CHAPTER X. **ONE-DIMENSIONAL GAS FLOW**

§97. Flow of gas through a nozzle

Let us consider steady flow of a gas out of a large vessel through a tube with variable cross-section (a ***nozzle***). We shall suppose that the gas flow is uniform over the cross-section at every point in the tube, and that the velocity is along the axis of the tube. For this to be so, the tube must not be too wide, and its cross-sectional area *S* must vary fairly slowly along its length. Thus all quantities characterizing the flow will be functions only of the coordinate along the axis of the tube. Under these conditions we can apply the relations obtained in §83, which are valid along streamlines, directly to the variation of quantities along the axis.

The mass of gas passing through a cross-section of the tube in unit time (the ***discharge***) is $Q = \rho v S$; this must evidently be constant along the tube:

$$Q = \rho v S = \text{constant} . \tag{97.1}$$

The linear dimensions of the vessel are supposed very large in comparison with the diameter of the tube. The velocity of the gas in the vessel may therefore be taken as zero, and accordingly all quantities with the suffix 0 in the formulae of §83 will be the values of those quantities in the vessel.

We have seen that the flux density $j = \rho v$ cannot exceed a certain limiting value j_*. It is therefore clear that the possible values of the total discharge *Q* have (for a given tube and a given state of the gas in the vessel) an upper limit $Q_{\max}$, which is easily determined. If the value j_* of the flux density were reached anywhere except at the narrowest point of the tube, we should have $j > j_*$ for cross-sections with smaller *S*, which is impossible. The value $j = j_*$ can therefore be attained only at the narrowest point of the tube; let the cross-sectional area there be $S_{\min}$. Then the upper limit to the total discharge is

$$Q_{\max} = \rho_* v_* S_{\min} = \sqrt{\gamma p_0 \rho_0} \left(\frac{2}{\gamma + 1} \right)^{(1+\gamma)/2(\gamma-1)} S_{\min} . \tag{97.2}$$

Let us first consider a nozzle which narrows continually towards its outer end, so that the minimum cross-sectional area is at that end (Fig. 70). By (97.1), the flux density *j* increases monotonically along the tube. The same is true of the gas velocity *v*, and the pressure accordingly falls monotonically. The greatest possible value of *j* is reached if *v* attains the

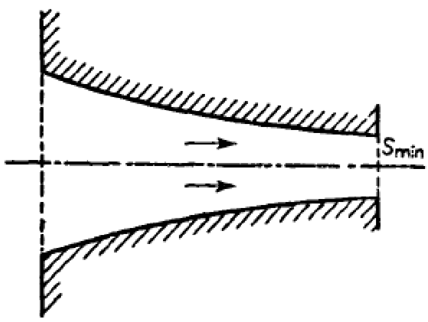

FIG. 70

value c just at the outer end of the tube, i.e., if $v_1 = c_1 = v_*$ (the suffix 1 denotes quantities pertaining to the outer end). At the same time, $p_1 = p_*$.

Let us now follow the change in the manner of outflow of the gas when the external pressure p_e diminishes. When this pressure decreases from p_0, the pressure inside the vessel, to p_*, the pressure p_1 at the outer end of the tube decreases also, and the two pressures p_1 and p_e remain equal; that is, the whole of the pressure drop from p_0 to p_e occurs in the nozzle. The velocity v_1 with which the gas leaves the tube, and the total discharge $Q = j_1 S_{\min}$, increases monotonically, however. For $p_e = p_*$ this velocity becomes equal to the local velocity of sound, and the discharge reaches the value $Q_{\max}$. When the external pressure decreases further, the pressure p_1 remains constant at p_*, and the fall of pressure from p_* to p_e occurs outside the tube, in the surrounding medium. In other words, the pressure drop along the tube cannot be greater than from p_0 to p_*, whatever the external pressure. For air ($p_* = 0.53 p_0$), the maximum pressure drop is $0.47 p_0$. The velocity at the end of the tube and the discharge also remain constant for $p_e < p_*$. Thus the *gas cannot acquire a supersonic velocity* in flowing through a nozzle of this kind.

The impossibility of achieving supersonic velocities by flow through a continually narrowing nozzle is due to the fact that a velocity equal to the local velocity of sound can be reached only at the very end of such a tube. It is clear that a supersonic velocity can be attained by means of a nozzle which first narrows and then widens again (Fig. 71). This is called a ***de Laval nozzle***.

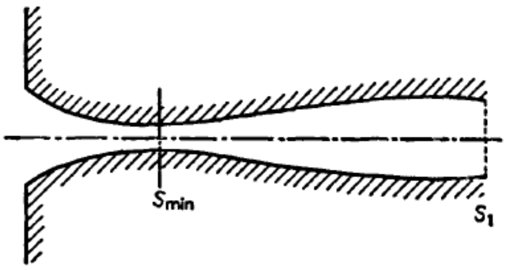

FIG. 71

The maximum flux density j_*, if reached, can again occur only at the narrowest cross-section, so that the discharge cannot exceed $S_{\min} j_*$. In the narrowing part of the nozzle, the flux density increases (and the pressure falls); the curve in Fig. 72 shows j as a function[1] of p, and the variation just described corresponds to the interval from c to b. If the maximum flux density is reached at the cross-section $S_{\min}$ (the point b in Fig. 72), the pressure continues to diminish in the widening part of the nozzle, while j begins to decrease

[1] According to formulae (83.15-83.17), the dependence is

also, corresponding to the segment *ba* of the curve. At the outer end of the tube j takes a definite value, $j_{1\max} = j_* S_{\min} / S_1$, and the pressure has the corresponding value, denoted in Fig. 72 by p_1', at some point d on the curve. If, however, only some point e is reached at

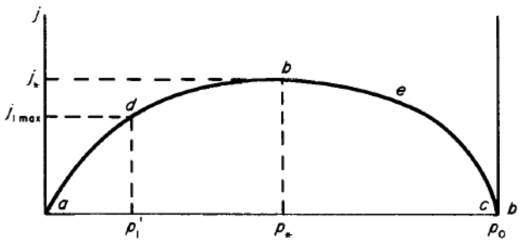

Fig. 72

the cross-section $S_{\min}$, the pressure increases in the widening part of the nozzle, corresponding to a return down the curve from e towards c. At first sight it might appear that we might pass discontinuously from *cb* to *ab*, without going through the point b, by the formation of a shock wave. This, however, is impossible, since the gas "entering" the shock wave cannot have a subsonic velocity.

Bearing in mind these results, let us now investigate the manner of variation in the outflow when the external pressure p_e is gradually increased. For small pressures, from zero to p_1', the pressure p_* and velocity $v_* = c_*$ are reached at the cross-section $S_{\min}$. In the widening part of the nozzle the velocity continues to increase, so that there results a supersonic flow of the gas, and the pressure accordingly continues decreasing, reaching the value p_1' at the outer end of the tube, whatever the pressure p_e. The pressure falls from p_1' to p_e outside the nozzle, in the rarefaction wave which leaves the edge of the tube mouth (see §112).

When p_e exceeds p_1' an oblique shock wave leaves the edge of the tube mouth, compressing the gas from p_1' to p_e (§112). We shall see, however, that a steady shock wave can leave a solid surface only if its intensity is not too great (§111). Hence, when the external pressure increases further, the shock wave soon begins to move into the nozzle, with separation occurring in front of it on the inner surface of the tube. For some value of p_e the shock wave reaches the narrowest cross-section and then disappears; the flow becomes everywhere subsonic, with separation on the walls of the widening part of the nozzle. All these complex phenomena are, of course, three-dimensional.

PROBLEM

A small amount of heat is supplied over a short segment of a tube to a gas in steady flow in the tube. Determine the change in the gas velocity when it passes through this segment. The gas is assumed polytropic.

$$j = \left(\frac{p}{p_0}\right)^{1/\gamma} \left\{ \frac{2\gamma}{\gamma-1} p_0 \rho_0 \left[1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma} \right] \right\}^{1/2}.$$

Solution . Let Sq be the amount of heat supplied per unit time, S being the cross-sectional area of the tube at the segment concerned. The mass flux density $j=\rho v$ and the momentum flux density $p+jv$ are the same on both sides of the heated segment; hence $\Delta p=-j\Delta v$, where Δ denotes the change in a quantity in passing through the segment. The difference in the energy flux density $(w^2+v^2/2)j$ is q. Writing $w=\dfrac{\gamma p}{(\gamma-1)\rho}=\dfrac{\gamma pv}{(\gamma-1)j}$, we obtain (supposing Δv and Δp small) $vj\Delta v+\dfrac{\gamma}{\gamma-1}(p\Delta v+v\Delta p)=q$. Eliminating Δp, we find $\Delta v=\dfrac{(\gamma-1)q}{\rho(c^2-v^2)}$. We see that, in subsonic flow, the supply of heat accelerates the flow ($\Delta v>0$), while in supersonic flow it retards it.

Writing the gas temperature as $T=\dfrac{\mu p}{T\rho}=\dfrac{\mu pv}{Rj}$ (R being the gas constant), we find

$$\Delta T=\frac{\mu}{Rj}(v\Delta p+p\Delta v)=\frac{\mu(\gamma-1)q}{Rj(c^2-v^2)}\left(\frac{c^2}{\gamma}-v^2\right).$$

for supersonic flow, this expression is always positive, and the gas temperature is increased; for subsonic flow, however, ΔT may be either positive or negative.

§98. Flow of a viscous gas in a pipe

Let us consider the flow of a gas in a pipe (with constant cross-section) so long that the friction of the gas against the walls, i.e., the viscosity of the gas, cannot be neglected. We shall suppose the walls to be thermally insulated, so that there is no heat exchange between the gas and the surrounding medium.

For gas velocities of the order of or exceeding the velocity of sound (the only case we shall discuss here), the gas flow in the pipe is, of course, turbulent if the radius of the pipe is not small. The turbulence of the flow is important, as regards our problem, only in one respect: we have seen in §43 that, in turbulent flow, the (mean) velocity is practically the same almost everywhere in the cross-section of the pipe, and falls rapidly to zero very close to the walls. We shall therefore suppose that the gas velocity v is a constant over the cross-section, and define it so that the product $S\rho v$ (S being the cross-sectional area) is equal to the total discharge through the cross-section.

Since the total discharge $S\rho v$ is constant along the pipe, and S is assumed constant, the **mass flux density** must also be constant:

$$j=\rho v=\text{constant}\ . \tag{98.1}$$

Next, since the pipe is thermally insulated, the **total energy flux** carried by the gas through any cross-section must also be constant. This flux is $S\rho v(w+v^2/2)$, and by (98.1) we have

$$w+\frac{1}{2}v^2=w+\frac{1}{2}j^2V^2=\text{constant}\ . \tag{98.2}$$

The **entropy** s of the gas does not, of course, remain constant, but increases as the gas

moves along the pipe, because of the internal friction. If x is the coordinate along the pipe, with x increasing downstream, we can write

$$\frac{ds}{dx} > 0 . \qquad (98.3)$$

We now differentiate (98.2) with respect to x. Since $dw = Tds + Vdp$, we have

$$T\frac{ds}{dx} + V\frac{dp}{dx} + j^2 V\frac{dV}{dx} = 0 .$$

Next, substituting

$$\frac{dV}{dx} = \left(\frac{\partial V}{\partial p}\right)_s \frac{dp}{dx} + \left(\frac{\partial V}{\partial s}\right)_p \frac{ds}{dx}, \qquad (98.4)$$

we obtain

$$\left[T + j^2 V\left(\frac{\partial V}{\partial s}\right)_s\right]\frac{ds}{dx} = -V\left[1 + j^2\left(\frac{\partial V}{\partial p}\right)_s\right]\frac{dp}{dx} . \qquad (98.5)$$

By a well-known formula of thermodynamics, $\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p}\left(\frac{\partial V}{\partial T}\right)_p$. The coefficient of thermal expansion is positive for gases. We therefore conclude, using (98.3), that the left-hand side of (98.5) is positive. The sign of the derivative dp/dx is therefore that of $-\left[1 + j^2\left(\frac{\partial V}{\partial p}\right)_s\right] = \left(\frac{v}{c}\right)^2 - 1$. We see that

$$\frac{dp}{dx} < 0 \quad \text{for} \quad v < c \text{, and } \frac{dp}{dx} > 0 \quad \text{for} \quad v > c . \qquad (98.6)$$

Thus, in subsonic flow, the pressure decreases downstream, as for an incompressible fluid. For supersonic flow, however, it increases.

We can similarly determine the sign of the derivative dv/dx. Since $j = v/V$ = constant, the sign of dv/dx is the same as that of dV/dx. The latter can be expressed in terms of the positive derivative ds/dx by means of (98.4) and (98.5). The result is that

$$\frac{dv}{dx} < 0 \quad \text{for} \quad v < c \text{, and } \frac{dv}{dx} < 0 \quad \text{for} \quad v > c . \qquad (98.7)$$

i.e., the velocity increases downstream for subsonic flow and decreases for supersonic flow.

Any two thermodynamic quantities for a gas flowing in a pipe are functions of one another, independent of (*inter alia*) the resistance law for the pipe. These functions depend on the constant j as a parameter, and are given by the equation $w + \frac{1}{2} j^2 V^2 = \text{constant}$, which is obtained by eliminating the velocity from the equations of conservation of mass and energy for the gas.

Let us ascertain the nature of the curves giving, for example, the entropy as a function of pressure. Rewriting (98.5) in the form

$$\frac{ds}{dp} = V \frac{\left(\frac{v}{c}\right)^2 - 1}{T + j^2 V \left(\frac{\partial V}{\partial s}\right)_p},$$

we see that, at the point where $v = c$, the entropy has an extremum. It is easy to see that s has a maximum. For the second derivative of s with respect to p at this point is

$$\left[\frac{d^2 s}{dp^2}\right]_{v=c} = -\frac{j^2 V \left(\frac{\partial^2 V}{\partial p^2}\right)_s}{T + j^2 V \left(\frac{\partial V}{\partial s}\right)_p} < 0\,;$$

we assume, as usual, that the derivative $\left(\frac{\partial^2 V}{\partial p^2}\right)_s$ is positive.

The curves giving s as a function of p are therefore as shown in Fig. 73. The region of subsonic velocities lies to the right of the maximum, and that of supersonic velocities to the left. When the parameter j increases, we go to lower curves. For, differentiating equation (98.2) with respect to j for constant p, we have

$$\frac{ds}{dj} = -\frac{jV^2}{T + j^2 V \left(\frac{\partial V}{\partial s}\right)_p} < 0\,.$$

We can draw an interesting conclusion from the above results. Let the gas velocity at the entrance to the pipe be less than that of sound. The entropy increases downstream, and the pressure decreases; this corresponds to a movement along the right-hand branch of the curve **s** = $s(p)$, from B towards O (Fig. 73). This can, however, continue only until the entropy reaches its maximum value. A further movement along the curve beyond O (i.e., into the

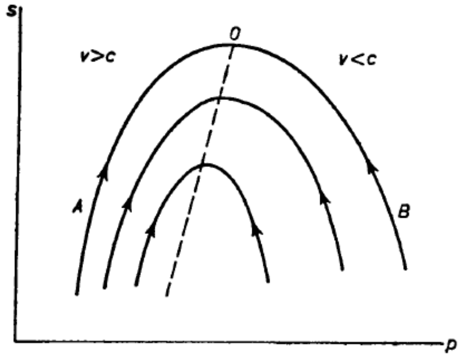

FIG. 73

region of supersonic velocities) is not possible, since the entropy of the gas would have to decrease as it moved along the pipe. The transition between the branches *BO* and *OA* cannot even be effected by a shock wave, since the gas entering a shock wave cannot move with subsonic velocity.

Thus we conclude that, if the gas velocity at the entrance to the pipe is less than that of sound, the flow remains subsonic everywhere in the pipe. The gas velocity becomes equal to the local velocity of sound only at the other end of the pipe, if at all (it does so if the pressure of the external medium into which the gas issues is sufficiently low).

In order that the gas should have supersonic velocities in the pipe, its velocity at the entrance must be supersonic. By the general properties of supersonic flow (the impossibility of propagating disturbances upstream), the flow will then be entirely independent of the conditions at the outlet of the pipe. In particular, the entropy will increase along the pipe in a quite definite manner, and its maximum value will be attained at a definite distance $x = l_k$ from the entrance. If the total length l of the pipe is less than l_k, the flow is supersonic throughout the pipe (corresponding to movement on the branch *AO* from *A* towards *O*). If, on the other hand, $l > l_k$, the flow cannot be supersonic throughout the pipe, nor can there be a smooth transition to subsonic flow, since we can move along the branch *OB* only in the direction shown by the arrow. In this case, therefore, a shock wave must necessarily be formed, which discontinuously changes the flow from supersonic to subsonic. The pressure is thereby increased, and we pass from the branch *AO* to *BO* without going through the point *O*. The flow is entirely subsonic beyond the discontinuity.

§99. One-dimensional similarity flow

An important class of one-dimensional non-steady gas flows is formed by flows occurring in conditions where there are characteristic velocities but not characteristic lengths. The simplest example of such a flow is given by gas flow in a semi-infinite cylindrical pipe terminated by a piston, when the piston begins to move with constant velocity.

Such a flow is defined by the velocity parameter and by parameters which give, say, the gas pressure and density at the initial instant. We can, however, form no combination of these parameters which has the dimensions of length or time. It therefore follows that the distributions of all quantities can depend on the coordinate *x* and the time *t* only through the ratio *x*/*t*, which has the dimensions of velocity. In other words, these distributions at various instants will be similar, differing only in the scale along the *x*-axis, which increases proportionally to the time. We can say that, if lengths are measured in a unit which increases proportionally to *t*, then the flow pattern does not change. This is called a ***similarity flow***.

The equation of **conservation of entropy** for a flow which depends on only one coordinate, *x*, is $\frac{\partial s}{\partial t} + v_x \frac{\partial s}{\partial x} = 0$. Assuming that all quantities depend only on $\xi = x / t$, and

noticing that in this case $\frac{\partial}{\partial x} = \frac{1}{t}\frac{d}{d\xi}$, $\frac{\partial}{\partial t} = -\frac{\xi}{t}\frac{d}{d\xi}$, we obtain $(v_x - \xi)s' = 0$ (the prime denoting differentiation with respect to ξ). Hence $s' = 0$, i.e., s = constant[2]; thus similarity flow in one dimension must be isentropic as well as adiabatic.

Likewise, from the y and z components of **Euler's equation**; $\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} = 0$, $\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} = 0$, we find that v_y and v_z are constants, which we can take as zero without loss of generality.

Next, the **equation of continuity** and the x-component of **Euler's equation** are

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x} = 0, \tag{99.1}$$

$$\frac{\partial c}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}; \tag{99.2}$$

here and henceforward we write v_x as v simply. In terms of the variable ξ, these equations become

$$(v-\xi)\rho' + \rho v' = 0, \tag{99.3}$$

$$(v-\xi)v' = -\frac{p'}{\rho} = -c^2 \frac{\rho'}{\rho}. \tag{99.4}$$

In the second equation we have put $p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho' = c^2 \rho'$, since the entropy is constant.

These equations have, first of all, the trivial solution v = constant, ρ = constant, i.e., a uniform flow with constant velocity. To find a non-trivial solution, we eliminate ρ' and v' from the equations, obtaining $(v-\xi)^2 = c^2$, whence $\xi = v \pm c$. We shall take the plus sign:

$$x/t = v + c; \tag{99.5}$$

this choice of sign means that we take the positive x-axis in a definite direction, selected in a manner shown later. Finally, putting $v - \xi = -c$ in (99.3), we obtain $c\rho' = \rho v'$, or $\rho dv = c d\rho$. The velocity of sound is a function of the thermodynamic state of the gas; taking as the fundamental thermodynamic quantities the entropy s and the density ρ, we can represent the velocity of sound as a function $c(\rho)$ of the density, for any given value of the constant entropy. With c understood as such a function, we can write

$$v = \int \frac{c\,d\rho}{\rho} = \int \frac{dp}{c\rho}. \tag{99.6}$$

This formula can also be written

---

[2] The assumption that $v_x - \xi = 0$ would contradict the other equations of motion; from (99.3) we should have v_x = constant, contrary to hypothesis.

$$v = \int \sqrt{-dpdV}\,, \tag{99.7}$$

in which the choice of independent variable remains open.

Formulae (99.5) and (99.6) give the required solution of the equations of motion. If the function $c(\rho)$ is known, then the velocity v can be calculated as a function of density from (99.6). Equation (99.5) then determines the density as an implicit function of x/t, and so the dependence of all the other quantities on x/t is determined also.

We can derive some general properties of the solution thus obtained. Differentiating equation (99.5) with respect to x, we have

$$t\frac{\partial\rho}{\partial x}\frac{d(v+c)}{d\rho} = 1\,. \tag{99.8}$$

For the derivative of $v + c$ we have, by (99.6),

$$\frac{d(v+c)}{d\rho} = \frac{c}{\rho} + \frac{dc}{d\rho} = \frac{1}{\rho}\frac{d(\rho c)}{d\rho}\,.$$

But

$$\rho c = \rho\sqrt{\frac{\partial p}{\partial\rho}} = \frac{1}{\sqrt{-\dfrac{\partial V}{\partial p}}}\,;$$

differentiating, we have

$$\frac{d(\rho c)}{d\rho} = c^2\frac{d(\rho c)}{dp} = \frac{1}{2}\rho^3 c^5\left(\frac{\partial^2 V}{\partial p^2}\right)_s\,. \tag{99.9}$$

Thus

$$\frac{d(v+c)}{d\rho} = \frac{1}{2}\rho^2 c^5\left(\frac{\partial^2 V}{\partial p^2}\right)_s > 0\,. \tag{99.10}$$

It therefore follows from (99.8) that $\dfrac{\partial\rho}{\partial x} > 0$ for t > 0. Since $\dfrac{\partial p}{\partial x} = c^2\dfrac{\partial\rho}{\partial x}$, we conclude that $\dfrac{\partial p}{\partial x} > 0$ also. Finally, we have $\dfrac{\partial v}{\partial x} = \dfrac{c}{\rho}\dfrac{\partial\rho}{\partial x}$, so that $\dfrac{\partial v}{\partial x} > 0$. The inequalities

$$\frac{\partial\rho}{\partial x} > 0\,,\ \frac{\partial p}{\partial x} > 0\,,\ \frac{\partial v}{\partial x} > 0 \tag{99.11}$$

therefore hold.

The meaning of these inequalities becomes clearer if we follow the variation of quantities, not along the x-axis for given t, but with time for a given gas element as it moves about. This variation is given by the total time derivative; for the density, for example, we have, using the equation of continuity, $\dfrac{d\rho}{dt} = \dfrac{\partial\rho}{\partial t} + v\dfrac{\partial\rho}{\partial x} = -\rho\dfrac{\partial v}{\partial x}$. By the third inequality

(99.11), this quantity is negative, and therefore so is $\dfrac{dp}{dt}$:

$$\frac{d\rho}{dt} < 0, \quad \frac{dp}{dt} < 0, \tag{99.12}$$

Similarly (using Euler's equation (99.2)) we can see that $\dfrac{dv}{dt} < 0$; this, however, does not mean that the magnitude of the velocity diminishes with time, since v may be negative.

The inequalities (99.12) show that the density and pressure of any gas element decrease as it moves. In other words, the gas is continually rarefied as it moves. Such a flow may therefore be called a ***non-steady rarefaction wave***.[3]

A rarefaction wave can be propagated only a finite distance along the x-axis; this is seen from the fact that formula (99.5) would give an infinite velocity for $x \to \pm\infty$, which is impossible.

Let us apply formula (99.5) to a plane bounding the region of space occupied by the rarefaction wave. Here x/t is the velocity of this boundary relative to the fixed coordinate system chosen. Its velocity relative to the gas itself is $(x/t) - v$ and is, by (99.5), equal to the local velocity of sound. This means that the boundaries of a rarefaction wave are **weak discontinuities**. The similarity flow in different cases is therefore made up of rarefaction waves and regions of constant flow, separated by surfaces of weak discontinuity. There may also, of course, be regions of constant flow separated by shock waves.

The choice of sign in (99.5) is now seen to correspond to the fact that these weak discontinuities are assumed to move in the positive x-direction relative to the gas. The inequalities (99.11) arise from this choice, but the inequalities (99.12), of course, do not depend on the direction of the x-axis.

We are usually concerned, in actual problems, with a rarefaction wave bounded on one side by a region where the gas is at rest. Let this region (I in Fig. 74) be to the right of the

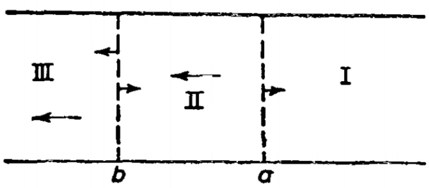

FIG. 74

rarefaction wave. Region II is the rarefaction wave, and region III contains gas moving with constant velocity. The arrows in the figure show the direction of motion of the gas, and of the weak discontinuities bounding the rarefaction wave; the discontinuity a always moves into the gas at rest, but the discontinuity b may move in either direction, depending on the

[3] This flow can occur only as the result of the presence of a singularity in the initial conditions (for example, the piston velocity changes discontinuously at t = 0). The opposite flow could occur only by the action of a compressive piston moving a particular manner.

velocity reached in the rarefaction wave (see Problem 2). We may give explicitly the relations between the various quantities in such a rarefaction wave, assuming that we have a polytropic gas. For an adiabatic process $\rho T^{1/(\gamma-1)} =$ constant. Since the velocity of sound is proportional to $\sqrt{T}$, we can write this relation as

$$\rho = \rho_0 (c / c_0)^{2/(\gamma-1)} \tag{99.13}$$

Substituting this expression in the integral (99.6), we obtain

$$v = \frac{2}{\gamma-1}\int dc = \frac{2}{\gamma-1}(c-c_0);$$

the constant of integration is chosen so that $c = c_0$ for $v = 0$ (we use the suffix 0 to refer to the point where the gas is at rest). We shall express all quantities in terms of *v*, bearing in mind that, with the above situation of the various regions, the gas velocity is in the negative *x*-direction, i.e., $v < 0$. Thus

$$c = c_0 - \frac{1}{2}(\gamma-1)|v|, \tag{99.14}$$

which determines the local velocity of sound in terms of the gas velocity. Substituting in (99.13), we find the density to be

$$\rho = \rho_0\left[1 - \frac{\gamma-1}{2}\frac{|v|}{c_0}\right]^{2/(\gamma-1)}, \tag{99.15}$$

and similarly the pressure is

$$p = p_0\left[1 - \frac{\gamma-1}{2}\frac{|c|}{c_0}\right]^{2\gamma/'(\gamma-1)}. \tag{99.16}$$

Finally, substituting (99.14) in formula (99.5), we obtain

$$|v| = \frac{2}{\gamma+1}\left(c_0 - \frac{x}{t}\right), \tag{99.17}$$

which gives *v* as a function of *x* and *t*.

The quantity *c* cannot be negative, by definition. We can therefore draw from (99.14) the important conclusion that the velocity must satisfy the inequality

$$|v| \le \frac{2c_0}{\gamma-1}; \tag{99.18}$$

when the velocity reaches this limiting value, the gas density (and also *p* and *c*) becomes zero. Thus a gas originally at rest and expanding non-steadily in a rarefaction wave can be accelerated only to velocities not exceeding $\frac{2c_0}{\gamma-1}$.

We have already mentioned, at the beginning of this section, a simple example of similarity flow, namely that which occurs in a cylindrical pipe in which a piston begins to move with constant velocity. If the piston moves out of the pipe, it creates a rarefaction, and a rarefaction wave of the kind described above is formed. If, however, the piston moves inwards, it compresses the gas in front of it, and the transition to the original lower pressure can occur only in a shock wave, which is in fact formed in front of a piston moving forward in a pipe (see the following Problems).[4]

PROMLEMS

[4] We may mention also an analogous similarity flow in three dimensions: the centrally symmetrical gas flow caused by a uniformly expanding sphere. A spherical shock wave, expanding with constant velocity, is formed in front of the sphere. Unlike what happens in the one-dimensional case, the velocity of the gas between the sphere and the shock is not constant; the equation which determines it as a f unction of the ratio *r*/*t* (and therefore the rate of propagation of the shock wave) cannot be integrated analytically. This problem has been discussed by L. I. Sedov (1945; see his book *Similarity and Dimensional Methods in Mechanics*, London 1959) and by G. I. Taylor, *Proceedings of the Royal Society*, A186, 273, 1946.

Problem 1. A gas is in a semi-infinite cylindrical pipe terminated by a piston. At an initial instant the piston begins to move into the pipe with constant velocity *U*. Determine the resulting flow, assuming the gas to be polytropic.

Solution. A shock wave is formed in front of the piston, and moves along the pipe. At the initial instant this shock and the piston are coincident, but at subsequent instants the shock is ahead of the piston, and a region of gas lies between them (region 2). In front of the shock wave (region 1), the gas pressure is equal to its initial value p_1, and its velocity relative to the pipe is zero. In region 2, the gas moves with constant velocity, equal to the velocity of the piston (Fig. 75). The difference in velocity between regions 1 and 2 is therefore also *U*, and, by formulae (85.7) and (89.1), we can write

$$U = \sqrt{(p_2 - p_1)(V_1 - V_2)}$$
$$= (p_2 - p_1)\sqrt{\frac{2V_1}{(\gamma-1)p_1 + (\gamma+1)p_2}}.$$

Hence we find the gas pressure p_2 between the piston and the shock wave to be given by

$$\frac{p_2}{p_1} = 1 + \frac{\gamma(\gamma+1)U^2}{4c_1^2} + \frac{\gamma U}{c_1}\sqrt{1 + \frac{(\gamma+1)^2 U^2}{16 c_1^2}}.$$

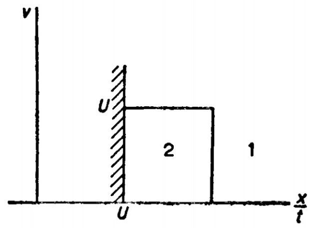

FIG. 75

Knowing p_2, we can calculate, from formulae (89.4), the velocity of the shock wave relative to the gas on each side of it. Since gas 1 is at rest, the velocity of the shock relative to it is equal to the rate of propagation of the shock in the pipe. If the *x* coordinate (along the pipe) is measured from the initial position of the piston (the gas being on the side $x > 0$), we find the position of the shock wave at time *t* to be

$$x = t\left\{\frac{1}{4}(\gamma+1)U + \sqrt{\frac{1}{16}(\gamma+1)^2 U^2 + c_1^2}\right\}$$

while the position of the piston is $x = Ut$.

Problem 2. The same as Problem 1, but for the case where the piston moves out of the pipe with velocity *U*.

Solution. The piston adjoins a region of gas (region 1 in Fig. 76a) which moves in the negative *x*-direction with constant velocity $-U$, equal to the velocity of the piston. Then follows a rarefaction wave (2), in which the gas moves in the negative *x*-direction, its velocity varying linearly from $-U$ to zero according to (99.17). The pressure varies according to (99.16) from $p_1 = p_0\left[1 - \frac{\gamma-1}{2}\frac{U}{c_0}\right]^{2\gamma/(\gamma-1)}$ in gas 1 to p_0 in the gas 3, which is at rest. The boundary of regions 1 and 2 is given by the condition $v = -U$; according to (99.17), we have $x = \left[c_0 - \frac{\gamma+1}{2}U\right]t = (c - U)t$, where *c* is the velocity of sound in gas 1. At the boundary of regions 2 and 3, $v = 0$, whence $x = c_0 t$. Both boundaries are weak discontinuities; the second is always propagated to the right (i.e., away from the piston), but the first may be propagated either to the right (as shown in Fig. 76a) or to the left (if the piston velocity $U > \frac{2c_0}{\gamma+1}$).

The flow pattern just described can occur only if $U < \frac{2c_0}{\gamma - 1}$. If $U > \frac{2c_0}{\gamma - 1}$, a vacuum is formed in front of the piston (the gas cannot follow the piston), which extends from the piston to the point $x = -\frac{2c_0 t}{\gamma - 1}$ (region 1 in Fig. 76b). At this point, $v = -\frac{2c_0}{\gamma - 1}$; then follow region 2, in which the velocity decreases to zero at the point $x = c_0 t$, and region 3, where the gas is at rest.

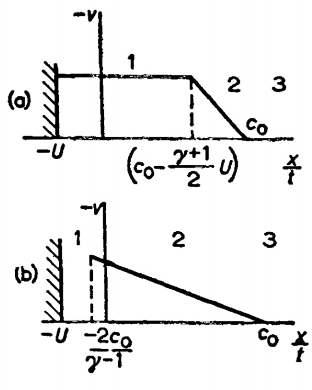

FIG. 76

Problem 3. A gas occupies a semi-infinite cylindrical pipe ($x > 0$) terminated by a valve. At time $t = 0$, the valve is opened, and the gas flows into the external medium, the pressure p_e in which is less than the initial pressure p_0 in the pipe. Determine the resulting flow.

Solution. Let $-v_e$ be the gas velocity which corresponds to the external pressure p_e according to formula (99.16); for $x = 0$ and $t > 0$,we must have $v = -v_e$. If $v_e < \frac{2c_0}{\gamma + 1}$, the velocity distribution shown in Fig. 77a results. For $v_e = \frac{2c_0}{\gamma + 1}$ (corresponding to a rate of outflow equal to the local velocity of sound at the end of the pipe: this is easily seen by putting $v = c$ in formula (99.14)), the region of constant velocity vanishes and the pattern shown in Fig. 77b is obtained. The quantity $\frac{2c_0}{\gamma + 1}$ is the greatest possible rate of outflow from the pipe in the conditions stated. If the external pressure p_e is such that

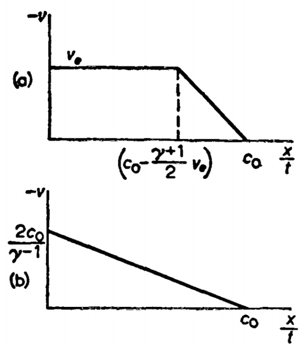

FIG. 77

$$p_e < p_0 \left[\frac{2}{\gamma + 1} \right]^{2\gamma/(\gamma-1)}, \tag{1}$$

the corresponding velocity v_e exceeds $\frac{2c_0}{\gamma + 1}$. In reality, the pressure at the pipe outlet would still be equal to the limiting value (the right-hand side of (1)), and the rate of outflow would be $\frac{2c_0}{\gamma + 1}$; the remaining pressure drop (to p_e) occurs in the external medium.

Problem 4. An infinite pipe is divided by a piston, on one side of which ($x < 0$) there is, at the initial instant, gas at pressure p_0, and on the other side a vacuum. Determine the motion of the piston as the gas expands.

Solution. A rarefaction wave is formed in the gas; one of its boundaries moves to the right with the piston, and the other moves to the left. The equation of motion of the piston is

$$m\frac{dU}{dt}=p_0\left[1-\frac{\gamma-1}{2}\frac{U}{c_0}\right]^{2\gamma/(\gamma-1)},$$

where U is the velocity of the piston and m its mass per unit area. Integrating, we obtain

$$U(t)=\frac{2c_0}{\gamma-1}\left\{1-\left[1+\frac{(\gamma+1)p_0t}{2mc_0}\right]^{-(\gamma-1)/(\gamma+1)}\right\}.$$

Problem 5. Determine the flow in an isothermal similarity rarefaction wave.

Solution. The isothermal velocity of sound is

$$c_T=\sqrt{\left(\frac{\partial p}{\partial\rho}\right)_T}=\sqrt{\frac{RT}{\mu}},$$

and for constant temperature $c_T=\text{constant}=c_{T_0}$. According to (99.5) and (99.6)·we therefore have

$$v=c_{T_0}\log\frac{\rho}{\rho_0}=c_{T_0}\log\frac{p}{p_0}=\frac{x}{t}-c_{T_0}.$$

Problem 6. Using **Burgers' equation** (§93), determine the structure due to dissipation in a weak discontinuity between a rarefaction wave and a gas at rest.

Solution. Let the gas at rest be to the left of the discontinuity, and the rarefaction wave to the right, so that the discontinuity moves to the left. Neglecting dissipation, we have in the first region $v=0$. In the second, the flow is described by (99.5) and (99.6) with the sign of c reversed, and v is small near the discontinuity; as far as terms of the first order in v, we have

$$\frac{x}{t}=v-c\cong-c_0+\left(1+\frac{\rho_0}{c_0}\frac{dc_0}{d\rho_0}\right)=-c_0+\alpha_0v,$$

where α is defined by (102.2), and the suffix 0, which denotes values for $v=0$, will be omitted henceforward.

As far as second-order small terms the velocity in the wave propagated to the left obeys equation (6) in§93, Problem 1, or Burgers' equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial\zeta}=\mu\frac{\partial^2u}{\partial\zeta^2},$$

where $\mu=ac^3$ and the unknown $u=\alpha v$ is expressed as a function of t and $\zeta=x+ct$; ζ measures the distance from the weak discontinuity at any instant. We have to find a continuous solution of this equation with the boundary conditions $u=\zeta/t$ for $\zeta\to\infty$, u =0 for $\zeta\to-\infty$, corresponding to flow without dissipation. According to the expression (96.1) for the expansion of a weak discontinuity, t should appear in the solution combined with ζ as $z=\zeta/\sqrt{t}$. Such a solution can satisfy the specified boundary conditions if

$$u(t,\zeta)=\frac{1}{\zeta}\psi(\zeta/\sqrt{t}).$$

The function ψ is related to ϕ in §93, Problem 2, by

$$-2\mu\log\phi=\int\psi(\zeta)d\zeta/\zeta=\int\psi(z)dz/z$$

so that ψ depends only on z, with

$$\psi(z)=-2\mu z\frac{d\log\phi(z)}{dz}$$

Equation (3) in that Problem becomes $2\mu\phi'' = -z\phi'$, whence

$$\phi(z) = \int e^{-z^2/4\mu}\,dz\,.$$

The solution which satisfies the boundary conditions is

$$u(z,\zeta) = \frac{2\mu z}{\zeta}\left[e^{z^2/4\mu}\int_z^\infty e^{-z^2/4\mu}\,dz\right]^{-1},$$

or finally

$$v(\zeta,t) = \sqrt{\frac{\mu}{\alpha^2 t}}\left[e^{\zeta^2/4\mu t}\int_{\zeta/2\sqrt{\mu t}}^\infty e^{-z^2}\,dz\right]^{-1}$$

which gives the structure of the weak discontinuity.