansion. As was mentioned before, no expansion shock can exist, and therefore the reflected wave appears as a fanlike system of wave fronts. Photographs illustrating these phenomena can be found in the papers of Liepmann and Ackeret.

If the boundary layer is or has artificially been made **turbulent** before the maximum velocity is reached, a strong compression shock is shown in the photographs. The shock is approximately normal to the wall and the wall pressure measurements show a rapid pressure rise caused by the presence of the shock wave. Sometimes the shock wave has a slight inclination to the flow direction, probably because the main flow is deflected by separation or by rapid increase of the thickness of the boundary layer. In such cases, some reflected expansion waves are also visible, as in the laminar case although the reflection is not sufficient to eliminate the pressure rise at the wall.

It has been known for a long time that in subsonic flow critical changes in the **drag and stalling characteristics** of airfoils depend on the relative location of the **transition point** between laminar and turbulent flow and the **separation point**. It is interesting to note that the same interplay between transition and separation also determines the formation of shock waves and therefore the drag and lift characteristics of airfoil sections and slender bodies at transonic speed. However, we are only at the beginning of an understanding of the phenomenon and have not yet reached the stage of practical conclusions.

13. THE THEORY OF TRANSONIC FLOW

In recent years great efforts have been made to compute the flow around bodies of various shapes in the transonic speed range, especially to construct **mixed solutions** of the flow equations consisting of supersonic and subsonic domains. One finds experimentally that the supersonic region expands rapidly when the Mach

Number approaches unity. It was found experimentally, in the case of an airfoil with 12 per cent thickness ratio, that the size of the supersonic region normal to the chord increased from zero to 28 per cent of the chord length when the Mach Number of the flow increased from 0.795 to 0.844, and reached 46 per cent of the chord length at M = 0.875. The corresponding values of the maximum local Mach Number on the airfoil surface were 1, 1.147, and 1.189, respectively. Theoretical calculations have been successful in some simple cases. They require, however, a large amount of analytical and numerical work.

The linear perturbation theory is not valid for the transonic range. In order to obtain a simplification of the flow equations they must be considered from a new point of view. It has been said that the linearized theory is based on the assumption that all velocities produced by the presence of a moving body are small in comparison to both the velocity of flight and the velocity of sound. However, if one follows exactly the process of deduction of the linearized equation, one notices that the assumption was made that the velocity perturbations must also be small in comparison with the difference between flight and sound velocities. In the transonic case, this assumption does not apply and must be replaced by the assumption that all velocities, including the velocity of the main flow, are only slightly different from the velocity of sound. The breakdown of the linearized theory is illustrated by the **obviously incorrect result** that it yields for M = 1, shown in Figure 26. It is seen

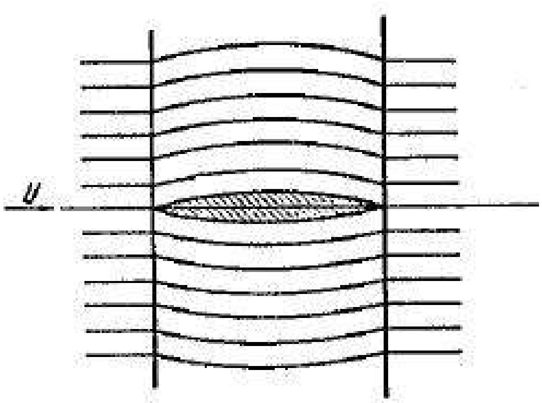

Figure 26 Flow at sonic velocity according to the linearized theory.

that all streamlines have identical curvatures. -According to elementary laws of dynamics, an infinite pressure difference should therefore exist between the surface of the airfoil and infinity. The new assumption leads to a simplification of the flow equations and promises to lead to useful results with a. moderate amount of mathematical work. Besides that, it gives a simple rule of **similarity** for transonic

flow around bodies and wings with similar thickness, camber, or angle of attack distribution. For example, in the case of two-dimensional flow around a symmetricairfoil, one obtains the rule that, in order to have a similar flow condition around airfoils with similar thickness distribution, one must change the quantity l – M - i.e., the difference between unity and the actual flight Mach Number, in proportion to the two-thirds power of the thickness/chord ratio.

According to this rule, the experimental drag curves for airfoil sections with different thickness ratio are reduced to one single curve according to the equation

$$C_D = \frac{1}{M^2}\left(\frac{t}{c}\right)^{5/3} D\left[\frac{(t/c)^{1/3}}{\sqrt{1-M^2}}\right].$$

It seems that this rule is confirmed by the few experiments that are available in the transonic range. The lift coefficient of an extremely thin (flat plate) airfoil should follow the law expressed by the equation

$$C_L = \frac{1}{M^2} i^{2/3} L\left[\frac{i^{1/3}}{\sqrt{1-M^2}}\right],$$

where i denotes the angle of attack. If the similarity rule applies at sound velocity, the drag of a symmetric two-dimensional airfoil at sound velocity is proportional to the five-thirds power and the lift coefficient of a thin airfoil to the two-thirds power of the thickness ratio and the angle of attack, respectively.

The problem of transonic flow raises many interesting questions that have not yet been decided either experimentally or theoretically. For example, it is questionable whether a **stationary solution** for the flow past a body is possible when the free-stream velocity is assumed to be exactly equal to sound velocity. It is evident that no such solution can exist if the stream is limited between parallel surfaces. Another problem, perhaps more important practically, is that of the **accelerated or decelerated motion** in the transonic range and at the velocity of sound. In the theory of incompressible fluids, the apparent mass of a body gives a measure of the air mass involved in the acceleration caused by the motion of the body. Now it is reasonable to assume that the corresponding effect increases rapidly when the velocity approaches the velocity of sound. The problem of accelerated motion has several practical applications. It enters, for example, into the theory of **wing oscillations** and **flutter**. It also enters into the evaluation of **dropping experiments**, undertaken for the determination of aerodynamic characteristics of

bodies in the transonic range, to replace wind-tunnel measurements, which become questionable in the transonic range.

However, from a practical point of view the most important questions remain: the determination of **wing shapes**, **plan forms**, and **sections** which cause a postponement of the critical phenomena - namely, the rise of drag and breakdown of lift at Mach Numbers approaching unity. It is known that one important method is the use of large **sweepback**. The fundamental idea of this method is the decrease of the **effective Mach Number**, which is assumed to be equal to the Mach Number corresponding to the component of the flight velocity normal to the leading edge. The detailed exploration of transonic phenomena on wings with sweepback is also important for supersonic flight, since the sweepback produces transonic conditions over certain parts of the wing even at Mach Numbers much larger than unity.

14. A SIMPLE METHOD OF RANGE PREDICTION FOR SUPERSONIC AIRPLANES

According to **Breguet's formula**, the range of an airplane depends on three quantities: (a) the consumption of propellants carried in the airplane and necessary to produce a unit amount of useful work, including thermodynamic, mechanical, and propulsion efficiency; (b) the lift/drag ratio of the airplane; and (c) the ratio of the initial flight weight of the airplane to the weight that remains after all the propellant is gone.

The first quantity depends on the future developments of propulsive units and will not be discussed in this paper. The third one is essentially a question of structural design and is also outside of our considerations. The second parameter - namely, the lift/drag ratio of the entire airplane – is primarily a problem of aerodynamics. In the case of supersonic airplanes, the aerodynamic efficiency depends to a great extent on the solution of the problem of volume - i.e., on the ability of the airplane designer to provide volume for fuel or propellants. Of course, the same point of view occurs also in the design of long-range transportation airplanes in the subsonic range. In supersonic flight the influence of the **volume requirement** on the aerodynamic efficiency of the plane is greatly exaggerated by the high wing loading. In other words, the supersonic airplane easily becomes a large body with small winglets, and, correspondingly, the lift/drag ratio becomes rather unfavorable.

The following simple calculation is crude but may illustrate the point (see Figure 27). Denote the wing area by A; the master cross section of the fuselage by S; the

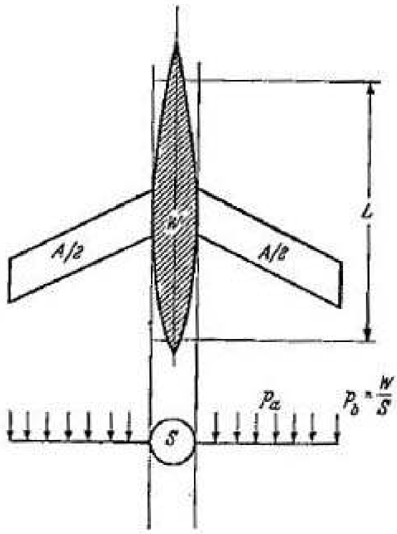

Figure 27 Wing-body combination.

drag and lift coefficients of the wing referred to the wing area by C_{D_w} and C_L, respectively; the drag coefficient of the fuselage referred to the master cross section by C_{D_f}. Then the resultant drag/lift ratio of the airplane is equal to

$$\frac{C_D}{C_L} = \frac{C_{D_w}}{C_L} + \frac{C_{D_f} S}{C_L A} . \tag{14.1}$$

Now one can express the condition that the lifting force be able to carry the weight of the body (the weight of the wing is neglected). Assume that the weight of the body is equal to the product *SLw*, where *L* is a kind of reduced length of the fuselage and *w* is the average specific weight of the fuselage full of pay load and fuel. Thus we have the equation

$$SLw = \rho \frac{U^2}{2} AC_L . \tag{14.2}$$

Taking into account the definition of the Mach Number and the formula for the velocity of sound, one obtains

$$SLw = \frac{\gamma M^2}{2} p_a AC_L \tag{14.3}$$

and substituting equation (14.3) in equation (14.1)

$$\frac{C_D}{C_L} = \frac{C_{D_w}}{C_L} + \frac{\gamma M^2}{2} \frac{p_a}{Lw} C_{D_f} . \tag{14.4}$$

In these equations p_a is the ambient pressure corresponding to the altitude of flight. Now *Lw* is obviously equal to the weight of the fuselage per square foot of the master cross section. One may call this quantity the **cross-sectional loading** of the fuselage. It is seen from equation (l4.4) that the value of the resulting drag/lift

ratio depends essentially on the ratio between ambient pressure and the cross-sectional loading of the fuselage. In other words, the range of the supersonic airplane can be greatly increased by the reduction of the ambient pressure and by the increase of the cross-sectional loading of the fuselage. Reduction of the ambient pressure means **high altitude**. Increase of the cross-sectional loading of the fuselage can be accomplished by denser loading, by use of high specific density fuels, and, finally, by the **increase of the overall dimensions** of the airplane. Obviously, the weight increases with the third power and the master cross section with the second power of the linear size. Consequently, the cross-sectional loading of the fuselage, everything else remaining similar, will increase in proportion to the linear dimensions of the airplane.

The results expressed by equation (14.4) are remarkable in that for the conventional **subsonic** speed range, flight altitude has no significant effect on the range of airplanes. Velocity and size also have only secondary influence. The effect of flight velocity on range in the **supersonic** case is expressed in equation (14.4) by the quantity $\frac{\gamma}{2} M^2 C_{D_f}$; this product has a flat minimum between $M = 1.5$ and $M = 2$.

If one proceeds to extremely high altitude, the **wing loading** decreases essentially and the proportion between wings and fuselage becomes more conventional. Therefore, for **large dimensions** and extremely **high altitudes**, the flying wing may again have fair possibilities, and the analysis must include the volume available in the wing, which was neglected in the above considerations.

The application of flying wings of usual design in supersonic flight may be restricted by the necessity of employing thin sections at supersonic speed. On the other hand, when **large sweepback** is used, this necessity is no longer imperative and allows more **freedom in the choice of sections**. The triangular wing also appears suitable for flying wing design and may combine small thickness/chord ratio with relatively large included volume.

The author does not want to commit himself to numerical range predictions on the basis of the approximate method presented in this section. In a real performance calculation, the period of climb and acceleration through the transonic speed range must be included.

The reader himself may substitute numerical values that he believes are realistic enough to give answers that are more than guesses. The author will be satisfied if his

general considerations have given some inspiration to further developments in supersonic aerodynamics.

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BIBLIOGRAPHY