§80. Acoustic streaming

One of the most interesting ways in which sound waves are affected by viscosity consists in the formation of **steady vortex flow** in a stationary sound wave when there are solid obstacles or solid boundary walls. This ***acoustic streaming*** occurs in the second approximation with respect to the wave amplitude; its characteristic feature is that the velocity in it (in the region outside a thin boundary layer) is independent of the viscosity, even though it originates from the viscosity (Rayleigh 1883).

The properties of acoustic streaming are most typically seen when the characteristic length in the problem (the size of the obstacles or of the flow region) is much less than the sound wavelength λ, but much greater than the penetration depth $\delta = \sqrt{2\nu/\omega}$ for viscous waves (§24):

$$\lambda \gg l \gg \delta . \qquad (80.1)$$

In view of the second condition, we can distinguish in the flow region a narrow ***acoustic boundary layer*** in which the velocity decreases from its value in the sound wave to zero at the solid surface. Since the velocity in this layer, as in the sound wave itself, is much less than that of sound, and the characteristic dimension δ is much less than λ according to (10.17), the flow there may be regarded as incompressible.

Let us consider the acoustic boundary layer at a plane solid wall (the *xz*-plane), assuming two-dimensional flow in the *xy*-plane (H. Schlichting 1932). The approximations resulting from the thinness of the boundary layer have been described in §39 and remain valid for the non-steady flow under consideration. The non-steadiness simply means that **Prandtl's equation** (39.5) includes time-derivative terms:

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} - \nu \frac{\partial^2 v_x}{\partial y^2} = U \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} ; \qquad (80.2)$$

the derivative dp/dx is expressed in terms of the flow velocity U (x, t) outside the boundary layer by means of (9.3). In the present case

$$U = v_0 \cos kx \cdot \cos \omega t = v_0 \cos kx \cdot \mathrm{Re}\, e^{-i\omega t} , \qquad (80.3)$$

where $k = \omega / c$; this corresponds to a plane stationary sound wave with frequency ω. The required velocity $\boldsymbol{v}$ in the boundary layer is expressed in terms of the stream function $\psi(x, y, t)$ by

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x},$$

and the continuity equation (39.6) is then satisfied automatically.

We shall solve equation (80.2) by successive approximations with respect to the small quantity v_0, the amplitude of the velocity fluctuations in the sound wave. In the first approximation, the quadratic terms are omitted altogether. The solution of the equation

$$\frac{\partial v^{(1)}{}_x}{\partial t} - \nu \frac{\partial^2 v^{(1)}{}_x}{\partial y^2} = -i\omega v_0 \cos kx \cdot e^{-i\omega t}$$

which satisfies the necessary conditions at $y = 0$ and $y = \infty$ is

$$v^{(1)}{}_x = \mathrm{Re}\left\{v_0 \cos kx \cdot e^{-i\omega x}(1-e^{-\kappa y})\right\},$$

where

$$\kappa = \sqrt{-\frac{i\omega}{\nu}} = \frac{1-i}{\delta}. \tag{80.4}$$

The corresponding stream function (satisfying the condition $\psi^{(1)} = 0$ at y = 0, which is equivalent to $v^{(1)}{}_y = 0$) is

$$\psi^{(1)} = \mathrm{Re}\left\{v_0 \cos kx \cdot \zeta^{(1)}(y) e^{-i\omega t}\right\}, \tag{80.5}$$

$$\zeta^{(1)}(y) = y + \frac{e^{-\kappa y}}{\kappa}.$$

In the next approximation, we write $\boldsymbol{v} = \boldsymbol{v}^{(1)} + \boldsymbol{v}^{(2)}$ and obtain for $\boldsymbol{v}^{(2)}$, from (80.2), the equation

$$\frac{\partial v^{(2)}{}_x}{\partial t} - \nu \frac{\partial^2 v^{(2)}{}_x}{\partial y^2} = U\frac{\partial U}{\partial x} - v^{(1)}{}_x \frac{\partial v^{(1)}{}_x}{\partial x} - v^{(1)}{}_y \frac{\partial v^{(1)}{}_x}{\partial y}. \tag{80.6}$$

The right-hand side contains terms with frequencies $\omega + \omega = 2\omega$ and $\omega - \omega = 0$. The latter give rise to time-independent terms in $\boldsymbol{v}^{(2)}$, which are the ones representing the steady flow in question; we shall take $\boldsymbol{v}^{(2)}$ to mean only this part of the velocity. The corresponding part of the stream function is written as

$$\psi^{(2)} = \frac{v_0^{\;2}}{c} \sin 2kx \cdot \zeta^{(2)}(y), \tag{80.7}$$

and we obtain for $\zeta^{(2)}(y)$ the equation

$$\delta^2 \zeta^{(2)0\,'''} = \frac{1}{2} - \frac{1}{2}\left|\zeta^{(1)\,'}\right|^2 + \frac{1}{2}\mathrm{Re}\left\{\zeta^{(1)}\zeta^{(1)\,''}\right\}, \tag{80.8}$$

the primes denoting differentiation with respect to *y*.

The solution of this equation must satisfy the conditions $\zeta^{(2)} = 0$, $\zeta^{(2)\,'}(0) = 0$, which are equivalent to $v^{(2)}{}_x = v^{(2)}{}_y = 0$ on the solid surface. The condition far from this surface can only be that $v^{(2)}{}_x$ tends to a finite value (not necessarily zero). Substitution of (80.5) in (80.8) and a twofold integration gives the following result for the derivative $\zeta^{(2)\,'}$:

$$\zeta^{(2)\,'}(y) = \frac{3}{8} - \frac{1}{8}e^{-2y/\delta} - e^{-y/\delta}\sin\frac{y}{\delta} - \frac{1}{4}e^{-y/\delta}\cos\frac{y}{\delta} + \frac{y}{4\delta}e^{-y/\delta}\left(\cos\frac{y}{\delta} - \sin\frac{y}{\delta}\right).$$

As $y \to \infty$, it tends to

$$\zeta^{(2)\,'}(\infty) = \frac{3}{8}, \tag{80.9}$$

corresponding to a velocity

$$v^{(2)}{}_x(\infty) = \frac{3v_0^{\;2}}{8c}\sin 2kx. \tag{80.10}$$

This demonstrates the effect described at the beginning of the section. We see that outside the boundary layer there is (in the second approximation with respect to v_0) a steady flow whose velocity is independent of the viscosity. Its value (80.10) serves as a

boundary condition for determining the main acoustic flow (see Problem).[1]

PROBLEM

Determine the acoustic streaming in the space between two plane-parallel walls (the planes $y = 0$ and $y = h$), where there is a stationary sound wave (80.3). The distance h between the planes, which acts as the characteristic length l, satisfies the conditions (80.1) (Rayleigh 1883).

Solution. Since the velocity $v^{(2)}$ of the required steady flow is much less than that of sound, the flow may be regarded as incompressible. Moreover, since v_0 is assumed infinitesimal in the sound wave (and therefore so is $v^{(2)} \sim v_0{}^2/c$), the quadratic terms in the equation of motion may be neglected.[2] Then equation (15.12) for the stream function reduces to

$$\Delta^2\psi^{(2)} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \psi^{(2)} = 0$$

(this arises from the viscosity term, but the viscosity itself does not appear in it). We shall seek $\psi^{(2)}$ in the form (80.7). According to the condition $h \ll \lambda$, the derivatives with respect to y are much larger than those with respect to x; neglecting the latter, we obtain for $\zeta^{(2)}(y)$ the equation

$$\zeta^{(2)\,\prime\prime\prime\prime} = 0\,. \qquad (1)$$

From the obvious symmetry of the problem, the flow is symmetrical about the plane $y = h/2$. Hence

$$v^{(2)}{}_x(x,y) = v^{(2)}{}_x(x,h-y)\,, \quad v^{(2)}{}_y(x,y) = -v^{(2)}{}_y(x,h-y)\,,$$

and therefore

$$\zeta^{(2)}(y) = -\zeta^{(2)}(h-y)\,.$$

A solution of equation (1) having this property is

$$\zeta^{(2)}(y) = A(y-h/2) + B(y-h/2)^3\,.$$

The constants A and B are determined by the boundary conditions $\zeta^{(2)}(0) = 0$, $\zeta^{(2)\prime}(0) = 3/8$. This gives for the stream function

$$\psi^{(2)} = \frac{3v_0{}^2}{16c}\sin 2kx\left[-(y-h/2) + \frac{(y-h/2)^3}{(h/2)^2}\right],$$

and thence the velocity distributions

$$v^{(2)}{}_x = -\frac{3v_0{}^2}{16c}\sin 2kx\left[1 - \frac{3(y-h/2)^2}{(h/2)^2}\right],$$

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1 The transverse velocity corresponding to the longitudinal velocity (80.10) is $v^{(2)}{}_y = -(3v_0{}^2k/4c)y\cos 2kx \ll v^{(2)}{}_x$. In solving the problem of flow outside the boundary layer, this arises automatically from the equation of continuity with the boundary condition $v^{(2)}{}_y = 0$ at $y = 0$.

2 That is, the ratio v_0/c is assumed much smaller than any of the other small parameters of the problem; in particular, $v_0/c \ll \delta/h$.

$$v^{(2)}{}_y = \frac{3v_0{}^2 k}{8c}\cos 2kx\left[1-\frac{h}{2}-\frac{(y-h/2)^3}{(h/2)^2}\right].$$

The velocity $v^{(2)}{}_x$ changes sign at a distance $\frac{1}{2}h\left(1-\frac{1}{\sqrt{3}}\right)=0.423\cdot\frac{h}{2}$ from the wall.

The flow described by these expressions consists of two series of vortices lying symmetrically about the median plane $y = h / 2$ and periodic in the x-direction, with period $\lambda/2$.

§81. Second viscosity

The second viscosity coefficient ζ (which we shall call simply the ***second viscosity***) is usually of the same order of magnitude as the viscosity coefficient η. There are, however, cases where ζ can take values considerably exceeding η. As we know, the second viscosity appears in processes which are accompanied by a **change in volume** (i.e., in density) of the fluid. In compression or expansion, as in any rapid change of state, the fluid ceases to be in thermodynamic equilibrium, and internal processes are set up in it which tend to restore this equilibrium. These processes are usually so rapid (i.e., their relaxation time is so short) that the restoration of equilibrium follows the change in volume almost immediately unless, of course, the rate of change of volume is very large.

It may happen, nevertheless, that the relaxation times of the processes of restoration of equilibrium are long, i.e., they take place comparatively slowly. For instance, if we are concerned with a liquid or gas which is a mixture of substances between which a chemical reaction occurs, there is a state of chemical equilibrium, characterized by the concentrations of the substances in the mixture, for any given density and temperature. If, for example, we compress the fluid, the state of equilibrium is destroyed, and a reaction begins, as a result of which the concentrations of the substances tend to take the equilibrium values corresponding to the new density and temperature. If this reaction is not rapid, the restoration of equilibrium occurs relatively slowly and does not immediately follow the compression. The latter process is then accompanied by internal processes which tend towards the equilibrium state. But the processes which establish equilibrium are irreversible; they increase the entropy, and therefore involve energy dissipation. Hence, if the relaxation time of these processes is long, a considerable dissipation of energy occurs when the fluid is compressed or expanded, and, since this dissipation must be determined by the second viscosity, we reach the conclusion that ζ is large.[3]

The intensity of the dissipative processes, and therefore the value of ζ, depend, of course, on the relation between the rate of compression or expansion and the relaxation time. If, for example, we have compression or expansion due to a sound wave, the second viscosity will depend on the frequency of the wave. Thus the second viscosity is not just a

[3] A slow process which results in a large ζ is often the transfer of energy from translatory degrees of freedom of a molecule to vibrational (intramolecular) degrees of freedom.

constant characteristic of the material concerned, but depends on the frequency of the motion in which it appears. The dependence of ζ on the frequency is called its ***dispersion***.

The following general method of discussing all these phenomena is due to L. I. Mandel'shtam and M. A. Leontovich (1937). Let ξ be some physical quantity characterizing the state of a body, and ξ_0 its value in the equilibrium state; ξ_0 is a function of density and temperature. For instance, in fluid mixtures ξ may be the concentration of one component, and then ξ_0 is the concentration in chemical equilibrium.

If the body is not in equilibrium, ξ will vary with time, tending to the value ξ_0. In states close to equilibrium the difference $\xi-\xi_0$ is small, and we can expand the rate of change $\dot{\xi}$ of ξ in a series of powers of this difference. The zero-order term is absent, since $\dot{\xi}$ must be zero in the equilibrium state, i.e. when $\xi=\xi_0$. Hence, as far as the first-order term, we have

$$\dot{\xi}=-\frac{\xi-\xi_0}{\tau}. \tag{81.1}$$

The proportionality coefficient must be negative, since otherwise ξ would not tend to a finite limit. The positive constant τ is of the dimensions of time, and may be regarded as the **relaxation time** for the process in question; the greater is τ, the more slowly the approach to equilibrium takes place.

In what follows we shall consider processes in which the fluid is subjected to a periodic adiabatic[4] compression and expansion, so that the variable part of the density (and of the other thermodynamic quantities) depends on the time through a factor $e^{-i\omega t}$; we are considering a sound wave in the fluid. Together with the density and other quantities, the position of equilibrium also varies, so that ξ_0 can be written as $\xi_0=\xi_{00}+\xi_0'$, where ξ_{00} is the constant value of ξ_0 corresponding to the mean density, and ξ_0' is a periodic part, proportional to $e^{-i\omega t}$. Writing the true value ξ in the form $\xi=\xi_{00}+\xi'$, we conclude from equation (81.1) that ξ' also is a periodic function of time, related to ξ_0' by

$$\xi'=\frac{\xi_0'}{1-i\omega\tau}. \tag{81.2}$$

Let us calculate the derivative of the pressure with respect to the density for the process in question. The pressure must now be regarded as a function of the density and of the value of ξ in the state concerned, and also of the entropy, which we suppose constant and, for brevity, omit. Then

$$\frac{\partial p}{\partial \rho}=\left(\frac{\partial p}{\partial \rho}\right)_{\xi}+\left(\frac{\partial p}{\partial \xi}\right)_{\rho}\frac{\partial \xi}{\partial \rho}.$$

In accordance with (81.2), we substitute here

[4] The change in the entropy (in states close to equilibrium) is of the second order of smallness. Hence, to the first order of accuracy, we can speak of an adiabatic process.

$$\frac{\partial \xi}{\partial \rho} = \frac{\partial \xi'}{\partial \rho} = \frac{1}{1-i\omega\tau}\frac{\partial \xi_0'}{\partial \rho} = \frac{1}{1-i\omega\tau}\frac{\partial \xi_0}{\partial \rho},$$

obtaining

$$\frac{\partial p}{\partial \rho} = \frac{1}{1-i\omega\tau}\left\{\left(\frac{\partial p}{\partial \rho}\right)_\xi + \left(\frac{\partial p}{\partial \xi}\right)_\rho \frac{\partial \xi_0}{\partial \rho} - i\omega\tau\left(\frac{\partial p}{\partial \rho}\right)_\xi\right\}.$$

The sum $\left(\frac{\partial p}{\partial \rho}\right)_\xi + \left(\frac{\partial p}{\partial \xi}\right)_\rho \frac{\partial \xi_0}{\partial \rho}$ is just the derivative of p with respect to ρ for a process which is so slow that the fluid remains in equilibrium; denoting it by $\left(\frac{\partial p}{\partial \rho}\right)_{eq}$, we have finally

$$\frac{\partial p}{\partial \rho} = \frac{1}{1-i\omega\tau}\left\{\left(\frac{\partial p}{\partial \rho}\right)_{eq} - i\omega\tau\left(\frac{\partial p}{\partial \rho}\right)_\xi\right\}. \tag{81.3}$$

Next, let p_0 be the pressure in a state of thermodynamic equilibrium; p_0 is related to the other thermodynamic quantities by the equation of state of the fluid, and is entirely determined when the density and entropy are given. The pressure p in a non-equilibrium state, however, differs from p_0, and is a function of ξ also. If the density is adiabatically increased by δp, the equilibrium pressure changes by $\delta p_0 = \left(\frac{\partial p}{\partial \rho}\right)_{eq}\delta\rho$, while the total increase in the pressure is $\left(\frac{\partial p}{\partial \rho}\right)\delta\rho$, with $\frac{\partial p}{\partial \rho}$ given by formula (81.3). Hence the difference $p - p_0$ between the true pressure and the equilibrium pressure, in a state where the density is $\rho + \delta\rho$, is

$$p - p_0 = \left[\frac{\partial p}{\partial \rho} - \left(\frac{\partial p}{\partial \rho}\right)_{eq}\right]\delta\rho = \frac{i\omega\tau}{1-i\omega\tau}\left[\left(\frac{\partial p}{\partial \rho}\right)_{eq} - \left(\frac{\partial p}{\partial \rho}\right)_\xi\right]\delta\rho .$$

We are here interested in the density changes due to the motion of the fluid. Then $\delta\rho$ is related to the velocity by the equation of continuity, which we write in the form $\frac{d(\delta\rho)}{dt} + \rho \operatorname{div} \boldsymbol{v} = 0$, where d/dt denotes the total time derivative. In a periodic motion we have $\frac{d(\delta\rho)}{dt} = -i\omega\delta\rho$, and therefore $\delta\rho = \frac{\rho}{i\omega}\operatorname{div}\boldsymbol{v}$. Substituting this expression in $p - p_0$, we obtain

$$p - p_0 = \frac{\tau\rho}{1-i\omega\tau}({c_0}^2 - {c_\infty}^2)\operatorname{div}\boldsymbol{v}, \tag{81.4}$$

where we have used the notation

$$c_0{}^2 = \left(\frac{\partial \rho}{\partial p}\right)_{eq}, \quad c_\infty{}^2 = \left(\frac{\partial \rho}{\partial p}\right)_{\xi}, \tag{81.5}$$

the significance of which will be explained below.

In order to relate these expressions to the viscosity of the fluid, we write down the stress tensor σ_{ik}. In this tensor the pressure appears in the term $-p\delta_{ik}$. Subtracting the pressure p_0 determined by the equation of state, we find that in a non-equilibrium state σ_{ik} contains an additional term

$$-(p-p_0)\delta_{ik} = \frac{\tau\rho}{1-i\omega\tau}(c_\infty{}^2 - c_0{}^2)\delta_{ik}\,\mathrm{div}\,\boldsymbol{v}.$$

Comparing this with the general expression (15.2) and (15.3) for the stress tensor, in which $\mathrm{div}\,\boldsymbol{v}$ appears in the term $\zeta\,\mathrm{div}\,\boldsymbol{v}$, we conclude that the presence of slow processes tending to establish equilibrium is macroscopically equivalent to the presence of a second viscosity given by

$$\zeta = \frac{\tau\rho(c_\infty{}^2 - c_0{}^2)}{1-i\omega\tau}. \tag{81.6}$$

These processes do not affect the ordinary viscosity η. For processes so slow that $\omega\tau << 1$, ζ is

$$\zeta_0 = \tau\rho(c_\infty{}^2 - c_0{}^2); \tag{81.7}$$

it increases with the relaxation time τ, in accordance with what was said above. For large frequencies, ζ depends on the frequency, i.e., it exhibits **dispersion**.

Let us now consider the question of how the presence of processes with long relaxation times (we shall speak of chemical reactions) affects the propagation of sound in a fluid. To do so, we might start from the equation of motion of a viscous fluid, with ζ given by formula (81.6). It is simpler, however, to consider a motion in which viscosity is neglected but the pressure p is given by the above formulae instead of by the equation of state. The general relations which we obtained in §64 then remain formally applicable. In particular, the wave number and the frequency are still related by $k = \omega / c$, where $c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)}$, and the derivative $\frac{\partial p}{\partial \rho}$ is now given by (81.3); the quantity *c,* however, no longer denotes the velocity of sound, being complex. Thus we obtain

$$k = \omega\sqrt{\frac{1-i\omega\tau}{c_0{}^2 - c_\infty{}^2 i\omega\tau}}. \tag{81.8}$$

The "wave number" given by this formula is complex. The meaning of this fact is easily seen. In a plane wave, all quantities depend on the coordinate x (the x-axis being in the direction of propagation) through a factor e^{ikx}. Writing k in the form $k = k_1 + ik_2$ with k_1, k_2 real, we have $e^{ikx} = e^{ik_1x}e^{-k_2x}$, i.e., besides the periodic factor e^{ik_1x} we have a

damping factor $e^{-k_2 x}$ (k_2 must, of course, be positive). Thus the complex nature of the wave number formally expresses the fact that the wave is damped, i.e., there is absorption of sound. The real part of the complex wave number gives the variation in phase of the wave with distance, and the imaginary part is the absorption coefficient.

It is not difficult to separate the real and imaginary parts of (81.8). In the general case of arbitrary ω the expressions for k_1 and k_2 are rather cumbersome, and we shall not write them out here. It is important that k_1 is a function of the frequency (as is k_2). Thus, if chemical reactions can occur in the fluid, the propagation of sound at sufficiently high frequencies is accompanied by dispersion.

In the limiting case of **low frequencies** ($\omega\tau \ll 1$), formula (81.8) gives to a first approximation $k = \omega / c_0$, corresponding to the propagation of sound with velocity c_0. This is as it should be, of course; the condition $\omega\tau \ll 1$ means that the period $1/\omega$ of the sound wave is large compared with the relaxation time, i.e., the establishment of chemical equilibrium follows the variations of density in the sound wave, and so the velocity of sound is determined by the equilibrium value of the derivative $\dfrac{\partial p}{\partial \rho}$. In the second approximation we have

$$k = \frac{\omega}{c_0} + \frac{i\omega^2\tau}{2c_0^{\,3}}(c_\infty^{\,2} - c_0^{\,2}), \tag{81.9}$$

i.e., damping occurs, with a coefficient proportional to the square of the frequency. Using (81.7), we can write the imaginary part of k in the form $k_2 = \dfrac{\omega^2\zeta_0}{2\rho c_0^{\,3}}$; this agrees with the ζ-dependent part of the absorption coefficient γ as given by (79.6), which was obtained without taking account of the dispersion.

In the opposite limiting case of **high frequencies** ($\omega\tau \gg 1$), we have in the first approximation $k = \omega / c_\infty$, i.e., the propagation of sound with velocity c_∞ - again a natural result, since for $\omega\tau \gg 1$ we can suppose that no reaction occurs during a single period, and the velocity of sound must therefore be determined by the derivative $\left(\dfrac{\partial p}{\partial \rho}\right)_\xi$ taken at constant concentration. The second approximation gives

$$k = \frac{\omega}{c_\infty} + \frac{i}{2\tau c_\infty^{\,3}}(c_\infty^{\,2} - c_0^{\,2}). \tag{81.10}$$

The damping coefficient is independent of the frequency. As we go from $\omega \ll 1/\tau$ to $\omega \gg 1/\tau$, this coefficient increases monotonically to the constant value given by formula (81.10). It should be noted that the quantity k_2 / k_1, which represents the amount of absorption over a distance of one wavelength, is small in both limiting cases ($k_2 / k_1 \ll 1$); it has a maximum at some intermediate frequency, namely $\omega = \dfrac{\sqrt{c_0 / c_\infty}}{\tau}$.

It is seen from (81.7) (e.g.) that

$$c_\infty > c_0\,, \tag{81.11}$$

since we must have $\zeta > 0$. The same result can be obtained by simple arguments based on **Le Chatelier's principle**. Let us suppose that the volume of the system is reduced, and the density increased, by some external agency. The system is thereby brought out of equilibrium, and according to Le Chatelier's principle processes must begin which tend to reduce the pressure. This means that $\dfrac{\partial p}{\partial \rho}$ will decrease, and, when the system returns to equilibrium, the value of $\dfrac{\partial p}{\partial \rho} = c^2$ will be less than in the non-equilibrium state.

In deriving all the above formulae we have assumed that there is only a single slow internal process of relaxation. Cases are also possible where several different such processes occur simultaneously. All the formulae can easily be generalized to cover such cases. Instead of a single quantity ξ, we now have several quantities $\xi_1, \xi_2, \ldots$ which characterize the state of the system, and a corresponding series of relaxation times $\tau_1, \tau_2, \ldots$. We choose the quantities ξ_n in such a way that each of the derivatives $\dot{\xi}_n$ depends only on the corresponding ξ_n, i.e., so that

$$\dot{\xi}_n = -\frac{\xi_n - \xi_{n0}}{\tau_n}\,. \tag{81.12}$$

Calculations entirely similar to the above then give

$$c^2 = c_\infty{}^2 + \sum_n \frac{a_n}{1 - i\omega\tau_n}\,, \tag{81.13}$$

where $c_\infty{}^2 = \left(\dfrac{\partial p}{\partial \rho}\right)_\xi$, and the constants a_n are

$$a_n = \left(\frac{\partial p}{\partial \xi_n}\right)\left(\frac{\partial \xi_n}{\partial \rho}\right)_{eq}\,. \tag{81.14}$$

If there is only one quantity ξ, formula (81.13) becomes (81.3), as it should.

(End of Chapter 8.)