CHAPTER VIII SOUND

§64. Sound waves

We proceed now to the study of the flow of compressible fluids, and begin by investigating small oscillations; an oscillatory motion with small amplitude in a compressible fluid is called a ***sound wave***. At each point in the fluid, a sound wave causes alternate compression and rarefaction.

Since the oscillations are small, the velocity **v** is small also, so that the term $(\boldsymbol{v}\cdot\text{grad})\boldsymbol{v}$ in Euler's equation may be neglected. For the same reason, the relative changes in the fluid density and pressure are small. We can write the variables p and ρ in the form

$$p = p_0 + p', \quad \rho = \rho_0 + \rho', \tag{64.1}$$

where ρ_0 and p_0 are the constant equilibrium density and pressure, and ρ' and p' are their variations in the sound wave ($\rho' << \rho_0$, $p' << p_0$). The equation of continuity $\frac{\partial\rho}{\partial t} + \text{div}(\rho\boldsymbol{v}) = 0$, on substituting (64.1) and neglecting small quantities of the second order (ρ', p' and $\boldsymbol{v}$ being of the first order), becomes

$$\frac{\partial(\rho_0+\rho')}{\partial t} + \text{div}((\rho_0+\rho')\boldsymbol{v}) = 0$$
$$\frac{\partial(\rho')}{\partial t} + \rho_0\,\text{div}(\boldsymbol{v}) = 0$$

$$\frac{\partial\rho'}{\partial t} + \rho_0\,\text{div}\,\boldsymbol{v} = 0. \tag{64.2}$$

Euler's equation

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\text{grad})\boldsymbol{v} = -\frac{1}{\rho}\text{grad}\,p$$

$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\text{grad})\boldsymbol{v} = -\frac{1}{\rho_0+\rho'}\text{grad}(p_0+p')$$
$$\frac{\partial\boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\text{grad}(p')$$

reduces, in the same approximation, to

$$\frac{\partial\boldsymbol{v}}{\partial t} + \frac{1}{\rho_0}\text{grad}\,p' = 0. \tag{64.3}$$

The condition that the linearized equations of motion (64.2) and (64.3) should be applicable to the propagation of sound waves is that the velocity of the fluid particles in the wave should be small compared with the velocity of sound: $v << c$. This condition can be obtained, for example, from the requirement that $\rho' << \rho_0$ (see formula (64.12) below).

Equations (64.2) and (64.3) contain the unknown functions $\boldsymbol{v}$, p' and ρ'. To eliminate one of these, we notice that a sound wave in an ideal fluid is, like any other motion in an ideal fluid, ***adiabatic***. Hence the small change p' in the pressure is related to the small change ρ' in the density by

$$p' = \left(\frac{\partial p}{\partial\rho_0}\right)_S \rho'. \tag{64.4}$$

Substituting for ρ' according to this equation in (64.2), we find

$$\frac{\partial p'}{\partial t} + \rho_0\left(\frac{\partial p}{\rho_0}\right)_S \text{div}\,\boldsymbol{v} = 0. \tag{64.5}$$

The two equations (64.3) and (64.5), with the unknowns $\boldsymbol{v}$ and p', give a complete

description of the sound wave.

In order to express all the unknowns in terms of one of them, it is convenient to introduce the ***velocity potential*** by putting $\boldsymbol{v} = \text{grad}\,\phi$. We have from equation (64.3)

$$p' = -\rho \frac{\partial \phi}{\partial t}\,, \tag{64.6}$$

which relates p' and ϕ (here, and henceforward, we omit for brevity the suffix in p_0 and ρ_0). We then obtain from (64.5) the equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \Delta\phi = 0\,, \tag{64.7}$$

which the potential ϕ must satisfy; here we have introduced the notation

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S}\,. \tag{64.8}$$

An equation having the form (64.7) is called a ***wave equation***. Applying the gradient operator to (64.7), we find that each of the three components of the velocity $\boldsymbol{v}$ satisfies an equation having the same form, and on differentiating (64.7) with respect to time we see that the pressure p' (and therefore ρ') also satisfies the wave equation.

Let us consider a sound wave in which all quantities depend on only one coordinate (x, say). That is, the flow is completely homogeneous in the yz-plane. Such a wave is called a ***plane wave***. The wave equation (64.7) becomes

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0\,. \tag{64.9}$$

To solve this equation, we replace x and t by the new variables $\xi = x - ct$, $\eta = x + ct$. It is easy to see that in these variables (64.9) becomes $\frac{\partial^2 \phi}{\partial \eta \partial \xi} = 0$. Integrating this equation with respect to ξ, we find $\frac{\partial \phi}{\partial \eta} = F(\eta)$, where $F(\eta)$ is an arbitrary function of η. Integrating again, we obtain $\phi = f_1(\xi) + f_2(\eta)$, where f_1 and f_2 are arbitrary functions of their arguments. Thus

$$\phi = f_1(x - ct) + f_2(x + ct)\,. \tag{64.10}$$

The distribution of the other quantities (p', ρ', $\boldsymbol{v}$) in a plane wave is given by functions having the same form.

To be definite, we shall discuss the density, $\rho' = f_1(x - ct) + f_2(x + ct)$. For example, let $f_2 = 0$, so that $\rho' = f_1(x - ct)$. The meaning of this solution is evident: in any plane x = constant the density varies with time, and at any given time it is different for different x, but it is the same for coordinates x and times t such that $x - ct$ = constant, or x = constant + ct. This means that, if at some instant $t = 0$ and at some point the fluid density has a certain

value, then after a time t the same value of the density is found at a distance ct along the x-axis from the original point. The same is true of all the other quantities in the wave. Thus the pattern of motion is propagated through the medium in the x-direction with a velocity c; c is called the ***velocity of sound***.

Thus $f_1(x-ct)$ represents what is called a ***travelling plane wave*** propagated in the positive direction of the x-axis. It is evident that $f_2(x+ct)$ represents a wave propagated in the opposite direction.

Of the three components of the velocity $\boldsymbol{v} = \operatorname{grad}\phi$ in a plane wave, only $v_x = \partial\phi/\partial x$ is not zero. Thus the fluid velocity in a sound wave is in the direction of propagation. For this reason sound waves in a fluid are said to be ***longitudinal***.

In a travelling plane wave, the velocity $v_x = v$ is related to the pressure p' and the density ρ' in a simple manner. Putting $\phi = f(x-ct)$, we find $v = \partial\phi/\partial x = f'(x-ct)$ and $p' = -\rho\partial\phi/\partial t = \rho c f'(x-ct)$. Comparing the two expressions, we find

$$v = \frac{p'}{\rho c}. \tag{64.11}$$

Substituting here from (64.4) $p' = c^2\rho'$, we find the relation between the velocity and the density variation:

$$v = \frac{c\rho'}{\rho}. \tag{64.12}$$

We may mention also the relation between the velocity and the temperature oscillations in a sound wave. We have $T' = \left(\frac{\partial T}{\partial p}\right)_S p'$ and, using the well-known thermodynamic formula $\left(\frac{\partial T}{\partial p}\right)_S = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p$ and formula (64.11), we obtain

$$T' = c\beta T\frac{v}{c_p}, \tag{64.13}$$

where $\beta = (1/V)(\partial V/\partial T)_p$ is the coefficient of thermal expansion.

Formula (64.8) gives the velocity of sound in terms of the adiabatic compressibility of the fluid. This is related to the isothermal compressibility by the thermodynamic formula

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \left(\frac{c_p}{c_v}\right)\left(\frac{\partial p}{\partial \rho}\right)_{T'}. \tag{64.14}$$

Let us calculate the velocity of sound in a perfect gas. The equation of state is $pV = p/\rho = RT/\mu$, where R is the gas constant and μ the molecular weight. We obtain for the velocity of sound the expression

$$c = \sqrt{\frac{\gamma RT}{\mu}}, \tag{64.15}$$

where γ denotes the ratio c_p/c_v. Since γ usually depends only slightly on the

temperature, the velocity of sound in the gas may be supposed proportional to the square root of the temperature.[1] For a given temperature it does not depend on the pressure.[2]

What are called ***monochromatic waves*** are a very important case. Here all quantities are just periodic (harmonic) functions of the time. It is usually convenient to write such functions as the real part of a complex quantity (see the beginning of §24). For example, we put for the velocity potential

$$\phi = \mathrm{Re}[\phi_0(x,y,z)e^{-i\omega t}], \qquad (64.16)$$

where ω is the frequency of the wave. The function ϕ_0 satisfies the equation

$$\Delta\phi_0 + \left(\frac{\omega^2}{c^2}\right)\phi_0 = 0, \qquad (64.17)$$

which is obtained by substituting (64.16) in (64.7).

Let us consider a monochromatic travelling plane wave, propagated in the positive direction of the x-axis. In such a wave, all quantities are functions of $x - ct$ only, and so the potential is of the form

$$\phi = \mathrm{Re}\{A\exp[-i\omega(t - x/c)]\}, \qquad (64.18)$$

where A is a constant called the ***complex amplitude***. Writing this as $A = ae^{i\alpha}$ with real constants a and α, we have

$$\phi = a\cos(\frac{\omega x}{c} - \omega t + \alpha). \qquad (64.19)$$

The constant a is called the ***amplitude*** of the wave, and the argument of the cosine is called the ***phase***. We denote by $\boldsymbol{n}$ a unit vector in the direction of propagation. The vector

$$\boldsymbol{k} = \left(\frac{\omega}{c}\right)\boldsymbol{n} = \left(\frac{2\pi}{\lambda}\right)\boldsymbol{n} \qquad (64.20)$$

is called the ***wave vector***, and its magnitude k the ***wave number***. In terms of this vector (64.18) can be written

$$\phi = \mathrm{Re}\{A\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}. \qquad (64.21)$$

Monochromatic waves are very important, because any wave whatsoever can be represented as a sum of superposed monochromatic plane waves with various wave vectors and frequencies. This decomposition of a wave into monochromatic waves is simply an expansion as a Fourier series or integral (called also ***spectral resolution***). The terms of this expansion are called the ***monochromatic components*** or ***Fourier components*** of the wave.

PROBLEMS

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[1] It is useful to note that the velocity of sound in a gas has the same order of magnitude as the mean thermal velocity of the molecules.

[2] The expression $c^2 = p/\rho$ for the velocity of sound in a gas was first derived by Newton (1687). The need for the factor γ was shown by Laplace.

Problem 1. Determine the velocity of sound in a nearly homogeneous two-phase system consisting of a vapour with small liquid droplets suspended in it (a "wet vapour"), or a liquid with small vapour bubbles in it. The wavelength of the sound is supposed large compared with the size of the inhomogeneities in the system.

Solution. In a two-phase system, p and T are not independent variables, but are related by the equation of equilibrium of the phases. A compression or rarefaction of the system is accompanied by a change from one phase to the other. Let x be the fraction (by mass) of phase 2 in the system. We have

$$\begin{cases} s = (1-x)s_1 + xs_2 \\ V = (1-x)V_1 + xV_2 \end{cases}, \tag{1}$$

where the suffixes 1 and 2 distinguish quantities pertaining to the pure phases 1 and 2. To calculate the derivative $(\partial V/\partial p)_S$, we transform it from the variables *p, s* to *p, x,* obtaining

$$\left(\frac{\partial V}{\partial p}\right)_S = \left(\frac{\partial V}{\partial p}\right)_x - \left(\frac{\partial V}{\partial x}\right)_p \left(\frac{\partial s}{\partial p}\right)_x \bigg/ \left(\frac{\partial s}{\partial x}\right)_p .$$

The substitution (1) then gives

$$\left(\frac{\partial V}{\partial p}\right)_S = x\left[\frac{dV_2}{dp} - \frac{V_2 - V_1}{s_2 - s_1}\frac{ds_2}{dp}\right] + (1-x)\left[\frac{dV_1}{dp} - \frac{V_2 - V_1}{s_2 - s_1}\frac{ds_1}{dp}\right]. \tag{2}$$

The velocity of sound is obtained from (1) and (2), using formula (64.8).

Expanding the total derivatives with respect to the pressure, introducing the latent heat of the transition from phase 1 to phase 2 ($q = T(s_2 - s_1)$), and using the Clapeyron-Clausius equation for the derivative *dp/dT* along the curve of equilibrium ($dp/dT = q/T(V_2 - V_1)$; see *SP1,* §82), we obtain the expression in the first brackets in (2) in the form

$$\left(\frac{\partial V_2}{\partial p}\right)_T + \frac{2T}{q}\left(\frac{\partial V_2}{\partial T}\right)_p (V_2 - V_1) - \frac{Tc_{p2}}{q^2}(V_2 - V_1)^2 .$$

The second bracket is transformed similarly.

Let phase 1 be the liquid and phase 2 the vapour; we suppose the latter to be a perfect gas, and neglect the specific volume V_1 in comparison with V_2. If $x \ll 1$ (a liquid containing some bubbles of vapour), the velocity of sound is found to be

$$c = q\mu \frac{pV_1}{RT\sqrt{c_{p1}T}}, \tag{3}$$

where R is the gas constant and μ the molecular weight. This velocity is in general very small; thus, when vapour bubbles form in a liquid (***cavitation***), the velocity of sound undergoes a sudden sharp decrease.

If $1 - x \ll 1$ (a vapour containing some droplets of liquid), we obtain

$$\frac{1}{c^2} = \frac{\mu}{RT} - \frac{2}{q} + \frac{c_{p2}T}{q^2}. \tag{4}$$

Comparing this with the velocity of sound in the pure gas (64.15), we find that here also the addition of a second phase reduces the value of *c*, though by no means so markedly.

As *x* increases from 0 to 1, the velocity of sound increases monotonically from the value (3) to the value (4). For $x = 0$ and $x = 1$ it changes discontinuously as we go from a one-phase system to a two-phase system. This has the result that, for values of x very close to zero or unity, the usual linear theory of sound is no longer applicable, even when the amplitude of the sound wave is small; the compressions and rarefactions produced by the wave are in this case accompanied by a change between a one-phase and a two-phase system, and the essential assumption of a constant velocity of sound no longer holds good.

Problem 2. Determine the velocity of sound in a gas heated to such a high temperature that the pressure of equilibrium black-body radiation becomes comparable with the gas pressure.

Solution. The pressure is $p = nT + akT^4/4$, and the entropy is

$$s = \frac{1}{m}\log\left(\frac{T^{3/2}}{n}\right) + a\frac{T^3}{n}.$$

In these expressions the first terms relate to the particles, and the second terms to the radiation; *n* is the number density of particles, *m* their mass, and $a = 4\pi^2/45\hbar c^3$. (see *SPl,* §63. The temperature is inenergy units, as elsewhere in this book..) The density of matter is not affected by the black-body radiation, so that $\rho = mn$. The velocity of sound, denoted here by *u* to distinguish it from that of light, is given by

$$u^2 = \frac{\partial(p,s)}{\partial(\rho,s)} = \frac{\partial(p,s)}{\partial(n,T)}\Bigg/\frac{\partial(\rho,s)}{\partial(n,T)},$$

where the derivatives have been written in Jacobian form. Evaluating the Jacobians, we have

$$u^2 = \frac{5T}{3m}\left[1 + \frac{2a^2T^6}{5n(n+2aT^3}\right].$$

§65. The energy and momentum of sound waves

Let us derive an expression for the energy of a sound wave. According to the general formula, the energy in unit volume of the fluid is $\rho\varepsilon + \rho v^2/2$. We now substitute $\rho = \rho_0 + \rho'$, $\varepsilon = \varepsilon_0 + \varepsilon'$, where the primed letters denote the deviations of the respective quantities from their values when the fluid is at rest. The term $\rho' v^2/2$ is a quantity of the third order. Hence, if we take only terms up to the second order, we have

$$\rho_0\varepsilon_0 + \rho'\frac{\partial(\rho\varepsilon)}{\partial\rho_0} + \frac{1}{2}\frac{\partial^2(\rho\varepsilon)}{\partial{\rho_0}^2} + \frac{1}{2}\rho_0 v^2.$$

The derivatives are taken at constant entropy, since the sound wave is adiabatic. From the thermodynamic relation $d\varepsilon = Tds - pdV = Tds + (p/\rho^2)d\rho$ we have

$[\partial(\rho\varepsilon)/\partial\rho]_s = \varepsilon + p/\rho = w$, and the second derivative is

$$[\partial^2(\rho\varepsilon)/\partial\rho^2]_s = \left(\frac{\partial w}{\partial\rho}\right)_s = \left(\frac{\partial w}{\partial p}\right)_s\left(\frac{\partial p}{\partial\rho}\right)_s = \frac{c^2}{\rho}.$$

Thus the energy in unit volume of the fluid is

$$\rho_0\varepsilon_0 + w_0\rho' + \frac{1}{2}\frac{c^2\rho'^2}{\rho_0} + \frac{1}{2}\rho_0 v^2.$$

The first term ($\rho_0\varepsilon_0$) in this expression is the energy in unit volume when the fluid is at rest, and does not relate to the sound wave. The second term ($w_0\rho'$) is the change in energy due to the change in the mass of fluid in unit volume. This term disappears in the total energy, which is obtained by integrating the energy over the whole volume of the fluid: since the total mass of fluid is unchanged, we have

$$\int \rho' dV = 0.$$

Thus the total change in the energy of the fluid caused by the sound wave is given by the integral

$$\int\left(\frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{c^2\rho'^2}{\rho_0}\right)dV.$$

The integrand may be regarded as the ***density*** E of sound energy

$$E = \frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{c^2\rho'^2}{\rho_0}. \qquad (65.1)$$

This expression takes a simpler form for a travelling plane wave. In such a wave $\rho' = \rho_0 v/c$, and the two terms in (65.1) are equal, so that

$$E = \rho_0 v^2. \qquad (65.2)$$

In general this relation does not hold. A similar formula can be obtained only for the (time) average of the total sound energy. It follows immediately from a well-known general theorem of mechanics, that the *mean total potential energy of a system executing small oscillations is equal to the mean total kinetic energy*. Since the latter is, in the case considered,

$$\frac{1}{2}\int \rho_0 \overline{v}^2 dV$$

we find that the mean total sound energy is

$$\int \overline{E} dV = \int \rho_0 \overline{v}^2 dV. \qquad (65.3)$$

Next, let us consider some volume of a fluid in which sound is propagated, and determine the flux of energy through the closed surface bounding this volume. The energy flux density in the fluid is, by (6.3), $\rho\mathbf{v}(v^2/2 + w)$. In the present case we can neglect the

term in v^2, which is of the third order. Hence the energy flux density in the sound wave is $\rho w \boldsymbol{v}$. Substituting $w = w_0 + w'$, we have $\rho w \boldsymbol{v} = w_0 \rho \boldsymbol{v} + \rho w' \boldsymbol{v}$. For a small change w' in the heat function we have $w' = (\partial w / \partial p)_s p' = p' / \rho$, and $\rho w \boldsymbol{v} = w_0 \rho \boldsymbol{v} + p' \boldsymbol{v}$. The total energy flux through the surface in question is

$$\oint (w_0 \rho \boldsymbol{v} + p' \boldsymbol{v}) \cdot d\boldsymbol{f} .$$

The first term here is the energy flux due to the change in the mass of fluid in the volume considered. We have already omitted the corresponding term $w_0 \rho'$ (which gives zero on integration over an infinite volume) in the energy density. Hence, to find the energy flux, whose density is given by (65.1), we should omit this term, and the energy flux is simply

$$\oint p' \boldsymbol{v} \cdot d\boldsymbol{f} .$$

We see that the sound energy density flux is represented by the vector

$$\boldsymbol{q} = p' \boldsymbol{v} . \tag{65.4}$$

It is easy to verify that the expected relation

$$\frac{\partial E}{\partial t} + \operatorname{div}(p' \boldsymbol{v}) = 0 \tag{65.5}$$

holds. In this form the equation gives the ***law of conservation of the sound energy***, with the vector (65.4) taking the part of the energy flux density.

In a travelling plane wave (left to right) the pressure variation is related to the velocity by $p' = c \rho_0 v$, where $v = v_x$ is taken with the appropriate sign. Introducing the unit vector $\boldsymbol{n}$ in the direction of propagation of the wave, we obtain

$$\boldsymbol{q} = c \rho_0 v^2 \boldsymbol{n} = cE\boldsymbol{n} . \tag{65.6}$$

Thus the energy flux density in a plane sound wave equals the energy density multiplied by the velocity of sound, a result which was to be expected.

Let us now consider a sound wave which, at any given instant, occupies a finite region of space nowhere bounded by solid walls (a ***wave packet***), and determine the total momentum of the fluid in the wave. The momentum of unit volume of fluid is equal to the mass flux density $\boldsymbol{j} = \rho \boldsymbol{v}$. Substituting $\rho = \rho_0 + \rho'$, we have $\boldsymbol{j} = \rho_0 \boldsymbol{v} + \rho' \boldsymbol{v}$. The density change is related to the pressure change by $\rho' = p' / c^2$. Using (65.4), we therefore obtain

$$\boldsymbol{j} = \rho_0 \boldsymbol{v} + \boldsymbol{q} / c^2 . \tag{65.7}$$

If the viscosity of the fluid is not significant in the phenomena under consideration, we can assume potential flow in a sound wave, and write $\boldsymbol{v} = \operatorname{grad} \phi$; it should be emphasized that this result is not a consequence of the approximations made in deriving the linear equations of motion in §64, since a solution such that $\operatorname{curl} \boldsymbol{v} = 0$ is an exact solution of Euler's equations. We therefore have $\boldsymbol{j} = \rho_0 \operatorname{grad} \phi + \boldsymbol{q} / c^2$. The total momentum in the

wave equals the integral $\int \boldsymbol{j} dV$ over the volume occupied by the wave. The integral of $\operatorname{grad}\phi$ can be transformed into a surface integral,

$$\int \operatorname{grad}\phi dV = \oint \phi d\boldsymbol{f},$$

and is zero, since ϕ is zero outside the volume occupied by the wave packet. Thus the total momentum of the wave packet is

$$\int \boldsymbol{j} dV = \frac{1}{c^2}\int \boldsymbol{q} dV. \tag{65.8}$$

This quantity is not, in general, zero. The existence of a non-zero total momentum means that there is a transfer of matter. We therefore conclude that the *propagation of a sound wave packet is accompanied by the transfer of fluid*. This is a second-order effect (since $\boldsymbol{q}$ is a second-order quantity).

Finally, let us consider a region of space unlimited in length but finite in cross-section (a ***wave train*** with finite aperture), and calculate the mean value of the pressure change p' in a sound wave. In the first approximation, corresponding to the usual linearized equations of motion, p' is a function which periodically changes sign, and the mean value of p' is zero. This result, however, may cease to hold if we go to higher approximations. If we take only second-order quantities, $\overline{p}'$ can be expressed in terms of quantities calculated from the linear sound equations, so that it is not necessary to solve directly the non-linear equations of motion obtained when terms of higher order are taken into account.

A characteristic property of the sound in question is that the difference between the velocity potentials ϕ at different points remains finite when the distance between them increases without limit (and the same is true of the difference in the values of ϕ at a given point in space at different times): this difference is

$$\phi_2 - \phi_1 = \oint \boldsymbol{v}\cdot d\boldsymbol{l}$$

which can be taken along any path between the points 1 and 2, and the property stated is obvious if we note that a path can be chosen which lies along the wave train but outside it.[3]

We therefore start from Bernoulli's equation: $w + v^2/2 + \partial\phi/\partial t = \text{constant}$, and average it with respect to time. The mean value of the time derivative $\partial\phi/\partial t$ is zero.[4] Putting also $w = w_0 + w'$ and including w_0 in the constant, we obtain

[3] Essentially similar arguments have been used in deriving (65.8) from the proposition that $\phi = 0$ everywhere far from a wave packet.

[4] By the general definition of the mean value, we have for the mean derivative of any function $f(t)$

$$\overline{df/dt} = \frac{1}{2T}\int_{-T}^{T}\frac{df}{dt}dt = \frac{f(T)-f(-T)}{2T}.$$

$\overline{w}'+\overline{v}^2/2=\text{constant}$. Since the constant is the same in all space, and w' and v are zero far from the wave train, the constant must evidently be zero, so that

$$\overline{w}'+\frac{1}{2}\overline{v}^2=0. \tag{65.9}$$

We next expand w' in powers of p', and take only the terms up to the second order:

$$w'=\left(\frac{\partial w}{\partial p}\right)_s p'+\frac{1}{2}\left(\frac{\partial^2 w}{\partial p^2}\right)_s p'^2;$$

since $\left(\frac{\partial w}{\partial p}\right)_s=\frac{1}{\rho}$, we have

$$w'=\frac{p'}{\rho_0}-\frac{p'^2}{2\rho_0{}^2}\left(\frac{\partial\rho}{\partial p}\right)_s=\frac{p'}{\rho_0}-\frac{p'^2}{2c^2\rho_0{}^2}.$$

Substituting this in (65.9) gives

$$\overline{p}'=-\frac{1}{2}\rho_0\overline{v}^2+\frac{\overline{p}'^2}{2\rho_0c^2}=-\frac{1}{2}\rho_0\overline{v}^2+\frac{\overline{\rho}'^2c^2}{2\rho_0}, \tag{65.10}$$

which determines the required mean value. The expression on the right is a second-order quantity, and is calculated by using the ρ' and v obtained from the solution of the linearized equations of motion. The mean density is

$$\overline{\rho}'=\left(\frac{\partial\rho}{\partial p}\right)_s\overline{p}'+\frac{1}{2}\left(\frac{\partial^2\rho}{\partial p^2}\right)_s\overline{p}'^2. \tag{65.11}$$

Since the cross-section of the wave train is finite, it cannot be regarded as exactly a plane wave, but if the linear size of the cross-section is sufficiently great relative to the sound wavelength, there may be a very close approximation to a plane wave. In a travelling plane wave, $v=c\rho'/\rho_0$, so that $\overline{v}^2=c^2\overline{\rho}'^2/\rho_0{}^2$, and the expression (65.10) is zero, i.e. the mean pressure variation in a plane wave is an effect of higher order than the second. The density variation $\overline{\rho}'=\frac{1}{2}\left(\frac{\partial^2\rho}{\partial p^2}\right)_s\overline{p}'^2$ is not zero, however.[5] In the same approximation, we have for the mean value of the momentum flux density tensor in a travelling plane wave $\overline{p}'+\overline{\rho v_i v_k}=\overline{p}'+\rho_0\overline{v_i v_k}$. The first term is zero. In the second term, we introduce the unit vector $\boldsymbol{n}$ in the direction of propagation of the wave (the same as the direction of $\boldsymbol{v}$, apart from sign), and, using (65.2), obtain for the momentum flux density

$$\overline{\Pi}_{ik}=\overline{E}n_in_k. \tag{65.12}$$

If the wave is propagated in the x-direction, only the component $\overline{\Pi}_{xx}=\overline{E}$ is not zero. Thus, in this approximation, there is only an x-component of the mean momentum flux, and this is transmitted in the x-direction.

It should be emphasized once more in connection with the discussion in the preceding paragraph that the wave train has a limited cross-section. For a strictly plane wave, the

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[5] We may mention that the derivative $(\partial^2\rho/\partial p^2)_s$ is in fact always negative, and therefore $\overline{\rho}'<0$ in a traveling wave.

results would not be valid; in particular, $\overline{p}'$ might not be zero even in the quadratic approximation (see §101, Problem 4). This arises, formally, because for a strictly plane wave (which cannot be "by-passed") it is not in general correct to say that the potential ϕ is finite in all space or at all times. The physical difference is the result of the possible occurrence (in a wave train with a limited cross-section) of a transverse flow which equalizes the mean pressure.

§66. Reflection and refraction of sound waves

When a sound wave is incident on the boundary between two different fluid media, it undergoes reflection and refraction. The motion in the first medium is a combination of two waves (the ***incident wave*** and the ***reflected wave***), whereas in the second medium there is only one, the ***refracted wave***.

The relation between these three waves is determined by the boundary conditions at the surface of separation.

Let us consider the reflection and refraction of a monochromatic longitudinal wave at a plane surface separating two media, which we take as the *yz*-plane. It is easy to see that all three waves have the same frequency ω and the same components k_y, k_z, of the wave vector, but not the same component k_x perpendicular to the plane of separation. For, in an infinite homogeneous medium, a monochromatic wave with constant $\boldsymbol{k}$ and ω satisfies the equations of motion. The presence of a boundary introduces only some boundary conditions, which in the case considered apply at $x = 0$, i.e. do not depend on the time or on the coordinates *y, z*. Hence the dependence of the solution on *t, y* and z remains the same in all space and time, i.e. ω, k_y, and k_z are the same as in the incident wave.

From this result we can immediately derive the relations which give the directions of propagation of the reflected and refracted waves. Let the plane of the incident wave be the *xy*-plane. Then $k_z = 0$ in the incident wave, and the same must be true of the reflected and refracted waves. Thus the directions of propagation of the three waves are coplanar.

Let θ be the angle between the direction of propagation of the wave and the *x*-axis. Then, from the equality of $k_y = (\omega / c)\sin\theta$ for the incident and reflected waves, it follows that

$$\theta_1 = \theta_1', \tag{66.1}$$

i.e., the angle of incidence θ_1 is equal to the angle of reflection θ_1'. From a similar equation for the incident and refracted waves it follows that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{c_1}{c_2}, \tag{66.2}$$

which relates the angle of incidence θ_1 to the angle of refraction θ_2 (c_1 and c_2 being the velocities of sound in the two media).

In order to obtain a quantitative relation between the intensities of the three waves, we write the respective velocity potentials as

$$\begin{cases} \phi_1 = A_1 \exp[i\omega\{(x/c_1)\cos\theta_1 + (y/c_1)\sin\theta_1 - t\}] \\ \phi_1' = A_1' \exp[i\omega\{(-x/c_1)\cos\theta_1 + (y/c_1)\sin\theta_1 - t\}] \\ \phi_2 = A_2 \exp[i\omega\{(x/c_2)\cos\theta_2 + (y/c_2)\sin\theta_2 - t\}] \end{cases}$$

On the surface of separation (x = 0) the pressure ($p = -\rho\partial\phi/\partial t$) and the normal velocities ($v_x = \partial\phi/\partial x$) in the two media must be equal; these conditions lead to the equations

$$\rho_1(A_1 + A_1') = \rho_2 A_2, \quad \frac{\cos\theta_1}{c_1}(A_1 - A_1') = \frac{\cos\theta_2}{c_2} A_2 .$$

The ***reflection coefficient*** R is defined as the ratio of the (time) average **energy flux densities** in the reflected and incident waves. Since the energy flux density in a plane wave is $c\rho v^2$, we have $R = c_1\rho_1\bar{v}_1'^2 / c_1\rho_1\bar{v}_1^2 = |A_1'|^2/|A_1|^2$. A simple calculation gives

$$R = \left(\frac{\rho_2\tan\theta_2 - \rho_1\tan\theta_1}{\rho_2\tan\theta_2 + \rho_1\tan\theta_1}\right)^2 . \tag{66.3}$$

The angles θ_1 and θ_2 are related by (66.2); expressing θ_2 in terms of θ_1, we can put the reflection coefficient in the form

$$R = \left(\frac{\rho_2 c_2\cos\theta_1 - \rho_1\sqrt{c_1^2 - c_2^2\sin^2\theta_1}}{\rho_2 c_2\cos\theta_1 + \rho_1\sqrt{c_1^2 - c_2^2\sin^2\theta_1}}\right)^2 . \tag{66.4}$$

For normal incidence (θ_1 = 0), this formula gives simply

$$R = \left(\frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1}\right)^2 . \tag{66.5}$$

For an angle of incidence such that

$$\tan^2\theta_1 = \frac{\rho_2^2 c_2^2 - \rho_1^2 c_1^2}{\rho_1^2(c_1^2 - c_2^2)} \tag{66.6}$$

the reflection coefficient is zero, i.e. the wave is totally refracted. This can happen if $c_1 > c_2$ but $\rho_2 c_2 > \rho_1 c_1$, or if both inequalities are reversed.

PROBLEM

Determine the pressure exerted by a sound wave on the boundary separating two fluids.

Solution. The sum of the total energy fluxes in the reflected and refracted waves must equal the incident energy flux. Taking the energy flux per unit area of the surface of separation, we can write this condition in the form

$$c_1 E_1\cos\theta_1 = c_1 E_1'\cos\theta_1 + c_2 E_2\cos\theta_2 ,$$

where E_1, E_1' and E_2 are the energy densities in the three waves. Introducing the reflection coefficient $R = \bar{E}_1'/\bar{E}_1$, we therefore have

$$\bar{E}_2 = \frac{c_1\cos\theta_1}{c_2\cos\theta_2}(1-R)\bar{E}_1 .$$

The required pressure p is determined as the x-component of the momentum lost per unit time by the sound wave (per unit area of the boundary). Using the expression (65.12) for the momentum flux density tensor in a sound wave, we find

$$p = \overline{E}_1 \cos^2\theta_1 + \overline{E}_1'\cos^2\theta_1 - \overline{E}_2\cos^2\theta_2 .$$

Substituting for $\overline{E}_2$, introducing R and using (66.2), we obtain

$$p = \overline{E}_1 \sin\theta_1 \cos\theta_1[(1+R)\cot\theta_1 - (1-R)cor\theta_2] .$$

For normal incidence ($\theta_1 = 0$), we find, using (66.5),

$$p = 2\overline{E}_1\left[\frac{\rho_1{}^2c_1{}^2 + \rho_2{}^2c_2{}^2 - 2\rho_1\rho_2c_1{}^2}{(\rho_1c_1+\rho_2c_2)^2}\right].$$