## §94. Shock waves in a relaxing medium

A considerable increase in the thickness of a shock wave may be caused by the presence in the gas of comparatively slow relaxation processes (slow chemical reactions, a slow energy transfer between different degrees of freedom of the molecule, and so on (Ya. B. Zel'dovich 1946) ).[1]

Let $\tau$ be a time of the order of magnitude of the **relaxation time**. Both the initial and the final state of the gas must be states of complete equilibrium; it is therefore immediately clear that the total thickness of the shock wave will be of the order of $\tau v_1$, the distance traversed by the gas in the time $\tau$. It is also found that, if the shock strength is above a certain limit, its structure becomes more complex; this may be seen as follows.

In Fig, 67 the continuous curve shows the shock adiabatic drawn through a given initial point 1, on the assumption that the final states of the gas are states of complete equilibrium; the slope of the tangent at the point 1 is given by the "equilibrium" velocity of sound, denoted in §81 by $c_0$. The dashed curve shows the shock adiabatic through the same point 1, on the assumption that the relaxation processes are "frozen" and do not occur. The slope of the tangent to this curve at the point 1 depends on the velocity of sound denoted in §81 by $c_\infty$.

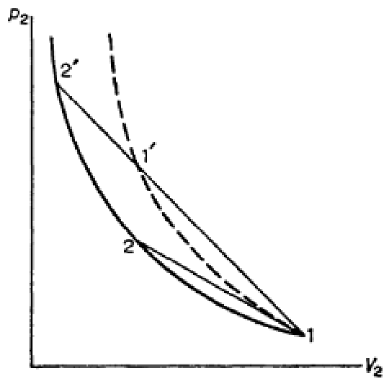


FIG. 67

If the velocity of the shock wave is such that $c_0 < v_1 < c_\infty$, the chord 12 lies as shown in Fig. 67 (the lower chord). In this case we have a simple increase in the shock thickness, all intermediate states between the initial state 1 and the final state 2 being represented in the $pV$-plane by points on the segment 12. This follows from the fact that (neglecting ordinary viscosity and thermal conduction) all the states through which the gas passes satisfy the equations of conservation of **mass**, $\rho v = j = \text{constant}$, and of **momentum**, $p + j^2 V = \text{constant}$ (cf. the similar but more detailed discussion in §129).

If, however, $v_1 > c_\infty$, the chord takes the position 11'2'. No point lying between 1 and 1' corresponds to any actual state of the gas; the first real point (after 1) is 1', which

[1] For example, in diatomic gases, for temperatures behind the shock wave ~ 1000-3000 K, the excitation of intramolecular vibrations is a slow relaxation process. At higher temperatures, the role of such a process falls to thermal dissociation of molecules into their constituent atoms.

corresponds to a state in which the relaxation equilibrium is no different from that in state 1. The compression of the gas from state 1 to state 1' occurs discontinuously, and afterwards (over distances $\sim v_1\tau$ ) it is gradually compressed to the final state 2'.

If the equilibrium and non-equilibrium shock adiabatics intersect (Fig. 68), there can exist shock waves of a further type: if the shock velocity is such that the chord 12 meets the adiabatics above their intersection, as in Fig. 68, the relaxation is accompanied by a pressure decrease from the value corresponding to the point 1' to that for point 2 (S. P. D'yakov 1954).[2]

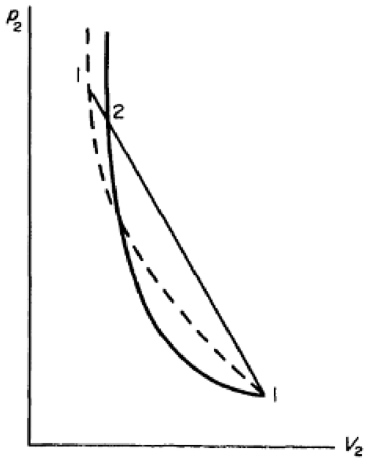


FIG. 68

**§95. The isothermal discontinuity**

The discussion of the structure of a shock wave in §93 involves the assumption that the viscosity and thermal conductivity are of the same order of magnitude, as is usually the case. The case where $\chi >> \nu$ is also possible, however. If the temperature is sufficiently high, additional heat is transferred by thermal radiation in equilibrium with the matter. Radiation has a much smaller effect on the viscosity (i.e., the momentum transfer), and so $\nu$ may be small compared with $\chi$. We shall now see that this inequality leads to a very important difference in the structure of the shock wave.

Neglecting terms which contain the viscosity, we can write equations (93.2) and (93.3), which determine the structure of the transition layer, as

$$p + j^2V = p_1 + j^2V_1, \tag{95.1}$$

$$\frac{\kappa}{j}\frac{dT}{dx} = w + \frac{1}{2}j^2V^2 - w_1 - \frac{1}{2}j^2V_1^2. \tag{95.2}$$

The right-hand side of (95.2) is zero only at the boundaries of the layer. Since the

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[2] Such a case might in principle occur in a dissociating polyatomic gas if in the equilibrium state sufficiently complete dissociation of the molecules into smaller parts occurs behind the shock wave. The dissociation increases the specific-heat ratio $\gamma$ and therefore reduces the limiting compression in the shock wave, if it is so complete that heating the gas does not require any appreciable expenditure of energy on continuing the dissociation.

temperature behind the shock wave must be higher than that in front of it, it follows that we have

$$\frac{dT}{dx} > 0, \tag{95.3}$$

everywhere in the transition layer, i.e., the temperature increases monotonically.

All quantities in the layer are functions of a single variable, the coordinate $x$, and therefore are functions of one another. Differentiating (95.1) with respect to $V$, we obtain

$$\left(\frac{\partial p}{\partial T}\right)_V \frac{dT}{dV} + \left(\frac{\partial p}{\partial V}\right)_T + j^2 = 0.$$

The derivative $\left(\frac{\partial p}{\partial T}\right)_V$ is always positive in gases. The sign of the derivative $\frac{dT}{dV}$ is therefore the reverse of that of the sum $\left(\frac{\partial p}{\partial V}\right)_T + j^2$. In state 1 we have $j^2 > -\left(\frac{\partial p_1}{\partial V_1}\right)_s$ (since $v_1 > c_1$), and, since the adiabatic compressibility is always less than the isothermal compressibility, $j^2 > -\left(\frac{\partial p_1}{\partial V_1}\right)_T$. On side 1, therefore, $\frac{dT_1}{dV_1} < 0$. If this derivative remains negative everywhere in the transition layer, then, as the gas is compressed ($V$ decreasing), the temperature increases monotonically, in accordance with (95.3), from side 1 to side 2. In other words, we have a shock wave whose thickness is much increased by the high thermal conductivity (possibly to such an extent that even to call it a shock wave is mere convention).

If, however, the shock is so strong (see (95.7)) that

$$j^2 < -\left(\frac{\partial p_2}{\partial V_2}\right)_T, \tag{95.4}$$

then we have in state 2 $\frac{dT_2}{dV_2} > 0$, so that the function $T(V)$ has a maximum somewhere between $V_1$ and $V_2$ (Fig. 69). It is clear that the transition from state 1 to state 2, with $V$ changing continuously, then becomes impossible, since the inequality (95.3) cannot be satisfied everywhere.

Consequently, we have the following pattern of transition from the initial state 1 to the final state 2. First comes a region where the gas is gradually compressed from the specific volume $V_1$ to some $V$' (the value for which $T(V') = T_2$ for the first time; see Fig. 69); the thickness of this region is determined by the thermal conductivity, and may be considerable. The compression from $V'$ *to* $V_2$ then occurs discontinuously, the temperature remaining constant at $T_2$. This may be called an ***isothermal discontinuity***.

Let us determine the variation of the pressure and density in an isothermal discontinuity,

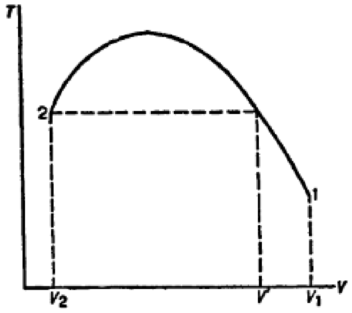



FIG. 69

assuming that we have a perfect gas. The condition of continuity of momentum flux (95.1), applied to the two sides of the discontinuity, gives

$$p' + j^2V' = p_2 j^2 V_2 .$$

For a perfect gas $V = \frac{RT}{\mu p}$; since $T' = T_2$, we have

$$p' + \frac{j^2RT_2}{\mu p'} = p_2 + \frac{j^2RT_2}{\mu p_2} .$$

This quadratic equation for $p'$ has the solutions $p' = p_2$ (trivial) and

$$p' = \frac{j^2RT_2}{\mu p_2} = j^2V_2 . \qquad (95.5)$$

We can express $j^2$ in the form (85.6), obtaining $p' = \frac{(p_2 - p_1)V_2}{V_1 - V_2}$, and, substituting $N_2/V_1$ from (89.1), we have for a polytropic gas

$$p' = \frac{1}{2}\left[(\gamma+1)p_1 + (\gamma-1)p_2\right]. \qquad (95.6)$$

Since we must have $p_2 > p_1$, we find that an isothermal discontinuity occurs only when the ratio of the pressures $p_2$ and $p_1$ satisfies

$$\frac{p_2}{p_1} > \frac{\gamma+1}{3-\gamma} \qquad (95.7)$$

(Rayleigh 1910). This condition can, of course, be obtained directly from (95.4).

Since, for a given temperature, the gas density is proportional to the pressure, the density ratio in an isothermal discontinuity is equal to the pressure ratio;

$$\frac{\rho'}{\rho} = \frac{V_2}{V'} = \frac{p'}{p_2}, \qquad (95.8)$$

and tends to $(\gamma-1)/2$ as $p_2$ increases.

## §96. Weak discontinuities

Besides surface discontinuities, at which the quantities $\rho, p, \mathbf{v}$ etc., are discontinuous, we can also have surfaces at which these quantities, though remaining continuous, are not regular functions of the coordinates. The irregularity may be of various kinds. For example,

the first spatial derivatives of $\rho, p, \boldsymbol{v}$ etc., may be discontinuous on a surface, or these derivatives may become infinite; or higher derivatives may behave in the same manner. We call such surfaces ***weak discontinuities***, in contrast to the **strong discontinuities** (shock waves and tangential discontinuities), in which the quantities $\rho, p, \boldsymbol{v}$ ,...themselves are discontinuous. Since these are continuous at a weak discontinuity, so are their tangential derivatives; only the normal derivatives are discontinuous.

It is easy to see from simple considerations that weak discontinuities are propagated relative to the gas (on either side of the surface) with the velocity of sound. For, since the functions $\rho, p, \boldsymbol{v}$ ,... themselves are continuous, they can be "smoothed" by modifying them only near the surface of discontinuity, and only by arbitrarily small amounts, in such a way that the smoothed functions have no singularity. The true distribution of the pressure, say, can thus be represented as a superposition of a perfectly smooth function $p_0$, free from all singularities, and a very small perturbation $p'$ of this distribution near the surface of discontinuity; and the latter, like any small perturbation, is propagated, relative to the gas, with the velocity of sound.

It must be emphasized that, for a shock wave, the smoothed functions would differ from the true ones by quantities which in general are not small, and the foregoing arguments are therefore invalid. If, however, the discontinuities in the shock wave are sufficiently small, those arguments are again applicable, and such a shock wave is propagated with the velocity of sound, a result which was obtained by another method in §86.

If the flow is steady in a given coordinate system, then the surface of discontinuity is at rest in that system, and the gas flows through it. The gas velocity component normal to the surface must equal the velocity of sound. If we denote by $\alpha$ the angle between the direction of the gas velocity and the tangent plane to the surface, then $v_n = v \sin\alpha = c$, or $\sin\alpha = c/v$, i.e., a surface of weak discontinuity intersects the streamlines at the **Mach angle**. In other words, a surface of weak discontinuity is one of the **characteristic surfaces**, a result which is entirely reasonable if we recall the physical significance of the latter: they are surfaces along which small perturbations are propagated (see §82). It is clear that, in steady flow of a gas, weak discontinuities can occur only at velocities not less than that of sound.

Weak discontinuities differ fundamentally from strong ones in the manner of their occurrence. We shall see that shock waves can be formed as a direct result of the gas flow, the boundary conditions being continuous (for instance, the formation of shock waves in a sound wave, §102). In contrast to this, weak discontinuities cannot occur spontaneously; they are always the result of some **singularity of the initial or boundary conditions** of the flow. These singularities may be of various kinds, like the weak discontinuities themselves. For example, a weak discontinuity may occur on account of the presence of angles on the

surface of a body past which the flow takes place; in this case the first spatial derivatives of the velocity are discontinuous. A weak discontinuity is also formed when the curvature of the surface of the body is discontinuous, without there being an angle; in this case the second spatial derivatives of the velocity are discontinuous, and so on. Finally, any singularity in the time variation of the flow results in a non-steady weak discontinuity.

The gas velocity component tangential to the surface of a weak discontinuity is always directed away from the point (e.g., an angle on the surface of a body) from which the perturbation begins which causes the discontinuity; we shall say that the discontinuity starts from this point. This is an example of the fact that, in a supersonic flow, perturbations are propagated downstream.

The presence of viscosity and thermal conduction results in a finite thickness of a weak discontinuity, which is therefore in reality a transition layer, like a shock wave. The thickness of the latter, however, depends only on its strength and is constant in time, whereas the thickness of a weak discontinuity increases with time after its formation. It is easy to determine the qualitative law governing this increase. To do so, we use the fact that the motion of a weak discontinuity follows the same equations as the propagation of any weak sound disturbance. When viscosity and thermal conduction occur, a perturbation which is initially concentrated in a small volume (a wave packet) expands as it moves in the course of time; the manner of this expansion has been determined in §79. We can therefore conclude that the thickness $\delta$ of a weak discontinuity is

$$\delta \sim \sqrt{ac^3 t}\,, \qquad (96.1)$$

where $t$ is the time from the formation of the discontinuity and $a$ the coefficient of the squared frequency in the sound absorption coefficient (79.6). If the discontinuity is at rest, then the time $t$ must be replaced by $l/c$, where $l$ is the distance from the point where the discontinuity starts (e.g., for a weak discontinuity starting from an angle on the surface of a body, $l$ is the distance from the vertex of the angle); consequently $\delta \sim \sqrt{ac^2 l}$.[3]

To conclude this section, we should make the following remark, analogous to the one at the end of §82. We stated there that, among the various perturbations of the state of a gas in motion, perturbations of entropy (at constant pressure) and vorticity are distinct in their properties. Such perturbations do not move relative to the gas, and are not propagated with

[3] We must emphasize, however, that the analogy with sound would not suffice for a quantitative determination of the structure of a weak discontinuity. The reason is that, to determine the law of sound damping, the amplitude may be assumed infinitesimal and the linearized equations of motion may be used. For weak discontinuities, as for weak shock waves (§93), the non-linearity of the equations has to be taken into account, since otherwise there would be no discontinuities. An example of such an analysis is given in §99, Problem 6.

the velocity of sound. Hence the surfaces at which the entropy and vorticity[4] are weakly discontinuous are at rest relative to the gas, and move with it relative to a fixed system of coordinates. Such discontinuities may be called ***weak tangential discontinuities***; they pass through streamlines, and are in this respect entirely analogous to the strong tangential discontinuities.

[4] A weak discontinuity of the vorticity implies a weak discontinuity of the velocity component tangential to the surface of discontinuity; for example, the normal derivatives of the tangential velocity may be discontinuous.