§100. Discontinuities in the initial conditions

One of the most important reasons for the occurrence of surfaces of discontinuity in a gas is the possibility of discontinuities in the initial conditions. These conditions (i.e., the initial distributions of velocity, pressure, etc.) may in general be prescribed arbitrarily. In particular, they need not be everywhere continuous, but may be discontinuous on various surfaces. For example, if two masses of gas at different pressures are brought together at some instant, their surface of contact will be a surface of discontinuity of the initial pressure distribution.

It is of importance that the discontinuities of the various quantities in the initial conditions (or, as we shall say, in the ***initial discontinuities***) can have any values whatever; no relation between them need exist. We know, however, that certain conditions must hold on stable surfaces of discontinuity in a gas; for instance, the discontinuities of density and pressure in a shock wave are related by the shock adiabatic. It is therefore clear that, if these conditions are not satisfied in the initial discontinuity, it cannot continue to be a discontinuity at subsequent instants. Instead, the initial discontinuity in general splits into several discontinuities, each of which is one of the possible types (shock wave, tangential discontinuity, weak discontinuity); in the course of time, these discontinuities move apart.[1]

During a short interval of time after the initial instant $t = 0$, the discontinuities formed from the initial discontinuity do not move apart to great distances, and the flow under consideration therefore takes place in a relatively small volume adjoining the surface of initial discontinuity. As usual, it suffices to consider separate portions of this surface, each of which may be regarded as plane. We need therefore consider only a plane surface of discontinuity, which we take as the yz-plane. It is evident from symmetry that the discontinuities formed from the initial discontinuity will also be plane, and perpendicular to the x-axis. The flow pattern will depend on the coordinate x only (and on the time), so that the problem is **one-dimensional**. There being no characteristic parameters of length and time, we have a **similarity problem**, and the results obtained in §99 can be used.

The discontinuities formed from the initial discontinuity must evidently move away from their point of formation, i.e., away from the position of the initial discontinuity. It is easy to see that either one shock wave, or one pair of weak discontinuities bounding a rarefaction wave, can move in each direction (the positive and negative x-direction). For, if there were, say, two shock waves formed at the same point at time $t = 0$ and both propagated in the positive x-direction, the leading one would have to move more rapidly than the other. According to the general properties of shock waves, however, the leading shock wave must move, relative to the gas behind it, with a velocity less than the velocity of sound c in that gas, and the following shock must move, relative to the same gas, with a velocity exceeding c (c being a constant in the region between the shock waves), i.e., it must overtake the other. For the same reason, a shock wave and a rarefaction wave cannot move in the same direction; to see this, it is sufficient to notice that weak discontinuities move with the velocity of sound relative to the gas on each side of them. Finally, two rarefaction waves formed at the same time cannot become separated, since the velocities of their backward fronts are the same.

As well as shock waves and rarefaction waves, a tangential discontinuity must, in general, be formed from an initial discontinuity. Such a discontinuity must occur if the transverse velocity components v_y, v_z are discontinuous in the initial discontinuity. Since these velocity components do not change in a shock or rarefaction wave, their discontinuities always occur at a tangential discontinuity, which remains at the position of the initial discontinuity; on each side of this discontinuity, v_y and v_z are constant (in reality, of course, the instability of a tangential velocity discontinuity causes its gradual smoothing into a turbulent region).

A tangential discontinuity must occur, however, even if v_y and v_z are continuous at the initial discontinuity (without loss of generality, we can, and shall, assume that they are zero). This is shown as follows. The discontinuities formed from the initial discontinuity

[1] A general discussion of this topic has been given by N. E. Kochin (1926).

must make it possible to go from a given state 1 of the gas on one side of the initial discontinuity to a given state 2 on the other side. The state of the gas is determined by three independent quantities, e.g., p, ρ, and $v_x = v$. It is therefore necessary to have three arbitrary parameters in order to go from state 1 to an arbitrary state 2 by some choice of the discontinuities. We know, however, that a shock wave, perpendicular to the stream, propagated in a gas whose thermodynamic state is given, is completely determined by one parameter (§85). The same is true of a rarefaction wave; as we see from formulae (99.14) – (99.16), when the state of the gas entering a rarefaction wave is given, the state of the gas leaving it is completely determined by one parameter. We have seen, moreover, that at most one wave (rarefaction or shock)can move in each direction. We therefore have at our disposal only two parameters, which are not sufficient.

The tangential discontinuity formed at the position of the initial discontinuity furnishes the third parameter required. The pressure is continuous there, but the density (and therefore the temperature and entropy) is not. The tangential discontinuity is stationary with respect to the gas on both sides of it and the arguments about the "***overtaking***" of two waves propagated in the same direction therefore do not apply to it.

The gases on the two sides of the tangential discontinuity do not mix, since there is no motion of gas through a tangential discontinuity; in all the examples given below, these gases may be different substances.

Figure 78 shows schematically all possible types of break-up of an initial discontinuity. The continuous line shows the variation of the pressure along the *x*-axis; the variation of the density would be given by a similar line, the only difference being that there would be a further jump at the tangential discontinuity. The vertical lines show the discontinuities formed, and the arrows show their direction of propagation and that of the gas flow. The coordinate system is always that in which the tangential discontinuity is at rest, together with the gas in the regions 3 and 3' which adjoin it. The pressures, densities and velocities of the gases in the extreme left-hand (1) and right-hand (2) regions are the values of these quantities at time $t = 0$ on each side of the initial discontinuity.

In the first case, which we write $I \to S_{\leftarrow} T S_{\to}$ (Fig. 78a), the initial discontinuity *I* gives two shock waves *S*, propagated in opposite directions, and a tangential discontinuity *T* between them. This case occurs when two masses of gas collide with a large relative velocity.

In the case $I \to S_{\leftarrow} T R_{\to}$ (Fig. 78b), a shock wave is propagated on one side of the tangential discontinuity, and a rarefaction wave *R* on the other side. This case occurs, for instance, if two masses of gas at relative rest ($v_2 - v_1 = 0$) and at different pressures are brought into contact at the initial instant. For, of all the cases shown in Fig. 78, the second is the only one in which gases 1 and 2 are moving in the same direction, and so the equation $v_1 = v_2$ is possible.

In the third case ($I \to R_{\leftarrow} T R_{\to}$, Fig. 78c), a rarefaction wave is propagated on each side of the tangential discontinuity. If gases 1 and 2 separate with a sufficiently great relative velocity $v_2 - v_1$, the pressure may decrease to zero in the rarefaction waves. We then have the pattern shown in Fig. 78d; a vacuum 3 is formed between regions 4 and 4'.

We can derive the analytical conditions which determine the manner in which the initial discontinuity breaks up, as a function of its parameters. We shall suppose in every case that $p_2 > p_1$, and take the positive *x*-direction from region 1 to region 2 (as in Fig. 78).

Since the gases on the two sides of the initial discontinuity may be of different substances, we shall distinguish them as gases 1 and 2.

(1) $I \to S_{\leftarrow} T S_{\to}$. If $p_3 = p_{3'}$, $v_3 = v_{3'}$, V_3 and $V_{3'}$ are the pressures, velocities and specific volumes in the resulting regions 3 and 3', then we have $p_3 > p_2 > p_1$, and the volumes V_3 and $V_{3'}$ are the abscissae of the points with ordinate p_3 on the shock adiabatics through (p_1, V_1) and (p_2, V_2), respectively. Since the gases in regions 3 and 3' are at rest in the coordinate system chosen, we can use formula (85.7) to give the velocities v_1 and v_2, which are in the positive and negative *x*-directions, respectively:

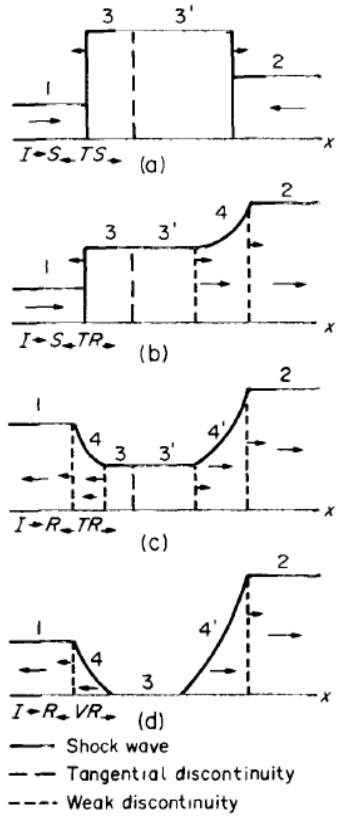

FIG. 78

$$v_1 = \sqrt{(p_3 - p_1)(V_1 - V_3)},$$
$$v_2 = -\sqrt{(p_3 - p_2)(V_2 - V_{3'})}.$$

The least value of p_3, for given p_1 and p_2, which does not contradict the initial assumption ($p_3 > p_2 > p_1$) is p_2. Since, moreover, the difference $v_1 - v_2$ is a monotonically increasing function of p_3, we find the required inequality

$$v_1 - v_2 > \sqrt{(p_2 - p_1)(V_1 - V')}, \qquad (100.1)$$

where V' denotes the abscissa of the point with ordinate p_2 on the shock adiabatic for gas 1 through (p_1, V_1) . Calculating V' from formula (89.1) (in which V_2 is replaced by V'), we obtain the condition (100.1)for a polytropic gas in the form

$$v_1 - v_2 > (p_2 - p_1)\sqrt{\frac{2V_1}{(\gamma_1 - 1)p_1 + (\gamma_1 + 1)p_2}}. \qquad (100.2)$$

It should be noted that the limits placed by (100.1) and (100.2) on the possible values of the velocity difference $v_1 - v_2$ clearly do not depend on the coordinate system chosen.

(2) $I \to S_{\leftarrow} TR_{\rightarrow}$. Here $p_1 < p_3 = p_{3'} < p_2$. For the gas velocity in region 1 we again have

$$v_1 = \sqrt{(p_3 - p_1)(V_1 - V_3)},$$

and the total change in velocity in the rarefaction wave 4 is, by (99.7),

$$v_2 = \int_{p_3}^{p_2} \sqrt{-dpdV}.$$

For given p_1 and p_2, p_3 can lie between them. Replacing p_3 in the difference

$v_2 - v_1$ by p_1 and then by p_2, we obtain the condition

$$-\int_{p_1}^{p_2} \sqrt{-dp dV} < v_1 - v_2 < \sqrt{(p_2 - p_1)(V_1 - V')} . \tag{100.3}$$

Here V' has the same significance as in the previous case; the upper limit of the difference $v_2 - v_1$ must be calculated for gas 1, and the lower limit for gas 2. For a polytropic gas we have

$$-\frac{2c_2}{\gamma_2 - 1}\left[1 - \left(\frac{p_1}{p_2}\right)^{(\gamma_2-1)/2\gamma_2}\right] < v_1 - v_2 < (p_2 - p_1)\sqrt{\frac{2V_1}{(\gamma_1 - 1)p_1 + (\gamma_1 + 1)p_2}} \tag{100.4}$$

where $c_2 = \sqrt{\gamma_2 p_2 V_2}$ is the velocity of sound in gas 2 in the state (p_2, V_2).

(3) $I \rightarrow R_{\leftarrow} T R_{\rightarrow}$. Here $p_2 > p_1 > p_3 = p_{3'} > 0$. By the same method we find the following condition for this case to occur:

$$-\int_0^{p_1} \sqrt{-dp dV} - \int_0^{p_2} \sqrt{-dp dV} < v_1 - v_2 < -\int_{p_1}^{p_2} \sqrt{-dp dV} . \tag{100.5}$$

The first integral in the first member is calculated for gas 1, and the others for gas 2. For a polytropic gas we find

$$-\frac{2c_1}{\gamma_1 - 1} - \frac{2c_2}{\gamma_2 - 1} < v_1 - v_2 < -\frac{2c_2}{\gamma_2 - 1}\left[1 - \left(\frac{p_1}{p_2}\right)^{(\gamma_2-1)/2\gamma_2}\right], \tag{100.6}$$

where $c_1 = \sqrt{\gamma_1 p_1 V_1}$, $c_2 = \sqrt{\gamma_2 p_2 V_2}$. If

$$v_1 - v_2 < -\frac{2c_1}{\gamma_1 - 1} - \frac{2c_2}{\gamma_2 - 1}, \tag{100.7}$$

a vacuum is formed between the rarefaction waves ($I \rightarrow R_{\leftarrow} V R_{\rightarrow}$).

The problem of a discontinuity in the initial conditions includes that of various collisions between plane surfaces of discontinuity. At the instant of collision, the two planes coincide, and form some initial discontinuity, which then leads to one of the patterns described above. The collision of two shock waves, for instance, results in two other shock waves, which move away from the tangential discontinuity remaining between them: $S_{\rightarrow} S_{\leftarrow} \rightarrow S_{\leftarrow} T S_{\rightarrow}$. When one shock wave overtakes another, there are two possibilities: $S_{\rightarrow} S_{\rightarrow} \rightarrow S_{\rightarrow} T S_{\rightarrow}$ and $S_{\rightarrow} S_{\rightarrow} \rightarrow R_{\leftarrow} T S_{\rightarrow}$. In either case a shock wave continues in the same direction.

The problem of the reflection and transmission of shock waves by a tangential discontinuity (boundary of two media) also comes under this heading. Here two cases are possible: $S_{\rightarrow} T \rightarrow S_{\leftarrow} T S_{\rightarrow}$ and $S_{\rightarrow} T \rightarrow R_{\leftarrow} T S_{\rightarrow}$. The wave transmitted into the second medium is always a shock (see also the following Problems).[2]

PROBLEMS

[2] For completeness we should mention that, when a shock wave collides with a weak discontinuity (a problem which is not of the similarity type considered here), the shock wave continues to be propagated in the same direction, but behind it there remain a weak discontinuity of the original kind and a weak tangential discontinuity (see the end of §96).

Problem 1. A plane shock wave is reflected from a rigid plane surface. Determine the gas pressure behind the reflected wave (H. Hugoniot 1885).

Solution. When a shock wave is incident on a rigid wall, a reflected shock wave is propagated away from the wall. We denote by the suffixes 1, 2 and 3, respectively, quantities pertaining to the undisturbed gas in front of the incident shock, the gas behind this shock (which is also the gas in front of the reflected shock) and the gas behind the reflected shock; see Fig. 79, where the arrows indicate the direction of motion of the shock waves and of the gas itself.

The gas in regions 1 and 3, which adjoin the wall, is at rest relative to the wall. The relative velocity of the gases on the two sides of the discontinuity is the same in both the incident and the reflected shock wave, and equal to the velocity of gas 2. Using formula (85.7) for the relative velocity, we therefore have $(p_2 - p_1)(V_1 - V_2) = (p_3 - p_2)(V_2 - V_3)$. The equation of the shock adiabatic (89.1) for each shock gives

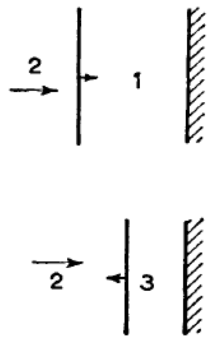

FIG. 79

$$\frac{V_2}{V_1} = \frac{(\gamma_1 + 1)p_1 + (\gamma - 1)p_2}{(\gamma - 1)p_1 + (\gamma + 1)p_2},$$

$$\frac{V_3}{V_2} = \frac{(\gamma_1 + 1)p_2 + (\gamma - 1)p_3}{(\gamma - 1)p_2 + (\gamma + 1)p_3}.$$

We can eliminate the specific volumes from these three equations, and the result is

$$(p_3 - p_2)^2\left[(\gamma + 1)p_1 + (\gamma - 1)p_2\right] = (p_2 - p_1)^2\left[(\gamma + 1)p_3 + (\gamma - 1)p_2\right].$$

This is a quadratic equation for p_3, which has the trivial root $p_3 = p_1$; cancelling $p_3 - p_1$, we obtain

$$\frac{p_3}{p_2} = \frac{(3\gamma - 1)p_2 - (\gamma - 1)p_1}{(\gamma - 1)p_2 - (\gamma + 1)p_1},$$

which determines p_3 from p_1 and p_2. In the limiting case of a very strong incident shock, the further compression of the gas in the reflected shock is given by

$$p_3 = \frac{(3\gamma - 1)}{\gamma - 1}p_2, \quad \frac{V_3}{V_1} = \frac{\gamma - 1}{\gamma},$$

while for a weak shock

$$p_3 - p_2 = p_2 - p_1$$

corresponding to the sound wave approximation.

Problem 2. Find the condition for a shock wave to be reflected from a plane boundary between two gases.

Solution. Let $p_1 < p_{2'}$, V_1, $V_{2'}$ be the pressures and specific volumes of the two media before the incidence of the shock wave (propagated in gas 2), at their surface of separation, and p_2, V_2 the values behind the shock wave. The condition for the reflected wave to be a shock wave is given by the inequality (100.2), in which we must now put

$$v_1 - v_2 = \sqrt{(p_2 - p_{2'})(V_{2'} - V_2)}.$$

Expressing all quantities in terms of the ratio of pressures p_2 / p_1 and the initial specific volumes V_1, $V_{2'}$, we obtain

$$\frac{V_1}{(\gamma_1 + 1)\dfrac{p_2}{p_1} + (\gamma_1 - 1)} < \frac{V_{2'}}{(\gamma_2 + 1)\dfrac{p_2}{p_1} + (\gamma_2 - 1)}.$$

§101. One-dimensional travelling waves

In discussing sound waves in §64, we assumed the amplitude of oscillations in the wave to be small. The result was that the equations of motion were linear and were easily solved. A particular solution of these equations is any function of $x \pm ct$ (a plane wave), corresponding to a ***travelling wave*** whose profile moves with velocity c, its shape remaining unchanged; by the ***profile*** of a wave we mean the distribution of density, velocity, etc., along the direction of propagation. Since the velocity v, the density ρ and the pressure p (and the other quantities) in such a wave are functions of the same quantity $x \pm ct$, they can be expressed as functions of one another, in which the coordinates and time do not explicitly appear ($p = p(\rho)$, $v = v(p)$, and so on).

When the wave amplitude is not necessarily small, these simple relations do not hold. It is found, however, that a general solution of the exact equations of motion can be obtained, in the form of a travelling plane wave which is a generalization of the solution $f(x \pm ct)$ of the approximate equations valid for small amplitudes. To derive this solution, we shall begin from the requirement that, for a wave with any amplitude, the velocity can be expressed as a function of the density.

In the absence of shock waves the flow is adiabatic. If the gas is homogeneous at some initial instant (so that, in particular, s = constant), then s = constant at all times, and we shall assume this in what follows. The pressure is thus a function of the density only.

In a plane sound wave propagated in the x-direction, all quantities depend on x and t only, and for the velocity we have $v_x = v$, $v_y = v_z = 0$. The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0,$$

and Euler's equation is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 .$$

Using the fact that v is a function of ρ only, we can write these equations as

$$\frac{\partial \rho}{\partial t} + \frac{d(\rho v)}{d\rho}\frac{\partial \rho}{\partial x} = 0, \qquad (101.1)$$

$$\frac{\partial v}{\partial t} + \left(v + \frac{1}{\rho}\frac{\partial p}{\partial v}\right)\frac{\partial v}{\partial x} = 0 . \qquad (101.2)$$

Since

$$\frac{\dfrac{\partial \rho}{\partial t}}{\dfrac{\partial \rho}{\partial x}} = -\left(\frac{\partial x}{\partial t}\right)_\rho ,$$

we have from (101.1)

$$\left(\frac{\partial x}{\partial t}\right)_\rho = \frac{d(\rho v)}{d\rho} = v + \rho\frac{dv}{d\rho},$$

and similarly from (101.2)

$$\left(\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho}\frac{dp}{dv} . \qquad (101.3)$$

Since the value of ρ uniquely determines that of v, the derivatives for constant ρ and constant v are the same, i.e., $\left(\frac{\partial x}{\partial t}\right)_\rho = \left(\frac{\partial x}{\partial t}\right)_v$, so that $\rho\frac{dv}{d\rho} = \frac{1}{\rho}\frac{dp}{dv} = \frac{c^2}{\rho}\frac{d\rho}{dv}$. Thus $\frac{dv}{d\rho} = \pm\frac{c}{\rho}$, whence

$$v = \pm\int\frac{c}{\rho}d\rho = \pm\int\frac{dp}{\rho c} . \qquad (101.4)$$

This gives the general relation between the velocity and the density or pressure in the wave.[3]

Next, we can combine (101.3) and (101.4) to give $\left(\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho}\frac{dp}{dv} = v \pm c(v)$ or, integrating,

$$x = t[v \pm c(v)] + f(v)\,, \tag{101.5}$$

where $f(v)$ is an arbitrary function of the velocity, and $c(v)$ is given by (101.4).

Formulae (101.4) and (101.5) give the required general solution (B. Riemann 1860). They determine the velocity (and therefore all other quantities) as an implicit function of x and t, i.e., the wave profile at every instant. For any given value of v, we have $x = at + b$, i.e., the point where the velocity has a given value moves with constant velocity; in this sense, the solution obtained is a travelling wave. The two signs in (101.5) correspond to waves propagated (relative to the gas) in the positive and negative x-directions.

The flow described by the solution (101.4) and (101.5) is often called a ***simple wave***, and we shall use this expression below. It should be noticed that the similarity flow discussed in §99 is a particular case of a simple wave, corresponding to $f(v) = 0$ in (101.5).

We can write out explicitly the relations for a simple wave in a polytropic gas; we assume that there is a point in the wave for which $v = 0$, as usually happens in practice. Since formula (101.4) is the same as (99.6), we have by analogy with formulae (99.14) - (99.16)

$$c = c_0 \pm \frac{\gamma - 1}{2} v\,, \tag{101.6}$$

$$\left.\begin{aligned} \rho &= \rho_0 \left(1 \pm \frac{\gamma-1}{2}\frac{v}{c_0}\right)^{2/(\gamma-1)} \\ p &= p_0 \left(1 \pm \frac{\gamma-1}{2}\frac{v}{c_0}\right)^{2/(\gamma-1)} \end{aligned}\right\}. \tag{101.7}$$

Substituting (101.6) in (101.5), we obtain

$$x = t\left(\pm c_0 + \frac{\gamma+1}{2} v\right) + f(v)\,. \tag{101.8}$$

It is sometimes convenient to write this solution in the form

$$v = F\left[x - \left(\pm c_0 + \frac{\gamma+1}{2} v\right)t\right], \tag{101.9}$$

where F is another arbitrary function.

From formulae (101.6) and (101.7) we again see (as in §99) that the velocity in a direction opposite to that of the propagation of the wave (relative to the gas itself) is of limited magnitude; for a wave propagated in the positive x-direction we have

$$-v \le \frac{2c_0}{\gamma - 1}\,. \tag{101.10}$$

A travelling wave described by formulae (101.4) and (101.5) is essentially different from the one obtained in the limiting case of small amplitudes. The velocity of a point in the wave profile is

$$u = v \pm c\,; \tag{101.11}$$

it may be conveniently regarded as a superposition of the propagation of a disturbance relative to the gas with the velocity of sound and the movement of the gas itself with velocity v. The velocity u is now a function of the density, and therefore is different for different points in the profile. Thus, in the general case of a plane wave with arbitrary amplitude, there is no definite constant "**wave velocity**". Since the velocities of different points in the wave profile are different, the profile changes its shape in the course of time.

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[3] In a wave with small amplitude we have $\rho = \rho_0 + \rho'$, and (101.4) gives in the first approximation $v = c_0\rho'/\rho_0$ (where $c_0 = c(\rho_0)$, i.e., the usual formula (64.12).

Let us consider a wave propagated in the positive x-direction, for which $u = v + c$. The derivative of $v + c$ with respect to the density has been calculated in §99; see (99.10). We have seen that $\frac{du}{d\rho} > 0$. The velocity of propagation of a given point in the wave profile therefore increases with the density. If we denote by c_0 the velocity of sound for a density equal to the equilibrium density ρ_0, then in compressions $\rho > \rho_0$ and $c > c_0$, while in rarefactions $\rho < \rho_0$ and $c < c_0$.

The inequality of the velocity of different points in the wave profile causes its shape to change in the course of time: the points of compression move forward and those of rarefaction are left behind (Fig. 80b). Finally, the profile may become such that the function $\rho(x)$ (for given t) is no longer one-valued; three different values of ρ correspond to some x (the dashed line in Fig. 80c).[4] This is, of course, physically impossible. In reality, discontinuities are formed where ρ is not one-valued, and ρ is consequently one-valued everywhere except at the discontinuities themselves. The wave profile then has the form shown by the continuous line in Fig. 80c. The surfaces of discontinuity are thus formed at points a wavelength apart.

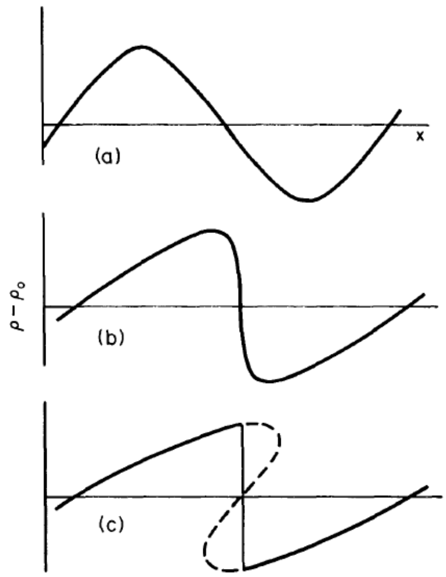

FIG. 80

When the discontinuities are formed, the wave ceases to be a simple wave. The cause of this can be briefly stated thus: when surfaces of discontinuity are present, the wave is reflected from them, and therefore ceases to be a wave travelling in one direction. The assumption on which the whole derivation is based, namely that there is a one-to-one relation between the various quantities, consequently ceases to be valid in general.

The presence of discontinuities (**shock waves**) results, as was mentioned in §85, in the dissipation of energy. The formation of discontinuities therefore leads to a marked damping of the wave. This is evident from Fig. 80. When the discontinuity is formed, the highest part of the wave profile is cut off. In the course of time, as the profile is bent over, its height becomes less, and the profile is smoothed to one with smaller amplitude, i.e., the wave is damped.

It is clear from the above that discontinuities must ultimately be formed in every simple wave which contains regions where the density decreases in the direction of propagation. The only case where discontinuities do not occur is a wave in which the density everywhere

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[4] This change in the wave profile is often referred to as ***turn-over***.

increases monotonically in the direction of propagation (such, for example, is the wave formed when a piston moves out of an infinite pipe filled with gas; see the Problems at the end of this section).

Although the wave is no longer a simple one when a discontinuity has been formed, the time and place of formation of the discontinuity can be determined analytically. We have seen that the occurrence of discontinuities is mathematically due to the fact that, in a simple wave, the quantities p, ρ and v become many-valued functions of x (for given t) at times greater than a certain definite value t_0, whereas for $t < t_0$ they are one-valued functions. The time t_0 is the time of formation of the discontinuity. It is evident from geometrical considerations that, at the instant t_0, the curve giving, say, v as a function of x becomes vertical at some point $x = x_0$, which is the point where the function is subsequently many-valued. Analytically, this means that the derivative $\left(\frac{\partial v}{\partial x}\right)_t$ becomes infinite, and $\left(\frac{\partial x}{\partial v}\right)_t$ becomes zero. It is also clear that, at the instant t_0, the curve $v = v(x)$ must lie on both sides of the vertical tangent, since otherwise $v(x)$ would already be many-valued. In other words, the point $x = x_0$ must be, not an extremum of the function $x(v)$, but a point of inflexion, and therefore the second derivative $\left(\frac{\partial^2 x}{\partial v^2}\right)_t$ must also vanish. Thus the place and time of formation of the shock wave are determined by the simultaneous equations

$$\left(\frac{\partial x}{\partial v}\right)_t = 0, \quad \left(\frac{\partial^2 x}{\partial v^2}\right)_t = 0. \qquad (101.12)$$

For a polytropic gas these equations are

$$t = -\frac{2f'(0)}{\gamma+1}, \quad f'(v) = 0, \qquad (101.13)$$

where $f(v)$ is the function appearing in the general solution (101.8).

These conditions require modification if the simple wave adjoins a gas at rest and the shock wave is formed at the boundary. Here also the curve $v = v(x)$ must become vertical, i.e., the derivative $\left(\frac{\partial x}{\partial v}\right)_t$ must vanish, at the time when the discontinuity occurs. The second derivative, however, need not vanish; the second condition here is simply that the velocity be zero at the boundary of the gas at rest, so that $\left(\frac{\partial x}{\partial v}\right)_t = 0$ for $v = 0$. From this condition we can obtain explicit expressions for the time and place of formation of the discontinuity. Differentiating (101.5), we obtain

$$t = -\frac{f'(0)}{\alpha_0}, \quad x = \pm c_0 t + f(0), \qquad (101.14)$$

where α_0 is the value, for $v = 0$, of the quantity α defined by formula (102.2). For a polytropic gas

$$t = -\frac{2f'(0)}{\gamma+1}. \qquad (101.15)$$

PROBLEMS

Problem 1. A gas is in a semi-infinite cylindrical pipe ($x > 0$) terminated by a piston. At time $t = 0$ the piston begins to move with a uniformly accelerated velocity $U = \pm\, at$. Determine the resulting flow, assuming the gas to be polytropic.

Solution. If the piston moves out of the pipe (U = - at), the result is a simple rarefaction wave, whose forward front is propagated to the right, through gas at rest, with velocity c_0; in the region $x > c_0 t$ the gas is at rest. At the surface of the piston, the gas and the piston must have the same velocity, i.e., we must have v = - at for $x = -(1/2)at^2$ (t > 0). This condition gives for the function $f(v)$ in (101.8)

$$f(-at) = -c_0 t + \frac{\gamma}{2} at^2 .$$

Hence we have

$$x - \left[c_0 + \frac{\gamma+1}{2} v\right] t = f(v)$$

$$= c_0 \frac{v}{a} + \frac{1}{2}\frac{\gamma v^2}{a},$$

whence

$$-v = \frac{c_0 + \frac{\gamma+1}{2}at}{\gamma} - \frac{\sqrt{\left[c_0 + \frac{\gamma+1}{2}at\right]^2 - 2a\gamma(c_0 t - x)}}{\gamma} . \qquad (1)$$

This formula gives the change in velocity over the region between the piston and the forward front $x = c_0 t$ of the wave (Fig. 81a) during the time interval t = 0 to $t = 2c_0/(\gamma - 1)a$. The gas velocity is everywhere to the left, like that of the piston, and decreases monotonically in magnitude in the positive x-direction; the density and pressure increase monotonically in that direction. For $t > 2c_0/(\gamma - 1)a$, the inequality (101.10) does not hold for the piston velocity, and so the gas can no longer follow the piston. A vacuum is then formed in a region adjoining the piston, beyond which the gas velocity decreases from $-2c_0/(\gamma - 1)$ to zero according to formula (1).

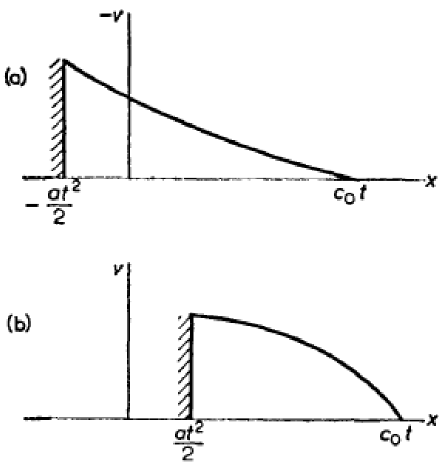

FIG. 81

If the piston moves into the pipe (U = at), a simple compression wave is formed; the corresponding solution is obtained by merely changing the sign of a in (1) (Fig. 81b). It is valid, however, only until a shock wave is formed; the time when this happens is determined from formula (101.15), and is

$$t = \frac{2c_0}{a(\gamma+1)} .$$

Problem 2. The same as Problem 1, but for the case where the piston moves in any manner.

Solution. Let the piston begin to move at time t = 0 according to the law $x = X(t)$ (with $X(0) = 0$); its velocity is $U = X'(t)$. The boundary condition on the piston ($v = U$ for $x = X$) gives $v = X'(t)$, $f(v) = X(t) - t\left[c_0 + \frac{\gamma+1}{2}X'(t)\right]$. If we now regard t as a parameter, these two equations determine the function $f(v)$ in parametric form. Denoting the parameter by τ, we can write the solution as

$$v = X'(\tau) , \quad x = X(\tau) + (t - \tau)\left[c_0 + \frac{\gamma+1}{2}X'(\tau)\right] \qquad (2)$$

which determines, in parametric form, the required function v (t, x) in the simple wave which is caused by the motion of the piston.

Problem 3. Determine the time and place of formation of the shock wave when the piston (Problem 1) moves according to the law $U = at^n$ $(n > 0)$.

Solution. If $a < 0$, i.e., the piston moves out of the pipe, a simple rarefaction wave results, in which no shock wave is formed. We therefore assume that $a > 0$, i.e., the piston moves into the pipe, causing a simple compression wave.

When the function $v(x, t)$ is given by the parametric formulae (2), and $X = a\tau^{n+1}/(n+1)$, the time and place of formation of the shock wave are given by the equations

$$\left.\begin{aligned} \left(\frac{\partial x}{\partial \tau}\right)_t &= -c_0 + \frac{1}{2}t\tau^{n-1}an(\gamma+1) - \frac{1}{2}a\tau^n[\gamma - 1 + n(\gamma+1)] = 0 \\ \left(\frac{\partial^2 x}{\partial \tau^2}\right)_t &= \frac{1}{2}t\tau^{n-2}an(n-1)(\gamma+1) - \frac{1}{2}an\tau^{n-1}[\gamma - 1 + n(\gamma+1)] = 0 \end{aligned}\right\} \tag{3}$$

where the second equation must be replaced by $\tau = 0$ if we are concerned with the formation of a shock wave at the forward front of the simple wave.

For $n = 1$ we find $\tau = 0$, $t = 2c_0 / a(\gamma+1)$, i.e., the shock wave is formed at the forward front at a finite time after the motion begins, in accordance with the results of Problem 1.

For $n < 1$, the derivative $\dfrac{\partial x}{\partial \tau}$ is of variable sign (and therefore the function $v(x)$ for given t is many-valued) for any $t > 0$. This means that a shock wave is formed at the piston as soon as it begins to move.

For $n > 1$ the shock wave is formed, not at the forward front of the simple wave, but at some intermediate point given by (3). Having determined τ and t from (3), we can then find the place of formation of the discontinuity from (2). The result is

$$t = \left(\frac{2c_0}{a}\right)^{1/n} \frac{1}{\gamma+1}\left[\frac{n+1}{n-1}\gamma + 1\right]^{(n-1)/n},$$

$$x = 2c_0\left(\frac{2c_0}{a}\right)^{1/n}\left[\frac{\gamma}{\gamma+1} + \frac{n-1}{n+1}\right]\frac{1}{(n-1)^{(n-1)/n}[\gamma - 1 + n(\gamma+1)]^{1/n}}.$$

Problem 4. For a plane (sound) wave with small amplitude, determine the time-averaged values of quantities in the approximation quadratic in the amplitude. The wave is emitted by a piston moving in accordance with some law $z = X(t)$, $U = X'(t)$, $X(0) = 0$, $\overline{X} = 0$, $\overline{U} = 0$.[5]

Solution. We start from the exact solution (101.9), which we write in an equivalent form with a different choice of argument:

$$v = F(t - x/u), \quad u = c_0 + \alpha_0 v, \tag{4}$$

where $\alpha_0 = \dfrac{\gamma+1}{2}$, or $v = F(\xi)$, where ξ is determined implicitly by the equation[6]

$$\xi = t - x/u(\xi). \tag{5}$$

We shall show that, in a calculation as far as second-order quantities, averaging over t is equivalent to averaging over ξ. For a given x,

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5 The solution follows that by L. A. Ostrovskii (1968).

6 For waves with small amplitude, (4) is valid for any (not necessarily polytropic) gas if α_0 is defined by (102.2).

$$dt = d\xi\left(1 - \frac{x}{u^2}\frac{du}{d\xi}\right) \cong d\xi\left(1 - \frac{x\alpha_0}{c_0{}^2}\frac{dv}{d\xi}\right);$$

in the denominator u^2, the small quantity $v << c_0$ can be neglected. The effect sought is due to cumulative non-linear distortions of the profile, and is found by solving (4) for v. Hence

$$\begin{aligned}\int_{t_1}^{t_2} v dt &= \int_{\xi_1}^{\xi_2}\left\{F - \frac{x\alpha_0}{c_0{}^2}F\frac{dF}{d\xi}\right\}d\xi \\ &= \int_{\xi_1}^{\xi_2} F d\xi - \frac{x\alpha_0}{c_0{}^2}\left[F^2(\xi_2) - F^2(\xi_1)\right]\end{aligned}.$$

The second term is always finite and makes no contribution when averaged over a long interval of time. Since also

$$\begin{aligned}\xi_2 - \xi_1 &\cong t_2 - t_1 + \frac{x\alpha_0}{c_0{}^2}(v_2 - v_1) \\ &\cong t_2 - t_1\end{aligned},$$

we reach the result that $\overline{v}^t = \overline{v}^\xi$, where the index beside the bar shows the variable over which the averaging is done (and will be omitted henceforward); the average over t is therefore independent of x.

For the problem with an oscillating piston, $F(\xi)$ is determined by equation (2), which may be written as

$$v(\tau) = X'(\tau), \quad \tau = \xi + X(\tau)/u(\tau)$$

or, since the oscillation amplitude is small,

$$\tau \cong \xi + \frac{1}{c_0}X(\xi), \quad v(\tau) \cong U(\xi) + \frac{1}{c_0}X(\xi)\frac{dU(\xi)}{d\xi}$$

Averaging the last expression gives

$$\overline{v} = \frac{1}{c_0}\overline{X\frac{dU}{d\xi}} = \frac{1}{c_0}\overline{\frac{d(XU)}{d\xi}} - \frac{1}{c_0}\overline{U^2}$$

and, since the mean value of a total derivative is zero,

$$\overline{v} = -\overline{U^2}/c_0.$$

To the same accuracy, the time-averaged mass flux density is

$$\overline{\rho v} = \rho_0\overline{v} + \overline{\rho' v} = \rho_0\overline{v} + \rho_0\overline{v^2}/c_0.$$

Using (6) and the relation (in the same approximation) $\overline{v^2} = \overline{U^2}$, we find that $\overline{\rho v} = 0$, as it should be, according to the conservation of mass, in the purely one-dimensional case, where no mass is transferred "sideways". The mean energy flux density is

$$\overline{q} = \overline{\rho w v} = w_0\overline{\rho v} + \rho_0\overline{w' v} = \overline{p' v} = \rho_0 c_0\overline{v^2}$$

(cf. §65); thus $\overline{q} = \rho_0 c_0\overline{U^2}$.

To calculate $\overline{p'}$ and $\overline{\rho'}$, we have to express p' and ρ' in terms of v as far as v^2 terms. From (101.7), or from (101.4) and (101.6) for a non-polytropic gas, we have

$$\frac{\rho'}{\rho_0} = \frac{v}{c_0} + (2-\alpha)\frac{v^2}{2c_0{}^2}, \quad p' = c^2\rho' + (\alpha - 1)\rho_0 v^2,$$

and on averaging[7]

[7] These formulae were derived, with more restrictive assumptions, by A. Eichenwald (1932).

$$\overline{\rho'} = -\frac{1}{2}\alpha\rho_0\frac{\overline{U^2}}{c_0^{\ 2}}, \quad \overline{p'} = -\frac{1}{2}(2-\alpha)\rho_0\overline{U^2}.$$

Note that $\overline{p'}$ here is not zero even in the quadratic approximation; cf. the end of §65.