## §70. Spherical waves

Let us consider a sound wave in which the distribution of density, velocity, etc., depends only on the distance from some point, i.e., is spherically symmetrical. Such a wave is called a ***spherical wave***.

Let us determine the general solution of the wave equation which represents a spherical wave. We take the wave equation for the velocity potential: $\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$. Since $\phi$ is a function only of the distance $r$ from the centre and of the time $t$, we have, using the expression for the Laplacian in spherical polar coordinates,

$$\frac{\partial^2\phi}{\partial t^2} = c^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right). \tag{70.1}$$

We write $\phi = f(r,t)/r$. Substituting, we have the following equation for $f$:

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial r^2}.$$

This is just the ordinary one-dimensional wave equation, with the radius $r$ as the coordinate. The solution of this equation has, as we know, the form $f = f_1(ct-r) + f_2(ct+r)$, where $f_1$ and $f_2$ are arbitrary functions. Thus the general solution of equation (70.1) has the form

$$\phi = \frac{f_1(ct-r)}{r} + \frac{f_2(ct+r)}{r}. \tag{70.2}$$

The first term is an **outgoing wave**, propagated in all directions from the origin. The second term is a wave **coming in to the centre**. Unlike a plane wave, whose amplitude remains constant, a spherical wave has an *amplitude which decreases inversely* as the distance from the centre. The intensity in the wave is given by the square of the amplitude, and falls off inversely as the square of the distance, as it should, since the total energy flux in the wave is distributed over a surface whose area increases as $r^2$.

The variable parts of the pressure and density are related to the potential by $p' = -\rho\frac{\partial\phi}{\partial t}$, $\rho' = -\frac{\rho}{c^2}\frac{\partial\phi}{\partial t}$, and their distribution is determined by formulae having the same form as (70.2). The (radial) velocity distribution, however, being given by the gradient of the potential, has the form

$$v = \frac{\partial}{\partial r}\left\{\frac{f_1(ct-r) + f_2(ct+r)}{r}\right\}. \tag{70.3}$$

If there is no source of sound at the origin, the potential (70.2) must remain finite for $r = 0$. For this to be so we must have $f_1(ct) = -f_2(ct)$, i.e.,

$$\phi = \frac{f(ct-r) - f(ct+r)}{r} \tag{70.4}$$

(a stationary spherical wave). If there is a source at the origin, on the other hand, the

potential of the outgoing wave from it is $\phi = f(ct-r)/r$; it need not remain finite at $r = 0$, since the solution holds only for the region outside sources.

A monochromatic stationary spherical wave has the form

$$\phi = Ae^{-i\omega t}\frac{\sin kr}{r}, \tag{70.5}$$

where $k = \omega / c$. An outgoing monochromatic spherical wave is given by

$$\phi = A\frac{e^{i(kr-\omega t)}}{r}. \tag{70.6}$$

It is useful to note that this expression satisfies the differential equation

$$\Delta\phi + k^2\phi = -4\pi A e^{-i\omega t}\delta(\boldsymbol{r}), \tag{70.7}$$

where on the right-hand side we have the delta function $\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$. For $\delta(\boldsymbol{r}) = 0$ everywhere except at the origin, and we return to the homogeneous equation (70.1); and, integrating (70.7) over the volume of a small sphere including the origin (where the expression (70.6) reduces to $Ae^{-i\omega t}/r$) we obtain $-4\pi Ae^{-i\omega t}$ on each side.

Let us consider an outgoing spherical wave, occupying a spherical shell outside which the medium is either at rest or very nearly so; such a wave can originate from a source which emits during a finite interval of time only, or from some region where there is a sound disturbance (cf. the end of §72, and §74, Problem 4). Before the wave arrives at any given point, the potential is $\phi \equiv 0$. After the wave has passed, the motion must die away; this means that must become constant. In an outgoing spherical wave, however, the potential is a function having the form $\phi = f(ct-r)/r$; such a function can tend to a constant only if the function *f* is zero identically. Thus the potential must be zero both before and after the passage of the wave.[1] From this we can draw an important conclusion concerning the distribution of condensations and rarefactions in a spherical wave.

The variation of pressure in the wave is related to the potential by $p' = -\rho\frac{\partial\phi}{\partial t}$. From what has been said above, it is clear that, if we integrate $p'$ over all time for a given *r*, the result is zero:

$$\int_{-\infty}^{\infty} p'\,dt = 0. \tag{70.8}$$

This means that, as the spherical wave passes through a given point, both **condensations** ($p'>0$) and **rarefactions** ($p' < 0$) will be observed at that point. In this respect a spherical wave is markedly different from a plane wave, which may consist of condensations or rarefactions only.

A similar pattern will be observed if we consider the manner of variation of $p'$ with distance at a given instant; instead of the integral (70.8) we now consider another which

[1] Unlike what happens for a plane wave, where we can have $\phi = \text{constant} \neq 0$ after the wave has passed.

also vanishes, namely

$$\int_0^\infty rp'\,dr = 0 . \qquad (70.9)$$

**PROBLEMS**

**Problem 1**. At the initial instant, the gas inside a sphere with radius *a* is compressed so that $\rho' = \text{constant} \equiv \Delta$; outside this sphere, $\rho' = 0$. The initial velocity is zero in all space. Determine the subsequent motion.

**Solution.** The initial conditions on the potential $\phi(r,t)$ are $\phi(r,0) = 0$, $\dot{\phi}(r,0) = F(r)$, where $F(r) = 0$ for $r > a$ and $F(r) = -c^2\Delta/\rho$ for $r < a$. We seek $\phi$ in the form (70.4). From the initial conditions we obtain $f(-r) - f(r) = 0$, $f'(-r) - f'(r) = rF(r)/c$. Hence $f'(r) = -f'(-r) = -rF(r)/2c$. Finally, substituting the value of $F(r)$, we find the following expressions for the derivative $f'(\xi)$ and the function $f(\xi)$ itself;

for $|\xi| > a$, $f'(\xi) = 0$, $f(\xi) = 0$;

for $|\xi| < a$, $f'(\xi) = c\xi\Delta/2\rho$, $f(\xi) = c(\xi^2 - a^2)\Delta/4\rho$,

which give the solution of the problem. If we consider a point with $r > a$, i.e., outside the region of the initial compression, we have for the density

for $t < (r-a)/c$, $\rho' = 0$;

for $(r-a)/c < t < (r+a)/c$, $\rho' = (r-ct)\Delta/2r$;

for $t > (r+a)/c$, $\rho' = 0$.

The wave passes the point considered during a time interval $2a/c$; in other words, the wave has the form of a **spherical shell** with thickness $2a$, which at time *t* lies between the spheres with radii $ct - a$ and $ct + a$. Within this shell the density varies linearly; in the outer part ($r > ct$), the gas is **compressed** ($\rho' > 0$), while in the inner part ($r < ct$) it is **rarefied** ($\rho' < 0$).

**Problem 2.** Determine the characteristic frequencies of centrally symmetrical sound oscillations in a spherical vessel with radius *a*.

**Solution.** From the boundary condition $\partial\phi/\partial r = 0$ for $r = a$ (where $\phi$ is given by (70.5)) we find $\tan ka = ka$, which determines the characteristic frequencies. The first (lowest) frequency is $\omega_1 = 4.49c/a$.

### §71. Cylindrical waves

Let us now consider a wave in which the distribution of all quantities is homogeneous in some direction (which we take as the *z*-axis) and has complete axial symmetry about that direction. This is called a ***cylindrical wave***, and in it we have $\phi = \phi(R,t)$, where *R* denotes the distance from the *z*-axis. Let us determine the general form of such an axially symmetrical solution of the wave equation. This can be done by starting from the general spherically symmetrical solution (70.2). *R* is related to *r* by $r^2 = R^2 + z^2$, so that $\phi$ as given by formula (70.2) depends on *z* when *R* and *t* are given. A function which depends on

$R$ and $t$ only and still satisfies the wave equation can be obtained by integrating (70.2) over all $z$ from $-\infty$ to $\infty$, or equally well from 0 to $\infty$. We can convert the integration over $z$ to one over $r$. Since $z = \sqrt{r^2 - R^2}$, $dz = rdr / \sqrt{r^2 - R^2}$. When $z$ varies from 0 to $\infty$, $r$ varies from $R$ to $\infty$. Hence we find the general axially symmetrical solution to be

$$\phi = \int_R^\infty \frac{f_1(ct - r)}{\sqrt{r^2 - R^2}} dr + \int_R^\infty \frac{f_2(ct + r)}{\sqrt{r^2 - R^2}} dr , \tag{71.1}$$

where $f_1$ and $f_2$ are arbitrary functions. The first term is an **outgoing** cylindrical wave, and the second an **ingoing** one.

Substituting in these integrals $ct \pm r = \xi$, can rewrite formula (71.1) as

$$\phi = \int_{-\infty}^{ct-R} \frac{f_1(\xi)}{\sqrt{(ct - \xi)^2 - R^2}} d\xi + \int_{ct+R}^{\infty} \frac{f_2(\xi)}{\sqrt{(\xi - ct)^2 - R^2}} d\xi . \tag{71.2}$$

We see that the value of the potential at time $t$ at the point $R$ in the outgoing cylindrical wave is determined by the values of $f_1$ at times from $-\infty$ to $t - R/c$; similarly, the values of $f_2$ which affect the ingoing wave are those at times from $t + R/c$ to infinity.

As in the spherical case, stationary waves are obtained when $f_1(\xi) = -f_2(\xi)$. It can be shown that a stationary cylindrical wave can also be represented in the form

$$\phi = \int_{ct-R}^{ct+R} \frac{F(\xi)}{\sqrt{R^2 - (\xi - ct)^2}} d\xi , \tag{71.3}$$

where $F(\xi)$ is another arbitrary function.

Let us derive an expression for the potential in a monochromatic cylindrical wave. The wave equation for the potential $\phi(R,t)$ in cylindrical polar coordinates is

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 .$$

In a monochromatic wave $\phi = e^{-i\omega t} f(R)$, and we have for the function $f(R)$ the equation

$$f'' + f'/R + k^2 f = 0 .$$

This is **Bessel's equation** of order zero. In a stationary cylindrical wave, $\phi$ must remain finite for $R = 0$; the appropriate solution is $J_0(kR)$, where $J_0$ is a Bessel function of the first kind. Thus, in a stationary cylindrical wave,

$$\phi = Ae^{-i\omega t} J_0(kR) . \tag{71.4}$$

For $R = 0$ the function $J_0$ tends to unity, so that the amplitude tends to the finite limit $A$. At large distances $R$, $J_0$ may be replaced by its asymptotic expression, and $\phi$ then takes the form

$$\phi = A\sqrt{\frac{2}{\pi}} \frac{\cos(kR - \pi/4)}{\sqrt{kR}} e^{-i\omega t} . \tag{71.5}$$

The solution corresponding to a monochromatic outgoing travelling wave is

$$\phi = Ae^{-i\omega t} H_0^{(1)}(kR), \tag{71.6}$$

where $H_0^{(1)}$ is the **Hankel function** of the first kind, of order zero. For $R \to 0$ this function has a **logarithmic singularity**:

$$\phi \cong \frac{2iA}{\pi} \log(kR) e^{-i\omega t}. \tag{71.7}$$

At large distances we have the asymptotic formula

$$\phi = A\sqrt{\frac{2}{\pi}} \frac{\exp[i(kR - \omega t - \pi/4)]}{\sqrt{kR}}. \tag{71.8}$$

We see that the *amplitude of a cylindrical wave diminishes* (*at large distances*) *inversely as the square root of the distance from the axis*, and the intensity therefore decreases as $1/R$.. This result is obvious, since the total energy flux is distributed over a cylindrical surface, whose area increases proportionally to $R$ as the wave is propagated.

An outgoing cylindrical wave differs from a spherical or plane wave in the important respect that it has a **forward front** but **no backward front**; once the sound disturbance has reached a given point, it does not cease, but diminishes comparatively slowly as $t \to \infty$. Suppose that the function $f_1(\xi)$ in the first term of (71.2) is different from zero only in some finite range $\xi_1 \le \xi \le \xi_1$. Then, at times such that $ct > R + \xi_2$, we have

$$\phi = \int_{\xi_1}^{\xi_2} \frac{f_1(\xi) d\xi}{\sqrt{(ct - \xi)^2 - R^2}}.$$

As $t \to \infty$, this expression tends to zero as

$$\phi = \frac{1}{ct} \int_{\xi_1}^{\xi_2} f_1(\xi) d\xi,$$

i.e., inversely as the time.

Thus the potential in an outgoing cylindrical wave, due to a source which operates only for a finite time, vanishes, though slowly, as $t \to \infty$. This means that, as in the spherical case, the integral of $p'$ over all time is zero:

$$\int_{-\infty}^{\infty} p' dt = 0. \tag{71.9}$$

Hence a cylindrical wave, like a spherical wave, must necessarily include both **condensations** and **rarefactions**.

## §72. The general solution of the wave equation

We shall now derive a general formula giving the solution of the wave equation in an infinite fluid for any initial conditions, i.e., giving the velocity and pressure distribution in the fluid at any instant in terms of their initial distribution.

We first obtain some auxiliary formulae. Let $\phi(x, y, z, t)$ and $\psi(x, y, z, t)$ be any two solutions of the wave equation which vanish at infinity. We consider the integral

$$I = \int (\phi\dot{\psi} - \psi\dot{\phi})dV ,$$

taken over all space, and calculate its time derivative. Since $\phi$ and $\psi$ satisfy the equations $\Delta\Phi - \ddot{\phi}/c^2 = 0$ and $\Delta\psi - \ddot{\psi}/c^2 = 0$, we have

$$\frac{dI}{dt} = \int (\phi\ddot{\psi} - \psi\ddot{\phi})dV = c^2 \int (\phi\Delta\psi - \psi\Delta\phi)dV = c^2 \int \mathrm{div}(\phi\,\mathrm{grad}\,\psi - \psi\,\mathrm{grad}\,\phi)dV .$$

The last integral can be transformed into an integral over an infinitely distant surface, and is therefore zero. Thus we conclude that $dI/dt = 0$, i.e., $I$ is independent of time:

$$I \equiv \int (\phi\dot{\psi} - \psi\dot{\phi})dV = \text{constant} . \tag{72.1}$$

Next, let us consider the following particular solution of the wave equation:

$$\psi = \delta[r - c(t_0 - t)]/r , \tag{72.2}$$

(where $r$ is the distance from some given point $O$, $t_0$ is some definite instant, and $\delta$ denotes the delta function), and calculate the integral of $\psi$ over all space. We have

$$\int \psi dV = \int_0^\infty \psi \cdot 4\pi r^2 dr = 4\pi \int_0^\infty r\delta[r - c(t_0 - t)]dr .$$

The argument of the delta function is zero for $r = c(t_0 - t)$ (we assume that $t_0 > t$). Hence, from the properties of the delta function, we find

$$\int \psi dV = 4\pi c(t_0 - t) . \tag{72.3}$$

Differentiating this equation with respect to time, we obtain

$$\int \dot{\psi} dV = -4\pi c . \tag{72.4}$$

We now substitute for $\psi$, in the integral (72.1), the function (72.2), and take $\phi$ to be the required general solution of the wave equation. According to (72.1), $I$ is a constant; using this, we write down the expressions for $I$ at the instants $t = 0$ and $t = t_0$, and equate the two. For $t = t_0$ the two functions $\psi$ and $\dot{\psi}$ are each different from zero only for $r = 0$. Hence, on integrating, we can put $r = 0$ in $\phi$ and $\dot{\phi}$ (i.e., take their values at the point $O$), and take $\phi$ and $\dot{\phi}$ outside the integral:

$$I = \phi(x, y, z, t_0)\int \dot{\psi} dV - \dot{\phi}(x, y, z, t_0)\int \psi dV ,$$

where $x$, $y$, $z$ are the coordinates of $O$. According to (72.3) and (72.4), the second term is zero for $t = t_0$, and the first term gives

$$I = -4\pi c\phi(x, y, z, t_0) .$$

Let us now calculate $I$ for $t = 0$. Putting $\dot{\psi} = \partial\psi/\partial t = -\partial\psi/\partial t_0$, and denoting by $\phi_0$ the value of the function $\phi$ for $t = 0$, we have

$$I = -\int \left( \phi_0 \frac{\partial \psi}{\partial t_0} + \dot{\phi}_0 \psi \right) dV = -\frac{\partial}{\partial t_0} \int \phi_0 \psi_{t=0} dV - \int \dot{\phi}_0 \psi_{t=0} dV \; .$$

We write the element of volume as $dV = r^2 dr do$, where $do$ is an element of solid angle, and then we obtain, by the properties of the delta function,

$$\int \phi_0 \psi_{t=0} dV = \int \phi_0 r \delta(r - ct_0) dr do = ct_0 \int \phi_{0,r=ct_0} do \; ;$$

the integral of $\dot{\phi}_0 \psi$ is similar. Thus

$$I = -\frac{\partial}{\partial t_0} \left( ct_0 \int \phi_{0,r=ct_0} do \right) - ct_0 \int \dot{\phi}_{0,r=ct_0} do \; .$$

Finally, equating the two expressions for *I* and omitting the suffix zero in $t_0$, we obtain

$$\phi(x, y, z, t) = \frac{1}{4\pi} \left\{ \frac{\partial}{\partial t} \left( t \int \dot{\phi}_{0,r=ct} do \right) + t \int \phi_{0,r=ct} do \right\} . \qquad (72.5)$$

This formula, called ***Poisson's formula***, gives the spatial distribution of the potential at any instant in terms of the distribution of the potential and its time derivative (or, equivalently, in terms of the velocity and pressure distribution) at some initial instant. We see that the value of the potential at time *t* is determined by the values of $\phi$ and $\dot{\phi}$ at time $t = 0$ on the surface of a sphere centred at *O*, with radius *ct*.

Let us suppose that, at the initial instant, $\phi_0$ and $\dot{\phi}_0$ are different from zero only in some finite region of space, bounded by a closed surface *C* (Fig. 44). We consider the

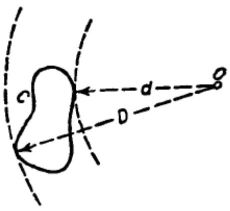


FIG. 44

values of 0 at subsequent instants at some point *O*. These values are determined by the values of $\phi_0$ and $\dot{\phi}_0$ at a distance *ct* from *O*. The spheres with radius *ct* pass through the region within the surface *C* only for $d / c \le t \le D / c$, where *d* and *D* are the least and greatest distances from the point *O* to the surface *C*. At other instants, the integrands in (72.5) are zero. Thus the motion at *O* begins at time $t = d/c$ and ceases at time $t = D/c$. The wave propagated from the region inside *C* has a forward front and a backward front. The motion begins when the forward front arrives at the point in question, while on the backward front particles previously oscillating come to rest.

## PROBLEM

Derive the formula giving the potential in terms of the initial conditions for a wave depending on only two coordinates, *x* and *y*.

**Solution.** An element of area of a sphere with radius *ct* can be written $df = c^2t^2do$, where *do* is an element of solid angle. The projection of *df* on the *xy*-plane is $dxdy = df\sqrt{(ct)^2 - \rho^2}\,/\,ct$, where $\rho$ is the distance of the point *x, y* from the centre of the sphere. Comparing the two expressions, we can write $do = dxdy\,/\,ct\sqrt{(ct)^2 - \rho^2}$. Denoting by *x*, *y* the coordinates of the point where we seek the value of $\phi$, and by $\xi, \eta$ the coordinates of a variable point in the region of integration, we can therefore replace *do* in the general formula (72.5) by $d\xi d\eta\,/\,ct\sqrt{(ct)^2 - (x-\xi)^2 - (y-\eta)^2}$, doubling the resulting expression because *dxdy* is the projection of two elements of area on opposite sides of the *xy*-plane. Thus

$$\phi(x,y,z,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\iint \frac{\phi_0(\xi,\eta)d\xi d\eta}{\sqrt{(ct)^2 - (x-\xi)^2 - (y-\eta)^2}} + \frac{1}{2\pi c}\iint \frac{\dot{\phi}_0(\xi,\eta)d\xi d\eta}{\sqrt{(ct)^2 - (x-\xi)^2 - (y-\eta)^2}}$$

where the integration is over a circle centred at *O,* with radius *ct.* If $\phi_0$ and $\dot{\phi}_0$ are zero except in a finite region *C* of the *xy*-plane (or, more exactly, except in a cylindrical region with its generators parallel to the *z*-axis), the oscillations at the point *O* (Fig. 44) begins at time *t* = *d/c,* where *d* is the least distance from *O* to a point in the region. After this time, however, circles with radius *ct* > *d* centred at *O* will always enclose part or all of the region *C*, and $\phi$ will tend only asymptotically to zero. Thus, unlike three-dimensional waves, the two-dimensional waves here considered have a **forward front** but **no backward front** (cf. §71).