## §67. Geometrical acoustics

A **plane wave** has the distinctive property that its direction of propagation and its amplitude are the same in all space. An arbitrary sound wave, of course, does not possess this property. However, cases can occur where a sound wave that is not plane may still be regarded as plane in any small region of space. For this to be so it is necessary that the *amplitude and the direction of propagation should vary only slightly over distances of the order of the wavelength*.

If this condition holds, we can introduce the idea of ***rays***, these being lines such that the tangent to them at any point is in the same direction as the direction of propagation; and we can say that the sound is propagated along the rays, and ignore its wave nature. The study of the laws of propagation of sound in such cases is the task of ***geometrical acoustics***. We may say that geometrical acoustics corresponds to the limit of short wavelengths, $\lambda \to 0$.

Let us derive the **basic equation** of geometrical acoustics, which determines the direction of the rays. We write the wave velocity potential as

$$\phi = ae^{i\psi}. \qquad (67.1)$$

In the case where the wave is not plane but geometrical acoustics can be applied, the amplitude $a$ is a slowly varying function of the coordinates and the time, while the wave phase $\psi$ is "almost linear" (we recall that in a plane wave $\psi = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \alpha$, with constant $\boldsymbol{k}$ and $\omega$). Over small regions of space and short intervals of time, the phase $\psi$ may be expanded in series; up to terms of the first order we have

$$\psi = \psi_0 + \boldsymbol{r}\cdot \operatorname{grad}\psi + t\frac{\partial \psi}{\partial t}.$$

In accordance with the fact that, in any small region of space (and during short intervals of time), the wave may be regarded as plane, we define the wave vector and the frequency at each point as

$$\boldsymbol{k} = \frac{\partial \psi}{\partial \boldsymbol{r}} \equiv \operatorname{grad}\psi, \quad \omega = -\frac{\partial \psi}{\partial t}. \qquad (67.2)$$

The quantity $\psi$ is called the ***eikonal***.

In a sound wave we have $\omega^2/c^2 = k^2 = k_x{}^2 + k_y{}^2 + k_z{}^2$. Substituting (67.2), we obtain the basic equation of geometrical acoustics;

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 - \frac{1}{c^2}\left(\frac{\partial \psi}{\partial t}\right)^2 = 0. \qquad (67.3)$$

If the fluid is not homogeneous, the coefficient $1/c^2$ is a function of the coordinates.

As we know from mechanics, the motion of material particles can be determined by means of the Hamilton-Jacobi equation, which, like (67.3), is a first-order partial differential equation. The quantity analogous to $\psi$ is the action $S$ of the particle, and the derivatives of the action determine the momentum $\boldsymbol{p} = \partial S / \partial \boldsymbol{r}$ and the Hamilton's function (the energy) $H = -\partial S / \partial t$ of the particle; these formulae are similar to (67.2). We know, also, that the Hamilton-Jacobi equation is equivalent to Hamilton's equations

$$\dot{\boldsymbol{p}} = -\partial H / \partial \boldsymbol{r}\,, \quad \boldsymbol{v} \equiv \dot{\boldsymbol{r}} = \partial H / \partial \boldsymbol{p}\,.$$

From the above analogy between the mechanics of a material particle and geometrical acoustics, we can write down similar equations for rays:

$$\dot{\boldsymbol{k}} = -\partial \omega / \partial \boldsymbol{r}\,, \quad \dot{\boldsymbol{r}} = \partial \omega / \partial \boldsymbol{k}\,.$$

In a homogeneous isotropic medium $\omega = ck$ with $c$ constant, so that $\dot{\boldsymbol{k}} = 0$, $\dot{\boldsymbol{r}} = c\boldsymbol{n}$ ($\boldsymbol{n}$ being a unit vector in the direction of $\boldsymbol{k}$), i.e. the rays are propagated in straight lines with a constant frequency $\omega$, as we should expect.

The frequency, of course, remains constant along a ray in all cases where the propagation of sound occurs under steady conditions, i.e., the properties of the medium at each point in space do not vary with time. For the total time derivative of the frequency, which gives its rate of variation along a ray, is

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \dot{\boldsymbol{r}} \cdot \frac{\partial \omega}{\partial \boldsymbol{r}} + \dot{\boldsymbol{k}} \cdot \frac{\partial \omega}{\partial \boldsymbol{k}}\,.$$

On substituting (67.4), the last two terms cancel, and in a steady state $\partial \omega / \partial t = 0$, so that $d\omega / dt = 0$.

In steady propagation of sound in an inhomogeneous medium at rest $\omega = ck$, where $c$ is a given function of the coordinates. The equations (67.4) give

$$\dot{\boldsymbol{r}} = c\boldsymbol{n}\,, \quad \dot{\boldsymbol{k}} = -k \operatorname{grad} c\,. \qquad (67.5)$$

The magnitude of the vector $\boldsymbol{k}$ varies along a ray simply according to $k = \omega / c$ (with $\omega$ constant). To determine the change in direction of $\boldsymbol{n}$ we put $\boldsymbol{k} = \omega \boldsymbol{n} / c$ in the second of (67.5):

$$\omega \dot{\boldsymbol{n}} / c - \frac{\omega \boldsymbol{n}}{c^2} (\dot{\boldsymbol{r}} \cdot \operatorname{grad} c) = -k \operatorname{grad} c\,,$$

whence

$$\frac{d\boldsymbol{n}}{dt} = -\operatorname{grad} c + \boldsymbol{n}(\boldsymbol{n} \cdot \operatorname{grad} c)\,.$$

Introducing the element of length along the ray $dl = c\ dt$, we can rewrite this

equation

$$\frac{d\boldsymbol{n}}{dl} = -\frac{1}{c}\operatorname{grad} c + \frac{\boldsymbol{n}(\boldsymbol{n}\cdot\operatorname{grad} c)}{c}. \qquad (67.6)$$

This equation determines the form of the rays; $\boldsymbol{n}$ is a unit vector tangential to a ray.[1]

If equation (67.3) is solved, and the eikonal $\psi$ is a known function of coordinates and time, we can then find also the distribution of sound intensity in space. In steady conditions, it is given by the equation div $\boldsymbol{q} = 0$ ($\boldsymbol{q}$ being the sound energy flux density), which must hold in all space except at sources of sound. Putting $\boldsymbol{q} = cE\boldsymbol{n}$, where $E$ is the sound energy density (see (65.6)), and remembering that $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{k} = \operatorname{grad}\psi$, we obtain the equation

$$\operatorname{div}\left(\frac{cE\operatorname{grad}\psi}{|\operatorname{grad}\psi|}\right) = 0, \qquad (67.7)$$

which determines the distribution of $E$ in space.

The second formula (67.4) gives the velocity of propagation of the waves from the known dependence of the frequency on the components of the wave vector. This is an important formula, which holds not only for sound waves, but for all waves (for example, we have already applied it to gravity waves in §12). We shall give here **another derivation of this formula**, which puts in evidence the meaning of the velocity which it defines. Let us consider a ***wave packet***, which occupies some finite region of space. We assume that its spectral composition includes monochromatic components whose frequencies lie in only a small range; the same is true of the components of their wave vectors. Let $\omega$ be some mean frequency of the wave, and $\boldsymbol{k}$ a mean wave vector. Then, at some initial instant, the wave is described by a function having the form

$$\phi = \exp(i\boldsymbol{k}\cdot\boldsymbol{r})f(\boldsymbol{r}). \qquad (67.8)$$

The function $f(\boldsymbol{r})$ is appreciably different from zero only in a region which is small (though it is large compared with the wavelength $1/k$). Its expansion as a Fourier integral contains, by the above assumptions, components having the form $\exp(i\boldsymbol{r}\cdot\Delta\boldsymbol{k})$, where $\Delta\boldsymbol{k}$ is small.

Thus each monochromatic component is, at the initial instant, proportional to

$$\phi_k = \text{constant}\times\exp[i(\boldsymbol{k}+\Delta\boldsymbol{k})\cdot\boldsymbol{r}]. \qquad (67.9)$$

The corresponding frequency is $\omega(\boldsymbol{k}+\Delta\boldsymbol{k})$ (we recall that the frequency is a

[1] As we know from differential geometry, the derivative d$\boldsymbol{n}$/d$l$ along the ray is equal to $\boldsymbol{N}/R$, where $\boldsymbol{N}$ is a unit vector along the principal normal and $R$ is the radius of curvature of the ray. The expression on the right-hand side of (67.6) is, apart from a factor $1/c$, the derivative of the velocity of sound along the principal normal; hence we can write the equation as $1/R = -(1/c)\boldsymbol{N}\cdot\operatorname{grad} c$. The rays bend towards the region where $c$ is smaller.

function of the wave vector). Hence the same component at time $t$ has the form

$$\phi_k = \text{constant} \times \exp[i(\boldsymbol{k} + \Delta\boldsymbol{k}) \cdot \boldsymbol{r} - i\omega(\boldsymbol{k} + \Delta\boldsymbol{k})t].$$

We use the fact that $\Delta\boldsymbol{k}$ is small, and put $\omega(\boldsymbol{k} + \Delta\boldsymbol{k}) \cong \omega + (\partial\omega / \partial\boldsymbol{k}) \cdot \Delta\boldsymbol{k}$. Then $\phi_k$ becomes

$$\phi_k = \text{constant} \times \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \exp[i\Delta\boldsymbol{k} \cdot (\boldsymbol{r} - t\frac{\partial\omega}{\partial\boldsymbol{k}})]. \qquad (67.10)$$

If we now sum all the monochromatic components, with all the $\Delta\boldsymbol{k}$ that occur in the wave packet, we see from (67.9) and (67.10) that the result is

$$\phi = \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] f(\boldsymbol{r} - t\frac{\partial\omega}{\partial\boldsymbol{k}}), \qquad (67.11)$$

where $f$ is the same function as in (67.8). A comparison with (67.8) shows that, after a time $t$, the amplitude distribution has moved as a whole through a distance $t\partial\omega / \partial\boldsymbol{k}$; the exponential coefficient of $f$ in (67.11) affects only the phase. Consequently, the velocity of the wave is

$$\boldsymbol{U} = \frac{\partial\omega}{\partial\boldsymbol{k}}. \qquad (67.11)$$

This formula gives the velocity of propagation for any dependence of $\omega$ on $\boldsymbol{k}$. When $\omega = ck$, with $c$ constant, it of course gives the usual result $U = \omega / k = c$. In general, when $\omega(k)$ is an arbitrary function, the velocity of propagation is a function of the frequency, and the direction of propagation may not be the same as that of the wave vector.

The velocity defined by (67.12) is called the ***group velocity*** of the wave, and the ratio $\omega / k$ the ***phase velocity***. However, it must be borne in mind that the phase velocity does not correspond to any actual physical propagation.

Regarding the derivation given here it should be noted that the motion of the wave packet without change of form, expressed by (67.11), is approximate, and results from the assumption that the range $\Delta\boldsymbol{k}$ is small. In general, when $U$ depends on $\omega$, a wave packet is "smoothed out" during its propagation, and the region of space which it occupies increases in size. It can be shown that the amount of this smoothing out is proportional to the squared magnitude of the range $\Delta\boldsymbol{k}$ of the wave vectors which occur in the composition of the wave packet

## PROBLEM

Determine the altitude variation in the amplitude of sound propagated in an isothermal atmosphere under gravity.

**Solution.** In an isothermal atmosphere (regarded as a perfect gas) the velocity of sound is constant. The energy flux density evidently decreases along a ray in inverse

proportion to the square of the distance $r$ from the source: $c\rho\overline{v}^2 \propto 1/r^2$. Hence it follows that the amplitude of the velocity fluctuations in the sound wave varies along a ray inversely as $r\sqrt{\rho}$; according to the barometric formula, $\rho \propto \exp(-\mu gz/RT)$, where $z$ is the altitude, $\mu$ the molecular weight of the gas and $R$ the gas constant.

## §68. Propagation of sound in a moving medium

The relation $\omega = ck$ between the frequency and the wave number is valid only for a monochromatic sound wave propagated in a medium at rest. It is not difficult to obtain a similar relation for a wave propagated in a **moving medium** (and observed in a fixed system of coordinates).

Let us consider a homogeneous flow with velocity $\boldsymbol{u}$. We take a fixed system $K$ of coordinates $x, y, z,$ and also a system $K'$ of coordinates $x', y', z'$ moving with velocity $\boldsymbol{u}$ relative to $K$. In the system $K'$ the fluid is at rest, and a monochromatic wave has the usual form $\phi = \text{constant} \times \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}' - kct)]$. The position vector $\boldsymbol{r}'$ in the system $K'$ is related to the position vector $\boldsymbol{r}$ in the system $K$ by $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{u}t$. Hence, in the fixed system of coordinates, the wave has the form $\phi = \text{constant} \times \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - (kc + \boldsymbol{k}\cdot\boldsymbol{u})t)]$. The coefficient of $t$ in the exponent is the frequency $\omega$ of the wave. Thus the frequency in a moving medium is related to the wave vector $\boldsymbol{k}$ by

$$\omega = ck + \boldsymbol{u}\cdot\boldsymbol{k}\,. \tag{68.1}$$

The velocity of propagation is

$$\frac{\partial\omega}{\partial\boldsymbol{k}} = c\frac{\boldsymbol{k}}{k} + \boldsymbol{u}\,; \tag{68.2}$$

this is the vector sum of the velocity $c$ in the direction of $\boldsymbol{k}$ and the velocity $\boldsymbol{u}$ with which the sound is "carried along" by the moving fluid.

Let us next determine the sound wave **energy density** in the moving medium. The total instantaneous energy density is

$$\frac{1}{2}(\rho+\rho')(\boldsymbol{u}+\boldsymbol{v})^2 + \frac{1}{2}c^2\frac{\rho'^2}{\rho} = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho' u^2 + \rho\boldsymbol{v}\cdot\boldsymbol{u} + \frac{1}{2}\rho v^2 + \rho'\boldsymbol{u}\cdot\boldsymbol{v} + c^2\frac{\rho'^2}{2\rho}$$

(cf. (65.1); the suffix 0 to the unperturbed quantities is omitted). The first term here is the energy of the unperturbed flow. The next two are first-order small quantities, but on averaging over time they give second-order quantities related to the energy of the mean flow due to the wave. All these are to be omitted, and the required energy density of the sound wave as such is given by the last three terms, in the brackets.

The velocity and the pressure change in a plane wave in the moving medium are related by

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{u})\boldsymbol{v} = \boldsymbol{k}c^2\frac{\rho'}{\rho},$$

which follows from the linearized Euler's equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = (\boldsymbol{u}\cdot\mathrm{grad})\boldsymbol{v} = -\frac{1}{\rho}\mathrm{grad}\, p\,.$$

With (68.1), we have finally as the sound energy density in the moving medium

$$E = E_0\frac{\omega}{\omega - \boldsymbol{k}\cdot\boldsymbol{u}}, \tag{68.3}$$

where $E_0 = c^2\rho'^2/\rho = p'^2/\rho c^2$ is the energy density in the frame of reference moving with the medium.[2]

Using formula (68.1), we can investigate what is called the ***Doppler effect***: the frequency of sound, as received by an observer moving relative to the source, is not the same as the frequency of oscillation of the source.

Let sound emitted by a source at rest (relative to the medium) be received by an observer moving with velocity ***u***. In a system *K′* at rest relative to the medium we have $k = \omega_0/c$, where $\omega_0$ is the frequency of oscillation of the source. In a system *K* moving with the observer, the medium moves with velocity –***u***, and the frequency of the sound is, by (68.1), $\omega = ck - \boldsymbol{u}\cdot\boldsymbol{k}$. Introducing the angle $\theta$ between the direction of the velocity ***u*** and that of the wave vector ***k***, and putting $k = \omega_0/c$, we find that the frequency of the sound received by the moving observer is

$$\omega = \omega_0\left[1 - \left(\frac{u}{c}\right)\cos\theta\right]. \tag{68.4}$$

The opposite case, to a certain extent, is the propagation in a medium at rest of a sound wave emitted from a moving source. Let ***u*** be now the velocity of the source. We change from the fixed system of coordinates to a system *K′* moving with the source; in the system *K′*, the fluid moves with velocity -***u***. In *K′*, where the source is at rest, the frequency of the emitted sound wave must equal the frequency $\omega_0$ of the oscillations of the source. Changing the sign of ***u*** in (68.1) and introducing the angle $\theta$ between the directions of ***u*** and ***k***, we have $\omega_0 = ck[1-(u/c)\cos\theta]$. In the original fixed system *K*, however, the frequency and the wave number are

[2] Equation (68.3) can be interpreted from a quantum standpoint as meaning simply that the number of sound quanta (phonons) $N = E/\hbar\omega = E_0/\hbar(\omega - \boldsymbol{k}\cdot\boldsymbol{u})$ is independent of the choice of reference frame.

related by $\omega = ck$. Thus we find

$$\omega = \frac{\omega_0}{[1-(u/c)\cos\theta]}. \qquad (68.5)$$

This formula gives the relation between the frequency $\omega_0$ of the oscillations of a moving source and the frequency $\omega$ of the sound heard by an observer at rest.

If the source is moving away from the observer, the angle $\theta$ between its velocity and the direction to the observer lies in the range $\pi/2 < \theta \le \pi$, so that $\cos\theta < 0$. It then follows from (68.5) that, if the source is *moving away* from the observer, the frequency of the sound heard is *less than* $\omega_0$.

If, on the other hand, the source is approaching the observer, then $0 \le \theta < \pi/2$, so that $\cos\theta > 0$, and the frequency $\omega > \omega_0$ increases with *u.* For $u\cos\theta > c$, according to formula (68.5) $\omega$ becomes negative, which means that the sound heard by the observer actually reaches him in the reverse order, i.e., *sound emitted by the source at any given instant arrives earlier than sound emitted at previous instants*.

As has been mentioned at the beginning of §67, the approximation of geometrical acoustics corresponds to the case of short wavelengths, i.e., large wave numbers. For this to be so the frequency of the sound must in general be large. In the acoustics of moving media, however, the latter condition need not be fulfilled if the velocity of the medium exceeds that of sound. For in this case *k* can be large even when the frequency is zero; from (68.1) we have for $\omega = 0$ the equation

$$ck = -\boldsymbol{u} \cdot \boldsymbol{k},$$

and this has solutions if u > c. Thus, in a medium moving with **supersonic** velocities, there can be steady small perturbations described (if *k* is sufficiently large) by geometrical acoustics. This means that such perturbations are propagated along rays.

Let us consider, for example, a homogeneous **supersonic** stream moving with constant velocity ***u***, whose direction we take as the *x*-axis. The vector ***k*** is taken to lie in the *xy*-plane, and its components are related by

$$(u^2 - c^2){k_x}^2 = c^2{k_y}^2, \qquad (68.7)$$

which is obtained by squaring both sides of equation (68.6). To determine the form of the rays, we use the equations of geometrical acoustics (67.4), according to which $\dot{x} = \partial\omega/\partial k_x$, $\dot{y} = \partial\omega/\partial k_y$. Dividing one of these equations by the other, we have $dy/dx = (\partial\omega/\partial k_y)/(\partial\omega/\partial k_x)$. This relation, however, is, by the rule of differentiation for implicit functions, just the derivative $-\partial k_x/\partial k_y$ taken at a constant frequency (in this case zero). Thus the equation which gives the form of the rays from the known relation between $k_x$ and $k_y$ is

$$\frac{dy}{dx} = -\frac{\partial k_x}{\partial k_y}. \qquad (68.8)$$

Substituting (68.7), we obtain

$$\frac{dy}{dx} = \pm\frac{c}{\sqrt{u^2 - c^2}}.$$

For constant $u$ this equation represents two straight lines intersecting the $x$-axis at angles $\pm\alpha$, where $\sin\alpha = c/u$.

We shall return to a detailed study of these rays in gas dynamics, where they are very important.

**PROBLEMS**

**Problem 1.** Derive an equation giving the form of sound rays propagated in a steadily moving medium with a velocity distribution $\boldsymbol{u}$(x, *y, z*), when $u \ll c$ everywhere. It is assumed that the velocity $\boldsymbol{u}$ varies appreciably only over distances large compared with the wavelength of the sound.

**Solution.** Substituting (68.1) in (67.4), we obtain the equations of propagation of the rays in the form

$$\dot{\boldsymbol{k}} = -(\boldsymbol{k}\cdot\mathrm{grad})\boldsymbol{u} - \boldsymbol{k}\times\mathrm{curl}\,\boldsymbol{u}, \quad \dot{\boldsymbol{r}} \equiv v = c\boldsymbol{k}/k + \boldsymbol{u}.$$

Using these equations, and also

$$\frac{d\boldsymbol{u}}{dt} \equiv \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{v}\cdot\mathrm{grad})\boldsymbol{u} = (\boldsymbol{v}\cdot\mathrm{grad})\boldsymbol{u} \cong \frac{c}{k}(\boldsymbol{k}\cdot\mathrm{grad})\boldsymbol{u},$$

we calculate the derivative d(*kv*)/d*t*, retaining only terms as far as the first order in **u**. The result is $d(k\boldsymbol{v})/dt = -kv\boldsymbol{n}\times\mathrm{curl}\,\boldsymbol{u}$, when $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{v}$. But $d(k\boldsymbol{v})/dt = \boldsymbol{n}d(kv)/dt + kvd\boldsymbol{n}/dt$. Since $\boldsymbol{n}$ and d$\boldsymbol{n}$/dt are perpendicular (because $\boldsymbol{n}^2 = 1$, and therefore $\boldsymbol{n}\cdot\dot{\boldsymbol{n}} = 0$), it follows from the above equations that $\dot{\boldsymbol{n}} = -\boldsymbol{n}\times\mathrm{curl}\,\boldsymbol{u}$. Introducing the element of length along the ray d*l* = *c*d*t*, we can write finally

$$\frac{d\boldsymbol{n}}{dl} = -\boldsymbol{n}\times\frac{\mathrm{curl}\,\boldsymbol{u}}{c}. \qquad (1)$$

This equation determines the form of the rays; $\boldsymbol{n}$ is a unit tangential vector (and is no longer in the same direction as $\boldsymbol{k}$).

**Problem 2.** Determine the form of sound rays in a moving medium with a velocity distribution $u_x = u(z)$, $u_y = u_z = 0$.

**Solution.** Expanding equation (1), Problem 1, we find $dn_x/dl = (n_z/c)du/dz$, $dn_y/dl = 0$; the equation for $n_z$ need not be written down, since $\boldsymbol{n}^2 = 1$. The

second equation gives $n_y = \text{constant} \equiv n_{y,0}$. In the first equation we write $n_z = dz/dl$, and then we have by integration $n_x = n_{x,0} + u(z)/c$. These formulae give the required solution.

Let us assume that the velocity $u$ is zero for $z$ = 0 and increases upwards ($du/dz$ > 0). If the sound is propagated "against the wind" ($n_x > 0$), its path is curved upwards; if it is propagated "with the wind" ($n_x < 0$), its path is curved downwards. In the latter case a ray leaving the point $z$ = 0 at a small angle to the $x$-axis (i.e., with $n_{x,0}$ close to unity) rises only to a finite altitude z = $z_{\max}$, which can be calculated as follows. At the altitude $z_{\max}$ the ray is horizontal, i.e., $n_z = 0$. Hence we have

$$n_x{}^2 + n_y{}^2 \cong n_{x,0}{}^2 + n_{y,0}{}^2 + 2n_{x,0}\frac{u}{c} = 1,$$

so that $2n_{x,0}u(z_{\max})/c = n_{z,0}{}^2$, whence we can determine $z_{\max}$ from the given function $u(z)$ and the initial direction $\boldsymbol{n_0}$ of the ray

**Problem 3.** Obtain the expression of Fermat's principle for sound rays in a steadily moving medium.

**Solution.** Fermat's principle is that the integral

$$\oint \boldsymbol{k}\cdot d\boldsymbol{l},$$

taken along a ray between two given points, is a minimum; $\boldsymbol{k}$ is supposed expressed as a function of the frequency $\omega$ and the direction $\boldsymbol{n}$ of the ray (see *Fields,* §53). This function can be found by eliminating $v$ and $k$ from the relations $\omega = ck + \boldsymbol{u}\cdot\boldsymbol{k}$ and $v\boldsymbol{n} = c\boldsymbol{k}/k + \boldsymbol{u}$. Fermat's principle then takes the form

$$\delta\oint \frac{\sqrt{(c^2-u^2)dl^2 + (\boldsymbol{u}\cdot d\boldsymbol{l})^2} - \boldsymbol{u}\cdot d\boldsymbol{l}}{c^2-u^2} = 0.$$

In a medium at rest, this integral reduces to the usual one, $\oint dl/c$.

### §69. Characteristic vibrations

Hitherto we have discussed only oscillatory motion in infinite media, and we have seen, in particular, that in such media waves with any frequency can be propagated.

The situation is very different when we consider a **fluid in a vessel with finite dimensions**. The equations of motion themselves (the wave equations) are of course unchanged, but they must now be supplemented by **boundary conditions** to be satisfied at the solid walls or at the free surface of the fluid. We shall consider here

only what are called ***free vibrations***, i.e., those which occur in the absence of variable external forces. Vibrations occurring as a result of external forces are called ***forced vibrations***.

The equations of motion for a finite fluid do not have solutions satisfying the appropriate boundary conditions for every frequency. Such solutions exist only for a series of definite frequencies $\omega$. In other words, in a medium with finite volume, free vibrations can occur only with certain frequencies. These are called the ***characteristic frequencies*** of the fluid in the vessel concerned.

The actual values of the characteristic frequencies depend on the size and shape of the vessel. In any given case there is an infinite number of characteristic frequencies. To find them, it is necessary to examine the equations of motion with the appropriate boundary conditions.

The order of magnitude of the first (i.e., smallest) characteristic frequency can be seen at once from **dimensional considerations**. The only parameter having the dimensions of length which appears in the problem is the linear dimension $l$ of the body. Hence it is clear that the wavelength $\lambda_1$ corresponding to the first characteristic frequency must be of the order of $l$, and the order of magnitude of the frequency $\omega$ itself is obtained by dividing the velocity of sound by the wavelength. Thus

$$\lambda_1 \sim l, \quad \omega_1 c / l . \tag{69.1}$$

Let us ascertain the nature of the motion in characteristic vibrations. If we seek a solution of the wave equation for the velocity potential (say) which is periodic in time, having the form $\phi = \phi_0(x, y, z)e^{-i\omega t}$, then we find for $\phi_0$ the equation

$$\Delta\phi_0 + \left(\frac{\omega^2}{c^2}\right)\phi_0 = 0 . \tag{69.2}$$

In an infinite medium, where no boundary conditions need be applied, this equation has both real and complex solutions. In particular, it has a solution proportional to $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, which gives the velocity potential in the form

$$\phi = \text{constant} \times \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] .$$

Such a solution represents a wave propagated with a definite velocity - a ***travelling wave***.

For a medium with finite volume, however, *complex solutions cannot in general exist*. This can be seen as follows. The equation satisfied by $\phi_0$ is real, and the boundary conditions are real also. Hence, if $\phi_0(x, y, z)$ is a solution of the equations of motion, the complex conjugate function $\phi_0^*$ is also a solution. Since, however, the solution of the equations for given boundary conditions is in general

unique[3] apart from a constant factor, we must have $\phi_0{}^* = \text{constant} \times \phi_0$, where the constant is complex and its modulus is unity. Thus $\phi_0$ must have the form $\phi_0 = f(x, y, z)e^{-i\alpha}$, the function *f* and the constant $\alpha$ being real. The potential $\phi$ thus has the form (taking the real part of $\phi_0 e^{-i\omega t}$)

$$\phi = f(x, y, z)\cos(\omega t + \alpha), \qquad (69.3)$$

i.e., it is the product of some function of the coordinates and a simple periodic function of the time

This solution has properties entirely different from those of a travelling wave. In the latter, the phase $\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \alpha$ of the oscillations at different points in space is different at any given instant, except only at points separated by a distance equal to the wavelength. In the wave represented by (69.3), all points are oscillating in the same phase $\omega t + \alpha$ at any given instant. Such a wave is obviously not "propagated"; it is called a ***stationary wave***. Thus the characteristic vibrations are stationary waves.

Let us consider a stationary plane sound wave, in which all quantities are functions of one coordinate only (*x*, say) and of time. Writing the general solution of $\partial^2\phi_0 / \partial x^2 + \omega^2\phi_0 / c^2 = 0$ in the form $\phi_0 = a\cos(\omega x / c + \beta)$, we have

$$\phi = a\cos(\omega tc + \alpha)\cos(\omega x / c + \beta) .$$

By an appropriate choice of the origins of *x* and *t*, we can make $\alpha$ and $\beta$ zero, so that

$$\phi = a\cos(\omega t)\cos(\omega x / c) . \qquad (69.4)$$

For the velocity and pressure in the wave we have

$$v = \partial\phi / \partial x = -(a\omega / c)\cos\omega t \sin\omega x / c ,$$

$$p' = -\rho\partial\phi / \partial t = \rho\omega \sin\omega t \cos\omega x / c .$$

At the points $x = 0$, $\pi c / \omega$, $2\pi c / \omega$, …, which are at a distance $\pi c / \omega = \lambda / 2$ apart, the velocity *v* is always zero; these points are called ***nodes*** of the velocity. The points midway between them ($x = \pi c / 2\omega$, $3\pi c / 2\omega$, …) are those at which the amplitude of the time variations of the velocity is greatest. These are called ***antinodes***. The pressure $p'$ evidently has nodes and antinodes in the reverse positions. Thus, in a stationary plane wave, the nodes of the pressure are the antinodes of the velocity, and vice versa.

An interesting case of characteristic vibrations is that of the vibrations of a gas in a vessel having a **small aperture** (a ***resonator***). In a closed vessel the smallest characteristic frequency is, as we know, of the order of *c*/*l,* where *l* is the linear dimension of the vessel. When there is a small aperture, however, new characteristic vibrations with considerably smaller frequency appear. These are due to the fact that,

[3] This may not be true when the vessel is highly symmetrical in form (e.g. a sphere).

if there is a pressure difference between the gas in the vessel and that outside, this difference can be equalized by the **motion of gas into or out of the vessel**. Thus oscillations appear which involve an exchange of gas between the resonator and the outside medium. Since the aperture is small, this exchange takes place only slowly, and hence the *period of the oscillations is large*, and the frequency correspondingly small (see Problem 2). The frequencies of the ordinary vibrations occurring in a closed vessel are practically unchanged by the presence of a small aperture.

**PROBLEMS**

**Problem 1**. Determine the characteristic frequencies of sound waves in a fluid contained in a cuboidal vessel.

**Solution.** We seek a solution of the equation (69.2) in the form

$$\phi_0 = \text{constant} \times \cos qx \cos ry \cos sz\ ,$$

where $q^2 + r^2 + s^2 = \omega^2 / c^2$. At the walls of the vessel we have the conditions $v_x = \partial\phi / \partial x = 0$ for $x = 0$ and $a$, $\partial\phi / \partial y = 0$ for $y = 0$ and $b$, $\partial\phi / \partial z = 0$ for $z = 0$ and $c$, where $a, b, c$ are the sides of the cuboid. Hence we find $q = m\pi / a$, $r = n\pi / b$, $s = p\pi / c$, where $m$, $n$, $p$ are any integers. Thus the characteristic frequencies are

$$\omega^2 = c^2\pi^2 (m^2 / a^2 + n^2 / b^2 + p^2 / c^2)\ .$$

**Problem 2.** A narrow tube with cross-sectional area $S$ and length $l$ is fixed to the aperture of a resonator. Determine the characteristic frequency.

**Solution .** Since the tube is narrow, in considering oscillations accompanied by the movement of gas into and out of the resonator we can suppose that only the gas in the tube has an appreciable velocity, while the **gas in the vessel is almost at rest**. The mass of gas in the tube is $S\rho l$, and the force on it is $S(p_0 - p)$, where $p$ and $p_0$ are the gas pressures inside and outside the resonator, respectively. Hence we must have $S\rho l\dot{v} = S(p - p_0)$, where $v$ is the gas velocity in the tube. The time derivative of the pressure is given by $\dot{p} = c^2 \dot{\rho}$, and the decrease per unit time in the gas density in the resonator ($-\dot{\rho}$) can be supposed equal to the mass of gas leaving the resonator per unit time ($S\rho v$) divided by the volume $V$ of the resonator. Thus we have $\dot{p} = -c^2 S\rho v / V$, whence

$$\ddot{p} = -c^2 S\rho\dot{v} / V = -c^2 S(p - p_0) / lV\ .$$

This equation gives $p - p_0 = cons\tan t \times \cos\omega_0 t$, where the characteristic frequency $\omega_0 = c\sqrt{S / lV}$. This is small compared with $c/L$ (where $L$ is the linear dimension of the vessel), and the wavelength is therefore large compared with $L$.

In solving this problem we have supposed that the linear amplitude of the oscillations of gas in the tube is small compared with its length $l$. If this were not so,

the oscillations would be accompanied by the outflow of a considerable fraction of the gas in the tube, and the linear equation of motion used above would be inapplicable.