§75. Sound excitation by turbulence

Turbulent velocity fluctuations also are a cause of sound excitation in the surrounding fluid. The present section will give the general theory of this effect (M. J. Lighthill 1952). We shall consider the case where the turbulence occupies a finite region V_0 surrounded by an infinite volume of fluid at rest. The turbulence itself is treated in terms of incompressible fluid theory, the density changes due to the fluctuations being neglected. This means that the velocity of the turbulent flow is assumed to be much less than that of sound (as was assumed throughout Chapter III).

We shall begin by deriving the general equation, taking into account not only the motion in the sound waves but also the flow in the turbulent region. The only difference from the derivation in §64 is that the non-linear term $(\boldsymbol{v}\cdot\text{grad})\boldsymbol{v}$ must be retained: although $\boldsymbol{v}$ is much less than c, it is much greater than the fluid velocity in the sound wave. We therefore have, instead of (64.3),

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\text{grad})\boldsymbol{v}+\frac{1}{\rho_0}\text{grad}\,p'=0\,.$$

Taking the divergence of this equation and using (64.5),

$$\frac{\partial p'}{\partial t}+\rho_0 c^2\,\text{div}\,\boldsymbol{v}=0\,,$$

we obtain

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2}-\Delta p'=\rho_0\frac{\partial}{\partial x_i}\left(v_k\frac{\partial v_i}{\partial x_k}\right).$$

The right-hand side of this equation can be transformed by means of the equation of continuity div $\boldsymbol{v}$ = 0 (the turbulence being regarded as incompressible), and the differentiation with respect to x_k taken outside the brackets. The final result is

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2}-\Delta p'=\rho\frac{\partial^2 T_{ik}}{\partial x_i\partial x_k}\,,\quad T_{ik}=v_i v_k\,, \tag{75.1}$$

the suffix in ρ_0 being again omitted. Outside the turbulent region, the expression on the right is a second-order small quantity and may be omitted, so that we return to the wave equation of sound propagation. The non-zero right-hand side in the volume V_0 acts as a source of sound. In that volume, $\boldsymbol{v}$ is the velocity of the turbulent flow.

Equation (75.1) is of the **retarded-potential** type. The solution which describes emission from a source is

$$p'(\boldsymbol{r},t)=\frac{\rho}{4\pi}\int\left[\frac{\partial^2 T_{ik}(\boldsymbol{r}_1,t)}{\partial x_{1i}\partial x_{1k}}\right]_{t-R/c}\frac{dV_1}{R}\,; \tag{75.2}$$

see *Fields,* §62. Here, $\boldsymbol{r}$ is the position vector of the point of observation, $\boldsymbol{r}_1$ that of a variable point in the region of integration, and $R=|\boldsymbol{r}-\boldsymbol{r}_1|$; the integrand is taken at the "retarded" time t - R/c. The integration in (75.2) is in practice to be taken only over the

volume V_0 in which the integrand is non-zero.

The majority of the energy of turbulent flow is at frequencies ~ u/l which correspond to the fundamental scale l of the turbulence; u is the characteristic velocity (see §33). These will evidently be also the main frequencies in the spectrum of sound waves excited. The corresponding wavelengths $\lambda \sim cl/u >> l$.

To determine the emission intensity, it is sufficient to consider the sound at distances much greater than the wavelength λ (in the wave region); these are also much greater than the linear size of the source, i.e., of the turbulent region.[1] The factor $1/R$ in the integrand may be replaced in this region by $1/r$ and taken outside the integral (r being the distance from the point of observation to an origin taken somewhere inside the source); we thereby neglect terms that decrease faster than $1/r$, which in any case do not contribute to the intensity of waves going to infinity. Thus

$$p'(\boldsymbol{r},t) = \frac{\rho}{4\pi r}\int \left[\frac{\partial^2 T_{ik}(\boldsymbol{r}_1,t)}{\partial x_{1i}\partial x_{1k}}\right]_{t-R/c} dV_1 \ . \tag{75.3}$$

The derivatives in the integrand are taken before the evaluation at $t - R/c$, that is, only with respect to the first argument of the $T_{ik}(\boldsymbol{r}_1,t)$. They may be replaced by derivatives of the functions $T_{ik}(\boldsymbol{r},t-R/c)$ taken with respect to both arguments, the derivatives with respect to the second argument being subtracted. The former are complete divergences, and their integrals give zero when transformed into integrals over distant closed surfaces, since $T_{ik}=0$ outside the turbulent region. The derivatives with respect to the variable coordinates $\boldsymbol{r}_1$ which appear in the argument $t - R/c$ may be replaced by those with respect to the coordinates of the point of observation $\boldsymbol{r}$, since $\boldsymbol{r}$ and $\boldsymbol{r}_1$ occur only as the difference $R = |\boldsymbol{r}-\boldsymbol{r}_1|$. We thus obtain

$$p'(\boldsymbol{r},t) = \frac{\rho}{4\pi r}\frac{\partial^2}{\partial x_i \partial x_k}\int T_{ik}(\boldsymbol{r}_1, t-R/c)dV_1 \ . \tag{75.4}$$

The time $t - R/c$ differs from $t - r/c$ by ~ l/c. This, however, is small compared with the periods l/u of the fundamental turbulent eddies. This allows the argument $t - R/c$ in the integrand to be replaced by $\tau \equiv t - r/c$.[2] Then, differentiating under the integral sign and noting that $\partial r/\partial x_i = n_i$ (where $\boldsymbol{n}$ is a unit vector along $\boldsymbol{r}$), we obtain

$$p'(\boldsymbol{r},t) = \frac{\rho}{4\pi c^2 r} n_i n_k \int \ddot{T}_{ik}(\boldsymbol{r}_1,\tau)dV_1 \ . \tag{75.5}$$

where a dot denotes differentiation with respect to τ.

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[1] In referring to orders of magnitude we make no distinction between the fundamental scale l and the size of the turbulent region, although the latter may be noticeably larger.

[2] Here, we do not consider the emission spectrum, but take only the principal frequencies which determine the total intensity. Note also that the substitution in question could not have been made at an earlier stage, in (75.3), since the integral would then be zero.

The tensor $\ddot{T}_{ik}$, like any symmetrical tensor with non-zero trace, can be put in the form

$$\ddot{T}_{ik} = (\ddot{T}_{ik} - \frac{1}{3}\ddot{T}_{ll}\delta_{ik}) + \frac{1}{3}\ddot{T}_{ll}\delta_{ik} \equiv Q_{ik} + Q\delta_{ik}\,, \tag{75.6}$$

where Q_{ik} is an "irreducible" tensor with zero trace, and Q is a scalar. Then the spherical wave (75.5) separates as a sum of two terms

$$p'(\boldsymbol{r},t) = \frac{\rho}{4\pi c^2 r}\left\{\int Q(\boldsymbol{r}_1,\tau)dV_1 + n_i n_k \int Q_{ik}(\boldsymbol{r}_1,\tau)dV_1\right\}, \tag{75.7}$$

which respectively represent the emission from ***monopole*** and ***quadrupole*** sources.

Let us next calculate the total emitted **intensity**. The sound energy flux density in the wave region is along $\boldsymbol{n}$ at every point, and its magnitude is $q = p'^2 / c\rho$. The total intensity is found by multiplying q by $r^2 do$ and integrating over all directions of $\boldsymbol{n}$.[3] In practice, however, we are interested not in the instantaneous fluctuating value of the intensity but in the time-averaged value (the turbulence being here assumed "steady"). The latter operation is carried out by writing the squares of the integrals as double integrals and averaging (denoted by angle brackets) under the integral signs. The result is

$$I = \frac{\rho_0}{60\pi c^5}\iint\langle Q(\boldsymbol{r}_1,\tau)Q(\boldsymbol{r}_2,\tau)\rangle dV_1 dV_2 + \frac{\rho_0}{30\pi c^5}\iint\langle Q_{ik}(\boldsymbol{r}_1,\tau)Q_{ik}(\boldsymbol{r}_2,\tau)\rangle dV_1 dV_2\,. \tag{75.8}$$

The cross product of the two terms in (75.7) disappears on integration over directions, and so the total intensity is the sum of the monopole and quadrupole emissions. In the present case, these two parts have in general the same order of magnitude.

Let us estimate this order of magnitude (or rather, determine the dependence of I on the turbulent flow parameters). The tensor components $T_{ik} \sim u^2$, where u is the characteristic velocity of the turbulent flow. Each differentiation with respect to time multiplies this order of magnitude by the characteristic frequency u/l. Hence $Q \sim u^4/l^2$. The correlation between the turbulent fluctuation velocities at different points extends to distances $\sim l$. The quantity of energy emitted as sound by unit mass of the turbulent medium per unit time is therefore

$$\varepsilon_s \sim \frac{1}{c^5}\frac{u^8}{l^4}l^3 = \frac{u^8}{c^5 l}\,. \tag{75.9}$$

This emission intensity is thus proportional to the eighth power of the turbulent flow velocity.

The turbulent flow is maintained by power supplied from some external source. In the "steady" case, this is equal to the energy dissipated per unit time. The latter is, per unit mass,

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[3] This integration is achieved by using the following expressions for the mean products of two and four components of $\boldsymbol{n}$: $\overline{n_i n_k} = (1/3)\delta_{ik}$, $\overline{n_i n_k n_l n_m} = (1/15)(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl})$.

$\varepsilon_d \sim u^3 / l$.[4] The acoustic efficiency may be defined as the ratio of the emitted power and the dissipated power;

$$\frac{\varepsilon_s}{\varepsilon_d} \sim \left(\frac{u}{c}\right)^5 . \tag{75.10}$$

The high power of u/c has the result that when $u/c \ll 1$ the effectiveness of turbulence as a sound source is low.

§76. The reciprocity principle

In deriving the equations of a sound wave in §64, it was assumed that the wave is propagated in a homogeneous medium. In particular, the density ρ_0 of the medium and the velocity of sound in it, c, were regarded as constants. In order to obtain some general relations applicable for an arbitrary **inhomogeneous** medium, we shall first derive the equation for the propagation of sound in such a medium.

We write the equation of continuity in the form $\frac{d\rho}{dt} + \rho \operatorname{div} \boldsymbol{v} = 0$. Since the propagation of sound is **adiabatic**, we have

$$\frac{d\rho}{dt} = \left(\frac{\partial \rho}{\partial p}\right)_S \frac{dp}{dt} = \frac{1}{c^2}\frac{dp}{dt} = \frac{1}{c^2}\left(\frac{\partial p}{\partial t} + \boldsymbol{v}\cdot \operatorname{grad} p\right),$$

and the equation of continuity becomes $\frac{\partial p}{\partial t} + \boldsymbol{v}\cdot \operatorname{grad} p + \rho c^2 \operatorname{div} \boldsymbol{v} = 0$.

As usual, we put $\rho = \rho_0 + \rho'$, where ρ_0 is now a given function of the coordinates. In the equation $p = p_0 + p'$, however, we must put as before $p_0 = \text{constant}$, since the pressure must be constant throughout a medium in equilibrium (in the absence of an external field, of course). Thus we have to within second-order quantities $\frac{\partial p'}{\partial t} + \rho_0 c^2 \operatorname{div} \boldsymbol{v} = 0$.

This equation is the same in form as equation (64.5), but the coefficient $\rho_0 c^2$ is a function of the coordinates. As in §64, we obtain Euler's equation in the form $\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0} \operatorname{grad} p'$. Eliminating $\boldsymbol{v}$, and omitting the suffix in ρ_0, we finally obtain the equation of propagation of sound in an inhomogeneous medium:

$$\operatorname{div} \frac{\operatorname{grad} p'}{\rho} - \frac{1}{\rho c^2}\frac{\partial^2 p'}{\partial t^2} = 0 . \tag{76.1}$$

If the wave is monochromatic, with frequency ω, we have $\ddot{p}' = -\omega^2 p'$, so that

$$\operatorname{div} \frac{\operatorname{grad} p'}{\rho} + \frac{\omega^2}{\rho c^2} p' = 0 . \tag{76.2}$$

[4] See (33.1). We here do not distinguish between u and Δu; the choice of the frame of reference in which the flow is considered is determined by the fact that the fluid is assumed to be at rest outside the turbulent region.

Let us consider a sound wave emitted by a **pulsating source** of small dimension; we have seen in § 74 that the emission is isotropic. We denote by A the point where the source is, and by $p_A(B)$ the pressure p' at a point B in the emitted wave.[5] If the same source is placed at B, it produces at A a pressure which we denote by $p_B(A)$. We shall derive the relation between $p_A(B)$ and $p_B(A)$.

To do so, we use equation (76.2), applying it first to the sound from a source at A and then to the sound from a source at B:

$$\operatorname{div}\frac{\operatorname{grad} p'_A}{\rho}+\frac{\omega^2}{\rho c^2}p'_A=0,\quad \operatorname{div}\frac{\operatorname{grad} p'_B}{\rho}+\frac{\omega^2}{\rho c^2}p'_B=0.$$

We multiply the first equation by p'_B and the second by p'_A and subtract. The result is

$$p'_B\operatorname{div}\frac{\operatorname{grad} p'_A}{\rho}-p'_A\operatorname{div}\frac{\operatorname{grad} p'_B}{\rho}=\operatorname{div}\left(\frac{p'_B\operatorname{grad} p'_A}{\rho}-\frac{p'_A\operatorname{grad} p'_B}{\rho}\right)=0.$$

We integrate this equation over the volume between an infinitely distant closed surface C and two small spheres C_A and C_B which enclose the points A and B respectively. The volume integral can be transformed into three surface integrals, and the integral over C is zero, since the sound field vanishes at infinity. Thus we obtain

$$\oint\limits_{C_A+C_B}\left(p'_B\frac{\operatorname{grad} p'_A}{\rho}-p'_A\frac{\operatorname{grad} p'_B}{\rho}\right)\cdot d\boldsymbol{f}=0. \tag{76.3}$$

Inside the small sphere C_A, the pressure p'_A in the wave from a source at A falls off rapidly with the distance from A, and the gradient $\operatorname{grad} p'_A$ is therefore large. The pressure p'_B due to a source at B is a slowly varying function of the coordinates in the region near the point A, which is at a considerable distance from B, so that the gradient $\operatorname{grad} p'_B$ is relatively small. When the radius of the sphere C_A is sufficiently small, therefore, we can neglect the integral

$$\oint\frac{p'_A}{\rho}\operatorname{grad} p'_B\cdot d\boldsymbol{f}$$

over C_A in comparison with

$$\oint\frac{p'_B}{\rho}\operatorname{grad} p'_A\cdot d\boldsymbol{f}$$

and in the latter the almost constant quantity p'_B can be taken outside the integral and replaced by its value at the point A. Similar arguments hold for the integrals over the sphere C_B, and as a result we obtain from (76.3) the relation

$$p'_B(A)\oint\limits_{C_A}\frac{\operatorname{grad} p'_A}{\rho}\cdot d\boldsymbol{f}=p'_A(A)\oint\limits_{C_B}\frac{\operatorname{grad} p'_B}{\rho}\cdot d\boldsymbol{f}.$$

[5] The dimension of the source must be small compared with the distance between A and B and with the wavelength

But $\frac{1}{\rho}\operatorname{grad} p' = -\frac{\partial \boldsymbol{v}}{\partial t}$, and this equation can therefore be rewritten

$$p'_B(A)\frac{\partial}{\partial t}\oint_{C_A} \boldsymbol{v}_A \cdot d\boldsymbol{f} = p'_A(B)\frac{\partial}{\partial t}\oint_{C_B} \boldsymbol{v}_B \cdot d\boldsymbol{f}\,.$$

The integral

$$\oint \boldsymbol{v}_A \cdot d\boldsymbol{f}$$

over C_A is the volume of fluid flowing per unit time through the surface of the sphere C_A, i.e., it is the rate of change of the volume of the pulsating source of sound. Since the sources at *A* and *B* are identical, it is clear that

$$\oint_{C_A} \boldsymbol{v}_A \cdot d\boldsymbol{f} = \oint_{C_B} \boldsymbol{v}_B \cdot d\boldsymbol{f}\,,$$

and consequently

$$p'_A(B) = p'_B(A)\,. \tag{76.4}$$

This equation constitutes the ***reciprocity principle***: the pressure at *B* due to a source at *A* is equal to the pressure at *A* due to a similar source at *B*. It should be emphasized that this result holds, in particular, for the case where the medium is composed of several different regions, each of which is homogeneous. When sound is propagated in such a medium, it is reflected and refracted at the surfaces separating the various regions. Thus the reciprocity principle is valid also in cases where the wave undergoes reflection and refraction on its path from *A* to *B*.

PROBLEM

Derive the reciprocity principle for dipole emission of sound by a source which oscillates without change of volume.

Solution. In this case the integral

$$\oint_{C_A} \boldsymbol{v}_A \cdot d\boldsymbol{f} = 0\,, \tag{1}$$

and the next approximation must be taken in calculating the integrals in (76.3). To do so, we write, as far as the first-order terms,

$$p'_B = p'_B(A) + \boldsymbol{r}\cdot\operatorname{grad} p'_B\,, \tag{2}$$

where $\boldsymbol{r}$ is the radius vector from *A*. In the integral

$$\oint_{C_A}\left(p'_B\frac{\operatorname{grad} p'_A}{\rho} - p'_A\frac{\operatorname{grad} p'_B}{\rho}\right)\cdot d\boldsymbol{f}\,, \tag{3}$$

the two terms are now of the same order of magnitude. Substituting here for p'_B from (2) and using (1), we get

$$\oint_{C_A} \left((\boldsymbol{r} \cdot \operatorname{grad} p'_B) \frac{\operatorname{grad} p'_A}{\rho} - p'_A \frac{\operatorname{grad} p'_B}{\rho} \right) \cdot d\boldsymbol{f} .$$

Next, we take the almost constant quantity $\operatorname{grad} p'_B = -\rho \dot{\boldsymbol{v}}_B$ outside the integral, replacing it by its value at *A:*

$$\rho_A \dot{\boldsymbol{v}}_B(A) \cdot \oint_{C_A} \left\{ \frac{p'_A}{\rho} d\boldsymbol{f} - \boldsymbol{r} \left(\frac{\operatorname{grad} p'_B}{\rho} \cdot d\boldsymbol{f} \right) \right\},$$

where ρ_A is the density of the medium at the point *A*. To calculate this integral, we notice that near a source the fluid can be supposed incompressible (see § 74), and hence we can write for the pressure inside the small sphere C_A, by (11.1), $p'_A = -\rho \dot{\phi} = \rho \dot{\boldsymbol{A}} \cdot \boldsymbol{r} / r^3$. In a monochromatic wave $\dot{\boldsymbol{v}} = -i\omega \boldsymbol{v}$, $\dot{\boldsymbol{A}} = -i\omega \boldsymbol{A}$; introducing also the unit vector $\boldsymbol{n}_A$ in the direction of the vector $\boldsymbol{A}$ for a source at *A*, we find that the integral (3) is proportional to $\rho_A \boldsymbol{v}_B(A) \cdot \boldsymbol{n}_A$. Similarly, the integral over the sphere C_B is proportional to $-\rho_B \boldsymbol{v}_A(B) \cdot \boldsymbol{n}_B$, with the same factor of proportionality. Equating the sum to zero, we find the required relation

$$\rho_A \boldsymbol{v}_B(A) \cdot \boldsymbol{n}_A = \rho_B \boldsymbol{v}_A(B) \cdot \boldsymbol{n}_B ,$$

which expresses the *reciprocity principle for dipole emission of sound*.