§77. Propagation of sound in a tube

Let us now consider the propagation of a sound wave in a long narrow tube. By a "narrow" tube we mean one whose width is small compared with the wavelength. The cross-section of the tube may vary along its length in both shape and area. It is important, however, that this variation should occur fairly slowly: the cross-sectional area S must vary only slightly over distances of the order of the width of the tube.

Under these conditions we can suppose that all quantities (velocity, density, etc.) are constant over any transverse cross-section of the tube. The direction of propagation of the wave can be supposed to coincide with that of the axis of the tube at all points. The equation for the propagation of such a wave is most conveniently derived by a method similar to that used in § 12 in deriving the equation for the propagation of gravity waves in channels.

In unit time a mass $S\rho v$ of fluid passes through a cross-section of the tube. Hence the mass of fluid in the volume between two transverse cross-sections at a distance dx apart decreases in unit time by

$$(S\rho v)_{x+dx} - (S\rho v)_x = \frac{\partial (S\rho v)}{\partial x} dx ,$$

the coordinate x being measured along the axis of the tube. Since the volume between the two cross-sections remains constant, the decrease must be due only to the change in density of the fluid. The change in density per unit time is $\partial\rho / \partial t$, and the corresponding decrease in the mass of fluid in the volume Sdx between the two cross-sections is $-S\left(\frac{\partial\rho}{\partial t}\right)dx$. Equating the two expressions, we obtain

$$S\frac{\partial\rho}{\partial t} = -\frac{\partial (S\rho v)}{\partial x} , \qquad (77.1)$$

which is the **equation of continuity** for flow in a pipe.

Next, we write down **Euler's equation**, omitting the term quadratic in the velocity:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} . \qquad (77.2)$$

We differentiate (77.1) with respect to time, regarding ρ on the right-hand side as independent of time, since the differentiation of ρ gives a term which involves $v\frac{\partial\rho}{\partial t} = v\frac{\partial\rho'}{\partial t}$ and is therefore of the second order of smallness. Thus $S\frac{\partial^2\rho}{\partial t^2} = -\frac{\partial}{\partial x}\left(S\rho\frac{\partial v}{\partial t}\right)$. Here we substitute the expression (77.2) for $\frac{\partial v}{\partial t}$, and express the derivative of the density on the left-hand side in terms of the derivative of the pressure by $\ddot{\rho} = \ddot{p} / c^2$.

The result is the following equation for the propagation of sound in a tube:

$$\frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial p}{\partial x}\right) - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 . \qquad (77.3)$$

In a monochromatic wave p depends on time through a factor $e^{-i\omega t}$, and (77.3) becomes

$$\frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial p}{\partial x}\right)+k^2 p=0. \tag{77.4}$$

where $k=\omega/c$ is the wave number.[1]

Finally, let us consider the problem of the **emission of sound from the open end** of a tube. The pressure difference between the gas in the end of the tube and that in the space surrounding the tube is small compared with the pressure differences within the tube. Hence the boundary condition at the open end of the tube is, with sufficient accuracy, that the pressure p should vanish. The gas velocity v at the end of the tube is not zero; let its value be v_0. The product Sv_0 is the volume of gas leaving the tube per unit time.

We can now regard the open end of the tube as a source of gas with strength Sv_0. The problem of the emission from a tube thus becomes equivalent to that of the emission by a pulsating body, which is solved by formula (74.10). In place of the time derivative V of the volume of the body we must now put Sv_0. Thus the total intensity of the sound emitted is

$$I=\frac{\rho S^2\overline{\dot{v}_0^{\,2}}}{4\pi c}. \tag{74.5}$$

PROBLEMS

Problem 1. Determine the transmission coefficient for sound passing from a tube with cross-section S_1 into one with cross-section S_2.

Solution. In the first tube we have two waves, the incident wave $p_1=a_1e^{i(kx-\omega t)}$ and reflected wave $p_1'=a_1'e^{-i(kx+\omega t)}$. In the second tube we have the transmitted wave $p_2=a_2e^{i(kx-\omega t)}$. At the point where the tubes join (x = 0), the pressures must be equal, and so must the volumes Sv of gas passing from one tube to the other per unit time. These conditions give $a_1+a_1'=a_2$, $S_1(a_1-a_1')=S_2a_2$, whence $a_2=2a_1S_1/(S_1+S_2)$. The ratio D of the energy flux in the transmitted wave to that in the incident wave is

$$D=\frac{S_2\overline{|v_2|^2}}{S_1\overline{|v_1|^2}}=\frac{4S_1S_2}{(S_1+S_2)^2}=1-\left(\frac{S_2-S_1}{S_2+S_1}\right)^2.$$

Problem 2. Determine the amount of energy emitted from the open end of a cylindrical tube.

Solution. In the boundary condition p = 0 at the open end of the tube, we can approximately neglect the emitted wave (we shall see that the intensity emitted from the end of the tube is small). Then we have the condition $p_1=-p_1'$, where p_1 and p_1' are the pressures in the incident wave and in the wave reflected back into the tube; for the velocities we have correspondingly $v_1=v_1'$, so that the total velocity at the end of the tube is $v_0=v_1+v_1'=2v_1$. The energy flux in the incident wave is $cS\rho\overline{v_1^{\,2}}=(1/4)cS\rho\overline{v_0^{\,2}}$. Using

[1] Here, and in the Problems, p denotes the variable part of the pressure, which we have previously denoted by p'.

(77.5), we obtain for the ratio of the emitted energy to the energy flux in the incident wave $D = S\omega^2 / \pi c^2$. For a tube with circular cross-section (radius *R*) we have $D = R^2\omega^2 / c^2$. Since, by hypothesis, $R << c/\omega$, it follows that $D << 1$.

Problem 3. One of the ends of a cylindrical pipe is covered by a membrane which executes a given oscillation and emits sound; the other end is open. Determine the way in which sound is emitted from the tube.

Solution. In the general solution

$$p = (ae^{ikx} + be^{-ikx})e^{-i\omega t}$$

we determine the constants *a* and *b* from the conditions $v = u = u_0 e^{-i\omega t}$, the given velocity of the membrane, at the closed end (*x* = 0),and *p* = 0 at the open end (*x* = *l*). These give $ae^{ikl} + be^{-ikl} = 0$, $a - b = c\rho u_0$. Determining *a* and *b,* we find the gas velocity at the open end of the tube to be $v_0 = u/\cos kl$. If the tube were absent, the intensity of the sound emitted by the oscillating membrane would be given by the mean square $S^2\overline{|\dot{u}|^2} = S^2\omega^2\overline{|u|^2}$, according to formula (74.10) with *Su* in place of $\dot{V}$; *S* is the cross-sectional area of the membrane. The emission from the end of the tube is proportional to $S^2\overline{|v_0|^2}\omega^2$. The amplification coefficient of the pipe is $A = S^2\overline{|v_0|^2} / S^2\overline{|u|^2} = 1/\cos^2 kl$. This becomes infinite for frequencies of oscillation of the membrane equal to the characteristic frequencies of the tube (***resonance***); in reality, of course, it remains finite because of effects which we have neglected (such as friction due to the emission of sound).

Problem 4. The same as Problem 3, but for a conical tube, with the membrane covering the smaller end.

Solution. The cross-section of the tube is $S = S_0 x^2$; let the values of the coordinate *x* which correspond to the smaller and larger ends be x_1, x_2, so that the length of the tube is $l = x_2 - x_1$. The general solution of equation (77.4) is $p = \dfrac{1}{x}(ae^{ikx} + be^{-ikx})e^{-i\omega t}$; *a* and *b* are determined from the conditions $v = u$ for $x = x_1$ and *p* = 0 for $x = x_2$. The amplification coefficient is found to be

$$A = \frac{S_0{}^2 x_2{}^4 \overline{|v_2|^2}}{S_0{}^2 x_1{}^4 \overline{|u|^2}} = \frac{k^2 x_2{}^2}{(\sin kl + kx_1 \cos kl)^2}.$$

Problem 5. The same as Problem 3, but for a tube whose cross-section varies exponentially along its length: $S = S_0 e^{\alpha x}$.

Solution. Equation (77.4) becomes $\dfrac{\partial^2 p}{\partial x^2} + \alpha\dfrac{\partial p}{\partial x} + k^2 p = 0$, whence

$$p = e^{-\alpha x/2}(ae^{imx} + be^{-imx})e^{-i\omega t},$$

with $m=\sqrt{k^2-\alpha^2/4}$. Determining a and b from the conditions $v=u$ for $x = 0$ and $p = 0$ for $x = l$, we find the amplification coefficient.

$$A=\frac{S_0{}^2 e^{2\alpha l}\overline{|v_0|^2}}{S_0{}^2\overline{|u|^2}}=\frac{e^{\alpha l}}{[(1/2)(\alpha/m)\sin ml+\cos ml]^2},$$

for $k>\alpha/2$ and

$$A=\frac{e^{\alpha l}}{[(1/2)(\alpha/m')\sinh m'l+\cosh m'l]^2},\quad m'=\sqrt{\alpha^2/4-k^2},$$

for $k<\alpha/2$.

§78. Scattering of sound

If there is some body in the path of propagation of a sound wave, then the sound is ***scattered***: besides the incident wave there appear other (scattered) waves, which are propagated in all directions from the scattering body. The scattering of a sound wave occurs simply on account of the presence of the body in its path. In addition, the incident wave causes the body itself to move, and this in turn brings about additional emission of sound by the body, i.e., further scattering. If, however, the density of the body is large compared with that of the medium in which the sound is propagated, and its compressibility is small, then the scattering due to the motion of the body forms only a small correction to the main scattering caused by the mere presence of the body. In what follows we shall neglect this correction, and therefore suppose the scattering body immovable.

We assume that the wavelength λ of the sound is large compared with the dimension l of the body; to calculate the properties of the scattered wave, we can then use formulae (74.8) and (74.11).[2] In doing so, we regard the scattered wave as being emitted by the body; the only difference is that, instead of a motion of the body in the fluid, we now have a motion of the fluid relative to the body. The two problems are clearly equivalent.

For the potential of the emitted wave we have obtained the expression $\phi=-\dot{V}/4\pi r-\dot{\boldsymbol{A}}\cdot\boldsymbol{r}/cr^2$. In this formula V was the volume of the body. In the present case, however, the volume of the body itself remains unchanged, and $\dot{V}$ must be taken not as the rate of change of the volume of the body, but as the volume of fluid which would enter, per unit time, the volume V_0 occupied by the body if the body were absent. For, in the presence of the body, this volume of fluid does not penetrate into V_0, which is equivalent to the emission of the same volume of fluid from V_0. The coefficient of $1/4\pi r$ in the first term of ϕ must, as we have seen in §74, be just the volume of fluid emitted from the origin per unit time. This volume is easily found. The change per unit time in the mass of fluid in a

[2] At the same time, the dimension of the body must be large in comparison with the displacement amplitude of fluid particles in the wave, since otherwise the fluid is not in general in potential flow.

volume equal to that of the body is $V_0\dot{\rho}$, where $\dot{\rho}$ gives the rate of change of the fluid density in the incident sound wave (since the wavelength is large compared with the dimension of the body, the density ρ may be supposed constant over distances of the order of this dimension; hence we can write the rate of change of the mass of fluid in V_0 as $V_0\dot{\rho}$ simply, where $\dot{\rho}$ is the same throughout the volume V_0.). The change in volume corresponding to a mass change $V_0\dot{\rho}$ is evidently $V_0\dot{\rho}/\rho$. Thus $\dot{V}$ in the expression for ϕ must be replaced by $V_0\dot{\rho}/\rho$. In an incident plane wave, the variable part ρ' of the density is related to the velocity by $\rho' = \rho v/c$; hence $\dot{\rho} = \dot{\rho}' = \rho\dot{v}/c$, and we can replace $V_0\dot{\rho}/\rho$ by $V_0\dot{v}/c$.

When the body moves in the fluid, the vector $\boldsymbol{A}$ is determined by formulae (11.5), (11.6): $4\pi\rho A_i = m_{ik}u_k + \rho V_0 u_i$. We must now replace the velocity $\boldsymbol{u}$ of the body by the reversed velocity $\boldsymbol{v}$ of the fluid in the incident wave which it would have at the position of the body if the latter were absent. Thus

$$A_i = -\frac{m_{ik}v_k}{4\pi\rho} - \frac{V_0 v_i}{4\pi}. \tag{78.1}$$

We finally obtain for the potential of the scattered wave

$$\phi_{SC} = -\frac{V_0\dot{v}}{4\pi c r} - \frac{\dot{\boldsymbol{A}}\cdot\boldsymbol{r}}{c r^2}, \tag{78.2}$$

the vector $\boldsymbol{A}$ being given by formula (78.1). Hence we have for the velocity distribution in the scattered wave

$$\boldsymbol{v}_{SC} = \frac{V_0\ddot{v}\boldsymbol{n}}{4\pi r c^2} + \frac{\boldsymbol{n}(\boldsymbol{n}\cdot\ddot{\boldsymbol{A}})}{r c^2} \tag{78.3}$$

(see §74), $\boldsymbol{n}$ being a unit vector in the direction of scattering.

The mean amount of **energy scattered** per unit time into a given solid angle element *do* is given by the energy flux, which is $c\rho\overline{\boldsymbol{v}_{SC}{}^2}do$. The **total scattered intensity** I_{SC} is obtained by integrating this expression over all directions. The integration of twice the product of the two terms in (78.3) gives zero, since this product is proportional to the cosine of the angle between the direction of scattering and the direction of propagation of the incident wave, and there remains (cf. (74.10) and (74.13))

$$I_{SC} = \frac{V_0{}^2\rho}{4\pi c^3}\overline{\ddot{\boldsymbol{v}}^2} + \frac{4\pi\rho}{3c^3}\overline{\ddot{\boldsymbol{A}}^2}. \tag{78.4}$$

The scattering is generally characterized by what is called the ***cross-section*** $d\sigma$, which is the ratio of the (time) average energy scattered into a given solid-angle element to the mean energy flux density in the incident wave. The ***total cross-section*** σ is the integral of $d\sigma$ over all directions of scattering, i.e., it is the ratio of the total scattered intensity to the incident energy flux density, and evidently has the dimensions of area.

The mean energy flux density in the incident wave is $c\rho\overline{\boldsymbol{v}^2}$. Hence the differential

scattering cross-section is

$$d\sigma = \frac{\overline{\boldsymbol{V}_{SC}^{\,2}}}{\overline{\boldsymbol{V}^2}} r^2 do \,. \tag{78.5}$$

The total cross-section is

$$\sigma = \frac{V_0^{\,2}}{4\pi c^4}\frac{\overline{\ddot{\boldsymbol{v}}^2}}{\overline{\boldsymbol{v}^2}} + \frac{4\pi}{3c^4}\frac{\overline{\ddot{\boldsymbol{A}}^2}}{\overline{\boldsymbol{v}^2}} \,. \tag{78.6}$$

For a monochromatic incident wave, the mean square second time derivative of the velocity is proportional to the fourth power of the frequency. Thus the cross-section for the scattering of sound by a body which is small compared with the wavelength is proportional to ω^4.

Finally, let us briefly discuss the opposite limiting case, where the wavelength of the scattered sound is small compared with the dimension of the body. In this case all the scattering, except for the scattering through very small angles, amounts to simple reflection from the surface of the body. The corresponding part of the total scattering cross-section is clearly equal to the area *S* of the cross-section of the body by a plane perpendicular to the direction of the incident wave. The scattering through small angles (of the order of λ / l), however, constitutes ***diffraction*** from the edges of the body. We shall not pause here to expound the theory of this phenomenon, which is entirely analogous to that of the diffraction of light (see *Fields,* §§60, 61). We shall only mention that, by **Babinet's principle**, the total intensity of diffracted sound is equal to the total intensity of reflected sound. Hence the diffraction part of the scattering cross-section is also equal to *S*, and the total cross-section is therefore 2*S*.

PROBLEMS

Problem 1. Determine the cross-section for the scattering of a plane sound wave by a solid sphere with radius *R* small compared with the wavelength.

Solution. The velocity at a given point in a plane wave is $v = a\cos\omega t$. In the case of a sphere (see §11, Problem 1), the vector $\boldsymbol{A}$ is $-(1/2)R^3\boldsymbol{v}$. For the differential cross-section we obtain

$$d\sigma = \frac{\omega^4 R^6}{9c^4}\left(1 - \frac{3}{2}\cos\theta\right)^2 do \,,$$

where θ is the angle between the direction of the incident wave and the direction of scattering. The scattered intensity is greatest in the direction $\theta = \pi$, which is opposite to the direction of incidence. The total cross-section is

$$\sigma = \frac{7\pi}{9}\left(\frac{R^3\omega^2}{c^2}\right)^2 . \tag{1}$$

Here (and also in Problems 3 and 4 below) it is assumed that the density ρ_0 of the sphere is large compared with the density ρ of the gas; if this were not so, it would be necessary to take account of the movement of the sphere by the pressure forces exerted on it by the

oscillating gas.

Problem 2. Determine the cross-section for the scattering of sound by a drop of fluid, taking into account the compressibility of the fluid and the motion of the drop caused by the incident wave.

Solution. When the pressure of the gas surrounding the drop changes adiabatically by p', the volume of the drop is reduced by $\frac{V_0}{\rho_0}\left(\frac{\partial\rho_0}{\partial p}\right)_S p' = \frac{V_0 c\rho v}{\rho_0 {c_0}^2}$, where ρ_0 is the density of the drop, c_0 the velocity of sound in the fluid, and ρ the density of the gas. In the expressions (78.2) and (78.3), we must now replace $V_0\ddot{v}/c$ by the difference $V_0(\ddot{v}/c - \ddot{v}c\rho/{c_0}^2\rho_0)$. Moreover, in the expression for $\boldsymbol{A}$ we must replace $-\boldsymbol{v}$ by the difference $\boldsymbol{u} - \boldsymbol{v}$, where $\boldsymbol{u}$ is the velocity acquired by the drop as a result of the action of the incident wave. For a sphere we have, using the results of §11, Problem 1, $\boldsymbol{A} = \frac{R^3\boldsymbol{v}(\rho-\rho_0)}{2\rho_0+\rho}$.

Substituting these expressions, we have the cross-section

$$d\sigma = \frac{\omega^4 R^6}{9c^4}\left\{\left(1-\frac{c^2\rho}{{c_0}^2\rho_0}\right) - 3\cos\theta\,\frac{\rho_0-\rho}{2\rho_0+\rho}\right\}^2 do\,.$$

The total cross-section is

$$\sigma = \frac{4\pi\omega^4 R^6}{9c^4}\left\{\left(1-\frac{c^2\rho}{{c_0}^2\rho_0}\right)^2 + \frac{3(\rho_0-\rho)^2}{(2\rho_0+\rho)^2}\right\}^2\,.$$

Problem 3. Determine the cross-section for the scattering of sound by a solid sphere with radius R much less than $\sqrt{\nu/\omega}$. The specific heat of the sphere is supposed so large that its temperature can be regarded as a constant.

Solution. In this case we have to take into account the effect of the gas viscosity on the motion of the sphere,

and the vector $\boldsymbol{A}$ must be modified as shown in §74, Problem 2. For $R\sqrt{\omega/\nu} << 1$ we have $\boldsymbol{A} = -\frac{3iR\nu\boldsymbol{v}}{2\omega}$.

The thermal conductivity of the gas also results in scattering of the same order. Let $T'_0\, e^{-i\omega t}$ be the temperature variation at a given point in the sound wave. The temperature distribution near a sphere is (see §52, Problem 2)

$$T' = T'_0\, e^{-i\omega t}\left[1-\frac{R}{r}e^{-(1-i)(r-R)\sqrt{\omega/2\chi}}\right]$$

(for $r = R$ we must have $T' = 0$). The amount of heat transferred from the gas to the sphere per unit time is (for $R\sqrt{\omega/\chi} << 1$) $q = 4\pi R^2\kappa\left(\frac{dT'}{dr}\right)_{r=R} = 4\pi R\kappa T'_0\, e^{-i\omega t}$. This transfer of

heat results in a change in the volume of the gas, which can be taken to affect the scattering like a corresponding effective change in the volume of the sphere, $\dot{V} = -4\pi R\chi\beta T'_0 e^{-i\omega t} = -4\pi R\chi(\gamma-1)v/c$, where β is the coefficient of thermal expansion of the gas and $\gamma = c_p/c_v$; we have used also formulae (64.13) and (79.2).

Taking account of both effects, we obtain the differential scattering cross-section

$$d\sigma = \left(\frac{\omega R}{c^2}\right)^2 \left[\chi(\gamma-1) - \frac{3}{2}v\cos\theta\right]^2 do .$$

The total cross-section is

$$\sigma = 4\pi\left(\frac{\omega R}{c^2}\right)^2 \left[\chi^2(\gamma-1)^2 + \frac{3}{4}v^2\right].$$

These formulae are valid only if the Stokes frictional force is small compared with the inertia force, i.e., $\eta R << M\omega$, where $M = 4\pi R^3\rho_0/3$ is the mass of the sphere; otherwise, the movement of the sphere by viscous forces becomes important.

Problem 4. Determine the mean force on a solid sphere which scatters a plane sound wave ($\lambda >> R$).

Solution. The momentum transmitted per unit time from the incident wave to the sphere, i.e., the required force, is the difference between the momentum in the incident wave and the total momentum flux in the scattered wave. From the incident wave an energy flux $\sigma c\overline{E}_0$ is scattered, where E_0 is the energy density in the incident wave; the corresponding momentum flux is obtained by dividing by *c,* and is therefore $\sigma\overline{E}_0$. In the scattered wave, the momentum flux into the solid angle element *do* is $\overline{E_{SC}}r^2 do = \overline{E_0}d\sigma$; projecting this on the direction of propagation of the incident wave (which is obviously the direction of the required force), and integrating over all angles, we obtain

$$\overline{E_0}\int \cos\theta d\sigma .$$

Thus the force on the sphere is

$$F = \overline{E_0}\int (1-\cos\theta)d\sigma .$$

Substituting for $d\sigma$ from Problem 1, we obtain

$$F = \frac{11\pi\omega^4 R^6 \overline{E_0}}{9c^4} .$$

§79. Absorption of sound

The existence of viscosity and thermal conductivity results in the dissipation of energy in sound waves, and the sound is consequently ***absorbed***, i.e., its intensity progressively diminishes. To calculate the rate of energy dissipation $\dot{E}_{mech}$, we use the following general arguments. The mechanical energy is just the maximum amount of work that can be done in

passing from a given non-equilibrium state to one of thermodynamic equilibrium. As we know from thermodynamics, the maximum work is obtained when the transition is reversible (i.e., without change of entropy), and is then $\dot{E}_{mech} = E_0 - E(S)$, where E_0 is the given initial value of the energy, and $E(S)$ is the energy in the equilibrium state with the same entropy S as the system had initially. Differentiating with respect to time, we obtain $\dot{E}_{mech} = -\dot{E}(S) = -\left(\dfrac{\partial E}{\partial S}\right)\dot{S}$. The derivative of the energy with respect to the entropy is the temperature. Hence $\partial E / \partial S$ is the temperature which the system would have if it were in thermodynamic equilibrium (with the given value of the entropy). Denoting this temperature by T_0, we therefore have $\dot{E}_{mech} = -T_0 \dot{S}$.

We use for S the expression (49.6), which gives the rate of change of the entropy due to both thermal conduction and viscosity. Since the temperature T varies only slightly through the fluid, and differs little from T_0, it can be taken outside the integral, and T_0 can be written as T simply:

$$\dot{E}_{mech} = -\frac{\kappa}{T}\int (\mathrm{grad}\, T)^2\, dV - \frac{1}{2}\eta \int \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2 dV - \zeta \int (\mathrm{div}\, \boldsymbol{v})^2\, dV\,. \tag{79.1}$$

This formula generalizes formula (16.3) to the case of a compressible fluid which conducts heat.

Let the x-axis be in the direction of propagation of the sound wave. Then $v_x = v_0 \cos(kx - \omega t)$, $v_y = v_z = 0$. The last two terms in (79.1) give

$$-\left(\frac{4}{3}\eta + \zeta\right)\int \left(\frac{\partial v_x}{\partial x}\right)^2 dV = -k^2\left(\frac{4}{3}\eta + \zeta\right) v_0^2 \int \sin^2(kx - \omega t)\, dV\,.$$

We are, of course, interested only in the time average; taking this average, we have $-k^2\left(\dfrac{4}{3}\eta + \zeta\right)\dfrac{v_0^2}{2}V_0$, where V_0 is the volume of the fluid.

Next we calculate the first term in (79.1). The deviation T' of the temperature in the sound wave from its equilibrium value is related to the velocity by formula (64.13), so that the temperature gradient is

$$\frac{\partial T}{\partial x} = \frac{\beta c T}{c_p}\frac{\partial v}{\partial x} = -\frac{\beta c T}{c_p} v_0 k \sin(kx - \omega t)\,.$$

For the time average of the first term in (79.1) we obtain $-\kappa c^2 T \beta^2 v_0^2 k^2 V_0 / 2 c_p^2$. Using the thermodynamic formulae

$$c_p - c_v = T\beta^2 \left(\frac{\partial p}{\partial \rho}\right)_T = T\beta^2 \frac{c_v}{c_p}\left(\frac{\partial p}{\partial \rho}\right)_S = T\beta^2 c^2 \frac{c_v}{c_p}, \tag{79.2}$$

we can rewrite this expression as $-\dfrac{1}{2}\kappa\left(\dfrac{1}{c_v} - \dfrac{1}{c_p}\right)k^2 v_0^2 V_0$.

Collecting the above results, we find the mean value of the energy dissipation:

$$\overline{\dot{E}_{mech}} = -\frac{1}{2} k^2 v_0{}^2 V_0 \left[\left(\frac{4}{3}\eta + \zeta \right) + \kappa \left(\frac{1}{c_v} - \frac{1}{c_p} \right) \right]. \tag{79.3}$$

The total energy of the sound wave is

$$\overline{E} = \frac{1}{2} \rho v_0{}^2 V_0 . \tag{79.4}$$

The damping coefficient derived in §25 for gravity waves gives the manner of decrease of the intensity with time. For sound, however, the problem is usually stated somewhat differently; a sound wave is propagated through a fluid, and its intensity decreases with the distance x traversed. It is evident that this decrease will occur according to a law $e^{-2\gamma x}$, and the amplitude will decrease as $e^{-\gamma x}$, where the ***absorption coefficient*** γ is defined by

$$\gamma = \frac{\overline{|\dot{E}_{mech}|}}{2c\overline{E}} . \tag{79.5}$$

Substituting here (79.3) and (79.4), we find the following expression for the sound absorption coefficient

$$\gamma = \frac{\omega^2}{2\rho c^3} \left[\left(\frac{4}{3}\eta + \zeta \right) + \kappa \left(\frac{1}{c_v} - \frac{1}{c_p} \right) \right] \equiv a\omega^2 . \tag{79.6}$$

We may point out that it is proportional to the square of the frequency of the sound.[3] This formula is applicable so long as the absorption coefficient determined by it is small: the amplitude must decrease relatively little over distances of the order of a wavelength (i.e., we must have $\gamma c / \omega << 1$). The above derivation is essentially founded on this assumption, since we have calculated the energy dissipation by using the expression for an undamped sound wave. For gases this condition is in practice always satisfied. Let us consider, for example, the first term in (79.6). The condition $\gamma c / \omega << 1$ means that $\nu\omega / c^2 << 1$. It is known from the kinetic theory of gases, however, that the **viscosity coefficient** ν for a gas is of the order of product of the **mean free path** l and the **mean thermal velocity** of the molecules; the latter is of the same order as the velocity of sound in the gas, so that $\nu \sim lc$. Hence we have

$$\frac{\nu\omega}{c^2} \sim \frac{l\omega}{c} = \frac{l}{\lambda} << 1 , \tag{79.7}$$

since we know that $l << \lambda$. The thermal-conduction term in (79.6) gives the same result, since $\chi \sim \nu$.

In liquids, the condition of small absorption is always fulfilled when the problem of

[3] M. A. Isakovich (1948) has shown that there must be a special absorption when sound is propagated in a two-phase system (an *emulsion)*. Because of the different thermodynamic properties of the two components, their temperature changes during the passage of the sound wave will in general be different. The resulting heat exchange between the components leads to an additional absorption of sound. On account of the relative slowness of this heat

sound absorption, as stated here, is significant at all. The absorption over one wavelength can become large only if the viscous forces are comparable with the pressure forces which occur when the substance is compressed. In these conditions, however, the Navier-Stokes equation itself (with the viscosity coefficients independent of frequency) becomes invalid and a considerable dispersion of sound, due to processes of internal friction, occurs.[4]

For absorption of sound, the relation between the wave number and the frequency can evidently be written

$$k = \frac{\omega}{c} + ia\omega^2, \qquad (79.8)$$

where a denotes the coefficient in (79.6). It is easy to see from this how the equation for a travelling sound wave must be modified in order to take absorption into account. To do so, we notice that, in the absence of absorption, the differential equation for (say) the pressure $p' = p'(x-ct)$ can be written $\frac{\partial p'}{\partial x} = -\frac{1}{c}\frac{\partial p'}{\partial t}$. The equation whose solution is $e^{i(kx-\omega t)}$, with k given by (79.8), must clearly be

$$\frac{\partial p'}{\partial x} = -\frac{1}{c}\frac{\partial p'}{\partial t} + a\frac{\partial^2 p'}{\partial t^2}. \qquad (79.9)$$

If we replace t by $\tau + x/c$, this equation becomes

$$\frac{\partial p'}{\partial x} = a\frac{\partial^2 p'}{\partial \tau^2},$$

i.e., a one-dimensional equation of thermal conduction.

The general solution of this equation can be written (see §51)

$$p'(x,\tau) = \frac{1}{2\sqrt{\pi a x}}\int p'_0(\tau')\exp\left[-\frac{(\tau'-\tau)^2}{4ax}\right]d\tau'. \qquad (79.10)$$

where $p'_0(\tau) = p'(0,\tau)$. If the wave is emitted during a finite time interval, this expression becomes, at sufficiently large distances from the source,

$$p'(x,\tau) = \frac{1}{2\sqrt{\pi a x}}\exp\left[-\frac{\tau^2}{4ax}\right]\int p'_0(\tau')d\tau'. \qquad (79.11)$$

In other words, the wave profile at large distances is Gaussian. Its width is of the order of $\sqrt{ax}$, i.e., it increases as the square root of the distance travelled by the wave, while the amplitude falls off inversely as $\sqrt{x}$. Hence we at once conclude that the total energy of the wave decreases as $1/\sqrt{x}$.

It is easy to derive analogous formulae for **spherical** waves; to do so, we must use the

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exchange, a considerable dispersion of the sound takes place comparatively quickly.

[4] A special case where strong absorption is possible but can be discussed by the usual methods is that of a gas with a thermal conductivity which is unusually large compared with its viscosity, on account of effects such as radiative transfer at very high temperatures (see Problem 3).

fact that for such a wave

$$\int p'\, dt = 0$$

(see (70.8)). Instead of (79.11) we now have

$$p'(r,\tau) = \text{constant} \times \frac{1}{r}\frac{\partial}{\partial \tau}\frac{\exp(-\tau^2/4ar)}{\sqrt{r}},$$

or

$$p'(r,\tau) = \text{constant} \times \frac{\tau}{r^{5/2}}\exp(-\tau^2/4ar). \tag{79.12}$$

Strong absorption must occur when a sound wave is reflected from a **solid wall**. The reason for this is the following (K. F. Herzfeld 1938; B. P. Konstantinov 1939). In a sound wave not only the density and the pressure, but also the temperature, undergo periodic oscillations about their mean values. Near a solid wall, therefore, there is a periodically fluctuating temperature difference between the fluid and the wall, even if the mean fluid temperature is equal to the wall temperature. At the wall itself, however, the temperatures of the wall and the adjoining fluid must be the same. As a result, a large temperature gradient is formed in a thin boundary layer of fluid, where the temperature changes rapidly from its value in the sound wave to the wall temperature. The presence of large **temperature gradients**, however, results in a large dissipation of energy by thermal conduction. For a similar reason, the fluid viscosity leads to strong absorption of sound when the wave is incident in an oblique direction. In this case the fluid velocity in the wave (in the direction of propagation) has a non-zero component tangential to the surface. At the surface itself, however, the fluid must completely "adhere". Hence a large tangential **velocity gradient**[5] must occur in the boundary layer of fluid, resulting in a large viscous dissipation of energy (see Problem 1).

PROBLEMS

Problem 1. Determine the fraction of energy that is absorbed when a sound wave is reflected from a solid wall. The density of the wall is supposed so large that the sound does not penetrate it, and the specific heat so large that the temperature of the wall may be supposed constant.

Solution. We take the plane of the wall as the plane $x = 0$, and the plane of incidence as the xy-plane. Let the angle of incidence (which equals the angle of reflection) be θ. The change in density in the incident wave at any given point on the surface ($x = y = 0$, say) is $\rho'_1 = Ae^{-i\omega t}$. The reflected wave has the same amplitude, so that $\rho'_2 = \rho'_1$ at the wall. The actual change in the fluid density, since both waves (incident and reflected) are propagated

[5] The normal velocity component is zero at the boundary because of the boundary conditions, whether or not viscosity is present.

simultaneously, is $\rho' = 2Ae^{-i\omega t}$. The fluid velocity in the wave is given by $\boldsymbol{v}_1 = c\rho'_1 \boldsymbol{n}_1 / \rho$, $\boldsymbol{v}_2 = c\rho'_2 \boldsymbol{n}_2 / \rho$. The total velocity on the wall, $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, is therefore $v = v_y = 2A \sin\theta \times ce^{-i\omega t} / \rho$ (or, more precisely, this is what the velocity is found to be when the correct boundary conditions at the wall in the presence of viscosity are not applied). The actual variation of the velocity v_y near the wall is determined by formula (24.13), and the energy dissipation due to viscosity by formula (24.14), in which the above expression for v must be substituted for $v_0 e^{-i\omega t}$.

The deviation T' of the temperature from its mean value (which is the temperature of the wall), if calculated without using the correct boundary conditions at the wall, would be found to be (see (64.13)) $T' = 2Ac^2 T\beta e^{-i\omega t} / c_p \rho$. In reality, however, the temperature distribution is determined by the equation of thermal conduction, with the boundary condition $T' = 0$ for $x = 0$, and is accordingly given by a formula entirely similar to (24.13).

On calculating the energy dissipation due to thermal conduction as the first term in formula (79.1), we obtain for the total energy dissipation per unit area of the wall

$$\bar{\dot{E}}_{mech} = \frac{A^2 c^2 \sqrt{2\omega}}{\rho} \left[\sqrt{\chi} \left(\frac{c_p}{c_v} - 1 \right) + \sqrt{\nu} \sin^2\theta \right].$$

The mean energy flux density incident on unit area of the wall from the incident wave is $c\rho \overline{v_1^2} \cos\theta = \dfrac{c^3 A^2}{2\rho} \cos\theta$. Hence the fraction of energy absorbed on reflection is

$$\frac{2\sqrt{2\omega}}{c\cos\theta} \left[\sqrt{\nu} \sin^2\theta + \sqrt{\chi} \left(\frac{c_p}{c_v} - 1 \right) \right].$$

This expression is valid only if its value is small (since in deriving it we have supposed the amplitudes of the incident and reflected waves to be the same). This condition means that the angle of incidence θ must not be too near $\pi / 2$.

Problem 2. Determine the coefficient of absorption of sound propagated in a cylindrical pipe.

Solution. The main contribution to the absorption is due to the presence of the walls. The absorption coefficient γ is equal to the energy dissipated at the walls per unit time and per unit length of the pipe, divided by twice the total energy flux through a cross-section of the pipe. A calculation similar to that given in Problem 1 leads to the result

$$\gamma = \frac{\sqrt{\omega}}{\sqrt{2}Rc} \left[\sqrt{\nu} + \sqrt{\chi} \left(\frac{c_p}{c_v} - 1 \right) \right],$$

where R is the radius of the pipe.

Problem 3. Find the dispersion relation for sound propagated in a medium with very high thermal conductivity.

Solution. In the presence of a large thermal conductivity the flow in a sound wave is not adiabatic. Hence, instead of the condition of constant entropy, we now have

$$\dot{s}' = \frac{\kappa \Delta T'}{\rho T}, \qquad (1)$$

which is the linearized form of equation (49.4) without the viscosity terms. As a second equation we take

$$\ddot{\rho}' = \Delta p', \qquad (2)$$

which is obtained by eliminating v from equations (64.2) and (64.3). Taking as the fundamental variables p' and T', we write ρ' and s' in the form

$$\rho' = \left(\frac{\partial \rho}{\partial T}\right)_p T' + \left(\frac{\partial \rho}{\partial p}\right)_T p', \quad s' = \left(\frac{\partial s}{\partial T}\right)_p T' + \left(\frac{\partial s}{\partial p}\right)_T p'.$$

We substitute these expressions in (1) and (2), and then seek T' and p' in a form proportional to $e^{i(kx-\omega t)}$. The compatibility condition for the resulting two equations for p' and T' can (by using various relations between the derivatives of thermodynamic quantities) be brought to the form

$$k^4 - k^2\left(\frac{\omega^2}{c_T{}^2} + \frac{i\omega}{\chi}\right) + \frac{i\omega^3}{\chi c_s{}^2} = 0, \qquad (3)$$

which gives the required relation between k and ω. We have here used the notation

$$c_s{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_s, \quad c_T{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_T = \frac{c_s H2}{\gamma},$$

where $\gamma = c_p / c_v$ is the ratio of specific heats.

In the limiting case of **low frequencies** ($\omega << c^2 / \chi$), equation (3) gives

$$k = \frac{\omega}{c_s} + i\frac{\omega^2 \chi}{2c_s}\left(\frac{1}{c_T{}^2} - \frac{1}{c_s{}^2}\right),$$

which corresponds to the propagation of sound with the ordinary "adiabatic" velocity c_s and a small absorption coefficient which is the second term in (79.6). This is as it should be, since the condition $\omega << c^2 / \chi$ means that, during one period, heat can be transmitted only over a distance $\sim \sqrt{\chi / \omega}$ (cf. (51.7)) which is small compared with the wavelength c / ω.

In the opposite limiting case of **large frequencies**, we find from (3)

$$k = \frac{\omega}{c_T} + i\frac{c_T}{2\chi c_s{}^2}\left(c_s{}^2 - c_T{}^2\right).$$

In this case the sound is propagated with the "isothermal" velocity c_T, which is always less than c_s. The absorption coefficient is again small compared with the reciprocal of the

wavelength, and is independent of the frequency and inversely proportional to the thermal conductivity.[6]

Problem 4. Determine the additional absorption, due to diffusion, of sound propagated in a mixture of two substances (I. G. Shaposhnikov and Z. A. Gol'dberg 1952).

Solution. The mixture contains an additional source of absorption of sound because the temperature and pressure gradients occurring in the sound wave result in irreversible processes of thermal diffusion and barodiffusion (but there is evidently no mass-concentration gradient, and therefore no mass transfer). This absorption is given by the term

$$\frac{1}{T\rho D}\left(\frac{\partial\mu}{\partial C}\right)_{p,T}\int \boldsymbol{i}^2 dV$$

in the rate of change of entropy (59.13); we here denote the concentration by *C* to distinguish it from *c*, the velocity of sound. The diffusion flux is

$$\boldsymbol{i} = -\rho D\left[\frac{k_T}{T}\operatorname{grad} T + \frac{k_p}{p}\operatorname{grad} p\right],$$

with k_p given by (59.10). A calculation similar to that given in §79, using various relations between the derivatives of thermodynamic quantities, leads to the result that there must be added to the expression (79.6) for the absorption coefficient a term

$$\gamma_D = \frac{D\omega^2}{2c\rho^2\left(\frac{\partial\mu}{\partial C}\right)_{p,T}}\left\{\left(\frac{\partial\rho}{\partial C}\right)_{p,T} + \frac{k_T}{c_p}\left(\frac{\partial\rho}{\partial T}\right)_{p,C}\left(\frac{\partial\mu}{\partial C}\right)_{p,T}\right\}^2.$$

Problem 5. Determine the cross-section for the absorption of sound by a sphere whose radius is small compared with $\sqrt{\nu/\omega}$.

Solution. The total absorption is composed of the effects of the **viscosity** and thermal conductivity of the gas. The former is given by the work done by the Stokes frictional force when gas moving in a sound wave flows round a sphere; as in §78, Problem 3, it is assumed that the sphere is not moved by this force. The effect of **conductivity** is given by the amount of heat *q* transferred from the gas to the sphere per unit time (§78, Problem 3): the energy dissipation when an amount of heat *q* is transferred, the temperature difference between the gas (far from the sphere) and the sphere being *T'*, is *qT'/T.* The total absorption cross-section is found to be

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[6] The second root of equation (3), which is quadratic in k^2, corresponds to thermal waves which are rapidly damped with increasing *x*. In the limit $\omega\chi << c^2$ this root gives $k = \sqrt{i\omega/\chi} = (1+i)\sqrt{\omega/2\chi}$, in agreement with (52.15). In the case $\omega\chi >> c^2$ we have $k = (1+i)\sqrt{\omega c_v/2\chi c_p}$.

$$\sigma = \frac{2\pi R}{c}\left[3\nu + 2\chi\left(\frac{c_p}{c_v} - 1\right)\right].$$