CHAPTER IX **SHOCK WAVES**

§82. Propagation of disturbances in a moving gas

When the velocity of a fluid in motion becomes comparable with or exceeds that of sound, effects due to the compressibility of the fluid become of prime importance. Such motions are in practice met with in gases. The dynamics of high-speed flow is therefore usually called ***gas dynamics***.

It should be mentioned first of all that, in gas dynamics, the Reynolds numbers involved are almost always very large. For the kinematic viscosity of a gas is, as we know from the kinetic theory of gases, of the order of the mean free path l of the molecules multiplied by the mean velocity of their thermal motion; the latter is of the same order as the velocity of sound, so that $\nu \sim cl$. If the characteristic velocity in a problem of gas dynamics is also of the order of c, then the Reynolds number $R \sim Lu/\nu \sim Lu/lc$, i.e., it is determined by the ratio of the dimension L to the mean free path l, which we know is very large.[1] As always occurs when R is very large, the viscosity has an important effect on the motion of the gas only in a very small region, and in what follows we shall (except where the contrary is specifically stated) regard the gas as an **ideal fluid**.

The flow of a gas is entirely different in nature according as it is ***subsonic*** or ***supersonic***, i.e., the velocity is less than or greater than that of sound. One of the most important distinctive features of supersonic flow is the fact that there can occur in it what are called ***shock waves***, whose properties we shall examine in detail in the following sections. Here we shall consider another characteristic property of supersonic flow, relating to the manner of propagation of small disturbances in the gas.

If a gas in steady motion receives a slight perturbation at any point, the effect of the perturbation is subsequently propagated through the gas with the velocity of sound (relative to the gas itself). The rate of propagation of the disturbance relative to a fixed system of coordinates is composed of two parts: firstly, the perturbation is "carried along" by the gas flow with velocity $\boldsymbol{v}$ and, secondly, it is propagated relative to the gas with velocity c in any direction $\boldsymbol{n}$. Let us consider, for simplicity, a uniform flow of gas with constant velocity $\boldsymbol{v}$, subjected to a small perturbation at some point O (fixed in space). The velocity $\boldsymbol{v} + c\boldsymbol{n}$ with which the perturbation is propagated from O (relative to the fixed system of coordinates) has different values for different directions of the unit vector $\boldsymbol{n}$. We obtain all its possible values by placing one end of the vector $\boldsymbol{v}$ at the point O and drawing a sphere with radius c centred at the other end. The vectors from O to points on this sphere give the possible magnitudes and directions of the velocity of propagation of the perturbation. Let us first suppose that $v < c$. Then the vector $\boldsymbol{v} + c\boldsymbol{n}$ can have any direction in space (Fig. 50a). That is, a disturbance which starts from any point in a subsonic flow will eventually reach every point in the gas.

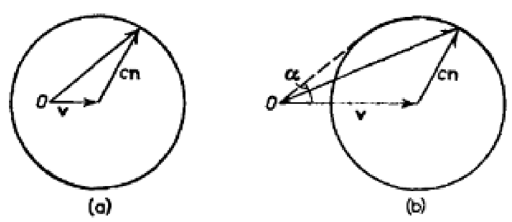


FIG. 50

[1] We shall not consider the problem of the motion of bodies in very rarefied gases, where the mean free path of the molecules is comparable with the dimension of the body. The problem is in essence not one of fluid dynamics, and must be examined by means of the kinetic theory of gases.

If, on the other hand, $v > c$, the direction of the vector $\mathbf{v} + c\mathbf{n}$ can lie, as we see from Fig. 50b, only in a cone with its vertex at O, which touches the sphere with its centre at the other end of the vector $\mathbf{v}$. If the aperture angle of the cone is 2α, then, as is seen from the figure,

$$\sin\alpha = \frac{c}{v}. \qquad (82.1)$$

Thus a disturbance starting from any point in a supersonic flow is propagated only downstream within a cone whose aperture angle decreases with the ratio c/v. A disturbance starting from O does not affect the flow outside this cone.

The angle α determined by equation (82.1) is called the ***Mach angle***. The ratio v/c itself, which often occurs in gas dynamics, is the ***Mach number*** M:

$$M = \frac{v}{c}. \qquad (82.2)$$

The surface bounding the region reached by a disturbance starting from a given point is called the ***Mach surface*** or ***characteristic surface***.

In the general case of an arbitrary steady flow, the Mach surface is not a cone throughout the volume. However, it can be asserted that, as before, this surface cuts the streamline through any point on it at the Mach angle. The value of the Mach angle varies from point to point with the velocities v and c. It should be emphasized here, incidentally, that, in flow with high velocities, the velocity of sound is different at different points: it varies with the thermodynamic quantities (pressure, density, etc.) of which it is a function.[2] The velocity of sound as a function of the coordinates is called the ***local velocity of sound***.

The properties of supersonic flow described above give it a character quite different from that of subsonic flow. If a subsonic gas flow meets any obstacle (if, for instance, it flows past a body), the presence of this obstacle affects the flow in all space, both upstream and downstream; the effect of the obstacle is zero only asymptotically at an infinite distance from it. A supersonic flow, however, is incident "blindly" on an obstacle; the effect of the latter extends only downstream,[3] and in all the remaining part of space upstream the gas flows as if the obstacle were absent.

In the case of steady two-dimensional flow of a gas, the **characteristic surfaces** can be replaced by ***characteristic lines*** (or simply ***characteristics***) in the plane of the flow. Through any point O in this plane there pass two characteristics (AA' and BB' in Fig. 51), which intersect the streamline through this point at the Mach angle. The downstream branches OA and OB of the characteristics may be said to ***leave*** the point O; they bound the region AOB of the flow where perturbations starting from O can take effect. The

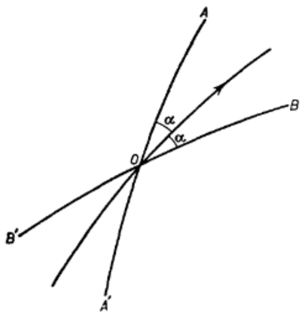


Fig. 51

[2] In the discussion of sound waves given in Chapter VIII, the velocity of sound could be regarded as constant.

[3] To avoid misunderstanding, we should mention that, if a shock wave is formed in front of the obstacle, this region is somewhat enlarged (see §122).

branches $B'O$ and $A'O$ may be said to ***reach*** the point O; the region $A'OB'$ between them is that which can affect the flow at O.

The concept of characteristics (surfaces in the three-dimensional case) has also a somewhat different aspect. They are rays along which disturbances are propagated which satisfy the conditions of geometrical acoustics. If, for example, a steady supersonic gas flow meets a fairly small obstacle, then a steady perturbation of the gas flow will be found along the characteristics which leave this obstacle. The same result was reached in §68 from a study of the geometrical acoustics of moving media.

When we speak of a perturbation of the state of the gas, we mean a slight change in any of the quantities characterizing its state: the velocity, pressure, density, etc. The following remark should be made on this point. *Perturbations in the values of the entropy of the gas* (for constant pressure) *and of its vorticity are not propagated with the velocity of sound*. These perturbations, once having arisen, do not move relative to the gas; relative to a fixed system of coordinates they move with the gas at the velocity appropriate to each point. For the entropy, this is an immediate consequence of the law of conservation (in an ideal fluid), which states that the entropy of any given volume element in the gas remains constant as the element moves about. The same result for the vorticity follows from the conservation of circulation.

Thus we can say that, *for perturbations of entropy and vorticity*, *the characteristics are the streamlines*. This, of course, does not affect the general validity of the statements made above about regions of influence, since they were based only on the existence of a maximum velocity of propagation (that of sound) of disturbances relative to the gas itself.

PROBLEM

Find the relations between small changes in the velocity and in the thermodynamic quantities when there is an arbitrary small perturbation in a uniform gas flow.

Solution. We denote the small changes in quantities during the perturbation by a prefixed δ (instead of a prime as in §64). In the approximation linear in these quantities, Euler's equation becomes

$$\frac{\partial \delta \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\mathrm{grad})\delta\boldsymbol{v} + \frac{1}{\rho}\mathrm{grad}\,\delta p = 0, \qquad (1)$$

where $\boldsymbol{v}$ is the constant unperturbed velocity; the equation of conservation of entropy is

$$\frac{\partial \delta s}{\partial t} + \boldsymbol{v}\cdot\mathrm{grad}\,\delta s = 0; \qquad (2)$$

the equation of continuity is

$$\frac{\partial \delta p}{\partial t} + \boldsymbol{v}\cdot\mathrm{grad}\,\delta p + \rho c^2\,\mathrm{div}\,\delta\boldsymbol{v} = 0. \qquad (3)$$

Here we have substituted $\delta\rho = \dfrac{\delta p}{c^2} + \left(\dfrac{\partial \rho}{\partial s}\right)_p \delta s$; the terms in δs disappear in accordance with (2). For a perturbation having the form $e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$ we get the algebraic equations

$$\begin{cases} (\boldsymbol{v}\cdot\boldsymbol{k}-\omega)\delta s = 0 \\ (\boldsymbol{v}\cdot\boldsymbol{k}-\omega)\delta\boldsymbol{v} + \boldsymbol{k}\dfrac{\delta p}{\rho} = 0 \\ (\boldsymbol{v}\cdot\boldsymbol{k}-\omega)\delta p + \rho c^2\boldsymbol{k}\cdot\delta\boldsymbol{v} = 0 \end{cases}.$$

These show that there are two possible types of perturbation.

In one type (**entropy-vortex wave**),

$$\omega = \boldsymbol{v}\cdot\boldsymbol{k}\,,\ \ \delta s \neq 0\,,\ \ \delta p = 0\,,\ \ \delta\rho = \left(\frac{\partial\rho}{\partial s}\right)_p \delta s\,,\ \ \boldsymbol{k}\cdot\delta\boldsymbol{v} = 0\,;$$

the vorticity $\operatorname{curl}\delta\boldsymbol{v} = i\boldsymbol{k}\times\delta\boldsymbol{v}$ is also not zero. The perturbations δs and $\delta\boldsymbol{v}$ in this wave are independent. The equation $\omega = \boldsymbol{v}\cdot\boldsymbol{k}$ signifies that the perturbation is carried along by the gas flow.

In the other type,

$$(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 = c^2k^2\,,\ \ \delta s = 0\,,\ \ \delta p = c^2\delta\rho\,,$$

$$(\omega - \boldsymbol{v}\cdot\boldsymbol{k})\delta p = \rho c^2\boldsymbol{k}\cdot\delta\boldsymbol{v}\,,\ \ \boldsymbol{k}\times\delta\boldsymbol{v} = 0$$

This is a **sound wave** whose frequency is shifted by the Doppler effect. When the perturbation of one quantity in this wave is specified, those of all others are determined.

§83. Steady flow of a gas

We can obtain immediately from **Bernoulli's equation** a number of general results concerning adiabatic steady flow of a gas. The equation is, for steady flow, $w + \frac{1}{2}v^2 = \text{constant}$ along each streamline; if we have potential flow, then the constant is the same for every streamline, i.e., at every point in the fluid. If there is a point on some streamline at which the gas velocity is zero, then we can write Bernoulli's equation as

$$w + \frac{1}{2}v^2 = w_0\,, \qquad (83.1)$$

where w_0 is the value of the ***heat function*** at the point where $v = 0$.

The **equation of conservation of entropy** for steady flow is $\boldsymbol{v}\cdot\operatorname{grad}s = v\frac{\partial s}{\partial l} = 0$, i.e., s is constant along each streamline. We can write this in a form analogous to (83.1):

$$s = s_0\,. \qquad (83.2)$$

We see from equation (83.1) that the velocity v is greater at points where the heat function w is smaller. The maximum value of the velocity (on the streamline considered) is found at the point where w is least. For constant entropy, however, we have $dw = \frac{dp}{\rho}$; since $\rho > 0$, the differentials dw and dp have like signs, and so w and p vary in the same sense. We can therefore say that the *velocity increases along a streamline when the pressure decreases*, and vice versa.

The smallest possible values of the pressure and the heat function (in adiabatic flow) are obtained when the absolute temperature $T = 0$. The corresponding pressure is $p = 0$, and the value of w for $T = 0$ can be arbitrarily taken as the zero of energy; then $w = 0$ for $T = 0$. We can now deduce from (83.1) that the greatest possible value of the velocity (for given values of the thermodynamic quantities at the point where $v = 0$) is

$$v_{\max} = \sqrt{2w_0}\,. \qquad (83.3)$$

This velocity can be attained when a gas flows steadily out into a vacuum.[4]

Let us now consider how the **mass flux density** $j = \rho v$ varies along a streamline. From Euler's equation $(\boldsymbol{v}\cdot\operatorname{grad})\boldsymbol{v} = -\frac{1}{\rho}\operatorname{grad}p$, we find that the relation $v\,dv = -dp/\rho$ between the differentials dv and dp holds along a streamline. Putting $dp = c^2d\rho$, we have

[4] In reality, of course, when there is a sharp fall in temperature the gas must condense and form a two-phase "fog".

$$\frac{d\rho}{dv} = -\frac{\rho v}{c^2} \qquad (83.4)$$

and, substituting in $d(\rho v) = \rho dv + v d\rho$, we obtain

$$\frac{d(\rho v)}{dv} = \rho\left(1 - \frac{v^2}{c^2}\right). \qquad (83.5)$$

From this we see that, as the velocity increases along a streamline, the mass flux density increases as long as the flow remains subsonic. In the supersonic range, however, the mass flux density diminishes with increasing velocity, and vanishes together with ρ when $v = v_{\max}$ (Fig. 52). This important

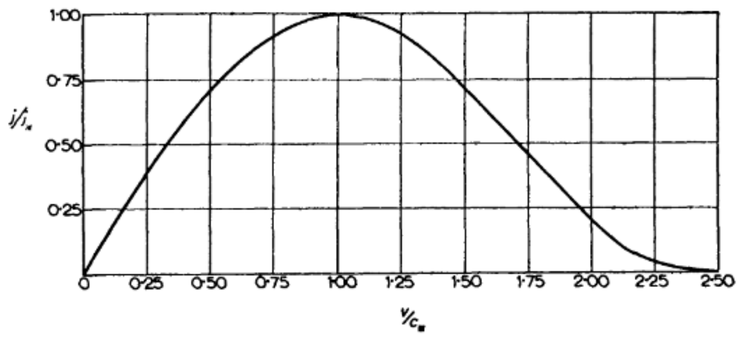


FIG. 52

$$j = \rho v$$
$$= \rho_0 c_0 \frac{\rho}{\rho_0}\frac{c_*}{c_0}\frac{v}{c_*}$$
$$= \rho_0 c_0 \left[1 - \frac{\gamma-1}{\gamma+1}\frac{v^2}{c_*^2}\right]^{1/(\gamma-1)} \left[\frac{2}{\gamma+1}\right]^{1/2} \frac{v}{c_*}$$
$$j_* = \rho_* c_*$$
$$= \rho_0 c_0 \frac{\rho_*}{\rho_0}\frac{c_*}{c_0}$$
$$= \rho_0 c_0 \left[\frac{2}{\gamma+1}\right]^{1/(\gamma-1)} \left[\frac{2}{\gamma+1}\right]^{1/2}$$
$$j / j_* = \left[1 - \frac{\gamma-1}{\gamma+1}\frac{v^2}{c_*^2}\right]^{1/(\gamma-1)} \left[\frac{2}{\gamma+1}\right]^{-1/(\gamma-1)} \frac{v}{c_*}$$

difference between subsonic and supersonic steady flows can be simply interpreted as follows. In a subsonic flow, the streamlines approach in the direction of increasing velocity. In a supersonic flow, however, they diverge in that direction.

The flux j has its maximum value j_* at the point where the gas velocity is equal to the local velocity of sound:

$$j_* = \rho_* c_*, \qquad (83.6)$$

where the asterisk suffix indicates values corresponding to this point. The velocity $v_* = c_*$ is called the ***critical velocity***. In the general case of an arbitrary gas, the critical values of quantities can be expressed in terms of their values at the point $v = 0$, by solving the simultaneous equations

$$\begin{cases} s_* = s_0 \\ w_* + \frac{1}{2}c_*^2 = w_0 \end{cases}. \qquad (83.7)$$

It is evident that, whenever $M = v / c < 1$, we have also $v / c_* < 1$, and if $M > 1$ then $v / c_* > 1$. Hence the ratio $M_* = v / c_*$ serves in this case as a criterion analogous to M, and is more convenient, since c_* is a constant, unlike c, which varies along the stream.

In applications of the general equations of gas dynamics, the case of a perfect gas is of particular importance. For a **perfect gas** we shall always assume (except where otherwise specified) that the *specific heat is a constant* independent of temperature in the range considered. Such a gas is often called a ***polytropic*** gas, and we shall use this term in order to emphasize that the assumption made goes much further than that of a perfect gas. The relations between the thermodynamic quantities for a polytropic gas are given by very simple formulae, and this often allows a complete solution of the equations of gas dynamics. We shall give here, for reference, the formulae in question, since they will be needed several times in what follows.

The **equation of state** for a perfect gas is

$$pV = \frac{p}{\rho} = \frac{RT}{\mu}, \qquad (83.8)$$

where R = 8.314×10^7 erg/deg · mol is the gas constant, and μ the molecular weight of the gas. The **velocity of sound** in a perfect gas is, as shown in §64, given by

$$c^2 = \frac{\gamma RT}{\mu} = \frac{\gamma p}{\rho}, \qquad (83.9)$$

where $\gamma = c_p / c_v$ is the **ratio of specific heats**, which always exceeds unity; for a polytropic gas it is constant. For monatomic gases γ = 5/3, and for diatomic gases γ = 7/5, at ordinary temperatures.[5]

The **internal energy** of a polytropic gas is, apart from an unimportant additive constant,

$$\varepsilon = c_v T = \frac{pV}{\gamma-1} = \frac{c^2}{\gamma(\gamma-1)}. \qquad (83.10)$$

For the **heat function** we have the analogous formulae

$$w = c_p T = \frac{\gamma pV}{\gamma-1} = \frac{c^2}{\gamma-1}. \qquad (83.11)$$

Here we have used the well-known relation $c_p - c_v = R/\mu$. Finally, the **entropy** of the gas is

$$s = c_v \log\left(\frac{p}{\rho^\gamma}\right) = c_p \log\left(\frac{p^{1/\gamma}}{\rho}\right). \qquad (83.12)$$

Let us now investigate **steady flow**, applying the general relations previously obtained to the case of a polytropic gas. Substituting (83.11) in (83.3), we find that the maximum velocity of steady flow is

$$v_{\max} = c_0\sqrt{\frac{2}{\gamma-1}}. \qquad (83.13)$$

For the critical velocity we obtain from the second equation (83.7)

$$\frac{c_*^{\,2}}{\gamma-1} + \frac{1}{2}c_*^{\,2} = w_0 = \frac{c_0^{\,2}}{\gamma-1},$$

whence[6]

$$c_* = c_0\sqrt{\frac{2}{\gamma+1}}. \qquad (83.14)$$

Bernoulli's equation (83.1), after substitution of the expression (83.11) for the heat function, gives the relation between the temperature and the velocity at any point on the streamline; similar relations for the pressure and density can then be obtained directly by means of the **Poisson adiabatic equation**:

$$\rho = \rho_0\left(\frac{T}{T_0}\right)^{1/(\gamma-1)}, \quad p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}. \qquad (83.15)$$

Thus we obtain the important results

[5] The name "polytropic" is derived from "polytropic process", i.e., one in which the pressure varies inversely as some power of the volume. For a gas with constant specific heats, such a process may be either isothermal, or adiabatic with $pV^\gamma = \text{constant}$ (Poisson adiabatic). The specific-heat ratio γ is called the *adiabatic index*.

[6] Figure 52 shows the ratio j/j_* as a function of v/c_* for air ($\gamma = 1.4$, $v_{\max} = 2.45c_*$).

$$\left.\begin{aligned} T &= T_0\left[1-\frac{\gamma-1}{2}\frac{v^2}{{c_0}^2}\right] = T_0\left(1-\frac{\gamma-1}{\gamma+1}\frac{v^2}{{c_*}^2}\right) \\ \rho &= \rho_0\left[1-\frac{\gamma-1}{2}\frac{v^2}{{c_0}^2}\right]^{1/(\gamma-1)} = \rho_0\left(1-\frac{\gamma-1}{\gamma+1}\frac{v^2}{{c_*}^2}\right)^{1/(\gamma-1)} \\ p &= p_0\left[1-\frac{\gamma-1}{2}\frac{v^2}{{c_0}^2}\right]^{\gamma/(\gamma-1)} = p_0\left(1-\frac{\gamma-1}{\gamma+1}\frac{v^2}{{c_*}^2}\right)^{\gamma/(\gamma-1)} \end{aligned}\right\}. \tag{83.16}$$

It is sometimes convenient to use these relations in a form which gives the velocity in terms of other quantities:

$$v^2 = \frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1-\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right] = \frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1-\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\right]. \tag{83.17}$$

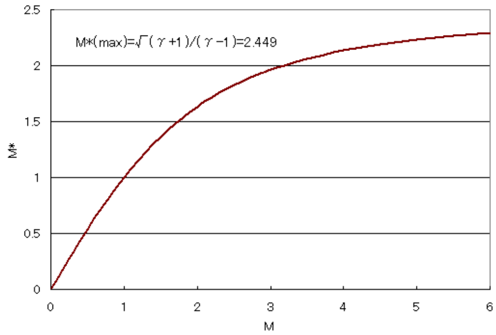
We may also give the relation between the velocity of sound and the velocity v:

$$c^2 = {c_0}^2 - \frac{\gamma-1}{2}v^2 = \frac{\gamma+1}{2}{c_*}^2 - \frac{\gamma-1}{2}v^2. \tag{83.18}$$

Hence we find that the numbers M and M_* are related by

$$ {M_*}^2 = \frac{\gamma+1}{\gamma-1+2/M^2}; \tag{83.19}$$

when M varies from 0 to ∞, ${M_*}^2$ varies from 0 to $(\gamma+1)/(\gamma-1)$.



Finally, we may give expressions for the critical temperature, pressure and density: they are obtained by putting $v = c_*$ in formulae (83.16):[7]

$$\left.\begin{aligned} T_* &= \frac{2T_0}{\gamma+1} \\ p_* &= p_0\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \\ \rho_* &= \rho_0\left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \end{aligned}\right\}, \tag{83.20}$$

In conclusion, it should be emphasized that the results derived above are **valid only for flow in which shock waves do not occur**. When shock waves are present, equation (83.2) does not hold; the entropy of the gas increases when a streamline passes through a shock wave. We shall see, however, that Bernoulli's equation (83.1) remains valid even when there are shock waves, since $w + v^2/2$ is a quantity which is conserved across a surface of discontinuity (§85); formula (83.14), for example, therefore remains valid also.

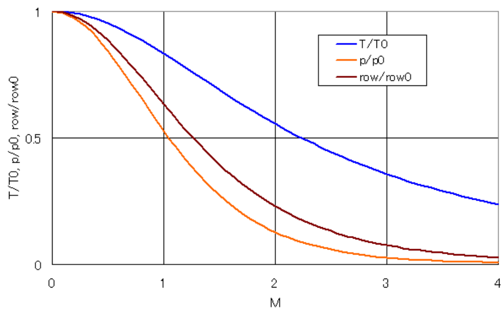
PROBLEM

Express the temperature, pressure and density along a streamline in terms of the Mach number.

Solution. Using the formulae obtained above, we find

[7] For air, e.g., ($\gamma = 1.4$), $c_* = \left(\frac{2}{\gamma+1}\right)^{1/2} c_0 = 0.913 c_0$, $p_* = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} p_0 = 0.5283 p_0$, $\rho_* = \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \rho_0 = 0.6339 \rho_0$, $T_* = \left(\frac{2}{\gamma+1}\right) T_0 = 0.8333 T_0$.

$$\begin{cases} \dfrac{T_0}{T} = 1 + \dfrac{\gamma - 1}{2} M^2 \\ \dfrac{p_0}{p} = \left[1 + \dfrac{\gamma - 1}{2} M^2\right]^{\gamma/(\gamma-1)} \\ \dfrac{\rho_0}{\rho} = \left[1 + \dfrac{\gamma - 1}{2} M^2\right]^{1/(\gamma-1)} \end{cases}.$$



Temperature, pressure and density along streamline.

§84. Surfaces of discontinuity

In the preceding chapters we have considered only flows such that all quantities (velocity, pressure, density, etc.) vary continuously. Flows are also possible, however, for which discontinuities in the distribution of these quantities occur.

A **discontinuity** in a gas flow occurs over one or more surfaces; the quantities concerned change discontinuously as we cross such a surface, which is called a ***surface of discontinuity***. In non-steady gas flow the surfaces of discontinuity do not in general remain fixed; here it should be emphasized, however, that the rate of motion of these surfaces bears no relation to the velocity of the gas flow itself. The gas particles in their motion may cross a surface of discontinuity.

Certain boundary conditions must be satisfied on surfaces of discontinuity. To formulate these conditions, we consider an element of the surface and use a coordinate system fixed to this element, with the x-axis along the normal.[8]

Firstly, the **mass flux** must be continuous; the mass of gas coming from one side must equal the mass leaving the other side. The mass flux through the surface element considered is ρv_x per unit area. Hence we must have $\rho_1 v_{1x} = \rho_2 v_{2x}$, where the suffixes 1 and 2 refer to the two sides of the surface of discontinuity.

The difference between the values of any quantity on the two sides of the surface will be denoted by enclosing it in square brackets; for example, $[\rho v_x] \equiv \rho_1 v_{1x} - \rho_2 v_{2x}$, and the condition just derived can be written

$$[\rho v_x] = 0. \qquad (84.1)$$

Next, the **energy flux** must be continuous. It is given by (6.3). We therefore obtain the condition

$$[\rho v_x (\frac{1}{2} v^2 + w)] = 0. \qquad (84.2)$$

Finally, the **momentum flux** must be continuous, i.e., the forces exerted on each other by the gases on the two sides of the surface of discontinuity must be equal. The momentum flux per unit area is (see §7) $pn_i + \rho v_i v_k n_k$. The normal vector $\boldsymbol{n}$ is along the x-axis. The continuity of the x-component of the momentum flux therefore gives the condition

$$[p + \rho {v_x}^2] = 0, \qquad (84.3)$$

while that of the y and z components gives

$$[\rho v_x v_y] = 0,\ [\rho v_x v_z] = 0. \qquad (84.4)$$

Equations (84.1)-(84.4) form a complete system of boundary conditions at a surface of discontinuity. From them we can immediately deduce the possibility of **two types of surface of discontinuity**.

In the **first type**, there is no mass flux through the surface. This means that $\rho_1 v_{1x} = \rho_2 v_{2x} = 0$. Since ρ_1 and ρ_2 are not zero, it follows that $v_{1x} = v_{2x} = 0$. The conditions (84.2) and (84.4) are then satisfied, and the condition (84.3) gives $p_1 = p_2$. Thus the normal velocity component and the gas pressure are continuous at the surface of discontinuity:

[8] If the flow is not steady, we consider an element of the surface during a short interval of time.

$$v_{1x} = v_{2x} = 0, \quad [p] = 0, \tag{84.5}$$

while the tangential velocities v_y, v_z and the density (as well as the other thermodynamic quantities except the pressure) may be discontinuous by any amount. We call this a ***tangential discontinuity***.

In the **second type**, the mass flux is not zero, and v_{1x} and v_{2x} are therefore also not zero. We then have from (84.1) and (84.4)

$$[v_y] = 0, \quad [v_z] = 0, \tag{84.6}$$

i.e., the tangential velocity is continuous at the surface of discontinuity. The pressure, the density (and the other thermodynamic quantities) and the normal velocity, however, are discontinuous, their discontinuities being related by (84.1)-(84.3). In the condition (84.2) we can cancel ρv_x by (84.1), and replace v^2 by $v_x{}^2$ since v_y and v_z are continuous. Thus the following conditions must hold at the surface of discontinuity in this case:

$$\left.\begin{aligned} [\rho v_x] &= 0 \\ [\tfrac{1}{2}v_x{}^2 + w] &= 0 \\ [p + \rho v_x{}^2] &= 0 \end{aligned}\right\}. \tag{84.7}$$

A discontinuity of this kind is called a ***shock wave***, or simply a ***shock***.

If we now return to the fixed coordinate system, we must everywhere replace v_x by the difference between the gas velocity component v_n normal to the surface of discontinuity and the velocity u of the surface itself, which is defined to be normal to the surface:

$$v_x = v_n - u. \tag{84.8}$$

The velocities v_n and u are taken in the fixed system. The velocity v_x is the velocity of the gas relative to the surface of discontinuity; we can also say that $-v_x = u - v_n$ is the rate of propagation of the surface relative to the gas. It should be noticed that, if v_x is discontinuous, this velocity has different values relative to the gas on the two sides of the surface.

We have already discussed (in §29) **tangential discontinuities**, at which the tangential velocity component is discontinuous, and we showed that, in an incompressible fluid, such discontinuities are unstable and must spread to form a turbulent region. A similar investigation for a compressible fluid shows that the same instability occurs, for any velocities (see Problem 1).

A particular case of tangential discontinuity is that of a ***contact discontinuity***, where the velocity is continuous, but not the density (and therefore the other thermodynamic quantities, except the pressure). The above remarks on instability do not relate to discontinuities of this kind.

PROBLEMS

Problem 1. Investigate the stability (with respect to infinitesimal perturbations) of tangential discontinuities in a homogeneous compressible medium (gas or liquid).

Solution. The calculations are similar to those in §29 for an incompressible fluid. As there, the z-axis is taken to be normal to the surface.

In medium 2 (velocity $v_2 = 0$, $z < 0$), the pressure satisfies the equation

$$\ddot{p}'_2 - c^2 \Delta p'_2 = 0,$$

instead of Laplace's equation (29.2) for an incompressible fluid. We seek p'_2 in the form

$$p'_2 = \text{constant} \times \exp(-i\omega t + iqx + i\kappa_2 z),$$

where the wave number of the surface "ripples" is denoted by q, instead of k as in §29; if κ_2 is complex, it must be chosen so that $\operatorname{Im}\kappa_2 < 0$. The wave

equation gives

$$\omega^2 = c^2(q^2 + \kappa_2{}^2) . \tag{1}$$

Instead of (29.7), we now find by the same procedure

$$p'_2 = \frac{\zeta\rho\omega^2}{i\kappa_2} .$$

In medium 1, moving with velocity $v_1 = v$ $(z > 0)$, we seek p'_1 in the form

$$p'_1 = \text{constant} \times \exp(-i\omega t + iqx - i\kappa_1 z) .$$

To simplify the derivations, we first assume that v also is in the x-direction. The relation between ω, q and κ_1 is

$$(\omega - vq)^2 = c^2(q^2 + \kappa_1{}^2) ; \tag{2}$$

cf. (68.1). Instead of (29.6), we now have

$$p'_1 = -\frac{\zeta(\omega - qv)^2}{i\kappa_1} ,$$

and the condition $p'_1 = p'_2$ gives

$$\frac{\kappa_1}{(\omega - qv)^2} + \frac{\kappa_2}{\omega^2} = 0 . \tag{3}$$

The assumption made above concerning the direction of v can be avoided if we note that the unperturbed velocity appears in the original linearized continuity equation and Euler's equation only as $\mathbf{v} \cdot \text{grad}$, in the terms $\mathbf{v} \cdot \text{grad}\, p'$ and $(\mathbf{v} \cdot \text{grad})\mathbf{v}'$, respectively. Hence, to change to an arbitrary direction of $\mathbf{v}$ (in the xy-plane) it is sufficient to replace $\mathbf{v}$ in (l)-(3) by $v\cos\phi$, where ϕ is the angle between $\mathbf{v}$ and $\mathbf{q}$; see the second footnote to §29.

Eliminating κ_1 and κ_2 from (l)-(3), we get the following **dispersion relation** for the perturbation frequency ω in terms of the wave number q:

$$\left[\frac{1}{\omega^2} - \frac{1}{(\omega - qv\cos\phi)^2}\right]\left[\frac{1}{c^2q^2} - \frac{1}{\omega^2} - \frac{1}{(\omega - qv\cos\phi)^2}\right] = 0 . \tag{4}$$

The zero of the first factor,

$$\omega = \frac{1}{2}qv\cos\phi , \tag{5}$$

is always real. The zeros of the second factor are

$$\omega = \frac{1}{2}qv\cos\phi \pm q\sqrt{\frac{1}{4}v^2\cos^2\phi + c^2 \pm c\sqrt{c^2 + v^2\cos^2\phi}} ; \tag{6}$$

they are real only if $v\cos\phi > v_k$, where

$$v_k = c\sqrt{8} . \tag{7}$$

Thus, when $v\cos\phi < v_k$, the dispersion relation has a pair of complex conjugate roots, one of which has $\operatorname{Im}\omega > 0$; the corresponding perturbations cause instability. When $v < v_k$, these are perturbations with any angle ϕ; when $v > v_k$, only those with $\cos\phi < v_k / k$ are unstable. *The tangential discontinuity is therefore always unstable*. The fact of instability (though not the perturbations which cause it) is evident from that which occurs in an incompressible fluid, coupled with the fact that v appears in the dispersion relation only in the combination $v\cos\phi$: whatever the value of v, there must be angles ϕ for which $v\cos\phi << c$, and the fluid therefore behaves like an incompressible one with respect to such perturbations.[9]

[9] The value (7) was derived by L. D. Landau (1944). The need to allow for the non-collinearity of $\mathbf{v}$ and $\mathbf{q}$ in this problem was noted by S. I. Syrovat'skii (1954).

Problem 2. A plane sound wave is incident on a tangential discontinuity in a homogeneous compressible medium. Determine the intensity of the waves reflected and refracted by the discontinuity (J. W. Miles 1957; H. S. Ribner 1957).

Solution. We take the coordinate axes as in Problem 1, the velocity $\boldsymbol{v}$ (in medium 1; $z > 0$) being in the x-direction. Let the sound wave be incident from the medium at rest (medium 2; $z < 0$), the direction of its wave vector $\boldsymbol{k}$ being specified by the angle θ between $\boldsymbol{k}$ and the z-axis and the angle ϕ between the projection $\boldsymbol{q}$ of $\boldsymbol{k}$ on the xy-plane and the velocity $\boldsymbol{v}$;

$$k_x = q\cos\phi\,,\quad k_y = q\sin\phi\,,\quad k_z = (\omega/c)\cos\theta\,,$$

$$q = (\omega/c)\sin\theta = k\sin\theta\,,$$

with $0 < \theta < -\pi/2$ (the wave is incident in the positive z-direction). In medium 2, we seek the pressure in the form

$$p'_2 = \exp(ik_x x + ik_y y - i\omega t)[\exp ik_z z + A\exp(-ik_z z)]\,,$$

where A is the reflected wave amplitude, the incident wave amplitude being arbitrarily taken as unity. In medium 1, there is just the refracted wave

$$p'_1 = B\exp(ik_x x + ik_y y + i\kappa z - i\omega t)\,,$$

where κ satisfies the equation

$$(\omega - vk_x)^2 = c^2(k_x{}^2 + k_y h2 + \kappa^2)\,;$$

cf. (2). The amplitudes A and B are found from the continuity conditions for the pressure and the vertical displacement of the fluid particles on either side of the discontinuity: $p'_1 = p'_2$ for $z = 0$, $\zeta_1 = \zeta_2 \equiv \zeta$. This gives two equations:

$$1 + A = B\,,\quad \frac{\kappa}{(\omega - vk_x)^2}B = \frac{k_z}{\omega^2}(1 - A)\,,$$

whence

$$A = \frac{(\omega - vk_x)^2/\kappa - \omega^2/k_z}{(\omega - vk_x)^2/\kappa + \omega^2/k_z}\,,\quad B = \frac{2(\omega - vk_x)^2/\kappa}{(\omega - vk_x)^2/\kappa + \omega^2/k_z}\,, \quad (8)$$

and the problem is thus solved. The sign of κ,

$$\kappa^2 = \left(\frac{\omega}{c}\right)^2\left[(1 - M\sin\theta\cos\phi)^2 - \sin^2\theta\right],\quad M = \frac{v}{c}\,,$$

must be chosen in accordance with the limiting conditions for $z \to \infty$: the velocity of the refracted wave is away from the discontinuity, i.e.,

$$U_z = \frac{\partial\omega}{\partial\kappa} = \frac{c^2\kappa}{\omega - vk_x} > 0\,.$$

These formulae show that **three types of reflection** are possible.

(1) When $M\cos\phi < \operatorname{cosec}\theta - 1$, κ is real, and since $\omega - vk_x > 0$ it follows from (9) that $\kappa > 0$. Then, from (8), $|A| < 1$, and the **reflected wave is weaker**.

(2) When $\operatorname{cosec}\theta - 1 < M\cos\phi < \operatorname{cosec}\theta + 1$, κ is imaginary and $|A| = 1$; there is total internal reflection of the sound wave.

(3) When $M\cos\phi > \operatorname{cosec}\theta + 1$, which can occur only if $M > 2$, κ is again real, but we must now take $\kappa < 0$. Then, from (8), $|A| > 1$, and the **reflected wave is stronger**. Moreover, the denominator of the expressions (8) may be zero for certain angles of incidence; the reflection coefficient then becomes infinite. Since this denominator is, apart from the notation, the same as the left-hand side of (3) in Problem 1, we can conclude immediately that the "resonance" angles of incidence are given by equations (5) and (6), the latter for $M > \sqrt{8}$. In turn the infinite reflection (and transmission) coefficient, i.e., the non-zero amplitude of the reflected wave when the incident wave amplitude tends to zero, signifies that there can be spontaneous emission of

sound by a surface of discontinuity: a perturbation (ripples) once formed on it will continue to emit sound waves indefinitely, being neither damped nor amplified; the energy carried away by the emitted sound is drawn from the whole of the medium in motion.

The **energy flux density** (averaged over time) in the refracted wave is

$$\overline{q}_2 = U_2\overline{E}_2 = \frac{c^2\kappa}{\omega - vk_x}\frac{\omega}{\omega - vk_x}\frac{|B|^2}{2\rho c^2},$$

with E_2 from (68.3). In case 3 we have $\kappa < 0$ and therefore $\overline{q}_2 < 0$: energy reaches the discontinuity from the moving medium, which acts as a source of amplification. When sound is spontaneously emitted, this incoming energy is equal to the energy carried away by the wave into the medium at rest.

In the solution given here, the instability of the surface of discontinuity is not taken into account. The fact that this statement of the problem is formally correct is the result of the linear independence of the sound waves and the unstable surface waves (which are damped as $z \to \pm\infty$). Physical correctness demands compliance with certain special (e.g., initial) conditions such that the surface waves are sufficiently weak.

§85. The shock adiabatic

Let us now investigate shock waves in detail.[10] We have seen that, in this type of discontinuity, the tangential component of the gas velocity is continuous. We can therefore take a coordinate system in which the surface element considered is at rest, and the tangential component of the gas velocity is zero on both sides.[11] Then we can write the normal component v_x ss v simply, and the conditions (84.7) take the form

$$\rho_1 v_1 = \rho_2 v_2 \equiv j, \tag{85.1}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2, \tag{85.2}$$

$$w_1 + \frac{1}{2}v_1^2 = w_2 + \frac{1}{2}v_2^2, \tag{85.3}$$

where j denotes the **mass flux density** at the surface of discontinuity. In what follows we shall always take j positive, with the gas going from side 1 to side 2. That is, we call **gas 1** the one into which the shock wave moves, and **gas 2** that which remains behind the shock. We call the side of the shock wave towards **gas 1** the ***front*** of the shock, and that towards **gas 2** the ***back***.

We shall derive a series of relations which follow from the above conditions. Using the specific volumes $V_1 = 1/\rho_1$, $V_2 = 1/\rho_2$, we obtain from (85.1)

$$v_1 = jV_1, \quad v_2 = jV_2, \tag{85.4}$$

and, substituting in (85.2),

$$p_1 + j^2 V_1 = p_2 + j^2 V_2, \tag{85.5}$$

or

$$j^2 = \frac{p_2 - p_1}{V_1 - V_2}. \tag{85.6}$$

This formula, together with (85.4), relates the rate of propagation of a shock wave to the pressures and densities of the gas on the two sides of the surface.

Since j^2 is positive, we see that either $p_2 > p_1$, $V_1 > V_2$, or $p_2 < p_1$,

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[10] One comment on terminology is needed. The shock wave is the discontinuity surface itself. Some authors, however, call this the shock front, while by the shock wave they mean the surface together with the gas flow behind it.

[11] This coordinate system is used everywhere in this chapter except §92.
A shock wave at rest is often called a ***compression discontinuity***. If the shock is perpendicular to the direction of flow, we have a ***normal shock***, otherwise an ***oblique shock***.

$V_1 < V_2$; we shall see below that *only the former case can actually occur*.

We may note the following useful formula for the velocity difference $v_1 - v_2$. Substituting (85.6) in $v_1 - v_2 = j(V_1 - V_2)$, we obtain[12]

$$v_1 - v_2 = \sqrt{(p_2 - p_1)(V_1 - V_2)} \,. \qquad (85.7)$$

Next, we write (85.3) in the form

$$w_1 + \frac{1}{2} j^2 V_1{}^2 = w_2 + \frac{1}{2} j^2 V_2{}^2 \,, \qquad (83.8)$$

and, substituting j^2 from (85.6), obtain

$$w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(p_1 - p_2) = 0 \,. \qquad (83.9)$$

If we replace the heat function w by $\varepsilon + pV$, where ε is the internal energy, we can write this relation as

$$\varepsilon_1 - \varepsilon_2 + \frac{1}{2}(V_1 - V_2)(p_1 + p_2) = 0 \,. \qquad (83.10)$$

These relations hold between the thermodynamic quantities on the two sides of the surface of discontinuity.

For given p_1, V_1, equation (85.9) or (85.10) gives the relation between p_2 and V_2. This relation is called the ***shock adiabatic*** or the *Hugoniot adiabatic* (W. J. M. Rankine 1870; H. Hugoniot 1885). It is represented graphically in the pV-plane (Fig. 53) by a curve passing through the given point (p_1, V_1) corresponding to the state of gas 1 in front of the shock wave, which we shall call the ***initial point***. It should be noted that the shock adiabatic cannot intersect the vertical line $V = V_1$ except at the initial point. For the existence of another intersection would mean that two different pressures satisfying (85.10) correspond to the same volume. For $V_1 = V_2$, however, we have from (85.10) also $\varepsilon_1 = \varepsilon_2$, and when the volumes and energies are the same the pressures must be the same. Thus the line $V = V_1$ divides the shock adiabatic into two parts, each of which lies entirely on one side of the line. Similarly, the shock adiabatic meets the horizontal line $p = p_1$ only at the point (p_1, V_1).

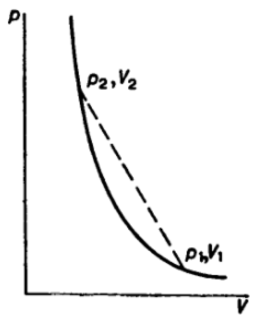

FIG. 53

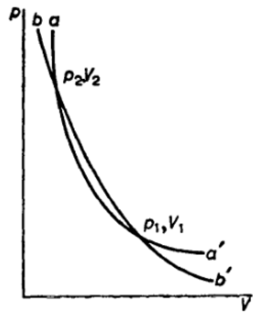

FIG. 54

Let aa' (Fig. 54) be the shock adiabatic through the point (p_1, V_1) as initial point. We take any point (p_2, V_2) on it and draw through that point another adiabatic bb', for which (p_2, V_2) is an initial point. It is evident that the pair of values (p_1, V_1) satisfies the equation of this adiabatic also. The

[12] Here we write the positive square root, since, as we shall see later (§87), we must have $v_1 - v_2 > 0$.

adiabatics aa' and bb' therefore intersect at the two points (p_1, V_1) and (p_2, V_2). It must be emphasized that the adiabatics are not identical, as would happen for Poisson adiabatics through a given point. This is a consequence of the fact that the equation of the shock adiabatic cannot be written in the form $f(p,V) = \text{constant}$, where f is some function, whereas the Poisson adiabatic, for example, can be written $s(p,V) = \text{constant}$. The Poisson adiabatics for a given gas form a one-parameter family of curves, but the shock adiabatic is determined by two parameters, the initial values p_1 and V_1. This has also the following important result: if two (or more) successive shock waves take a gas from state 1 to state 2 and from there to state 3, the transition from state 1 to state 3 cannot in general be effected by the passage of any one shock wave.

For a given initial thermodynamic state of the gas (i.e., for given p_1 and V_1), the shock wave is defined by only one parameter; for instance, if the pressure p_2 behind the shock is given, then V_2 is determined by the Hugoniot adiabatic, and the flux density j and the velocities v_1 and v_2 are then given by formulae (85.4) and (85.6). It should be mentioned, however, that we are here considering the shock wave in a coordinate system in which the gas is moving normal to the surface. If the shock wave may be situated obliquely to the direction of flow, another parameter is needed; for example, the value of the velocity component tangential to the surface.

The following convenient graphical interpretation of formula (85.6) may be mentioned. If the point (p_1, V_1) on the shock adiabatic (Fig. 53) is joined by a chord to any other point (p_2, V_2) on it, then $\dfrac{p_2 - p_1}{V_2 - V_1} = -j^2$ is just the slope of this chord relative to the axis of abscissae. Thus j, and therefore the velocity of the shock wave, are determined at each point of the shock adiabatic by the slope of the chord joining that point to the initial point.

Like the other thermodynamic quantities, the entropy is discontinuous at a shock wave. By the law of increase of entropy, the entropy of a gas can only increase during its motion. Hence the entropy s_2 of the gas which has passed through the shock wave must exceed its initial entropy s_1:

$$s_2 > s_1. \qquad (85.11)$$

We shall see below that this condition places very important restrictions on the manner of variation of all quantities in a shock wave.

The following fact should be emphasized. The presence of shock waves results in an increase in entropy in those flows which can be regarded as motions of an ideal fluid in all space, the viscosity and thermal conductivity being zero. The increase in entropy signifies that the motion is irreversible, i.e., energy is dissipated. Thus the discontinuities are a means by which energy can be dissipated in the motion of an ideal fluid. It follows that d'Alembert's paradox (§11) does not arise when bodies move in an ideal fluid in such a way as to cause shock waves. In such cases there is a drag force.

The true mechanism by which the entropy increases in shock waves lies, of course, in **dissipative processes** occurring in the very thin layers which actual shock waves are (see §93). It should be noticed, however, that the amount of this dissipation is entirely determined by the laws of conservation of mass, energy and momentum, when they are applied to the two sides of such layers; the width of the layers is just such as to give the increase in entropy required by these conservation laws.

The increase in entropy in a shock wave has another important effect on the motion: even if we have potential flow in front of the shock wave, the *flow behind it is in general rotational*. We shall return to this matter in §114.

§86. Weak shock waves

Let us consider a shock wave in which the discontinuity in every quantity is small; we call this a ***weak shock wave***. We transform the relation (85.9) by expanding in powers of the small differences $s_2 - s_1$ and $p_2 - p_1$. We shall see that the first- and second-order terms in $p_2 - p_1$ then cancel; we must therefore carry the expansion with respect to $p_2 - p_1$ as far as the third order. In the expansion with respect to $s_2 - s_1$, only the first-order terms need be retained. We have

$$w_2 - w_1 = \left(\frac{\partial w}{\partial s_1}\right)_p (s_2 - s_1) + \left(\frac{\partial w}{\partial p_1}\right)_s (p_2 - p_1) + \frac{1}{2}\left(\frac{\partial^2 w}{\partial p_1{}^2}\right)_s (p_2 - p_1)^2 + \frac{1}{6}\left(\frac{\partial^3 w}{\partial p_1{}^3}\right)_s (p_2 - p_1)^3$$

By the thermodynamic relation $dw = Tds + Vdp$ we have for the derivatives

$$\left(\frac{\partial w}{\partial s}\right)_p = T, \quad \left(\frac{\partial w}{\partial p}\right)_s = V.$$

Hence

$$w_2 - w_1 = T_1(s_2 - s_1) + V_1(p_2 - p_1) + \frac{1}{2}\left(\frac{\partial V}{\partial p_1}\right)_s (p_2 - p_1)^2 + \frac{1}{6}\left(\frac{\partial^2 V}{\partial p_1{}^2}\right)_s (p_2 - p_1)^3$$

The volume V_2 need be expanded only with respect to $p_2 - p_1$, since the second term of equation (85.9) already contains the small difference $p_2 - p_1$, and an expansion with respect to $s_2 - s_1$ would give a term of the form $(s_2 - s_1)(p_2 - p_1)$, which is of no interest. Thus

$$V_2 - V_1 = \left(\frac{\partial V}{\partial p_1}\right)_s (p_2 - p_1) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial p_1{}^2}\right)_s (p_2 - p_1)^2.$$

Substituting this expansion in (85.9), we obtain

$$s_2 - s_1 = \frac{1}{12T_1}\left(\frac{\partial^2 V}{\partial p_1{}^2}\right)_s (p_2 - p_1)^3. \qquad (86.1)$$

Thus the discontinuity of entropy in a weak shock wave is of the third order of smallness relative to the discontinuity of pressure.

The adiabatic compressibility $-\left(\frac{\partial V}{\partial p}\right)_s$ almost always decreases with increasing pressure, i.e., the second derivative[13]

$$\left(\frac{\partial^2 V}{\partial p^2}\right)_s > 0. \qquad (86.2)$$

It should be emphasized, however, that this is not a thermodynamic relation, and it is therefore possible, in principle, that the derivative might be negative.[14] We shall find several times in what follows that the sign of the derivative $\left(\frac{\partial^2 V}{\partial p^2}\right)_s$ is very important in gas dynamics. In future we shall assume it to be positive.

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[13] For a polytropic gas $(\partial^2 V/\partial p^2)_s = (\gamma+1)V/\gamma^2 p^2$. This expression can be most simply obtained by differentiating the Poisson adiabatic equation $pV^\gamma = \text{constant}$.

[14] This may be true, for instance, near a gas-liquid critical point. The case where (86.2) does not hold can also be simulated by the shock adiabatic for a medium which has a phase transition (and for which the adiabatic therefore has a kink); see Ya. B. Zel'dovich and Yu. P. Raizer, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena,* New York 1966, 1967, Chapter I §19, Chapter XI §20.

Let us draw through the point 1 (p_1, V_1) in the pV-plane two curves, the **shock adiabatic** and the **Poisson adiabatic**. The equation of the latter is $s_2 - s_1 = 0$. By comparing this with the equation (86.1) of the shock adiabatic near the point 1, we see that the two curves have contact of the second order at this point, both the first and the second derivatives being equal. In order to decide the relative position of the two curves near the point 1, we use the fact that, according to (86.1) and (86.2), we must have $s_2 > s_1$ on the shock adiabatic for $p_2 > p_1$, while on the Poisson adiabatic $s_2 = s_1$. The abscissa of a point on the shock adiabatic must therefore exceed that of a point on the Poisson adiabatic having the same ordinate p_2. This follows at once from the fact that, by the well-known thermodynamic formula $\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p$, the entropy increases with the volume at constant pressure for all substances which expand on heating, i.e., which have $\left(\frac{\partial V}{\partial T}\right)_s$ positive. We can similarly deduce that, for $p_2 < p_1$, the abscissa of a point on the Poisson adiabatic exceeds that of the corresponding point on the shock adiabatic. Thus, near the point of contact, the two curves lie as shown in Fig. 55 (*HH'* being the shock adiabatic and *PP'* the Poisson adiabatic)[15], both being concave upwards, by (86.2).

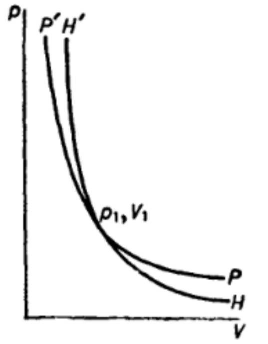

FIG. 55

For small $p_2 - p_1$ and $V_2 - V_1$, formula (85.6) can be written, in the first approximation, as $j^2 = -\left(\frac{\partial p}{\partial V}\right)_s$ (we take the derivative for constant entropy, since the tangents to the two adiabatics at the point 1 coincide). The velocities v_1 and v_2 are, in the same approximation, equal:

$$v_1 = v_2 = v = jV = \sqrt{-V^2\left(\frac{\partial p}{\partial V}\right)_s} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \quad p_2 - p_1 .$$

This is just the velocity of sound *c*. Thus the rate of propagation of weak shock waves is, in the first approximation, the velocity of sound:

$$v = c . \tag{86.3}$$

From the properties of the shock adiabatic near the point 1 derived above we can deduce a number of important consequences. Since we must have $s_2 > s_1$ in a shock wave, it follows that $p_2 > p_1$, i.e., the point 2 (p_2, V_2) must lie above the point 1. Moreover, since the chord 12 has a greater slope

15 If $(\partial V / \partial T)_p$ is negative, the relative position is reversed.

than the tangent to the adiabatic at the point 1 (Fig. 53), and the slope of the tangent is equal to the derivative $(\partial p/\partial V_1)_{s_1}$, we have $j^2 > -(\partial p/\partial V_1)_{s_1}$. Multiplying both sides of this inequality by V_1^2, we find

$$j^2 V_1^2 = v_1^2 > -V_1^2 \left(\frac{\partial p}{\partial V_1}\right)_{s_1} = \left(\frac{\partial p}{\partial \rho_1}\right)_{s_1} = c_1^2,$$

where c_1 is the velocity of sound corresponding to the point 1. Thus $v_1 > c_1$. Finally, from the fact that the chord 12 has a smaller slope than the tangent at the point 2, it follows in like manner that $v_2 < c_2$.[16]

When the derivative $(\partial^2 V/\partial p^2)_s$ is negative, the condition $s_2 > s_1$ for weak shock waves implies that $p_2 < p_1$, while the velocities again satisfy $v_1 > c_1$, $v_2 < c_2$.

§87. The direction of variation of quantities in a shock wave

The results of §86 show that, if the derivative $(\partial^2 V/\partial p^2)_s$ is assumed positive, it can be demonstrated very simply that for weak shocks the condition of increasing entropy ($s_2 > s_1$) necessarily means that

$$p_2 > p_1, \tag{87.1}$$

$$v_1 > c_1, \quad v_2 < c_2. \tag{87.2}$$

From the remark made concerning (85.6) it follows that, if $p_2 > p_1$, then

$$V_1 > V_2, \tag{87.3}$$

and, since $v_1/V_1 = v_2/V_2 = j$, also[17]

$$v_1 > v_2. \tag{87.4}$$

The inequalities (87.1) and (87.3) signify that, when the gas passes through the shock wave, it is compressed, the pressure and density increasing. The inequality $v_1 > c_1$ means that the shock wave moves supersonically relative to the gas ahead of it; clearly, therefore, no perturbations starting from the shock wave can penetrate into that gas. In other words, the presence of the shock has no effect on the state of gas in front of it.

We shall now show that all the inequalities (87.1)-(87.4) hold for shock waves with *any* intensity, if it is again assumed that $(\partial^2 V/\partial p^2)_s$ is positive.[18]

The quantity j^2 gives the slope of the chord from the initial point 1 on the shock adiabatic to any point 2 ($-j^2$ is the slope of this chord to the V-axis). We shall show first that the direction of variation of j^2 as the point 2 moves along the adiabatic is in a definite relation to that of the entropy s_2.

We differentiate the relations (85.5) and (85.8) with respect to the quantities pertaining to gas 2, assuming the state of gas 1 to be unchanged. This means that p_1, V_1 and w_1 are regarded as constants, while p_2, V_2, w_2 and j are differentiated. From (85.5) we obtain

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[16] This argument is valid only near point 1, where the slope of the tangent to the shock adiabatic at point 2 differs from the derivative $(\partial p_2/\partial V_2)_{s_2}$ only by second-order small quantities.

[17] If we change to a frame of reference in which gas 1 (in front of the shock wave) is at rest, and the shock is moving, then the inequality $v_1 > v_2$ means that the gas behind the shock wave moves (with velocity $v_1 - v_2$) in the same direction as the shock itself.

[18] These inequalities were derived for shock waves with any intensity in a polytropic gas by E. Jouguet (1904) and G. Zemplén (1905). The proof given below for any medium is due to L. D. Landau (1944).

$$dp_2 + j^2 dV_2 = (V_1 - V_2) d(j^2), \qquad (87.5)$$

and from (85.8)

$$dw_2 + j^2 V_2 dV_2 = \frac{1}{2}(V_1^2 - V_2^2) d(j^2),$$

or, expanding the differential dw_2

$$T_2 ds_2 + V_2 (dp_2 + j^2 dV_2) = \frac{1}{2}(V_1^2 - V_2^2) d(j^2).$$

Substituting in this equation from (87.5), we obtain

$$T_2 ds_2 = \frac{1}{2}(V_1 - V_2)^2 d(j^2). \qquad (87.6)$$

Hence we see that

$$\frac{d(j^2)}{ds_2} > 0, \qquad (87.7)$$

i.e., j^2 increases with s_2.

We now show that there can be no point on the shock adiabatic at which it touches any line drawn from the point 1 (such as the point O is in Fig. 56).

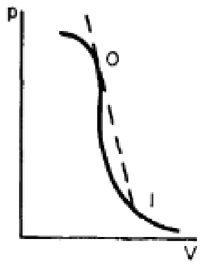

FIG. 56

At such a point the slope of the chord from the point 1 is a minimum, and j^2 has a corresponding maximum, so that $\frac{d(j^2)}{dp_2} = 0$. We see from (87.6) that in this case we also have $\frac{ds_2}{dp_2} = 0$. Next, we calculate $\frac{d(j^2)}{dp_2}$ at any point on the shock adiabatic; substituting in (87.5) the differential dV_2 in the form $dV_2 = \left(\frac{\partial V_2}{\partial p_2}\right)_{s_2} dp_2 + \left(\frac{\partial V_2}{\partial s_2}\right)_{p_2} ds_2$ and ds_2 in the form given by (87.6), and dividing by dp_2, we obtain

$$\frac{d(j^2)}{dp_2} = \frac{1 + j^2 \left(\frac{\partial V_2}{\partial p_2}\right)_{s_2}}{(V_1 - V_2)\left[1 - \frac{1}{2}\frac{j^2}{T_2}(V_1 - V_2)\left(\frac{\partial V_2}{\partial s_2}\right)_{p_2}\right]}. \qquad (87.8)$$

Hence it follows that, for this to be zero, we must have

$$1 + j^2 \left(\frac{\partial V_2}{\partial p_2}\right)_{s_2} = 1 - \frac{v_2^2}{c_2^2} = 0,$$

i.e., $v_2 = c_2$; conversely, if $v_2 = c_2$, it follows that $\frac{d(j^2)}{dp_2} = 0$; the only other possibility would be for both the numerator and the denominator in (87.8) to vanish, but these are two different functions of the point 2 on the shock adiabatic, and so their simultaneous vanishing would be an improbable

accident.[19]

Thus, of the three equations

$$\frac{d(j^2)}{dp_2}=0\,,\quad \frac{ds_2}{dp_2}=0\,,\quad v_2=c_2\,, \tag{87.9}$$

each implies the other two and all three would hold at the point *O* (Fig. 56). On account of the third equation, *O* will be called an ***acoustic point***. Finally, we have for the derivative of $v_2{}^2/c_2{}^2$ at the point *O*

$$\frac{d}{dp_2}\left(\frac{v_2{}^2}{c_2{}^2}\right)=-\frac{d}{dp_2}\left[j^2\left(\frac{\partial V_2}{\partial p_2}\right)_{s_2}\right]=-j^2\left(\frac{\partial^2 V_2}{\partial p_2{}^2}\right)_{s_2}.$$

In view of the assumption that $(\partial^2 V/\partial p^2)_s$ is positive, we therefore have **at the acoustic point**

$$\frac{d(v_2/c_2)}{dp_2}<0. \tag{87.10}$$

It is now easy to show that **such a point cannot exist on the shock adiabatic**. At points just above the initial point 1, $v_2/c_2<1$ (see the end of §86). The equation $v_2/c_2=1$ can therefore be satisfied only by an increase in v_2/c_2; that is, at the acoustic point we should necessarily have $\frac{d(v_2/c_2)}{dp_2}>0$, whereas by (87.10) the converse is true. In a similar manner, we can show that the ratio v_2/c_2 also cannot become equal to unity on the part of the shock adiabatic below the point 1.

From the **impossibility of the existence of acoustic points**, which has just been demonstrated, we can at once deduce from the graph of the shock adiabatic that the slope of the chord from the point 1 (p_1,V_1) to the point 2 (p_2,V_2) decreases as point 2 moves up the curve, and j^2 correspondingly increases. From this property of the shock adiabatic and the inequality (87.7) it follows immediately that s_2 likewise increases, and the necessary condition $s_2>s_1$ implies that $p_2>p_1$ also.

It is also easy to see that, on the upper part of the shock adiabatic, the inequalities $v_2<c_2$, $v_1>c_1$ hold. The former follows at once from the fact that it holds near the point 1, and the ratio v_2/c_2 can never become equal to unity. The second inequality follows from the fact that every chord from the point 1 to a point 2 above it is steeper than the tangent to the adiabatic at the point 1, since the curve cannot behave as shown in Fig. 56.

The condition $s_2>s_1$ and all three inequalities (87.1), (87.2) are therefore satisfied on the upper part of the shock adiabatic. On the lower part, however, none of these conditions holds. They are consequently equivalent, and if one is satisfied so are all the others.

In the preceding discussion we have everywhere assumed that the derivative $(\partial^2 V/\partial p^2)_s$ is positive. If this derivative could change sign, it would no longer be possible to draw from the necessity of $s_2>s_1$ any general conclusions concerning inequalities for the other quantities.

§88. Evolutionary shock waves

The derivation of the inequalities (87.1)-(87.4) in §§86 and 87 involved a particular assumption concerning the thermodynamic properties of the medium, namely that $(\partial^2 V/\partial p^2)_s$ is positive. It is most important,

[19] It should be emphasized, to avoid misunderstanding, that $d(j^2)/dp_2=0$ itself is not a further independent function of the point 2, since it is determined by (87.8).

however, to note that the inequalities

$$v_1 > c_1, \quad v_2 < c_2 \qquad (88.1)$$

for the velocities can be obtained by quite different arguments, which show that shock waves in which the inequalities (88.1) do not hold cannot exist, even if their existence would not be disproved by the purely thermodynamic arguments given above.[20]

The point is that we have still to discuss the subject of the **stability of shock waves**. The most general necessary condition for stability is that any infinitesimal perturbation of the initial state (at some instant $t = 0$) should cause only a quite definite infinitesimal change in the flow, at least during a sufficiently short interval t. The latter restriction means that the condition in question is not a sufficient one. For example, if an initial small perturbation grows, even exponentially as $e^{\gamma t}$ with a positive constant γ, it remains small for a time $t \le 1/\gamma$, although ultimately it will disrupt the flow pattern concerned. A perturbation that does not satisfy the necessary condition stated is the splitting of a shock wave into two or more surfaces of discontinuity. It is evident that the change in the flow here is immediately not small, although for small t (when the two discontinuities have not moved far apart) it occupies only a short range of distances δx.

Any initial small perturbation is defined by some number of independent parameters. Its subsequent development is governed by a set of linearized boundary conditions which must be satisfied at the surface of discontinuity. The necessary condition for stability stated above will be satisfied if the number of these equations is the same as the number of unknown parameters in them; the boundary conditions then determine the subsequent development of the perturbation, which remains small for small $t > 0$. If the number of equations is greater or less than the number of independent parameters, then the problem of the small perturbation has either no solution or an infinity of solutions. Either case would show that the initial assumption (that the perturbation is small if t is small) is incorrect, and would therefore contradict the condition imposed. The condition thus formulated is called the condition for the flow to be ***evolutionary***.

Let us suppose that a shock wave is subjected to an infinitesimal displacement in a direction perpendicular to its plane.[21] The displacement is accompanied by infinitesimal perturbations in the gas pressure, velocity, etc., on both sides of the surface of discontinuity. These perturbations near the shock are then propagated away from it with the velocity of sound (relative to the gas); this, however, does not apply to the perturbation in the entropy, which is transmitted only with the gas itself. Thus an arbitrary perturbation of the type in question can be regarded as consisting of sound disturbances propagated in gases 1 and 2 on both sides of the shock wave, and a perturbation of the entropy; the latter, which moves with the gas, will evidently occur only in gas 2 behind the shock. In each of the sound disturbances, the changes in the various quantities are related by certain formulae which follow from the equations of motion (as in any sound wave, §64), and therefore any such disturbance is specified by only one parameter.

Let us now compute the number of possible sound disturbances. It depends on the relative magnitudes of the gas velocities v_1, v_2 and the sound velocities c_1, c_2. We take the direction of motion of the gas (from 1 to 2) as the positive direction of the x-axis. The rate of propagation of the disturbance in gas 1 relative to the stationary shock wave is $u_1 = v_1 \pm c_1$, and in gas 2 it is $u_2 = v_2 \pm c_2$. Since these disturbances must be propagated

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[20] It should be mentioned at the same time that (at least for weak shock waves) these thermodynamic arguments lead to the conditions (88.1) also when $(\partial^2 V/\partial p^2)_s < 0$ and the shock wave is a rarefaction wave, not a compression wave, as has been noted at the end of §86.

[21] The following proof of the inequalities (88.1) is due to L.D. Landau (1944).

away from the shock wave, it follows that $u_1 < 0$, $u_2 > 0$.

Let us suppose that $v_1 > c_1$, $v_2 < c_2$. Then it is clear that both values $u_1 = v_1 \pm c_1$ are positive, while only $v_2 + c_2$ of the two values of u_2 is positive. This means that the sound disturbances in which we are interested cannot exist in gas 1, while in gas 2 there can be only one, which is propagated relative to the gas with velocity c_2. The calculation in other cases is similar.

The result is shown in Fig. 57, where each arrow corresponds to one sound disturbance, propagated relative to the gas in the direction shown by the arrow. Each sound disturbance is defined, as stated above, by one parameter. Furthermore, in all four cases there are two other parameters, one determining the entropy perturbation propagated in gas 2 and one determining the displacement of the shock wave. For each of the four cases in Fig. 57, the number in a circle shows the total number of parameters, thus obtained, which define an arbitrary perturbation arising from the displacement of the shock wave.

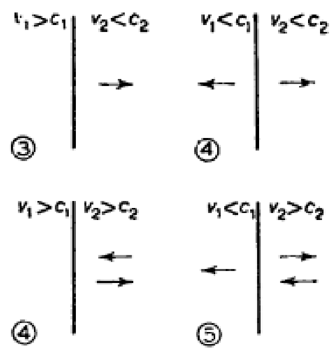


FIG. 57

The number of boundary conditions which must be satisfied by a perturbation on the surface of discontinuity is three (the continuity of the mass, energy and momentum fluxes). In all except the first of the cases shown in Fig. 57, the number of independent parameters available exceeds the number of equations. It is seen that only the shock waves which satisfy the conditions (88.1) are evolutionary. These conditions are therefore necessary ones for the existence of shock waves, whatever the thermodynamic properties of the gas. An artificially created discontinuity that did not satisfy these conditions would immediately disintegrate into other discontinuities.[22]

An evolutionary shock wave is stable with respect to the perturbation type considered and in the ordinary sense of the word. If the movement of the shock (and therefore the perturbation of all other quantities) is sought in a form proportional to $e^{-i\omega t}$, then it is evident a priori that the value of ω uniquely determined by the boundary conditions can only be zero, since the problem involves no parameters having the dimension of reciprocal time which might determine a non-zero value of ω.

We shall return in §90 to the question of shock wave stability.

§89. Shock waves in a polytropic gas

Let us apply the general relations obtained in the previous sections to

[22] For each of the non-evolutionary cases shown in Fig. 57, the perturbation is under-determined: the number of arbitrary parameters exceeds the number of equations. In magnetohydrodynamics, shock waves may be non-evolutionary because the perturbation is either under- or over-determined (see *ECM*, §73).

shock waves in a polytropic gas. The heat function of such a gas is given by the simple formula (83.11). Substituting this expression in (85.9), we have after a simple transformation

$$\frac{V_2}{V_1} = \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}. \qquad (89.1)$$

Using this formula, we can determine any of the quantities p_1, V_1, p_2, V_2 from the other three. The ratio V_2/V_1 is a monotonically decreasing function of the ratio p_2/p_1, tending to the finite limit $\frac{\gamma-1}{\gamma+1}$. The curve showing p_2 as a function of V_2 for given p_1, V_1 (the shock adiabatic) is represented in Fig. 58. It is a rectangular hyperbola with asymptotes $\frac{V_2}{V_1} = \frac{\gamma-1}{\gamma+1}$, $\frac{p_2}{p_1} = -\frac{\gamma-1}{\gamma+1}$. As we know, only the upper part of the curve, above the point $V_2/V_1 = p_2/p_1 = 1$, has any real significance; it is shown in Fig. 58 (for $\gamma = 1.4$) by a continuous line.

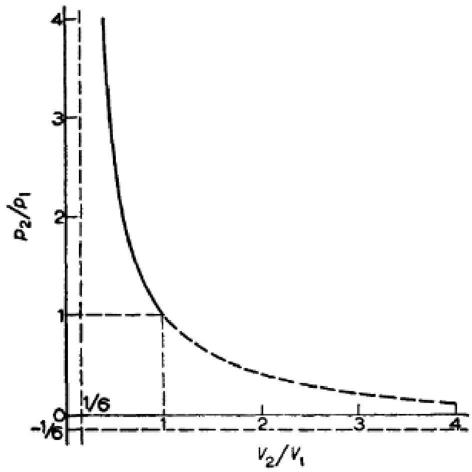


FIG. 58

For the ratio of the temperatures on the two sides of the discontinuity we find, from the equation of state for a perfect gas $\frac{T_2}{T_1} = \frac{p_2 V_2}{p_1 V_1}$, that

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}. \qquad (89.2)$$

For the flux density *j* we obtain from (85.6) and (89.1)

$$jh2 = \frac{(\gamma-1)p_1 + (\gamma+1)p_2}{2V_1}, \qquad (89.3)$$

and then for the velocities of propagation of the shock wave relative to the gas before and behind it

$$\left.\begin{aligned} v_1^2 &= \frac{1}{2} V_1 \{(\gamma-1)p_1 + (\gamma+1)p_2\} = \frac{1}{2}\frac{c_1^2}{\gamma}\left[\gamma - 1 + (\gamma+1)\frac{p_2}{p_1}\right] \\ v_2^2 &= \frac{1}{2} V_1 \frac{\{(\gamma+1)p_1 + (\gamma+1)p_2\}^2}{(\gamma-1)p_1 + \gamma+1)p_2} = \frac{1}{2}\frac{c_2^2}{\gamma}\left[\gamma - 1 + (\gamma+1)\frac{p_1}{p_2}\right] \end{aligned}\right\} \qquad (89.4)$$

their difference being

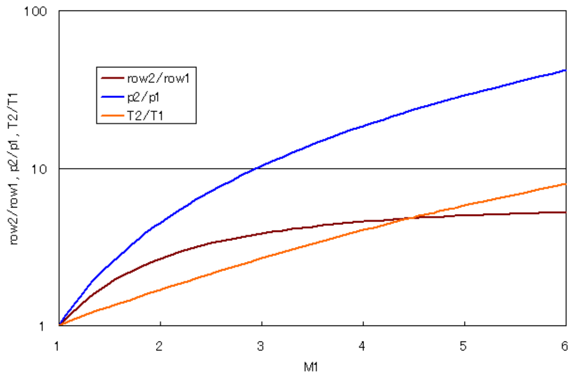
$$v_1 - v_2 = \frac{\sqrt{2V_1(p_2 - p_1)}}{\sqrt{(\gamma-1)p_1 + (\gamma+1)p_2}}. \qquad (89.5)$$

There are some formulae useful in applications, which express the ratios of densities, pressures and temperatures in a shock wave in terms of the Mach number $M_1 = v_1 / c_1$. These formulae are easily derived from the foregoing results:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)M_1^{\ 2}}{(\gamma-1)M_1^{\ 2}+2}, \tag{89.6}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^{\ 2}}{\gamma+1} - \frac{\gamma-1}{\gamma+1}, \tag{89.7}$$

$$\frac{T_2}{T_1} = \frac{\left\{2\gamma M_1^{\ 2}-(\gamma-1)\right\}\left\{(\gamma-1)M_1^{\ 2}+2\right\}}{(\gamma+1)^2 M_1^{\ 2}}. \tag{89.8}$$



The Mach number M_2 is given in terms of M_1 by

$$M_2^{\ 2} = \frac{2+(\gamma-1)M_1^{\ 2}}{2\gamma M_1^{\ 2}-(\gamma-1)}. \tag{89.9}$$

This is symmetrical in M_1 and M_2: it may be written in the form

$$2\gamma M_1^{\ 2} M_2^{\ 2} - (\gamma-1)(M_1^{\ 2}+M_2^{\ 2}) = 2.$$

We can give limiting results for **very strong shock waves**, in which $(\gamma-1)p_2$ is very large compared with $(\gamma+1)p_1$. From (89.1) and (89.2) we have

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1}, \quad \frac{T_2}{T_1} = \frac{(\gamma-1)p_2}{(\gamma+1)p_1}. \tag{89.10}$$

The ratio T_2 / T_1 increases to infinity with p_2 / p_1, i.e., the temperature discontinuity in a shock wave, like the pressure discontinuity, can be arbitrarily great. The density ratio, however, tends to a constant limit; e.g., for a monatomic gas the limit is $\rho_2 = 4\rho_1$, and for a diatomic gas $\rho_2 = 6\rho_1$. The velocities of propagation of a strong shock wave are

$$v_1 = \sqrt{\frac{\gamma+1}{2} p_2 V_1}, \quad v_2 = \sqrt{\frac{(\gamma-1)^2}{2(\gamma+1)} p_2 V_1}. \tag{89.11}$$

They increase as the square root of the pressure p_2.

Lastly, there are relations for **weak shock waves**, which are the leading terms in expansions in powers of the small quantity $z \equiv \dfrac{p_2 - p_1}{p_1}$:

$$\left.\begin{aligned} M_1 - 1 = 1 - M_2 &= \frac{(\gamma+1)z}{4\gamma} \\ \frac{c_2}{c_1} &= 1 + \frac{(\gamma-1)z}{2\gamma} \\ \frac{\rho_2}{\rho_1} &= 1 + \frac{z}{\gamma} - \frac{(\gamma-1)z^2}{2\gamma} \end{aligned}\right\} \tag{89.12}$$

These are the terms giving the first correction to the acoustic approximation.

PROBLEMS

Problem 1. Derive the formula $v_1 v_2 = c_*^{\ 2}$, where c_* is the critical velocity (L. Prandtl).

Solution. Since $w + \dfrac{1}{2}v^2$ is continuous at a shock wave, we can define a critical velocity which is the same for gases 1 and 2 by

$$\frac{\gamma p_1}{(\gamma-1)\rho_1} + \frac{1}{2}v_1^{\ 2} = \frac{\gamma p_2}{(\gamma-1)\rho_2} + \frac{1}{2}v_2^{\ 2} = \frac{\gamma+1}{2(\gamma-1)} c_*^{\ 2};$$

cf. (83.7). Determining p_2/ρ_2 and p_1/ρ_1 from these equations and substituting in

$$v_1 - v_2 = \frac{p_2}{\rho_2 v_2} - \frac{p_1}{\rho_1 v_1}$$

(obtained by combining (85.1) and (85.2)), we obtain

$$\frac{\gamma+1}{2\gamma}(v_1 - v_2)\left(1 - \frac{{c_*}^2}{v_1 v_2}\right) = 0 .$$

Since $v_1 \neq v_2$, this gives the required relation.

Problem 2. Determine the value of the ratio p_2/p_1, for given temperatures T_1, T_2 at a shock wave in a perfect gas with a variable specific heat.

Solution. For such a gas, we can say only that w (like ε) is a function of temperature alone, and that p, V and T are related by the equation of state $pV = RT/\mu$. Solving equation (85.9) for p_2/p_1, we obtain

$$\frac{p_2}{p_1} = \frac{\mu}{RT_1}(w_2 - w_1) - \frac{T_2 - T_1}{2T_1} + \sqrt{\left[\frac{\mu(w_2 - w_1)}{RT} - \frac{T_2 - T_1}{2_1}\right]^2 + \frac{T_2}{T_1}} ,$$

where $w_1 = w(T_1)$, $w_2 = w(T_2)$.

§90. Corrugation instability of shock waves

The conditions for a shock wave to be **evolutionary** are necessary, but not sufficient, to ensure that it is stable. It may be unstable with respect to perturbations having periodicity on the surface of discontinuity and thus forming "**ripples**" or "**corrugations**" on that surface; such perturbations have already been discussed in §29 for the case of tangential discontinuities.[23] We shall show how this topic may be investigated for shock waves in any medium (S. P. D'yakov 1954).

Let a shock wave be at rest on the plane $x = 0$, with the fluid passing through it from left to right, in the positive x-direction. Let the surface of discontinuity undergo a perturbation in which the points on the surface are displaced in the x-direction by a small amount

$$\zeta = \zeta_0 \exp(ik_y y - i\omega t) , \qquad (90.1)$$

where k_y is the wave number of the ripple. This surface ripple perturbs the flow behind the shock wave in the region $x > 0$; that in front of the discontinuity, in $x < 0$, is not perturbed, because of its supersonic velocity.

Any perturbation of the flow is composed of an entropy-vortex wave and a sound wave (see §82, Problem). In each, the dependence of quantities on time and coordinates is given by a factor having the form $\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t)$ with the same frequency ω as in (90.1). It is evident from symmetry that the wave vector $\boldsymbol{k}$ is in the xy-plane; its y-component is the same as k_y in (90.1), but the x-component is different for the **two types of perturbation**.

In the **entropy-vortex wave**, $\boldsymbol{k}\cdot\boldsymbol{v}_2 = \omega$, so that $k_x = \omega/v_2$, where v_2 is the unperturbed gas velocity beyond the discontinuity. In this wave there is no pressure perturbation; the specific volume perturbation arises from the entropy perturbation, $\delta V^{(ent)} = \left(\frac{\partial V}{\partial s}\right)_p \delta s$, and the velocity perturbation satisfies the condition

$$\boldsymbol{k}\cdot\delta\boldsymbol{v}^{(ent)} = \frac{\omega}{v_2}\delta {v_x}^{(ent)} + k_y \delta {v_y}^{(ent)} = 0 . \qquad (90.2)$$

In the **sound wave** in the moving gas, the relation between the frequency and the wave vector is $(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2 = c^2 k^2$ (see (68.1)); k_x in this wave is

[23] Instability with respect to such perturbations is called ***corrugation instability***.

therefore given by

$$(\omega - k_x v_2)^2 = c_2{}^2(k_x{}^2 + k_y{}^2) . \qquad (90.3)$$

The perturbations of the pressure, the specific volume, and the velocity are related by

$$\delta p^{(s)} = -\left(\frac{c_2}{V_2}\right)^2 \delta V^{(s)} , \qquad (90.4)$$

$$(\omega - v_2 k_x)\delta \boldsymbol{v}^{(s)} = V_2 \boldsymbol{k} \delta p^{(s)} . \qquad (90.5)$$

The total perturbation is a linear combination of perturbations of each type:

$$\delta v = \delta v^{(ent)} + \delta v^{(s)} , \quad \delta V = \delta V^{(ent)} + \delta V^{(s)} , \quad \delta p = \delta p^{(s)} . \qquad (90.6)$$

It has to satisfy certain boundary conditions on the perturbed surface of discontinuity.

Firstly, the tangential velocity component must be continuous on this surface, and the discontinuity of the normal component must be expressible in terms of the perturbed pressure and density by (85.7). These conditions are

$$\boldsymbol{v}_1 \cdot \boldsymbol{t} = (\boldsymbol{v}_2 + \delta \boldsymbol{v}) \cdot \boldsymbol{t} ,$$

$$v_1 \cdot n - (v_2 + \delta v) \cdot n = \sqrt{(p_2 - p_1 + \delta p)(V_1 - V_2 - \delta V)} ,$$

where $\boldsymbol{t}$ and $\boldsymbol{n}$ are unit vectors along the tangent and normal to the surface of discontinuity (Fig. 59). As far as first-order small quantities, the components

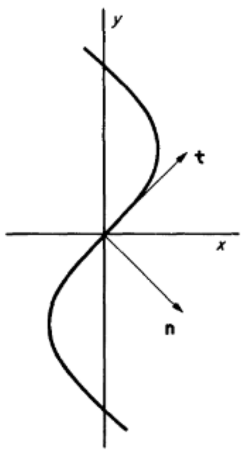


FIG. 59

of these vectors in the xy-plane are $\boldsymbol{t}(ik\zeta, 1)$ and $\boldsymbol{n}(1, -ik\zeta)$; the expression $ik\zeta$ arises as the derivative $\partial\zeta / \partial y$. To the same accuracy, the boundary conditions for the velocity are

$$\delta v_y = ik\zeta(v_1 - v_2) , \quad \delta v_x = \frac{1}{2}(v_2 - v_1)\left[\frac{\delta p}{p_2 - p_1} - \frac{\delta V}{V_1 - V_2}\right]. \qquad (90.7)$$

Next, the perturbed values $p_2 + \delta p$ and $V_2 + \delta V$ must satisfy the same Hugoniot adiabatic equation as the unperturbed p_2 and V_2. This gives the relation between δp and δV :

$$\delta p = \frac{dp_2}{dV_2} dV , \qquad (90.8)$$

the derivative being taken along the adiabatic.

Lastly, one further relation arises from that between the mass flux through the surface of discontinuity and the pressure and density discontinuities there. For the unperturbed surface, this relation is given by (85.6); for the perturbed surface, the corresponding one is

$$\frac{1}{V_1^2}(\boldsymbol{v}_1\cdot\boldsymbol{n}-\boldsymbol{u}\cdot\boldsymbol{n})^2=\frac{p_2-p_1+\delta p}{V_1-V_2-\delta V},$$

where $\boldsymbol{u}$ is the velocity of points on the surface. In the first approximation with respect to small quantities we have $\boldsymbol{u}\cdot\boldsymbol{n}=-i\omega\zeta$; expanding the equation also in powers of δp and δV , we find

$$\frac{2i\omega}{v_1}\zeta=\frac{\delta p}{p_2-p_1}+\frac{\delta V}{V_1-V_2}. \qquad (90.9)$$

The equations (90.2), (90.4), (90.5), and (90.7)-(90.9) form a set of eight linear algebraic equations for the eight quantities ζ, δp, $\delta V^{(ent)}$, $\delta V^{(s)}$, $\delta v_{x,y}{}^{(ent)}$ and $\delta v_{x,y}{}^{(s)}$.[24] The compatibility condition of these equations (expressed by the vanishing of the determinant of their coefficients) is

$$\frac{2\omega v_2}{v_1}\left(k_y{}^2+\frac{\omega^2}{v_2{}^2}\right)-\left(\frac{\omega^2}{v_1v_2}+k_x{}^2\right)(\omega-v_2k_y)(1+h)=0, \qquad (90.10)$$

where for brevity $h=j^2\left(\frac{dV_2}{dp_2}\right)$ and j as usual denotes $\frac{v_1}{V_1}=\frac{v_2}{V_2}$. In (90.10) k_x is to be taken as the function of k_y and ω that is determined by (90.3).

The instability condition is that perturbations exist which grow exponentially with time, and they must decay exponentially away from the surface of discontinuity (i.e., as $x\to\infty$); the latter condition means that the perturbation source is the shock wave itself, and not something outside it. The wave is thus unstable if (90.10) has solutions for which

$$\operatorname{Im}\omega>0,\ \operatorname{Im}k_x>0. \qquad (90.11)$$

The analysis of equation (90.10) to ascertain the conditions for such solutions to exist is fairly laborious. We shall not give it here, but mention only the final result.[25] The corrugation instability of a shock wave occurs if

$$j^2\frac{dV_2}{dp_2}<-1, \qquad (90.12)$$

or

$$j^2\frac{dV_2}{dp_2}>1+2\frac{v_2}{c_2}; \qquad (90.13)$$

the derivative is, as already stated, taken along the shock adiabatic for given p_1 and V_1.[26]

The conditions (90.12) and (90.13) correspond to the existence of complex roots of (90.10), satisfying (90.11). Under certain conditions, however, this equation may also have roots with real ω and k_x , corresponding to actual undamped sound and entropy waves leaving the discontinuity, i.e., to the spontaneous emission of sound by the surface of discontinuity. This will be referred to as a special form of shock wave instability, although there is here no instability in the literal sense; the perturbation (ripples), once created on the surface, continues indefinitely to emit waves without being either damped or amplified; the energy carried away by the emitted waves is drawn from the whole of the medium in

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[24] All these equations are taken for x = 0, and the enumerated quantities in them may be regarded as the constant amplitudes, without the variable exponential factors.

[25] The analysis is reported by S. P. D'yakov, *Zhurnal eksperimentalnoi i teoreticheskoi fiziki* 27, 288,1954. In §91, a less rigorous but more easily understandable derivation of the conditions (90.12) and (90.13) is given.

[26] The derivation of (90.12) and (90.13) uses only the obligatory condition (88.1), not the inequality $p_2>p_1$. These instability conditions therefore apply also to rarefaction shocks, which can exist if $(\partial^2V/\partial p^2)_s<0$.

motion.[27]

To determine the conditions for this phenomenon to occur, we transform (90.10) by using the angle θ between $\boldsymbol{k}$ and the x-axis; then

$$\left.\begin{aligned} c_2 k_x &= \omega_0 \cos\theta \\ c_2 k_y &= \omega_0 \sin\theta \\ \omega &= \omega_0\left[1+\frac{v_2}{c_2}\cos\theta\right] \\ {\omega_0}^2 &= {c_2}^2({k_x}^2+{k_y}^2) \end{aligned}\right\}, \tag{90.14}$$

where ω_0 is the sound frequency in coordinates moving with the gas behind the shock wave, and we obtain an equation quadratic in $\cos\theta$:

$$\begin{aligned} &\frac{{v_2}^2}{{c_2}^2}\left[\frac{4}{1+h}+\frac{v_1}{v_2}-1\right]\cos^2\theta+\frac{2v_2}{c_2}\left[\frac{3+(v_2/c_2)^2}{1+h}-1\right]\cos\theta \\ &+\frac{2\left[1+(v_2/c_2)^2\right]}{1+h}-\left(1+\frac{v_1 v_2}{{c_2}^2}\right)=0 \end{aligned}. \tag{90.15}$$

The velocity of the sound wave in the gas moving with velocity v_2, relative to the discontinuity surface at rest, is $v_2+c_2\cos\theta$. The sound wave leaves the surface if this sum is positive, i.e., if

$$-\frac{v_2}{c_2}<\cos\theta<1, \tag{90.16}$$

(values of $\cos\theta<0$ correspond to cases where $\boldsymbol{k}$ is towards the discontinuity but the transport of the sound wave by the moving gas makes it still a wave that leaves the discontinuity). The spontaneous emission of sound by the shock wave occurs if (90.15) has a root in this range. A simple analysis yields the following inequalities determining the range of the instability:[28]

$$\frac{1-\dfrac{{v_2}^2}{{c_2}^2}-\dfrac{v_1 v_2}{{c_2}^2}}{1-\dfrac{{v_2}^2}{{c_2}^2}+\dfrac{v_1 v_2}{{c_2}^2}}<j^2\frac{dV_2}{dp_2}<1+2\frac{v_2}{c_2}; \tag{90.17}$$

the lower and upper limits here correspond almost exactly to those in (90.16). The range (90.17) is adjacent to and extends the instability range (90.13).

The origin of shock wave instability in the range (90.17) can also be approached from a somewhat different standpoint by considering the reflection from the discontinuity surface of sound incident on it from the compressed-gas side. Since the shock is moving with supersonic velocity relative to the gas in front of it, the sound does not penetrate into that gas. In the gas behind the shock, we have not only the incident sound wave but also a reflected sound wave and an entropy-vortex wave (and ripples are formed on the discontinuity surface itself). The problem of determining the reflection coefficient is formulated similarly to the instability problem. The only difference is that the boundary conditions contain, as well as the amplitudes to be determined for the (reflected) waves leaving the discontinuity, the specified amplitude of the incident sound wave. Instead of a set of homogeneous algebraic equations, we now have a set of **inhomogeneous** equations in which the inhomogeneous terms involve the incident wave

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[27] Compare the analogous situation for tangential discontinuities (§84, Problem 2).

[28] This instability also was noted by S. P. D'yakov (1954); the correct value of the lower limit in (90.17) was given by V. M. Kontorovich (1957).

amplitude. The solution is given by expressions whose denominators contain the determinant of the homogeneous equations, whose vanishing gives the dispersion relation (90.10) for spontaneous perturbations. The fact that in the range (90.17) this equation has real roots for $\cos\theta$ means that there are certain values of the reflection angle (and therefore of the incidence angle) at which the reflection coefficient becomes infinite. This is another way of stating the possibility of spontaneous emission of sound, i.e., emission with no externally incident sound wave.

The same applies to the transmission coefficient for sound incident from the front on the surface of discontinuity. Here, there is no reflected wave; behind the surface, there are transmitted sound and entropy-vortex waves. In the range (90.17), the transmission coefficient can become infinite.[29]

A few words may be said about some types of shock adiabatic, possible in principle, which have instability regions as discussed above.[30]

The condition (90.12) requires the derivative dp_2/dV_2 to be negative, and the shock adiabatic at point 2 must be inclined to the abscissa axis less steeply than the chord 12 through that point; this is the opposite of the usual case (Fig. 53, §85). For this to be possible, the adiabatic must have the form shown in Fig. 60. The instability condition (90.12) is satisfied on the section *ab*.

The condition (90.13) requires dp_2/dV_2 to be positive, and the slope of the adiabatic must be sufficiently small. In Fig. 60, this is true on certain sections of the adiabatic immediately adjacent to the points *a* and *b*, thus extending the instability range. The condition (90.13) may also be satisfied on a section (*cd* in Fig. 61) of an adiabatic that does not have a section of the type *ab*.

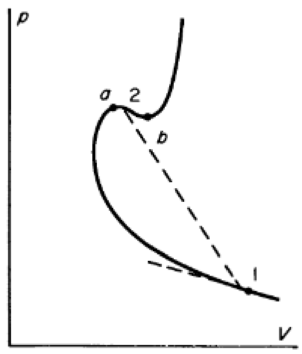

FIG. 60

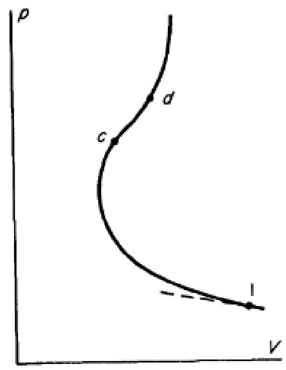

FIG. 61

The condition (90.17) is even less stringent than (90.13), and further extends the instability range on Hugoniot adiabatics having $dp_2/dV_2 > 0$. Moreover, the lower limit in (90.17) may be negative, so that instability of this type can in principle occur also on some sections of ordinary-type adiabatics having $dp_2/dV_2 < 0$ everywhere.

The **ultimate behaviour** of shock waves having corrugation instability is closely related to the following noteworthy fact. When the condition (90.12)

[29] The calculation of the sound reflection and transmission coefficients at a shock wave for any direction of incidence and in any media is given by S. P. D'yakov, *Soviet Physics JETP* 6, 729, 739, 1958; V. M. Kontorovich, *ibid.* 1180; *Soviet Physics Acoustics 5,* 320, 1959.

[30] In a polytropic gas $h = -(c_1/v_1)^2$, as is easily seen from the expressions derived in §89. None of the conditions (90.12), (90.13), and (90.17) is then satisfied, and so the shock wave is stable. Of course, weak shock waves in any medium are also stable.

or (90.13) is satisfied, the equations of fluid dynamics have **more than one solution** (C. S. Gardner 1963). For two states 1 and 2 of the medium, related by (85.1)-(85.3), the shock wave is usually the only solution of the (one-dimensional) problem of a flow which takes the medium from state 1 to state 2. It is found that, if one of the conditions (90.12) and (90,13) is satisfied, the solution of the problem is **not unique**: the transition from state 1 to state 2 can be brought about not only in a shock wave but also through a more complex system of waves. This **second or decay solution** consists of a weaker shock wave, a following contact discontinuity, and an isentropic non-steady rarefaction wave (see §99 below) propagated in the opposite direction relative to the gas behind the shock; in the shock wave, the entropy increases from s_1 to a value $s_3 < s_2$, and the further increase to s_2 takes place abruptly in the contact discontinuity. This is the picture for the type shown in Fig. 78b (§100); the inequality (86.2) is assumed satisfied.[31]

The question of **what determines the choice of one of the two solutions** in specific problems of fluid dynamics is **not yet resolved**. If the decay solution is chosen, this would mean that the instability of the shock wave with spontaneous amplification of surface ripples does not occur at all. It appears, however, that the choice may not be related to this instability, since the non-uniqueness of the solution is not limited by the conditions (90.12) and (90.13).[32]

PROBLEMS

Problem 1. A plane sound wave is incident normally from the rear (the compressed-gas side) on a shock wave. Determine the sound reflection coefficient.

Solution. We consider the process in coordinates for which the shock wave is at rest and the gas moves through it in the positive *x*-direction; the incident sound wave is propagated in the negative *x*-direction. For normal incidence (and therefore normal reflection) the velocity in the reflected entropy wave is $\delta v^{(ent)} = 0$. The pressure perturbation is $\delta p = \delta p^{(s)} + \delta p^{(0)}$, where the superscripts 0 and *s* refer to the incident and reflected sound waves, respectively. The velocity $\delta v_x \equiv \delta v$ is

$$\delta v = \frac{V_2}{c_2}(\delta p^{(s)} - \delta p^{(0)});$$

the difference appears instead of the sum, because of the opposite directions of propagation of the two waves. The second boundary condition (90.7) has its previous form (but now with $\delta V = \delta V^{(0)} + \delta V^{(s)} + \delta V^{(ent)}$; with (90.8) and (85.6), it can be written as

$$\delta v = -\frac{1-h}{2j}(\delta p^{(s)} + \delta p^{(0)}).$$

Equating the two expressions for δv, we obtain as the required ratio of the pressure amplitudes in the reflected and incident sound waves

$$\frac{\delta p^{(s)}}{\delta p^{(0)}} = -\frac{1-2M_2-h}{1+2M_2-h}, \qquad (1)$$

where $M_2 = \dfrac{v_2}{c_2}$. This becomes infinite at the upper boundary of the range

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[31] This is shown for the range (90.13) in Gardner's paper (*Physics of Fluids 6,* 1366,1963). A more general treatment including the range (90.12) has been given by N. M. Kuznetsov (*Soviet Physics JETP* 61, 275,1985), who also discusses shock adiabatics for which $(\partial^2 V/\partial p^2)_s$ is not positive and the decay solutions are made up of other sets of waves.

[32] It seems that the non-uniqueness range extends on the shock adiabatic somewhat beyond the limits of the instability range given by these conditions; see Kuznetsov's paper cited in the last footnote.

(90.17).

For a polytropic gas, $h = -\frac{1}{M_1^2}$. When the shock wave is weak ($p_2 - p_1 << p_1$), the ratio (1) tends to zero as $(p_2 - p_1)^2$, and in the opposite case of a strong shock it tends to a constant,

$$\frac{\delta p^{(s)}}{\delta p^{(0)}} \cong -\frac{\sqrt{\gamma} - \sqrt{2\gamma - 2}}{\sqrt{\gamma} + (2\gamma - 2)}.$$

PROBLEM 2. A plane sound wave is incident normally from the front on a shock wave. Determine the sound transmission coefficient.[33]

Solution. The perturbation in gas 1 in front of the shock wave is

$$\delta p_1 = \delta p^{(0)}, \ \delta V_1 = \delta V^{(0)} = -\frac{V_1^2}{c_1^2}\delta p_1, \ \delta v_1 = \frac{V_1}{c_1}\delta p_1,$$

and in gas 2 behind it

$$\delta p_2 = \delta p^{(s)}, \ \delta V_2 = \delta V^{(s)} + \delta V^{(ent)}, \ \delta v_2 = \frac{V_2}{c_2}\delta p_2,$$

the superscripts 0, *s* and *ent* refer to the incident sound wave, the transmitted sound wave, and the transmitted entropy wave, respectively. The perturbations δp_2 and δV_2 are related in a way which follows from the equation of the shock adiabatic: if this equation is written as $V_2 = V_2(p_2; p_1, V_1)$, then

$$\begin{aligned}\delta V_1 &= \left(\frac{\partial V_2}{\partial p_2}\right)_H \delta p_2 + \left(\frac{\partial V_2}{\partial V_1}\right)_H \delta V_1 + \left(\frac{\partial V_2}{\partial p_1}\right)_H \delta p_1 \\ &= \left(\frac{\partial V_2}{\partial p_2}\right)_H \delta p_2 + \left[-\frac{V_1^2}{c_1^2}\left(\frac{\partial V_2}{\partial V_1}\right)_H + \left(\frac{\partial V_2}{\partial p_1}\right)_H\right]\delta p_1\end{aligned},$$

where the suffix *H* to the derivatives means that they are taken along the Hugoniot adiabatic.[34] The boundary condition (90.7) is now replaced by

$$\begin{aligned}\delta v_2 - \delta v_1 &= -\frac{1}{2}(v_1 - v_2)\left[\frac{\delta p_2 - \delta p_1}{p_2 - p_1} - \frac{\delta V_2 - \delta V_1}{V_1 - V_2}\right] \\ &= -\frac{1}{2j}\left[\delta p_2 - \delta p_1 - j^2(\delta V_2 - \delta V_1)\right]\end{aligned}.$$

Equating the two expressions for $\delta v_2 - \delta v_1$, we obtain as the required ratio of amplitudes in the transmitted and incident sound waves

$$\frac{\delta p^{(s)}}{\delta p^{(0)}} = \frac{(1 + M_1)^2 + q}{1 + 2M_2 - h}, \tag{2}$$

where *h* is as before and

$$q = j^2\left[-\left(\frac{V_1}{c_1}\right)^2\left(\frac{\partial V_2}{\partial V_1}\right)_H + \left(\frac{\partial V_2}{\partial p_1}\right)_H\right].$$

For a **polytropic** gas,

$$q = -\frac{\gamma - 1}{\gamma + 1}\frac{(M_1 - 1)^2}{M_1^2}$$

and the transmission coefficient is

[33] This problem was discussed for a polytropic gas by D. I. Blokhintsev (1945) and J. M. Burgers (1946).

[34] The derivative $(\partial V_2 / \partial p_2)_H$ is what was previously denoted by dV_2 / dp_2 simply, it being implied that the derivative is taken at constant p_1 and V_1.

$$\frac{\delta p^{(s)}}{\delta p^{(0)}} = \frac{(1+M_1)^2}{1+2M_2+1/M_1{}^2}\left[1-\frac{\gamma-1}{\gamma+1}\left(1-\frac{1}{M_1}\right)^2\right].$$

For a **weak shock**, this gives

$$\frac{\delta p^{(s)}}{\delta p^{(0)}} \cong 1+\frac{\gamma+1}{2\gamma}\frac{p_2-p_1}{p_1},$$

and in the opposite case of a **strong shock**

$$\frac{\delta p^{(s)}}{\delta p^{(0)}} \cong \frac{1}{\gamma+\sqrt{2\gamma(\gamma-1)}}\frac{p_2}{p_1}.$$

In either case, the pressure amplitude in the transmitted sound wave is greater than in the incident wave.