§73. The lateral wave

The reflection of a spherical wave from the surface separating two media is of particular interest in that it may be accompanied by an unusual phenomenon, the appearance of a ***lateral wave***.

Let Q (Fig. 45) be the source of a spherical sound wave in medium 1, at a distance l from the infinite plane surface separating media 1 and 2. The distance l is arbitrary, and need not be large compared with the wavelength λ. Let the densities of the two media be ρ_1, ρ_2, and the velocities of sound in them c_1, c_2. We suppose first that $c_1 > c_2$; then, at distances from the source large compared with λ, the motion in medium 1 will be a superposition of two outgoing waves. One of these is the spherical wave emitted by the source (the ***direct wave***); its potential is

$$\phi_1{}^0 = \frac{e^{ikr}}{r}, \tag{73.1}$$

where r is the distance from the source, and the amplitude is arbitrarily taken to be unity. We shall, for brevity, omit the factor $e^{-i\omega t}$ from all expressions in the present section.

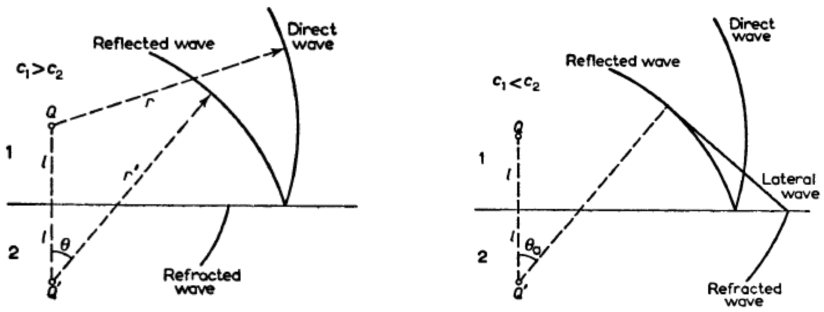

FIG. 45

The wave surfaces of the second (***reflected***) wave are spheres centred at Q', the image of the source Q in the plane of separation; this is the locus of points P reached at a given time by rays which leave Q simultaneously and are reflected from the plane in accordance with the laws of geometrical acoustics (in Fig. 46, the ray QAP with angles of incidence and reflection θ is shown). The amplitude of the reflected wave decreases inversely as the distance r' from the point Q' (which is sometimes called an ***imaginary source***), but depends also on the angle θ, as if each ray were reflected with the coefficient corresponding to the reflection of a plane wave at the given angle of incidence θ. In other words, at large distances the reflected wave is given by the formula

$$\phi_1{}' = \frac{e^{ikr'}}{r'}\left[\frac{\rho_2 c_2 \cos\theta - \rho_1\sqrt{c_1{}^2 - c_2{}^2 \sin^2\theta}}{\rho_2 c_2 \cos\theta + \rho_1\sqrt{c_1{}^2 - c_2{}^2 \sin^2\theta}}\right]; \tag{73.2}$$

cf. formula (66.4) for the reflection coefficient for a plane wave. This formula, which is clearly valid for large r', can be rigorously derived by the method shown below.

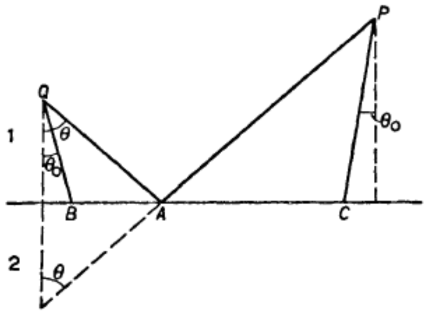

FIG. 46

A more interesting case is that where $c_1 < c_2$. Here, besides the ordinary reflected wave (73.2), **another wave** appears in the first medium. The chief properties of this wave can be seen from the following simple considerations.

The ordinary reflected ray *QAP* (Fig. 46) obeys **Fermat's principle** in the sense that it is the quickest path from *Q* to *P*, among paths lying entirely in medium 1 and involving a single reflection. When $c_1 < c_2$, however, Fermat's principle is also satisfied by another path, where the ray is incident on the boundary at the critical angle of total internal reflection θ_0 ($\sin\theta_0 = c_1 / c_2$), then is propagated in medium 2 along the boundary, and finally returns to medium 1 at the angle θ_0. The path is *QBCP* in Fig. 46, and it is evident that $\theta > \theta_0$. It is easy to see that this path also has the extremal property: the time taken to traverse it is less than for any other path from *Q* to *P* lying partly in medium 2.

The geometrical locus of points *P* reached at the same time by rays which simultaneously leave *Q* along the path *QB*, and then return to medium 1 at various points *C*, is evidently a conical surface whose generators are perpendicular to lines drawn from the imaginary source Q' at an angle θ_0.

Thus, if $c_1 < c_2$, together with the ordinary reflected wave, which has a spherical front, there is propagated in medium 1 another wave, which has a conical front extending from the plane of separation (where it meets the refracted wave front in medium 2) to the point where it touches the spherical front of the reflected wave; this occurs along the line of intersection with a cone having semi-angle θ_0 and axis QQ' (Fig. 45). This conical wave is called the ***lateral wave***.

It is easy to see by a simple calculation that the time along the path *QBCP* (Fig. 46) is less than along the path *QAP* to the same point *P*. This means that a *sound signal from the source Q reaches an observer at P first as the lateral wave*, and only later as the ordinary reflected wave.

It must be borne in mind that the lateral wave is an effect of **wave acoustics**, despite the fact that it follows the above simple interpretation in terms of the concepts of **geometrical acoustics**. We shall see below that the amplitude of the lateral wave tends to zero in the limit $\lambda \to 0$.

Let us now make a quantitative calculation. The propagation of a monochromatic sound wave from a point source is described by equation (70.7):

$$\Delta\phi + k^2\phi = -4\pi\delta(\boldsymbol{r}-\boldsymbol{l}), \tag{73.3}$$

where $k = \omega / c$ and $\boldsymbol{l}$ is the position vector of the source. The coefficient of the delta function is chosen so that the direct wave has the form (73.1). In what follows we take a system of coordinates with the *xy*-plane as the plane of separation and the *z*-axis along *QQ'*, with the first medium in $z > 0$. At the surface of separation the pressure and the *z*-component of the velocity, or (equivalently) $\rho\phi$ and $\partial\phi / \partial z$, must be continuous.

Using the general Fourier method, we obtain the solution in the form

$$\phi = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_\kappa(z)\exp[i(\kappa_x x + \kappa_y y)]d\kappa_x d\kappa_y , \tag{73.4}$$

where

$$\phi_\kappa(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\exp[-i(\kappa_x x + \kappa_y y)]dxdy . \tag{73.5}$$

From the symmetry relative to the *xy*-plane it is evident that ϕ_κ can depend only on the quantity $|\kappa| = \sqrt{\kappa_x{}^2 + \kappa_y{}^2}$. Using the formula

$$J_0(u) = \frac{1}{2\pi}\int_0^{2\pi}\cos(u\sin\phi)d\phi ,$$

we can therefore write (73.4) as

$$\phi = \frac{1}{2\pi}\int_0^{\infty}\phi_\kappa(z)J_0(\kappa R)\kappa d\kappa , \tag{73.6}$$

where $R = \sqrt{x^2 + y^2}$ is the cylindrical coordinate (the distance from the *z*-axis).

It is convenient for the subsequent calculations to transform this formula into one in which the integral is taken from $-\infty$ to ∞, expressing the integrand in terms of the **Hankel function** $H_0{}^{(1)}(u)$. The latter has a **logarithmic singularity** at $u = 0$; if we agree to go from positive to negative real u by passing above the point $u = 0$ in the complex *u*-plane, then $H_0{}^{(1)}(-u) = H_0{}^{(1)}(ue^{i\pi}) = H_0{}^{(1)}(u) - 2J_0(u)$. Using this relation, we ca rewrite (73.6) as

$$\phi = \frac{1}{4\pi}\int_{-\infty}^{\infty} \phi_\kappa(z) H_0{}^{(1)}(\kappa R)\kappa d\kappa \,. \tag{73.7}$$

From equation (73.3) we find for the function ϕ_κ the equation

$$\frac{d^2\phi_\kappa}{dz^2} - \left(\kappa^2 - \frac{\omega^2}{c^2}\right)\phi_\kappa = -4\pi\delta(z-l)\,. \tag{73.8}$$

The delta function on the right-hand side of the equation can be eliminated by imposing on the function $\phi_\kappa(z)$ (satisfying the homogeneous equation) the boundary conditions at $z = l$:

$$\left.\begin{aligned} [\phi_\kappa(z)]_{l+} - [\phi_\kappa(z)]_{l-} &= 0 \\ [d\phi_\kappa / dz]_{l+} - [d\phi_\kappa / dz]_{l-} &= -4\pi \end{aligned}\right\} \tag{73.9}$$

The boundary conditions at $z = 0$ are

$$\left.\begin{aligned} [\rho\phi_\kappa]_{0+} - [\rho\phi_\kappa]_{0-} &= 0 \\ [d\phi_\kappa / dz]_{0+} - [d\phi_\kappa / dz]_{0-} &= 0 \end{aligned}\right\} \tag{73.10}$$

We seek a solution in the form

$$\left.\begin{aligned} \phi_\kappa &= Ae^{-\mu_1 z} \text{ \_for\_ } z > l \\ \phi_\kappa &= Be^{-\mu_1 z} + Ce^{\mu_1 z} \text{ \_for\_} l > z > 0 \\ \phi_\kappa &= De^{\mu_2 z} \text{ \_for\_ } 0 > z \end{aligned}\right\} \tag{73.11}$$

Here

$$\mu_1{}^2 = \kappa^2 - k_1{}^2, \ \mu_2{}^2 = \kappa^2 - k_2{}^2, (k_1 = \omega / c_1, k_2 = \omega / c_2),$$

and we must put

$$\left.\begin{aligned} \mu &= +\sqrt{\kappa^2 - k^2} \text{ \_for\_ } \kappa > k \\ \mu &= -i\sqrt{k^2 - \kappa^2} \text{ \_for\_ } \kappa < k \end{aligned}\right\} \tag{73.12}$$

The first of these is necessary so that ϕ should not increase without limit as $z \to \infty$, and the second so that ϕ should represent an outgoing wave. The conditions (73.9) and (73.10) give four equations which determine the coefficients *A, B, C* and *D.* A simple calculation gives

$$\left.\begin{aligned} B &= C\frac{\mu_1\rho_2 - \mu_2\rho_2}{\mu_1\rho_2 + \mu_2\rho_1} \\ C &= \frac{2\pi e^{-l\mu_1}}{\mu_1} \\ D &= C\frac{2\rho_1\mu_1}{\mu_1\rho_2 + \mu_2\rho_1} \\ A &= B + Ce^{2l\mu_1} \end{aligned}\right\} \tag{73.13}$$

For $\rho_2 = \rho_1$, $c_2 = c_1$ (i.e., when all space is occupied by one medium), *B* is zero and $A = Ce^{2l\mu_1}$; the corresponding term in ϕ is evidently the direct wave (73.1), and the reflected wave in which we are interested is therefore

$$\phi_1{}' = \frac{1}{4\pi}\int B(\kappa)e^{-z\mu_1}H_0{}^{(1)}(\kappa R)\kappa d\kappa\,. \qquad (73.14)$$

In this expression the path of integration has to be specified. It passes above the singular point $\kappa = 0$ (in the complex κ-plane), as already mentioned. The integrand also has singular points (branch points) at $\kappa = \pm k_1, \pm k_2$, where μ_1 or μ_2 vanishes. In accordance with the conditions (73.10), the contour must pass below the points $+k_1$, $+k_2$, and above the points $-k_1$, $-k_2$.

Let us investigate the resulting expression for large distances from the source. Replacing the Hankel function by its asymptotic expression, we obtain

$$\phi_1{}' = \int_C \frac{\mu_1\rho_2 - \mu_2\rho_1}{\mu_1(\mu_1\rho_2 + \mu_2\rho_1)}\sqrt{\frac{\kappa}{2i\pi R}}\exp[-(z+l)\mu_1 + i\kappa R]d\kappa\,. \qquad (73.15)$$

Figure 47 shows the path of integration C for the case $c_1 > c_2$. The integral can be

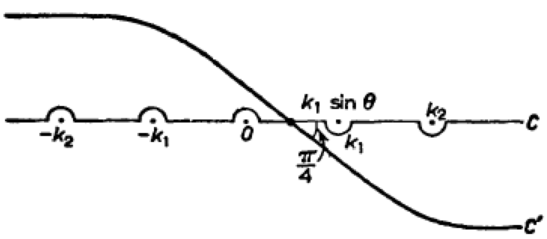

FIG. 47

calculated by means of the saddle-point method. The exponent $i[(z+l)\sqrt{k_1{}^2 - \kappa^2} + \kappa R]$ has an extremum at the point where

$$\frac{\kappa}{\sqrt{k_1{}^2 - \kappa^2}} = \frac{R}{z+l} = \frac{r'\sin\theta}{r'\cos\theta} = \tan\theta\,,$$

i.e., $\kappa = k_1\sin\theta$, where θ is the angle of incidence (see Fig. 45). On changing to the path of integration C' which passes through this point at an angle of $\pi/4$ to the axis of abscissae, we obtain formula (73.2).

In the case $c_1 < c_2$ (i.e., $k_1 > k_2$), the point $\kappa = k_1\sin\theta$ lies between k_2 and k_1 if $\sin\theta > k_2/k_1 = c_1/c_2 = \sin\theta_0$, i e., if $\theta > \theta_0$ (Fig. 45). In this case the contour C' must make a loop round the point k_2, and we have, besides the ordinary reflected wave (73.2), a wave $\phi_1{}''$ given by the integral (73,15) taken around the loop, which we call C'' (Fig. 48). This is the ***lateral wave***. The integral is easily calculated if the point $k_1\sin\theta$ is not close to

k_2, i.e., if the angle θ is not close to the internal-reflection angle θ_0.[1]

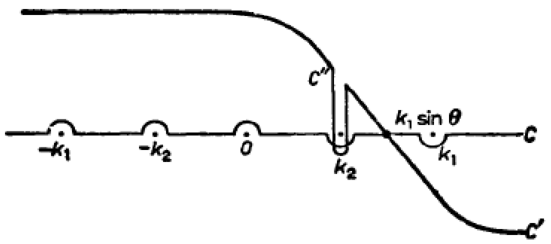

FIG. 48

Near the point $\kappa = k_2$, μ_2 is small; we expand the coefficient of the exponential in the integrand of (73.15) in powers of μ_2. The zero-order term has no singularity at $\kappa = k_2$, and its integral round *C*" is zero. Hence we have

$$\phi_1'' = -\int_{C''} \frac{2\mu_2\rho_1}{\mu_1^2\rho_2}\sqrt{\frac{\kappa}{2\pi i r}}\exp[-(x+l)\mu_1 + i\kappa R]d\kappa . \tag{73.16}$$

Expanding the exponent in powers of $\kappa - k$ and integrating round the loop *C*", we have after a simple calculation the following expression for the potential of the lateral wave:

$$\phi_1'' = \frac{2i\rho_1 k_2 \exp[ik_1 r'\cos(\theta_0 - \theta)]}{r'^2 \rho_2 k_1^2 \sqrt{\cos\theta_0 \sin\theta \sin^3(\theta_0 - \theta)}} . \tag{73.17}$$

In accordance with the previous results, the wave surfaces are the cones

$$r'\cos(\theta - \theta_0) = R\sin\theta_0 + (z+l)\cos\theta_0 = \text{constant} .$$

In a given direction, the wave amplitude decreases inversely as the square of the distance r'. We see also that this wave disappears in the limit $\lambda \to 0$. For $\theta \to \theta_0$, the expression (73.17) ceases to be valid; in actual fact, the amplitude of the lateral wave in this range of θ decreases with distance as $r'^{-5/4}$.

§74. The emission of sound

A body oscillating in a fluid causes a periodic compression and rarefaction of the fluid near it, and thus produces sound waves. The energy carried away by these waves is supplied from the kinetic energy of the body. Thus we can speak of the emission of sound by oscillating bodies. In what follows we shall always suppose that the velocity *u* of the

[1] For an investigation of the lateral wave for all values of θ see L. Brekhovskikh, *Zhurnal tekhnicheskoi fiziki* **18**,455,1948. This paper gives also the next term in the expansion of the ordinary reflected wave in powers of λ / R. We may mention here that, for angles θ close to θ_0 (in the case $c_1 < c_2$), the ratio of the correction term to the leading term falls off with distance as $(\lambda / R)^{1/2}$, and not as λ / R.

oscillating body is small compared with the velocity of sound. Since $u \sim a\omega$, where a is the linear amplitude of the oscillations, this means that $a \ll \lambda$.[2]

In the general case of a body of arbitrary shape oscillating in any manner, the problem of the emission of sound waves must be solved as follows. We take the velocity potential ϕ as the fundamental quantity; it satisfies the **wave equation**

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 . \tag{74.1}$$

At the surface of the body, the normal component of the fluid velocity must be equal to the corresponding component of the velocity ***u*** of the body:

$$\frac{\partial\phi}{\partial n} = u_n . \tag{74.2}$$

At large distances from the body, the wave must become an **outgoing spherical wave**. The solution of equation (74.1) which satisfies these boundary conditions and the condition at infinity determines the sound wave emitted by the body.

Let us consider the two limiting cases in more detail. We suppose first that the frequency of oscillation of the body is so large that the length of the emitted wave is very small compared with the dimension l of the body:

$$\lambda \ll l . \tag{74.3}$$

In this case we can divide the surface of the body into portions whose dimensions are so small that they may be approximately regarded as plane, but yet are large compared with the wavelength. Then we may suppose that each such portion emits a plane wave, in which the fluid velocity is simply the normal component u_n of the velocity of that portion of the surface. But the mean energy flux in a plane wave is (see §65) $c\rho\overline{v}^2$, where v is the fluid velocity in the wave. Putting $v = u_n$ and integrating over the whole surface of the body, we reach the result that the mean energy emitted per unit time by the body in the form of sound waves, i.e., the total intensity of the emitted sound, is

$$I = c\rho\oint \overline{u}_n^{\,2}\, df . \tag{74.4}$$

It is independent of the frequency of the oscillations (for a given velocity amplitude).

Let us now consider the opposite limiting case, where the length of the emitted wave is large compared with the dimension of the body:

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[2] The amplitude of the oscillations is in general supposed small in comparison with the dimensions of the body also, since otherwise we do not have potential flow near the body (cf. §9). This condition is unnecessary only for pure pulsations, when the solution (74.7) used below is really a direct deduction from the equation of continuity.

$$\lambda >> l\ . \tag{74.5}$$

Then we can neglect the term $(1/c^2)\partial^2\phi/\partial t^2$, in the general equation (74.1), near the body (at distances small compared with the wavelength). For this term is of the order of $\omega^2\phi/c^2 \sim \phi/\lambda^2$, whereas the second derivatives with respect to the coordinates are, in this region, of the order of ϕ/l^2.

Thus the flow near the body satisfies **Laplace's equation**, $\Delta\phi = 0$. This is the equation for potential flow of an incompressible fluid. Consequently the fluid near the body moves as if it were **incompressible**. *Sound waves proper, i.e., compression and rarefaction waves, occur only at large distances from the body.*

At distances of the order of the dimension of the body and smaller, the required solution of the equation $\Delta\phi = 0$ cannot be written in a general form, but depends on the actual shape of the oscillating body. At distances large compared with *l*, however (though still small compared with λ, so that the equation $\Delta\phi = 0$ remains valid), we can find a general form of the solution by using the fact that ϕ must decrease with increasing distance. We have already discussed such solutions of Laplace's equation in §11. As there, we write the general form of the solution as

$$\phi = -\frac{a}{r} + \boldsymbol{A}\cdot\mathrm{grad}\frac{1}{r}, \tag{74.6}$$

where *r* is the distance from an origin anywhere inside the body. Here, of course, the distances involved must be large compared with the dimension of the body, since we cannot otherwise restrict ourselves to the terms in ϕ which decrease least rapidly as *r* increases. We have included both terms in (74.6), although it must be borne in mind that the first term is sometimes absent (see below).

Let us ascertain in what cases this term –*a*/*r* is non-zero. We found in §11 that a potential –*a*/*r* results in a non-zero value $4\pi\rho a$ of the mass flux through a surface surrounding the body. In an incompressible fluid, however such a mass flux can occur only if the total volume of fluid enclosed within the surface changes. In other words, there must be a change in the volume of the body, as a result of which the fluid is either **expelled** from or "**sucked** into" the volume of space concerned. Thus the first term in (74.6) appears in cases where the emitting body undergoes pulsations during which its volume changes.

Let us suppose that this is so, and determine the total intensity of the emitted sound. The volume $4\pi a$ of the fluid which flows through the closed surface must, by the foregoing argument, be equal to the change per unit time in the volume *V* of the body, i.e. to the derivative d*V*/d*t* (the volume *V* being a given function of the time): $4\pi a = \dot{V}$. Thus, at

distances r such that $l << r << \lambda$, the motion of the fluid is given by the function $\phi = -\dot{V}(t)/4\pi r$. At distances $r >> \lambda$, however (i.e., in the ***wave region***), ϕ must represent an outgoing spherical wave, i.e., must have the form

$$\phi = -\frac{f(t-r/c)}{r}. \qquad (74.7)$$

Hence we conclude at once that the emitted wave has, at all distances large compared with l the form

$$\phi = -\frac{\dot{V}(t-r/c)}{4\pi r}, \qquad (74.8)$$

which is obtained by replacing the argument t of $\dot{V}(t)$ by $t - r/c$.

The **velocity** $\boldsymbol{v} = \operatorname{grad}\phi$ is directed at every point along the position vector, and its magnitude is $v = \partial\phi/\partial r$. In differentiating (74.8) for distances $r >> \lambda$, only the derivative of the numerator need be taken, since differentiation of the denominator would give a term of higher order in $1/r$, which we neglect. Since $\partial\dot{V}(t-r/c)/\partial r = -(1/c)\ddot{V}(t-r/c)$, we obtain

$$\boldsymbol{v} = \frac{\ddot{V}(t-r/c)}{4\pi c r}\boldsymbol{n}, \qquad (74.9)$$

where $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{r}$.

The **intensity** of the sound is given by the square of the velocity, and is here independent of the direction of emission, i.e., the emission is isotropic. The mean value of the total energy emitted per unit time is

$$I = \rho c\oint \overline{v^2}\,df = \frac{\rho}{16c\pi^2}\oint\frac{\overline{\ddot{V}^2}}{r^2}\,df,$$

where the integration is taken over a closed surface surrounding the origin. Taking this surface to be a sphere with radius r, and noticing that the integrand depends only on the distance from the origin, we have finally

$$I = \frac{\rho\overline{\ddot{V}^2}}{4\pi c}. \qquad (74.10)$$

This is the total intensity of the emitted sound. We see that it is given by the squared second time derivative of the volume of the body.

If the body executes harmonic pulsations with frequency ω, the second time derivative of the volume is proportional to the frequency and velocity amplitude of the oscillations, and its mean square is proportional to the square of the frequency. Thus the intensity of emission is proportional to the square of the frequency for a given velocity amplitude of points on the surface of the body. For a given amplitude of the oscillations, however, the velocity amplitude is itself proportional to the frequency, so that the intensity of emission is proportional to ω^4.

Let us now consider the emission of sound by a body oscillating **without change of volume**. Only the second term then remains in (74.6); we write it $\phi = \mathrm{div}[\boldsymbol{A}(t)/r]$. As in the preceding case, we conclude that the general form of the solution at all distances $r >> l$ is $\phi = \mathrm{div}[\boldsymbol{A}(t - r/c)/r]$. That this expression is in fact a solution of the wave equation is seen immediately, since the function $\boldsymbol{A}(t - r/c)/r$ is a solution, and therefore so are its derivatives with respect to the coordinates. Again differentiating only the numerator, we obtain (for distances $r >> \lambda$)

$$\phi = -\dot{\boldsymbol{A}}(t - r/c) \cdot \boldsymbol{n}/cr . \tag{74.11}$$

To calculate the velocity $\boldsymbol{v} = \mathrm{grad}\,\phi$, we need again differentiate only $\boldsymbol{A}$. Hence we have, by the familiar rules of vector analysis for differentiation with respect to a scalar argument

$$\boldsymbol{v} = -\frac{\ddot{\boldsymbol{A}}(t - r/c) \cdot \boldsymbol{n}}{c^2 r} \mathrm{grad}(t - r/c) ,$$

and, substituting grad $(t - r/c)$ = - (l/c)grad r = - $\boldsymbol{n}/c$, we have finally

$$\boldsymbol{v} = \frac{\boldsymbol{n}(\boldsymbol{n} \cdot \ddot{\boldsymbol{A}})}{c^2 r} . \tag{74.12}$$

The intensity is now proportional to the squared cosine of the angle between the direction of emission (i.e., the direction of $\boldsymbol{n}$) and the vector $\ddot{\boldsymbol{A}}$; this is called ***dipole emission***. The total emission is given by the integral

$$I = \frac{\rho}{c^3} \oint \overline{\frac{(\boldsymbol{n} \cdot \ddot{\boldsymbol{A}})^2}{r^2}} df$$

We again take the surface of integration to be a sphere with radius r, and use spherical polar coordinates with the polar axis in the direction of the vector $\ddot{\boldsymbol{A}}$. A simple integration gives finally for the total emission per unit time

$$I = \frac{4\pi\rho}{3c^2} \overline{\ddot{\boldsymbol{A}}^2} . \tag{74.13}$$

The components of the vector $\boldsymbol{A}$ are linear functions of the components of the velocity $\boldsymbol{u}$ of the body (see §11). Thus the intensity is here a quadratic function of the second time derivatives of the velocity components.

If the body executes harmonic oscillations with frequency ω, we conclude (reasoning as in the previous case) that the intensity is proportional to ω^4 for a given value of the velocity amplitude. For a given linear amplitude of the oscillations of the body, the velocity amplitude is proportional to the frequency, and therefore the intensity is proportional to ω^6.

In an entirely similar manner we can solve the problem of the emission of **cylindrical** sound waves by a cylinder with any cross-section pulsating or oscillating perpendicularly to

its axis. We shall give here the corresponding formulae, with a view to later applications.

Let us first consider small pulsations of a cylinder, and let $S = S(t)$ be its (variable) cross-sectional area. At distances r from the axis of the cylinder such that $l << r << \lambda$, where l is the transverse dimension of the cylinder, we have similarly to (74.8)

$$\phi = \frac{\dot{S}(t)}{2\pi} \log fr , \tag{74.14}$$

where $f(t)$ is a function of time, and the coefficient of logfr is chosen so as to obtain the correct value for the mass flux through a coaxial cylindrical surface. In accordance with the formula for the potential of an outgoing cylindrical wave (the first term of formula (71.2)), we now conclude that at all distances $r >> l$ the potential is given by

$$\phi = -\frac{c}{2\pi}\int_{-\infty}^{t-r/c} \frac{\dot{S}(t')dt'}{\sqrt{c^2(t-t')^2 - r^2}} . \tag{74.15}$$

As $r \to 0$ the leading term of this expression is the same as (74.14), and the function $f(t)$ in the latter equation is automatically determined (we suppose that the derivative $\dot{S}(t)$ tends sufficiently rapidly to zero as $t \to -\infty$). For very large values of r, on the other hand (the **wave region**), the values of $t - t' \sim r/c$ are the most important in the integral (74.15). We can therefore put, in the denominator of the integrand,

$$(t-t')^2 - r^r / c^2 \cong (2r/c)(t - t' - r/c) \text{,}$$

obtaining

$$\phi = -\frac{c}{2\pi\sqrt{2r}}\int_{-\infty}^{t-r/c} \frac{\dot{S}(t')dt'}{\sqrt{c(t-t') - r}} . \tag{74.16}$$

Finally, the **velocity** $v = \partial\phi / \partial r$. To effect the differentiation, it is convenient to substitute $t - t' - r/c = \xi$:

$$\phi = -\frac{1}{2\pi}\sqrt{\frac{c}{2r}}\int_{0}^{\infty} \frac{\dot{S}(t - r/c - \xi)}{\sqrt{\xi}} d\xi ;$$

the limits of integration are then independent of r. The factor $1/\sqrt{r}$ in front of the integral need not be differentiated, since this would give a term of higher order in $1/r$. Differentiating under the integral sign and then returning to the variable t', we obtain

$$v = \frac{1}{2\pi\sqrt{2r}}\int \frac{\ddot{S}(t')dt'}{\sqrt{c(t-t') - r}} . \tag{74.17}$$

The intensity is given by the product $2\pi r\rho c v^2$. It should be noticed that here, unlike what happens for the spherical case, the intensity at any instant is determined by the behaviour of

the function $S(t)$ at all times from $-\infty$ to $t - r/c$.

Finally, for **translatory oscillations of an infinite cylinder** in a direction perpendicular to its axis, the potential at distances r such that $l << r << \lambda$ has the form

$$\phi = \operatorname{div}(\boldsymbol{A} \log fr), \tag{74.18}$$

where $\boldsymbol{A}(t)$ is determined by solving Laplace's equation for the flow of an incompressible fluid past a cylinder. Hence we again conclude that, at all distances $r >> l$,

$$\phi = -\operatorname{div} \int_{-\infty}^{t-r/c} \frac{\boldsymbol{A}(t')dt'}{\sqrt{(t-t')^2 - r^2/c^2}}. \tag{74.19}$$

In conclusion, we must make the following remark. We have here entirely neglected the effect of the viscosity of the fluid, and accordingly have supposed that there is potential flow in the emitted wave. In reality, however, we do not have potential flow in a fluid layer with thickness $\sim \sqrt{v/\omega}$ round the oscillating body (see §24). Hence, if the above formulae are to be applicable, it is necessary that the *thickness of this layer should be small in comparison with the dimension l of the body*:

$$\sqrt{\frac{v}{\omega}} << l. \tag{74.20}$$

This condition may not hold for small frequencies or small dimensions of the body.

PROBLEMS

Problem 1. Determine the total intensity of sound emitted by a sphere executing small (harmonic) translatory oscillations with frequency ω, the wavelength being comparable in magnitude with the radius R of the sphere.

Solution. We write the velocity of the sphere in the form $\boldsymbol{u} = \boldsymbol{u}_0 e^{-i\omega t}$, then ϕ depends on the time through a factor $e^{-i\omega t}$ also, and satisfies the equation $\Delta\phi + k^2\phi = 0$, where $k = \omega / c$. We seek a solution in the form $\phi = \boldsymbol{u} \cdot \operatorname{grad} f(r)$, the origin being taken at the instantaneous position of the centre of the sphere. For f we obtain the equation $\boldsymbol{u} \cdot \operatorname{grad}(\Delta f + k^2 f) = 0$, whence $\Delta f + k^2 f = \text{constant}$. Apart from an unimportant additive constant, we therefore have $f = Ae^{ikr}/r$. The constant A is determined from the condition $\partial\phi/\partial r = u_r$, for $r = R$, and the result is

$$\phi = u \cdot r e^{ik(r-R)} \left(\frac{R}{r}\right)^3 \frac{ikr - 1}{2 - 2ikR - k^2 R^2}.$$

Thus we have **dipole emission**. At sufficiently large distances from the sphere, we can neglect unity in comparison with *ikr,* and ϕ takes the form (74.11), the vector $\dot{\boldsymbol{A}}$ being

$$\dot{\boldsymbol{A}} = -\boldsymbol{u}e^{ik(r-R)}R^3 \frac{i\omega}{2-2ikR-k^2R^2} \ .$$

Noticing that $\overline{(\mathrm{Re}\,\ddot{\boldsymbol{A}})^2} = \frac{1}{2}\overline{\left|\ddot{\boldsymbol{A}}\right|^2}$, we obtain for the total emission, by (74.13),

$$I = \frac{2\pi\rho}{3c^3}\left|\boldsymbol{u}_0\right|^2 \frac{R^6\omega^4}{4+(\omega R/c)^4} \ .$$

For $\omega R/c << 1$, this expression becomes $I = \dfrac{\pi\rho R^6\left|\boldsymbol{u}_0\right|^2\omega^4}{6c^3}$, a result which could also be obtained by directly substituting in (74.13) the expression $\boldsymbol{A} = \frac{1}{2}R^3\boldsymbol{u}$ from §11, Problem 1.

For $\omega R/c >> 1$ we have $I = \dfrac{2\pi\rho c R^2\left|\boldsymbol{u}_0\right|^2}{3}$, corresponding to formula (74.4).

The **drag force** acting on the sphere is obtained by integrating over the surface of the sphere the component of the pressure forces ($p' = -\rho(\dot{\phi}')_{r=R}$) in the direction of $\boldsymbol{u}$, and is

$$\boldsymbol{F} = \frac{4\pi}{3}\rho\omega R^3\boldsymbol{u}\frac{-k^3R^3 + i(2+k^2R^2)}{4+k^4R^4} ;$$

see the end of §24 concerning the meaning of a complex drag force.

Problem 2. The same as Problem 1, but for the case where the radius R of the sphere is comparable in magnitude with $\sqrt{\nu/\omega}$, whilst $\lambda >> R$.

Solution. If the dimension of the body is not large compared with $\sqrt{\nu/\omega}$, then the emitted wave must be investigated not from the equation $\Delta\phi = 0$, but from the equation of motion of an incompressible viscous fluid. The appropriate solution of this equation for a sphere is given by formulae (1) and (2) in §24, Problem 5. At great distances the first term in (1), which diminishes exponentially with r, may be omitted. The second term gives the velocity $\boldsymbol{v} = -b(\boldsymbol{u}\cdot\mathrm{grad})\,\mathrm{grad}(1/r)$. Comparison with (74.6) shows that

$$\boldsymbol{A} = -b\boldsymbol{u} = \frac{1}{2}R^3\left[1 - \frac{3}{(i-1)\kappa} - \frac{3}{2i\kappa^2}\right]\boldsymbol{u} \ ,$$

where $\kappa = R\sqrt{\omega/2\nu}$, i.e., $\boldsymbol{A}$ differs from the corresponding expression for an ideal fluid by the factor in brackets. The result is

$$I = \frac{\pi\rho R^6}{6c^3}\omega^4\left(1 + \frac{3}{\kappa} + \frac{9}{2\kappa^2} + \frac{9}{2\kappa^3} + \frac{9}{4\kappa^4}\right)\left|\boldsymbol{u}_0\right|^2 .$$

For $\kappa >> 1$ this becomes the formula given in Problem 1, while for $\kappa << 1$ we obtain

$$I = \frac{3\pi\rho R^2\nu^2\omega^2\left|u_0\right|^2}{2c^3} ,$$

i.e., the emission is proportional to the second, and not the fourth, power of the frequency.

Problem 3. Determine the intensity of sound emitted by a sphere executing small (harmonic) pulsations with any frequency.

Solution. We seek a solution of the form $\phi = \frac{au}{r} e^{ik(r-R)}$, R being the equilibrium radius of the sphere, and determine the constant a from the condition $\left[\frac{\partial \phi}{\partial r}\right]_{r=R} = u = u_0 e^{-i\omega t}$ (where u is the radial velocity of points on the surface of the sphere):

$$a = \frac{R^2}{ikR - 1}.$$

The intensity is $I = \frac{2\pi\rho c|\boldsymbol{u}_0|^2 k^2 R^4}{1 + k^2 R^2}$. For $kR << 1$, $I = \frac{2\pi\rho\omega^2 R^4 |\boldsymbol{u}_0|^2}{c}$, in accordance with (74.10), while for $kR >> 1$, $I = 2\pi\rho c R^2 |\boldsymbol{u}_0|^2$, in accordance with (74.4).

Problem 4. Determine the nature of the wave emitted by a sphere (with radius R) executing small pulsations, when the radial velocity of points on the surface is any function $u(t)$ of the time.

Solution. We seek a solution in the form $\phi = f(t')/r$, where $t' - t - (r-R)/c$, and determine f from the boundary condition $\partial\phi/\partial r = u(t)$ for $r = R$. This gives the equation $df/dt + cf(t)/R = -Rcu(t)$. Solving this linear equation and replacing t by t' in the solution for f, we obtain

$$\phi(r,t) = -\frac{cR}{r} e^{-ct'/R} \int_{-\infty}^{t'} u(\tau) e^{c\tau/R} d\tau . \qquad (1)$$

If the oscillations of the sphere cease at some instant, say $t = 0$ (i.e., $u(\tau) = 0 \quad 0$ for $\tau > 0$), then the potential at a distance r from the centre will have the form $\phi = \text{constant} \times e^{-\alpha/R}$ after the instant $t = (r — R)/c$, i.e., it will diminish exponentially.

Let T be the time during which the velocity $u(t)$ changes appreciably. If $T>>R/c$, i.e., if the wavelength of the emitted waves $\lambda \sim cT >> R$, then we can take the slowly varying factor $u(\tau)$ outside the integral in (1), replacing it by u (t'). For distances $r >> R$, we then obtain $\phi = -(R^2/r)u(t - r/c)$, in accordance with formula (74.8). If, on the other hand, $T << R/c$, we obtain in a similar manner

$$\phi = -\frac{cR}{r}\int_{-\infty}^{t'} u(\tau) d\tau , \quad v = \frac{\partial\phi}{\partial r} = \frac{R}{r} u(t') ,$$

in accordance with formula (74.4).

Problem 5. Determine the motion of an ideal compressible fluid when a sphere with radius R executes in it an arbitrary translatory motion, with velocity small compared with that of sound.

Solution. We seek a solution in the form

$$\phi = d\,\mathrm{iv}\left[\frac{f(t')}{r}\right],$$

where r is the distance from the origin, taken at the position of the centre of the sphere at the time $t' = t - (r - R)/c$; since the velocity $\boldsymbol{u}$ of the sphere is small compared with the velocity of sound, the movement of the origin may be neglected. The fluid velocity is

$$\boldsymbol{v} = \mathrm{grad}\,\phi = \frac{3(\boldsymbol{f}\cdot\boldsymbol{n})\boldsymbol{n} - \boldsymbol{f}}{r^3} + \frac{3(\boldsymbol{f}'\cdot\boldsymbol{n})\boldsymbol{n} - \boldsymbol{f}'}{cr^2} + \frac{(\boldsymbol{f}''\cdot\boldsymbol{n})\boldsymbol{n}}{c^2 r}, \qquad (2)$$

where $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{r}$, and the prime denotes differentiation with respect to the argument of $\boldsymbol{f}$. The boundary condition is $v_r = \boldsymbol{u}\cdot\boldsymbol{n}$ for $r = R$, whence $\boldsymbol{f}''(t) + (2c/R)\boldsymbol{f}'(t) + (2c^2/R^2)\boldsymbol{f}(t) = Rc^2\boldsymbol{u}(t)$. Solving this equation by variation of the parameters, we obtain for the function $\boldsymbol{f}(t)$ the general expression

$$\boldsymbol{f}(t) = cR^2 e^{-ct/R}\int_{-\infty}^{t} \boldsymbol{u}(\tau)\sin\frac{c(t-\tau)}{R}\, e^{c\tau/R}\, d\tau. \qquad (3)$$

In substituting in (1), we must replace t by t'. The lower limit is taken as $-\infty$ so that $\boldsymbol{f}$ shall be zero for $t = -\infty$.

Problem 6. A sphere with radius R begins at time $t = 0$ to move with constant velocity $\boldsymbol{u}_0$. Determine the sound intensity emitted at the instant when the motion begins.

Solution. Putting in formula (3) of Problem 5 $\boldsymbol{u}(\tau) = 0$ for $\tau < 0$ and $\boldsymbol{u}(\tau) = \boldsymbol{u}_0$ for $\tau > 0$, and substituting in formula (2) (retaining only the last term, which decreases least rapidly with increasing r), we find the fluid velocity far from the sphere

$$v = -n(n\cdot u_0)\frac{\sqrt{2R}}{r} e^{-ct'/R}\sin\left(\frac{ct'}{R} - \frac{\pi}{4}\right),$$

where $t' > 0$. The total intensity diminishes with time according to

$$I = \frac{8\pi}{3} c\rho R^2 {u_0}^2 e^{-2ct'/R}\sin^2\left(\frac{ct'}{R} - \frac{\pi}{4}\right).$$

The total amount of energy emitted is

$$\frac{1}{2}\pi\rho R^3 {u_0}^2.$$

Problem 7. Determine the intensity of sound emitted by an infinite cylinder, with radius R, executing harmonic pulsations with wavelength $\lambda >> R$.

Solution. According to formula (74.14), we find first of all that, at distances $r << \lambda$ (in Problems 7 and 8 r is the distance from the axis of the cylinder), the potential is $\phi = Ru\log kr$, where $u = u_0 e^{-i\omega t}$ is the velocity of points on the surface of the cylinder. From a comparison with formulae (71.7) and (71.8), we now find that at large distances the potential has the form $\phi = -Ru\sqrt{\frac{i\pi}{2kr}}e^{ikr}$. The velocity is therefore

$$\boldsymbol{v} = Ru\sqrt{\frac{\pi k}{2ir}}\boldsymbol{n}e^{ikr},$$

where $\boldsymbol{n}$ is a unit vector perpendicular to the axis of the cylinder, and the intensity per unit length of the cylinder is

$$I = \frac{1}{2}\pi^2\rho\omega R^2|u_0|^2.$$

Problem 8. Determine the intensity of sound emitted by a cylinder executing harmonic translatory oscillations in a direction perpendicular to its axis.

Solution. At distances $r << \lambda$ we have $\phi = -\operatorname{div}(R^2\boldsymbol{u}\log kr)$; cf. formula (74.18) and §10, Problem 3. Hence we conclude that at large distances

$$\phi = R^2\sqrt{\frac{i\pi}{2k}}\operatorname{div}\left(\frac{e^{ikr}\boldsymbol{u}}{\sqrt{r}}\right) = -R^2\boldsymbol{u}\cdot\boldsymbol{u}\sqrt{\frac{\pi k}{2ir}}e^{ikr},$$

whence the velocity is

$$v = -kR^2\sqrt{\frac{i\pi k}{2r}}\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{u})e^{ikr}.$$

The intensity is proportional to the squared cosine of the angle between the directions of oscillation and emission. The total intensity is

$$I = \frac{\pi^2}{4c^2}\rho\omega^3R^4|\boldsymbol{u}_0|^2.$$

Problem 9. Determine the intensity of sound emitted by a plane surface whose temperature varies periodically with frequency $\omega << c^2/\chi$, where χ is the thermometric conductivity of the fluid.

Solution. Let the variable part of the temperature of the surface be $T'_0\,e^{-i\omega t}$. These temperature oscillations cause a damped thermal wave (52.15) in the fluid;

$$T' = T'_0\,e^{-i\omega t}e^{-(1-i)\sqrt{\omega/2\chi}x},$$

and the fluid density therefore oscillates also; $\rho' = (\partial\rho/\partial T)_p T' = -\rho\beta T'$, where β is the coefficient of thermal expansion. This, in turn, results in the occurrence of a motion

determined by the equation of continuity: $\rho \partial v / \partial x = -\partial \rho' / \partial t = -i\omega\rho\beta T'$. At the solid surface the velocity $v_x = v = 0$, and far from the surface it tends to the limit

$$v = -i\omega\beta \int_0^\infty T' \, dx = \frac{1-i}{\sqrt{2}} \beta \sqrt{\omega\chi} T'_0 \, e^{-i\omega t}.$$

This value is reached at distances $\sim \sqrt{\chi / \omega}$, which are small compared with c / ω, and we thus have a boundary condition on the resulting sound wave. Hence we find the intensity per unit area of the surface to be

$$I = \frac{1}{2} c\rho\beta^2 \omega\chi |T'_0|^2.$$

Problem 10. A point source emitting a spherical wave is at a distance *l* from a solid wall which totally reflects sound and bounds a half-space occupied by fluid. Determine the ratio of the total intensity of sound emitted by the source to that which would be found in an infinite medium, and the dependence of the intensity on direction for large distances from the source.

Solution. The sum of the direct and reflected waves is given by a solution of the wave equation such that the normal velocity component $v_n = \partial\phi / \partial n$ is zero at the wall. Such a solution is

$$\phi = \left(\frac{e^{ikr}}{r} + \frac{e^{ikr'}}{r'} \right) e^{-i\omega t}$$

(we omit the constant factor, for brevity), where *r* is the distance from the source *O* (Fig. 49),

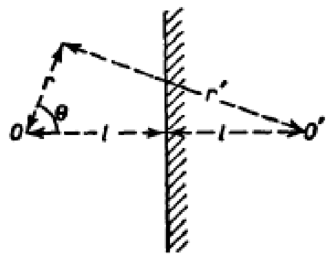

Fig. 49

and *r*' is the distance from a point *O*' which is the image of *O* in the wall. At large distances from the source we have $r' \cong r - 2l\cos\theta$, so that

$$\phi = \frac{e^{i(kr - \omega t)}}{r} \left(1 + e^{-2ikl\cos\theta} \right).$$

The dependence of the intensity on direction is given by a factor $\cos^2(kl\cos\theta)$.

To determine the total intensity, we integrate the energy flux $\overline{\boldsymbol{q}} = \overline{p' \boldsymbol{v}} = -\overline{\rho\dot{\phi}\,\mathrm{grad}\,\phi}$ (see (65.4)) over the surface of a sphere with arbitrarily small radius, centred at *O*. This gives

$2\pi\rho k\omega\left(1+\frac{1}{2kl}\sin 2kl\right)$. In an infinite medium, on the other hand, we should have simply a spherical wave $\phi = e^{i(kr-\omega t)}/r$, with a total energy flux $2\pi\rho k\omega$. Thus the required ratio of intensities is

$$1+\frac{1}{2kl}\sin 2kl\,.$$

Problem 11. The same as Problem 10, but for a fluid bounded by a free surface.

Solution. At the free surface the condition $p' = -\rho\dot{\phi} = 0$ must hold; in a monochromatic wave this is equivalent to $\phi = 0$. The corresponding solution of the wave equation is

$$\phi = \left(\frac{e^{ikr}}{r}-\frac{e^{ikr'}}{r'}\right)e^{-i\omega t}\,.$$

At large distances from the source, the intensity is given by a factor $\sin^2(kl\cos\theta)$. The required ratio of intensities is

$$1-\frac{1}{2kl}\sin 2kl\,.$$