§91. Shock wave propagation in a pipe

Let us consider the propagation of a shock wave in a medium occupying a long pipe with variable cross-section. The aim is to ascertain the influence of the changing area of the shock wave on its velocity (G. B. Whitham 1958).

We shall assume that the area $S(x)$ of the pipe cross-section varies only slowly along its length (in the x-direction), i.e., only slightly over distances of the order of the pipe width. This enables us to apply the ***hydraulic approximation*** already used in §77; all quantities in the flow may be assumed constant over any cross-section of the tube, and the velocity axial; the flow is thus regarded as **quasi-one-dimensional**. Such a flow is described by the equations

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0\,, \tag{91.1}$$

$$\frac{\partial p}{\partial t}+v\frac{\partial p}{\partial x}-c^2\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)=0\,, \tag{91.2}$$

$$S\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v S)=0\,. \tag{91.3}$$

The first is **Euler's equation**, the second is the **adiabatic equation**, and the third is the **continuity equation** in the form (77.1).

To elucidate the question raised, it is sufficient to consider a pipe in which the variation of $S(x)$ not only is slow but also remains relatively small in magnitude over the whole length. Then the flow perturbations due to the non-constancy of the cross-section will also be small, and equations (91.1)-(91.3) can be linearized. Lastly, initial conditions are to be imposed, which exclude the occurrence of any extraneous perturbations that might influence the motion of the shock wave, since we are interested only in the perturbations due to the change in $S(x)$. This can be achieved by assuming that the shock wave moves originally with constant velocity along a pipe with constant cross-section, which begins to vary only to the right of a certain point taken as $x = 0$.

The **linearized equations** (91.1)-(91.3) are

$$\frac{\partial \delta v}{\partial t}+v\frac{\partial \delta v}{\partial x}+\frac{1}{\rho}\frac{\partial \delta p}{\partial x}=0\,,$$

$$\frac{\partial \delta p}{\partial t}+v\frac{\partial \delta p}{\partial x}-c^2\left(\frac{\partial \delta\rho}{\partial t}+v\frac{\partial \delta\rho}{\partial x}\right)=0\,,$$

$$\frac{\partial \delta\rho}{\partial t}+v\frac{\partial \delta\rho}{\partial x}+\rho\frac{\partial \delta v}{\partial x}+\frac{\rho v}{S}\frac{\partial \delta S}{\partial x}=0\,,$$

where the letters without prefix denote the constant values in the homogeneous flow in the uniform part of the pipe, and δ denotes the changes in these quantities in the pipe with variable cross-section. Multiplying the first and third equations by ρc and c^2, respectively, and then adding all three, we can write

$$\left[\frac{\partial}{\partial t}+(v+c)\frac{\partial}{\partial x}\right](\delta p+\rho c\delta v)=-\frac{\rho v c^2}{S}\frac{\partial \delta S}{\partial x}. \qquad (91.4)$$

The general solution of this equation is given by the sum of the general solution of the homogeneous equation and a particular solution of the equation as it stands. The former is $F(x - vt - ct)$, where F is any function; this describes sound disturbances coming from the left. In the uniform part $x < 0$, however, there are none, and we must therefore put $F \equiv 0$. The solution is then the integral of the inhomogeneous equation

$$\delta p+\rho c\delta v=-\frac{\rho v c^2}{v+c}\frac{\delta S}{S}. \qquad (91.5)$$

The shock wave moves from left to right with velocity $v_1 > c_1$ in the medium at rest with specified values of p_1 and ρ_1. The motion behind the shock in medium 2 is given by the solution (91.5) throughout the part of the pipe to the left of the point reached by the discontinuity at a given instant. After the shock has passed, all quantities in each cross-section of the pipe remain constant in time and equal to the values they had at the moment of passage: the pressure p_2, the density ρ_2, and the velocity $v_1 - v_2$ (in accordance with the notation used in this chapter, v_2 denotes the gas relative to the moving shock wave; its velocity relative to the pipe walls is then $v_1 - v_2$). With this notation, and again separating the variable parts of these quantities, the equation (91.5) can be written

$$\frac{\delta S}{S}=-\frac{v_1-v_2+c_2}{\rho_2(v_1-v_2)c_2{}^2}\left[\delta p_2+\rho_2 c_2(\delta v_1-\delta v_2)\right]. \qquad (91.6)$$

All the quantities $\delta v_1, \delta v_2, \delta p_2$ can be expressed in terms of one of them, say δv_1. For this purpose, we write the varied relations (85.1), (85.2) at the discontinuity (for given p_1 and ρ_1):

$$\rho_1\delta v_1=v_2\delta\rho_2+\rho_2\delta v_2,\quad 2j(\delta v_1-\delta v_2)=\delta p_2+v_2{}^2\delta\rho_2,$$

where $j=\rho_1 v_1=\rho_2 v_2$ is the unperturbed mass flux; and we use also the relation

$$\delta p_2=\left(\frac{dp_2}{d\rho_2}\right)\delta\rho_2,$$

where the derivative is taken along the Hugoniot adiabatic. The calculation gives the following final relation between the change δv_1 in the shock wave velocity relative to the gas at rest in front of it and the change δS in the cross-sectional area of the pipe:

$$-\frac{1}{S}\frac{\delta S}{\sigma v_1}=\frac{v_1-v_2+c_2}{v_1 c_2}\left[\frac{1+2v_2/c_2-h}{1+h}\right], \qquad (91.7)$$

again with the notation

$$h=-\left(\frac{j}{\rho_2}\right)^2\frac{d\rho_2}{dp_2}=j^2\frac{dV_2}{dp_2}. \qquad (91.8)$$

The coefficient of the square brackets in (91.7) is positive. The sign of $\delta v_1/\delta S$ is therefore determined by that of the expression in the brackets. For all stable shocks it is

positive, and so $\delta v_1 / \delta S < 0$. When either of the conditions (90.12), (90.13) for corrugation instability is satisfied, however, the expression in the brackets becomes negative, and so $\delta v_1 / \delta S > 0$.

This leads to an **intuitive interpretation of the origin of the instability**. Figure 62 shows the corrugated surface of the shock wave, moving to the right; the arrows show schematically the direction of the streamlines. As the shock moves, the area δS increases on the parts of the surface that project forwards, and decreases on those that lag behind. When $\delta v_1 / \delta S < 0$, this retards the forward projections and accelerates the lagging parts, so that the surface tends to smooth out. When $\delta v_1 / \delta S > 0$, on the other hand, the perturbation of the surface is intensified, the forward projections moving further forward, and the lagging parts being left further behind.[1]

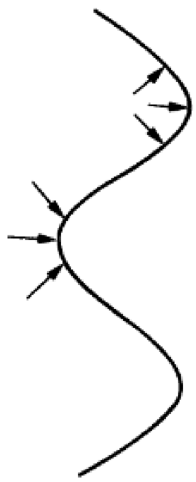


FIG. 62

§92. Oblique shock waves

Let us consider a steady shock wave, and abandon the system of coordinates used hitherto, in which the gas velocity is perpendicular to the shock surface element considered. The streamlines can intersect the surface of such a shock wave at any angle, and in so doing are refracted: the tangential component of the gas velocity is unchanged, while the normal component is, according to (87.4), diminished: $v_{1t} = v_{2t}$, $v_{1n} > v_{2n}$. It is therefore clear that the streamlines approach the shock wave as they pass through it (cf. Fig. 63). Thus the streamlines are always refracted in a definite direction in passing through the shock wave.

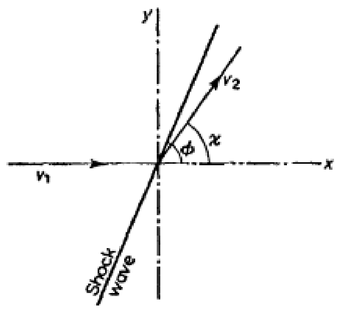


FIG. 63

We take as the x-axis the direction of the gas velocity $\boldsymbol{v}_1$ in front of the shock wave; let ϕ be the angle between the surface of discontinuity and the x-axis (Fig. 63).

[1] The expression (91.7) for any (not polytropic) medium and its relationship to the conditions for corrugation instability of shock waves were noted by S. G. Sugak, V. E. Fortov, and A. L. Ni (1981).

The possible values of ϕ are restricted only by the condition that the normal component of $\boldsymbol{v}_1$ be greater than the velocity of sound c_1. Since $v_{1n} = v \sin\phi$, it follows that ϕ can have any value in the range between $\pi/2$ and the Mach angle α_1:

$$\alpha_1 < \phi < \pi/2\,, \quad \sin\alpha_1 = \frac{c_1}{v_1} \equiv \frac{1}{M_1}\,.$$

The motion behind a shock wave may be either subsonic or supersonic (only the normal velocity component need be less than the velocity of sound c_2); the motion in front of it is necessarily supersonic. If the gas flow on both sides is supersonic, every disturbance must be propagated along the surface in the direction of the tangential component of the gas velocity. In this sense we can speak of the "direction" of a shock wave, and distinguish shock waves leaving and reaching any point (as we did for characteristics, the motion near which is always supersonic; see §82). If the motion behind the shock is subsonic, there is strictly no meaning in speaking of its "direction", since disturbances can be propagated in all directions on its surface.

We shall derive a relation between the two components of the gas velocity after it has passed through an oblique shock wave, supposing that we have a polytropic gas. The continuity of the velocity component tangential to the shock means that $v_1 \cos\phi = v_{2x} \cos\phi + v_{2y} \sin\phi$, or

$$\tan\phi = \frac{v_1 - v_{2x}}{v_{2y}}\,. \tag{92.1}$$

Next we use formula (89.6), in which v_1 and v_2 denote the velocity components normal to the plane of the shock wave and must be replaced by $v_1 \sin\phi$ and $v_{2x} \sin\phi - v_{2y} \cos\phi$, so that

$$\frac{v_{2x} \sin\phi - v_{2y} \cos\phi}{v_1 \sin\phi} = \frac{\gamma - 1}{\gamma + 1} + \frac{2{c_1}^2}{(\gamma+1){v_1}^2 \sin^2\phi}\,. \tag{92.2}$$

We can eliminate the angle ϕ from these two relations. After some simple transformations, we obtain the following formula which determines the relation between v_{2x} and v_{2y} (for given v_1 and c_1):

$${v_{2y}}^2 = (v_1 - v_{2x})^2 \, \frac{\dfrac{2}{\gamma+1}\left(v_1 - \dfrac{{c_1}^2}{v_1}\right) - (v_1 - v_{2x})}{v_1 - v_{2x} + \dfrac{2}{\gamma+1}\dfrac{{c_1}^2}{v_1}}\,. \tag{92.3}$$

This formula can be more intelligibly written by introducing the critical velocity. According to Bernoulli's equation and the definition of the critical velocity, we have

$$w_1 + \frac{1}{2}{v_1}^2 = \frac{1}{2}{v_1}^2 + \frac{1}{\gamma-1}{c_1}^2 = \frac{\gamma+1}{2(\gamma-1)}{c_*}^2\,,$$

(see §89, Problem 1), whence

$$c_*^2 = \frac{(\gamma-1)v_1^2 + 2c_1^2}{\gamma+1}. \tag{92.4}$$

Using this in (92.3), we obtain

$$v_{2y}^2 = (v_1 - v_{2x})^2 \frac{v_1 v_{2x} - c_*^2}{\frac{2}{\gamma+1} v_1^2 - v_1 v_{2x} + c_*^2}. \tag{92.5}$$

Equation (92.5) is called the equation of the ***shock polar*** (A. Busemann 1931). Figure 64 shows a graph of the function $v_{2y}(v_{2x})$; it is a cubic curve, called a

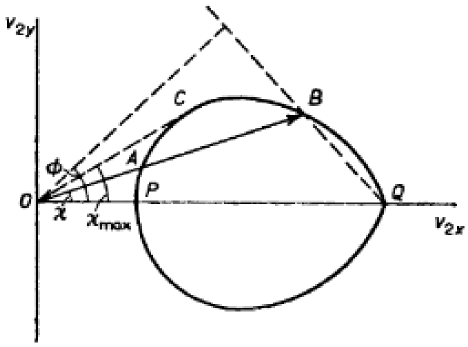
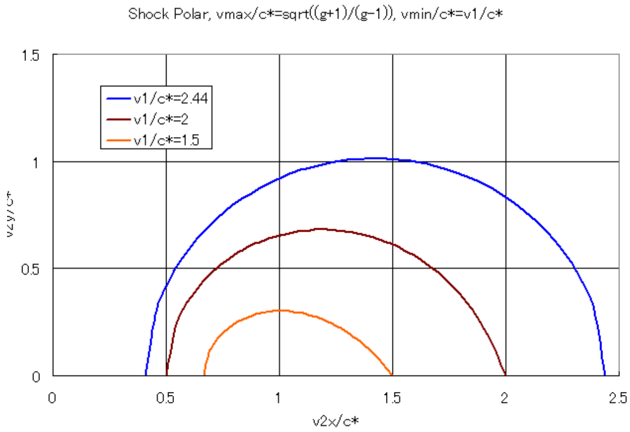


FIG. 64



strophoid. It crosses the axis of abscissae at the points P and Q, corresponding to $v_{2x} = c_*^2 / v_1$ and $v_{2x} = v_1$.[2] A line (OB in Fig. 64) drawn from the origin at an angle χ to the axis of abscissae gives, by the length of the segment between O and the point where it intersects the shock polar, the gas velocity behind a discontinuity which turns the stream through an angle χ. There are two such intersections (A, B), i.e., two different shock waves correspond to a given value of χ. The direction of the shock wave also can be immediately determined from the shock polar: it is given by the direction of the perpendicular from the origin to the line QB or QA (Fig. 64 shows the angle ϕ for a shock corresponding to the point B). As χ decreases, the point A approaches P, corresponding to a normal shock ($\phi = \pi/2$) with $v_2 = c_*^2 / v_1$. The point B approaches Q; the intensity of the shock (velocity discontinuity) tends to zero, and the angle ϕ tends, as it should, to the Mach angle α_1; the tangent to the shock polar at Q makes an angle $\pi/2 + \alpha_1$ with the axis of abscissae.

From the shock polar we can immediately derive the important result that the angle of deviation χ of the stream at the shock wave cannot exceed a certain maximum

[2] The strophoid actually continues in two branches from the point $v_{2x} = v_1$ (which is a double point) to infinite $|v_{2y}|$; these are not shown in Fig. 64. They have a common vertical asymptote $v_{2x} = c_*^2 / v_1 + 2v_1/(\gamma+1)$. The points on these branches have no physical significance; they would give values for v_{2x} and v_{2y} such that $v_{2n}/v_{1n} > 1$, which is impossible.

value χ_{max}, corresponding to the tangent from O to the curve. This quantity is, of course, a function of the Mach number $M_1 = v_1 / c_1$, but we shall not give the expression for it, which is very **cumbersome**. For $M_1 = 1$, $\chi_{max} = 0$; as M_1 increases, χ_{max} increases monotonically, and tends to a finite limit as $M_1 \to \infty$. It is easy to discuss the two limiting cases. If the velocity v_1 is near to c_*, then v_2 is so also, and the angle χ is small; the equation (92.5) of the shock polar can then be written in the approximate form[3]

$$\chi^2 = \frac{(\gamma+1)(v_1 - v_2)^2 (v_1 + v_2 - 2c_*)}{2{c_*}^3}, \qquad (92.6)$$

where we have put $v_{2x} \cong v_2$, $v_{2y} \cong c_* \chi$ in view of the smallness of χ. Hence we easily find[4]

$$\chi_{max} = \frac{4\sqrt{\gamma+1}}{3\sqrt{3}} \left(\frac{v_1}{c_*} - 1 \right)^{3/2} = \frac{8\sqrt{2}}{3\sqrt{3}(\gamma+1)} (M_1 - 1)^{3/2}. \qquad (92.7)$$

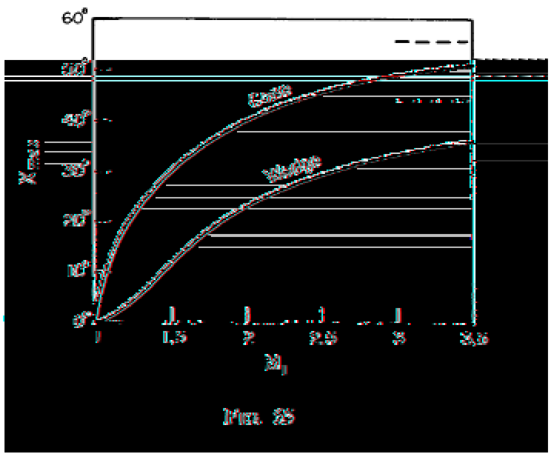
In the opposite limiting case $M_1 \to \infty$, the shock polar degenerates to a circle

$${v_{2y}}^2 = (v_1 - v_{2x}) \left(v_{2x} - \frac{\gamma - 1}{\gamma + 1} v_1 \right).$$

It is easy to see that we then have

$$\chi_{max} = \sin^{-1} \frac{1}{\gamma}. \qquad (92.8)$$

Figure 65 shows a graph of χ_{max} as a function of M_1 for air ($\gamma = 1.4$); the broken horizontal line gives the limiting value $\chi_{max}(\infty) = 45.6°$. The upper curve is a similar graph for flow past a cone (see §113).



The circle $v_2 = c_*$ cuts the axis of abscissae between the points P and Q (Fig. 64), and therefore divides the shock polar into two parts corresponding to subsonic and

[3] It is easily seen that equation (92.6) holds also for any (non-polytropic) gas, provided that $(\gamma+1)$ is replaced by $2\alpha_*$ from (102.2

[4] It may be noted that this dependence of χ_{max} on $M_1 - 1$ is in agreement with the general similarity law (126.7) for transonic flow.

supersonic gas velocities behind the discontinuity. The point where this circle crosses the polar lies to the right of, but very close to, the point *C;* the whole segment *PC* therefore corresponds to transitions to subsonic velocities, while *CQ* (except for a very small segment near *C*) corresponds to transitions to supersonic velocities

The pressure and density changes in an oblique shock wave depend only on the velocity components normal to it. The ratios p_2 / p_1 and ρ_2 / ρ_1 for given M_1 and ϕ are therefore obtained from (89.6) and (89.7) on simply replacing M_1 by $M_1 \sin\phi$:

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma}{\gamma+1}(M_1^2 \sin^2\phi - 1), \tag{92.9}$$

$$\frac{\rho_2 - \rho_1}{\rho_1} = \frac{2(M_1^2 \sin^2\phi - 1)}{(\gamma-1)M_1^2 \sin^2\phi + 2}. \tag{92.10}$$

These increase monotonically when ϕ increases from α_1 (for which $p_2 / p_1 = \rho_2 / \rho_1 = 1$) to $\pi / 2$, i.e., as we move along the shock polar from *Q* to *P*.

We may state for reference the formula giving the angle of deviation χ of the velocity in terms of M_1 and ϕ:

$$\cot\chi = \tan\phi\left[\frac{(\gamma+1)M_1^2}{2(M_1^2 \sin^2\phi - 1)} - 1\right], \tag{92.11}$$

and the formula giving the number $M_2 = v_2 / c_2$ in terms of M_1 and ϕ:

$$M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2 \sin^2\phi - (\gamma-1)} + \frac{2M_1^2 \cos^2\phi}{2+(\gamma-1)M_1^2 \sin^2\phi}; \tag{92.12}$$

when $\phi = \pi / 2$, the latter becomes (89.9).

The two shock waves determined by the shock polar for a given deviation angle χ are said to belong to the ***weak*** and ***strong*** families. A shock wave of the strong family (the segment *PC* of the polar) is **strong** (the ratio p_2 / p_1 is large), makes a large angle ϕ with the direction of the velocity $\mathbf{v}_1$, and converts the flow from supersonic to subsonic. A shock wave of the **weak** family (the segment *QC*) is weak, is inclined at a smaller angle to the stream, and almost always leaves the flow supersonic.

As an illustration. Fig. 66 shows the deviation angle χ as a function of the angle ϕ of the discontinuity surface for air ($\gamma = 1.4$), for several values of M_1, including the limit $M_1 \to \infty$. The branches shown as continuous curves correspond to shock waves of the weak family, and the broken curves to those of the strong family. The broken line $\chi = \chi_{\max}$ is the locus of the points with the maximum deviation angle for each M_1; the continuous line $M_2 = 1$ divides the regions of supersonic and subsonic flow behind the discontinuity. The narrow region between these two lines corresponds

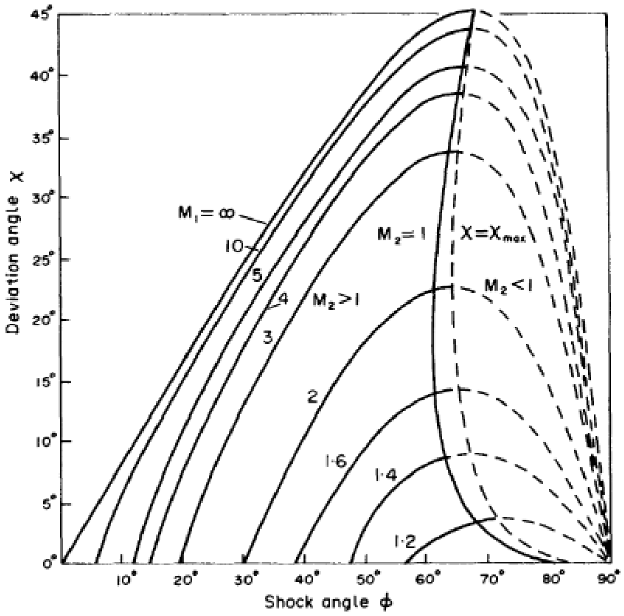


FIG. 66

to shock waves which belong to the weak family but nevertheless convert the flow from supersonic to subsonic. The difference between the values of ϕ on $\chi = \chi_{\max}$ and $M_2 = 1$ (for a given M_1) nowhere exceeds 4.5°; that between $\chi_{\max}$ and the value χ_s on the line $M_2 = 1$ (again for a given M_1) does not exceed 0.5°.[5]

§93. The thickness of shock waves

Hitherto we have regarded shock waves as geometrical surfaces with zero thickness. We shall now consider the structure of actual surfaces of discontinuity, and we shall see that shock waves in which the discontinuities are small are in reality transition layers with finite thickness, the thickness diminishing as the magnitude of the discontinuities increases. If the discontinuities are not small, the change occurs so sharply that the concept of thickness is meaningless in the macroscopic theory.

To determine the structure and thickness of the transition layer we must take account of the **viscosity** and **thermal conductivity** of the gas, which we have hitherto neglected.

The relations (85.1)-(85.3) for a shock wave were obtained from the constancy of the fluxes of mass, momentum and energy. If we consider a surface of discontinuity as a layer with finite thickness, these conditions must be written, not as the equality of the quantities concerned on the two sides of the discontinuity, but as their constancy throughout the thickness of the layer. The first condition, (85.1), is unchanged;

$$\rho v \equiv j = \text{constant} . \qquad (93.1)$$

In the other two conditions additional fluxes of momentum and energy, due to internal

[5] Detailed graphs and diagrams for the shock polar (with $\gamma = 1.4$) are given by H. W. Liepmann and A. Roshko, *Elements of Gasdynamics,* New York 1957; K. Oswatitsch, *Gas Dynamics*, New York 1956.

friction and thermal conduction, must be taken into account.

The momentum flux density (in the x-direction) due to internal friction is given by the component $-\sigma'_{xx}$ of the viscous stress tensor; according to the general expression (15.3) for this tensor, we have $\sigma'_{xx} = \left(\frac{4}{3}\eta + \zeta\right)\frac{dv}{dx}$. The condition (85.2) then becomes[6]

$$p + \rho v - (\frac{4}{3}\eta + \zeta)\frac{dv}{dx} = \text{constant}\ .$$

As in §85, we introduce the specific volume V in place of the velocity $v = jV$. The constant on the right can be expressed in terms of the limiting values of the quantities at large distances in front of the shock (side 1). Then

$$p - p_1 + j^2(V - V_1) - (\frac{4}{3}\eta + \zeta) j \frac{dV}{dx} = 0\ . \qquad (93.2)$$

Next, the energy flux density due to thermal conduction is $-\kappa\frac{dT}{dx}$. That due to internal friction is $-\sigma'_{xi}\, v_i = -\sigma'_{xx}\, v = -(\frac{4}{3}\eta + \zeta)v\frac{dv}{dx}$. Thus the condition (85.3) can be written

$$\rho v(w + \frac{1}{2}v^2) - (\frac{4}{3}\eta + \zeta)v\frac{dv}{dx} - \kappa\frac{dT}{dx} = \text{constant}\ .$$

Again putting $v = jV$, and expressing the constant in terms of quantities with the suffix 1, we can obtain the final form

$$w - w_1 + \frac{1}{2}j^2(V^2 - V_1{}^2) - j(\frac{4}{3}\eta + \zeta)V\frac{dV}{dx} - \left(\frac{\kappa}{j}\right)\frac{dT}{dx} = 0\ . \qquad (93.3)$$

We shall here consider shock waves in which all the discontinuities are small. Then all the differences $V - V_1$, $p - p_1$, etc., between the values inside and outside the transition layer are also small. It is seen from the relations obtained below that $1/\delta$ (where δ is the thickness of the discontinuity) is of the first order in the small quantity $p_2 - p_1$. Thus differentiation with respect to x increases the order of smallness by one; for example, dp/dx is a second-order small quantity.

We multiply (93.2) by $(V + V_1)/2$ and subtract it from (93.3). The result is

$$(w - w_1) - \frac{1}{2}(p - p_1)(V + V_1) = \frac{\kappa}{j}\frac{dT}{dx}\ ; \qquad (93.4)$$

the third-order term in $(V - V_1)dV/dx$ is omitted. We expand the left-hand side of (93.4) in powers of $p - p_1$ and $s - s_1$, taking the pressure and the entropy as the basic

[6] The positive x-direction is the direction of motion of the gas through the shock wave at rest. If we use a frame of reference in which the gas in front of the shock is at rest, the shock wave itself moves in the negative x-direction.

independent variables. The terms of first and second order in $p - p_1$ are zero (cf. the derivation of (86.1)), and omitting the higher-order terms gives just $T(s - s_1)$. We write

$$\frac{dT}{dx} = \left(\frac{\partial T}{\partial p}\right)_s \frac{dp}{dx} + \left(\frac{\partial T}{\partial s}\right)_p \frac{ds}{dx}.$$

The *ds*/*dx t*erm can be omitted, being a third-order small quantity (see below), and we thus obtain an expression for *s*(*x*) in terms of *p*(*x*):

$$T(s - s_1) = \frac{\kappa}{j}\left(\frac{\partial T}{\partial p}\right)_s \frac{dp}{dx}. \qquad (93.5)$$

Note that $s - s_1$ in the transition layer is a second-order small quantity, whereas the total discontinuity $s_2 - s_1$ is (as shown in §86) third-order with respect to the pressure discontinuity $p_2 - p_1$. The reason is that, as we shall show below, the pressure in the transition layer varies monotonically from p_1 to p_2, whereas the entropy *s*(*x*), which is determined by the derivative *dp*/*dx*, has a maximum within the layer.

An equation for *p*(*x*) could be obtained by making a similar expansion of (93.2) and (93.3) and then combining the two. We will, however, use a different and more instructive method, which allows a clearer understanding of the origin of the various terms in the equation.

It has been shown in §79 that a monochromatic weak disturbance of the state of the gas (a sound wave) is damped in the course of propagation, at a rate proportional to the square of the frequency: $\gamma = a\omega^2$. The positive coefficient *a* can be expressed in terms of the viscosity and the thermal conductivity by (79.6). It was also shown that (for any plane sound, wave) this damping can be described by including an extra term in the linearized equation of motion; see (79.9). In this equation, we replace the second time derivative by the second coordinate derivative and change the sign of *dp'*/*dx* (corresponding to wave propagation in the negative *x*-direction[7]), and write it as

$$\frac{\partial p'}{\partial t} - c\frac{\partial p'}{\partial x} = ac^3 \frac{\partial^2 p'}{\partial x^2}, \qquad (93.6)$$

where *p'* is the variable part of the pressure.

To take account of the slight non-linearity, we have to include a term in $p'\dfrac{\partial p'}{\partial x}$:

$$\frac{\partial p'}{\partial t} - c\frac{\partial p'}{\partial x} - \alpha_p p'\frac{\partial p'}{\partial x} = ac^3 \frac{\partial^2 p'}{\partial x^2}. \qquad (93.7)$$

The coefficient α_p in the non-linear term is found by an appropriate expansion of the equations of fluid dynamics for an ideal (dissipationless) fluid; the result is

[7] This direction propagation is chosen in accordance with the comment in the last footnote.

$$\alpha_p = \frac{1}{2}\frac{c^3}{V^2}\left(\frac{\partial^2 V}{\partial p^2}\right)_s \tag{93.8}$$

(see the Problem).[8]

Equation (93.7) describes the propagation of disturbances in a medium with slight dissipation and slight non-linearity. In the case of a weak shock wave, it describes the propagation in a frame of reference where the unperturbed gas (in front of the shock) is at rest. We need to find a solution with a steady (time-independent) profile in which the pressure far from the wave ($x \to \pm\infty$) has limiting values p_2 and p_1; the difference $p_2 - p_1$ is the pressure discontinuity.[9]

A steady profile is described by a solution having the form

$$p'(x,t) = p'(x + v_1 t), \tag{93.9}$$

where v_1 is the velocity. Substitution in (93.7) gives

$$\frac{d}{d\xi}\left[(v_1 - c)p' - \frac{1}{2}\alpha_p p'^2 - ac^3\frac{dp'}{d\xi}\right] = 0, \quad \xi = x + v_1 t,$$

whose first integral is

$$ac^3\frac{dp'}{d\xi} = -\frac{1}{2}\alpha_p p'^2 + (v_1 - c)p' + \text{constant}. \tag{93.10}$$

The quadratic trinomial on the right must be zero for the values of p' which correspond to the limiting conditions at infinity, where $dp'/d\xi = 0$. These values are $p_2 - p_1$ and 0, if p' is measured from the unperturbed value p_1 in front of the shock. The trinomial must therefore be representable as

$$-\frac{1}{2}\alpha_p\left[p' - (p_2 - p_1)\right]p',$$

the constant v_1 being given in terms of p_1 and p_2 by

$$v_1 = c + \frac{1}{2}\alpha_p(p_2 - p_1). \tag{93.11}$$

Equation (93.10) takes the following form for the pressure p itself:

$$ac^3\frac{dp}{d\xi} = -\frac{1}{2}\alpha_p(p - p_1)(p - p_2).$$

The solution of this equation satisfying the necessary conditions is

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[8] With a new unknown function $u = -p'\alpha_p$, a new independent variable (instead of x) $\zeta = x + ct$, and the notation $\mu = ac^3$, the equation (93.7) can be brought to the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \zeta} = \mu\frac{\partial^2 u}{\partial \zeta^2}, \tag{93.7a}$$

known as ***Burgers' equation*** (J. M. Burgers 1940).

[9] It will be seen later (§102) that in the absence of dissipation the non-linear effects distort the profile as propagation proceeds, the front becoming gradually steeper. This in turn enhances the dissipative effects, which tend to make the profile less steep (to reduce the gradients of the variable quantities). The mutual compensation of these opposite tendencies makes possible the propagation with a steady profile in a

$$p = \frac{1}{2}(p_1 + p_2) + \frac{1}{2}(p_2 - p_1)\tanh\frac{(p_2 - p_1)(x + v_1 t)}{4ac^3/\alpha_p}.$$

This solves the problem. Returning to the frame of reference in which the shock wave is at rest, we have as the formula for the pressure variation in it

$$p - \frac{1}{2}(p_2 + p_1) = \frac{1}{2}(p_2 - p_1)\tanh\frac{x}{\delta}, \tag{93.12}$$

where

$$\delta = \frac{8aV^2}{(p_2 - p_1)\left(\frac{\partial^2 V}{\partial p^2}\right)_s}. \tag{93.13}$$

Almost the whole change from p_1 to p_2 occurs over a distance of the order of δ, which may be called the ***thickness*** of the shock wave. We see that this is less for stronger shocks, i.e., greater pressure discontinuities.[10]

The variation of the entropy across the discontinuity is obtained from (93.5) and (93.12):

$$s - s_1 = \frac{\kappa}{16caVT}\left(\frac{\partial T}{\partial p}\right)_s\left(\frac{\partial^2 V}{\partial p^2}\right)_s (p_2 - p_1)^2 \frac{1}{\cosh^2(x/\delta)}. \tag{93.14}$$

From this we see that the entropy does not vary monotonically, but has a **maximum** inside the shock, at $x = 0$. For $x = \pm\infty$ this formula gives $s = s_1$ in either case; this is because the total entropy change $s_2 - s_1$ is of the third order in $p_2 - p_1$ (cf. (86.1)), whereas $s - s_1$ is of the second order.

Formula (93.12) is quantitatively valid only for sufficiently small differences $p_2 - p_1$. We can, however, use (93.13) qualitatively to determine the order of magnitude of the thickness in cases where the difference $p_2 - p_1$ is of the same order of magnitude as p_1 and p_2 themselves. The velocity of sound in the gas is of the same order as the thermal velocity v of the molecules. The kinematic viscosity is, as we know from the kinetic theory of gases, $\nu \sim lv \sim lc$, where l is the mean free path of the molecules. Hence $a \sim l/c^2$; an estimate of the thermal conduction term gives the same result. Finally, $\left(\frac{\partial^2 V}{\partial p^2}\right)_s \sim \frac{V}{p^2}$, and $pV \sim c^2$. Using these relations in (93.13), we obtain

$$\delta \sim l. \tag{93.15}$$

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non-linear dissipative medium.

[10] For a shock wave propagated in a mixture there is also a contribution to the thickness from diffusion processes in the transition layer, calculated by S. P. D'yakov, *Zhurnal eksperimental noi i teoreticheskoi fiziki* 27, 283, 1954.

Weak shock waves remain stable with respect to transverse modulation (see the eighth footnote to §90) even when their dissipative structure is taken into account (M. D. Spektor, *JETP Letters* 35, 221, 1983).

Thus *the thickness of a strong shock is of the same order of magnitude as the mean free path of the gas molecules*.[11] In macroscopic gas dynamics, however, where the gas is treated as a continuous medium, the mean free path must be taken as zero. It follows that *the methods of gas dynamics cannot strictly be used* alone to investigate the internal structure of strong shock waves.

PROBLEMS

Problem 1. Determine the non-linearity coefficient α_p in (93.7) for sound wave propagation in a gas.

Solution. The exact equations of one-dimensional gas flow without dissipation are

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0\,. \qquad (1)$$

We expand these as far as second-order small terms, putting

$$p=p_0+p', \quad \rho=\rho_0+\frac{p'}{c^2}+\frac{1}{2}p'^2\left(\frac{\partial^2\rho}{\partial p^2}\right)_s. \qquad (2)$$

The second-order terms can be simplified by bringing all of them to a form containing the product $p'\frac{\partial p'}{\partial x}$. To do so, we note that, for a wave propagated in the negative x-direction with velocity c, differentiations with respect to t and x/c are equivalent, and $v=-\frac{p'}{c\rho_0}$. Then equations (1) and (2) become

$$\frac{\partial v}{\partial t}+\frac{1}{\rho}\frac{\partial p'}{\partial x}=0, \qquad (3)$$

$$\frac{\partial v}{\partial x}+\frac{1}{\rho c^2}\frac{\partial p'}{\partial t}=c\rho\left(\frac{\partial^2 V}{\partial p^2}\right)_s p'\frac{\partial p'}{\partial x}; \qquad (4)$$

the suffix zero in the constant equilibrium values has been omitted. We have used also the relation

$$\left(\frac{\partial^2\rho}{\partial p^2}\right)_s=\frac{2}{\rho c^4}-\rho^2\left(\frac{\partial^2 V}{\partial p^2}\right)_s, \qquad (5)$$

where $V=1/\rho$ is the specific volume. Differentiation of (3) with respect to x and (5) with respect to t, followed by subtraction, gives

$$\left(\frac{1}{c}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p'=c^2\rho^2\left(\frac{\partial^2 V}{\partial p^2}\right)_s\frac{\partial}{\partial x}\left(p'\frac{\partial p'}{\partial x}\right).$$

To the same accuracy, we replace $\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}$ on the left by $2\frac{\partial}{\partial x}$. Lastly, cancelling $\frac{\partial}{\partial x}$ on each side and comparing the result with (93.7), we get α_p in accordance with

[11] A strong shock wave causes a considerable increase in temperature; l denotes the mean free path for some

(93.8).

An equation for v can be obtained directly from (93.7) without repeating calculations similar to the above. The sum of the first-order terms on the left of (93.7) contains the operator $\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$, which is to be regarded as a first-order term; it gives zero when applied to $p'(x, t)$ in the linear approximation. We thus obtain an equation for $v(x, t)$ in the required approximation on simply replacing p' in (93.7) according to the linear relation $p' = -\rho c v$:

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} + \alpha_v v \frac{\partial v}{\partial x} = ac^3 \frac{\partial^2 v}{\partial x^2}, \qquad (6)$$

where

$$\alpha_v = \frac{1}{2}\frac{c^4}{V^3}\left(\frac{\partial^2 V}{\partial p^2}\right)_s .$$

This α_v is dimensionless; for a polytropic gas, $\alpha_v = (\gamma + 1)/2$.

Problem 2. Use a non-linear substitution to convert **Burgers' equation** (93.7a) to a linear thermal conduction equation (E. Hopf 1950).

Solution. By the substitution

$$u(\zeta, t) = -2\mu \frac{\partial}{\partial \zeta} \log \phi(\zeta, t), \qquad (1)$$

equation (93.7a) is brought to the form

$$2\mu \frac{\partial}{\partial \zeta}\left[\frac{1}{\phi}\left(-\frac{\partial \phi}{\partial t} + \mu \frac{\partial^2 \phi}{\partial \zeta^2}\right)\right] = 0,$$

whence

$$\frac{\partial \phi}{\partial t} - \mu \frac{\partial^2 \phi}{\partial \zeta^2} = \phi \frac{df(t)}{dt}, \qquad (2)$$

where $\frac{df}{dt}$ is an arbitrary function of t. By making the change $\phi \to \phi e^f$ (which does not affect the function $u(\zeta, t)$ sought), we convert this equation to the required form

$$\frac{\partial \phi}{\partial t} = \mu \frac{\partial^2 \phi}{\partial \zeta^2}. \qquad (3)$$

The solution with the initial condition $\phi(\zeta, 0) = \phi_0(\zeta)$ is given by (51.3):

$$\phi(\zeta, t) = \frac{1}{2}\frac{1}{\sqrt{\pi \mu t}} \int \phi_0(\zeta') \exp\left[-\frac{(\zeta - \zeta')^2}{4\mu t}\right] d\zeta'. \qquad (4)$$

The initial function $\phi_0(\zeta)$ is related to the initial value of $u(\zeta, t)$ by

mean temperature of the gas in the shock.

$$\log \phi_0(\xi) = -\frac{1}{2\mu}\int_0^{\zeta} u_0(\zeta) d\xi \ , \qquad (5)$$

the lower limit of the integral being chosen arbitrarily.