§102. Formation of discontinuities in a sound wave

A travelling plane sound wave, being an exact solution of the equations of motion, is also a **simple wave**. We can use the general results obtained in § 101 to derive some properties of small-amplitude sound waves in the second approximation (the first approximation being that which gives the ordinary linear wave equation).

We must notice first of all that a discontinuity must ultimately appear in each wavelength of a sound wave. This leads to a very marked **damping** of the wave, as shown in § 101. It must be remarked, however, that this happens only for a sufficiently **strong sound wave**; a weak sound wave is damped by the usual effects of viscosity and thermal conduction before the effects of higher order in the amplitude can develop.

The distortion of the wave profile has another effect also. If the wave is purely harmonic at some instant, it ceases to be so at later instants, on account of the change in shape of the profile. The motion, however, remains periodic, with the same period as before. When the wave is expanded in a Fourier series, terms with frequencies $n\omega$ (n being integral and ω being the fundamental frequency) appear, as well as that with frequency ω. Thus the distortion of the profile as the sound wave is propagated may be regarded as the appearance in it of higher harmonics in addition to the fundamental frequency.

The velocity u of points in the wave profile (the wave being propagated in the positive x-direction) is obtained, in the first approximation, by putting in (101.11) $v = 0$, i.e., $u = c_0$, corresponding to the propagation of the wave with no change in its profile. In the next approximation we have

$$u = c_0 + \rho' \frac{\partial u}{\partial \rho_0} = c_0 + \frac{\partial u}{\partial \rho_0} \frac{\rho_0 v}{c_0},$$

or, using the expression (99.10) for the derivative $\dfrac{\partial u}{\partial \rho}$,

$$u = c_0 + \alpha_0 v, \qquad (102.1)$$

where we have put for brevity[1]

$$\alpha = \frac{c^4}{2V^3} \left(\frac{\partial^2 V}{\partial p^2} \right)_s. \qquad (102.2)$$

For a polytropic gas, $\alpha = \dfrac{\gamma + 1}{2}$, and formula (102.1) agrees with the exact formula (see (101.8)) for the velocity u.

In the general case of arbitrary amplitude, the wave is no longer simple after the discontinuities have appeared. A small-amplitude wave, however, is still simple in the second approximation even when discontinuities are present. This can be seen as follows. The changes in velocity, pressure and specific volume in a shock wave are related by $v_1 - v_2 = \sqrt{(p_2 - p_1)(V_1 - V_2)}$. The change in the velocity v over a segment of the x-axis in a simple wave is

$$v_2 - v_1 = \int_{p_1}^{p_2} \sqrt{-\frac{\partial V}{\partial p}} dp.$$

A simple calculation, using an expansion in series, shows that these two expressions differ only by terms of the third order (it must be borne in mind that the change in entropy at a discontinuity is of the third order of smallness, while in a simple wave the entropy is constant). Hence it follows that, as far as terms of the second order, a sound wave on either side of a discontinuity in it remains simple, and the appropriate boundary condition is satisfied at the discontinuity itself. In higher approximations this is no longer true, on account of the appearance of waves reflected from the surface of discontinuity.

Let us now derive the condition which determines the location of the discontinuities in a travelling sound wave (again in the second approximation). Let u be the velocity of the

[1] In §93, Problem 1, this quantity was denoted by α_0.

discontinuity relative to a fixed coordinate system, and v_1, v_2 the velocities of the gases on each side of it. Then the condition that the mass flux be continuous is $\rho_1(v_1-u)=\rho_2(v_2-u)$, whence $u=\dfrac{\rho_1 v_1-\rho_2 v_2}{\rho_1-\rho_2}$. As far as the second-order terms, this is equal to the derivative $\dfrac{d(\rho v)}{d\rho}$ at the point where v is equal to $\frac{1}{2}(v_1+v_2)$. Since, in a simple wave, $\dfrac{d(\rho v)}{d\rho}=v+c$, we have, by (102.1),

$$u=c_0+\frac{1}{2}\alpha_0(v_1+v_2)\,. \qquad (102.3)$$

From this we can obtain the following simple geometrical condition which determines the position of the shock wave. In Fig. 82 the curve shows the velocity profile corresponding to the simple wave; let *ae* be the discontinuity, and x_s its position. The difference of the shaded areas *abc* and *cde* is the integral

$$\int_{v_1}^{v_2}(x-x_s)dv$$

taken along the curve *abcde*. In the course of time, the wave profile moves; let us calculate the time derivative of the above integral. Since the velocity *dx*/*dt* of points in the wave profile is given by formula (102.1), and the velocity dx_s/dt of the discontinuity by (102.3), we have

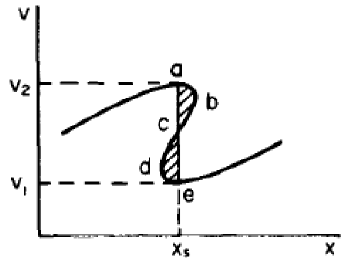

FIG. 82

$$\frac{d}{dt}\int_{v_1}^{v_2}(x-x_s)dv=\alpha\left[\int_{v_1}^{v_2}vdv-\frac{1}{2}(v_1+v_2)\int_{v_1}^{v_2}dv\right]=0\,;$$

in differentiating the integral, we must notice that, although the limits of integration v_1 and v_2 also vary with time, $x-x_s$ always vanishes at the limits, and so we need only differentiate the integrand.

Thus the integral $\int(x-x_s)dv$ remains constant in time. Since it is zero at the instant when the shock wave is formed (the points *a* and *e* then coinciding), it follows that we always have

$$\int_{abcde}(x-x_s)dv=0\,. \qquad (102.4)$$

Geometrically this means that the areas *abc* and *cde* are equal, a condition which determines the position of the discontinuity.

The formation of discontinuities in a sound wave is an example of the spontaneous occurrence of shock waves in the absence of any singularity in the external conditions of the flow. It must be emphasized that, although a shock wave can appear spontaneously at a particular instant, it cannot disappear in the same manner. Once formed, a shock wave decays only asymptotically as the time becomes infinite.

Let us consider a single one-dimensional **compression pulse**, in which a shock wave has already been formed, and ascertain how this shock will finally be damped. In the later stages of its propagation, a sound pulse containing a shock wave will have a triangular velocity profile, the linear profile remaining linear as it changes shape.[2]

[2] Here and later, a distribution profile of the velocity v is mentioned, simply in order not to complicate the formulae. A quantity having greater practical interest is the excess pressure p', which differs from v only by a

Let the profile be given at some instant (which we take as $t = 0$) by the triangle ABC in Fig. 83a; the values of quantities at this instant are denoted by the suffix 1.[3] If the points in this profile moved with the velocities (102.1), we should obtain after time t a profile $A'B'C'$ (Fig. 83b).

In reality, the discontinuity moves to E, and the actual profile will be $A'DE$. The areas $DB'F$ and $C'FE$ are equal, by (102.4), and therefore the area $A'DE$ of the new profile is equal to the area ABC of the original profile. Let l be the length of the sound pulse at time t, and Δv the velocity discontinuity in the shock wave. During a time t, the point B moves a distance $\alpha t(\Delta v)_1$ relative to C; the tangent of the angle $B'AC'$ is therefore $(\Delta v)_1/[l_1+\alpha t(\Delta v)_1]$, and we obtain the condition of equal areas ABC and $A'DE$ in the form

$$l_1(\Delta v)_1 = \frac{l^2(\Delta v)_1}{l_1+\alpha t(\Delta v)_1},$$

whence

$$\left.\begin{aligned} l &= l_1\sqrt{1+\frac{\alpha(\Delta v)_1 t}{l_1}} \\ \Delta v &= \frac{(\Delta v)_1}{\sqrt{1+\dfrac{\alpha(\Delta v)_1}{l}}} \end{aligned}\right\} \tag{102.5}$$

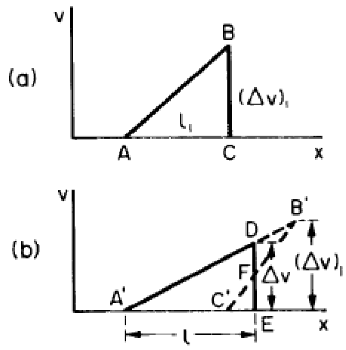

FIG. 83

The total energy of a travelling sound pulse (per unit area of its front) is

$$E = \rho\int v^2 dx = \frac{E_1}{\sqrt{1+\dfrac{\alpha(\Delta v)_1 t}{l}}}. \tag{102.6}$$

For $t\to\infty$ the strength of the shock wave and its energy decrease asymptotically as $1/\sqrt{t}$ (or, equivalently, as $1/\sqrt{x}$ with the distance $x = ct$). The pulse length increases as $\sqrt{t}$. Note also that the limiting slope of the profile $\Delta v/l \to 1/\alpha t$ is independent of the shock strength and of the pulse length.

Let us now consider the limiting properties (at large distances from the source) of shock waves formed in cylindrical and spherical sound waves (L. D. Landau 1945). We take first the cylindrical case.

At sufficiently large distances r from the axis, any small section of such a wave may be regarded as plane. The velocity of any point in the wave profile is then given by formula (102.1). If, however, we wish to use this formula to follow the motion of any point in the wave profile over long intervals of time, we must take into account the fact that the amplitude of a cylindrical wave falls off with distance as $1/\sqrt{r}$, even in the first approximation. This means that, at any given point in the profile, v is not constant, as it is for a plane wave, but decreases as $1/\sqrt{r}$. If v_1 is the value of v (for a given point in the profile) at a (large) distance r_1, we can put $v = v_1\sqrt{r_1/r}$. Thus the velocity u of points in the wave profile is

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constant factor, $p' = c/\rho_0 c_0$; similar results are valid for it. The sign of v is the same as that of p', so that $v > 0$ and $v < 0$ correspond to compression and rarefaction, respectively. The rate of movement of points in the profile is expressed in terms of p' by

$$u = c_0(1+v_0 p'/p_0),\quad v = \alpha p/\rho c^2;$$

for a polytropic gas, $v = (\gamma+1)/2\gamma$.

[3] The suffix 0 denoting the equilibrium values will be omitted.

$$u = c + \alpha v_1 \sqrt{r_1 / r} \ . \qquad (102.7)$$

The first term is the ordinary velocity of sound, and corresponds to movement of the wave without change in the shape of the profile (apart from the general decrease of the amplitude as $1/\sqrt{r}$, that is, taking as the profile the distribution of $v\sqrt{r}$). The second term results in a distortion of the profile. The amount δr of additional movement of points in the profile during a time $(r - r_1)/c$ is found by integrating over *dr*/*c*:

$$\delta r = 2\alpha \frac{v_1}{c} \sqrt{r_1} (\sqrt{r} - \sqrt{r_1}) \ . \qquad (102.8)$$

The distortion of the profile of a cylindrical wave increases more slowly than for a plane wave, where δx is proportional to the distance *x* traversed by the wave, but here too it does of course lead ultimately to the formation of discontinuities. Let us consider shock waves formed in a single cylindrical sound pulse which has reached a large distance from the source (the axis).

The cylindrical case is distinguished from the plane case primarily by the fact that a single pulse cannot consist of compression only or rarefaction only; if the sound wave front is followed by a region of compression, this in turn must be followed by a region of rarefaction (see §71).[4] The point of maximum rarefaction will lag behind all those to the rear of it, and the profile therefore turns over to form a discontinuity. Thus, in a cylindrical sound pulse, **two shock waves** are formed. In the leading one, the velocity increases abruptly from zero; then follows a region in which the compression gradually decreases into a rarefaction, after which the pressure again increases discontinuously in the second shock. A cylindrical sound pulse is, however, distinctive (in comparison with the plane and spherical cases) also in that it cannot have a backward front; *v* tends to zero only asymptotically. This has the result that in the rear discontinuity *v* increases not to zero but only to some negative non-zero value, afterwards tending asymptotically to zero. This leads to a profile of the kind shown in Fig. 84.

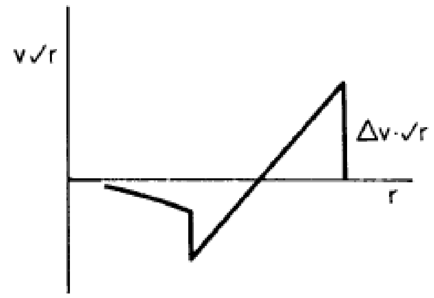

FIG. 84

The manner of the final damping of the shock waves with time (or, equivalently, with the distance *r* from the axis) can be found in the same way as for the plane case discussed above. It is seen from the previous result that the limiting form corresponds to the time when the displacement δr at the top of the profile becomes large in comparison with the "original" pulse width l_1 (by which is meant, for example, the distance from the leading shock wave to the point where $v = 0$). This displacement on the path from r_1 to $r << r_1$ is

$$\delta r \cong \frac{2\alpha}{c} (\Delta v)_1 \sqrt{r_1 r} \ ,$$

where $(\Delta v)_1$ is the "original" discontinuity (at distance r_1) on the leading shock. The "final" slope of the linear part of the profile between the shock waves is then $\cong \sqrt{r_1} (\Delta v)_1 / \delta r \cong c / 2\alpha\sqrt{r}$. The condition of constant area of the profile gives $l_1 \sqrt{r_1} (\Delta v)_1 = l^2 c / \alpha\sqrt{r}$, whence $l \propto r^{1/4}$, instead of $l \propto x^{1/2}$ in the plane case. The limiting decrease of Δv in the leading shock is then given by $l\sqrt{r}\Delta v = \text{constant}$, i.e.,

[4] This type of configuration will be the one considered. It pertains, in particular, to the application of the results to shock waves formed in supersonic motion of a body with finite size (§122).

$$\Delta v \propto r^{-3/4}. \tag{102.9}$$

Lastly, let us consider the spherical case.[5] The general decrease in amplitude of the outgoing sound wave takes place as $1/r$, where r is now the distance from the centre. Repeating the arguments given above for the cylindrical case, we find as the velocity of points in the wave profile

$$u = c + \alpha v_1 r_1 / r, \tag{102.10}$$

and hence the displacement δr of points in the profile on the path from r_1 to r:

$$\delta r = \frac{\alpha v_1 r_1}{c} \log \frac{r}{r_1}. \tag{102.11}$$

We see that the profile distortion in a spherical wave increases with distance only logarithmically, much more slowly than in the plane case or even the cylindrical one.

Spherical propagation of a compression sound wave must be accompanied, as in the cylindrical case, by a following rarefaction (see §70). Here also, two discontinuities must be formed (but a single spherical pulse can have a backward front, in which case v increases discontinuously to zero).[6] By the same method as before, we find the limiting relations for the increase in the pulse length and the decrease in the strength of the shock wave:

$$l \propto \sqrt{\log \frac{r}{a}}, \quad \Delta v \propto \frac{1}{r}\sqrt{\log \frac{r}{a}},. \tag{102.12}$$

where a is constant having the dimension of length.[7]

PROBLEMS

Problem 1. At the initial instant, the wave profile consists of an infinite series of "teeth", as shown in Fig. 85.[8] Determine how the profile and energy of the wave change with time.

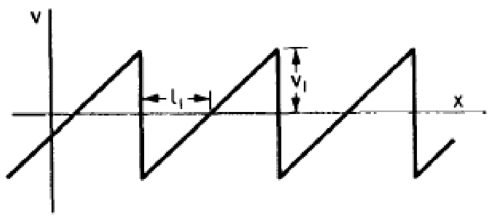

FIG. 85

Solution. It is evident that, at subsequent instants, the wave profile will have the same form, with l_0 unchanged but the height v_t less than v_1. Let us consider one "tooth": at time $t = 0$, the ordinate through the point where $v = v_t$ cuts off a part $v_t l_1 / v_1$ of the base of the triangle. During a time t, this point moves forward a distance $\alpha v_t t$. The condition that the base of the triangle be unchanged in length is $v_t l_1 / v_1 + \alpha t v_t = l_1$, whence $v_t = v_1 / (1 + \alpha v_1 t / l_1)$. As $t \to \infty$, the wave amplitude diminishes as $1/t$. The energy is $E = E_0 / (1 + \alpha v_1 t / l_1)^2$, i.e., it diminishes as $1/t^2$ for $t \to \infty$.

Problem 2. Determine the intensity of the second harmonic formed by the distortion of the

[5] For example, a shock wave formed in an explosion and considered at large distances from the source.

[6] Since in practice a gas always exhibits ordinary sound absorption due to thermal conduction and viscosity, the slowness of the distortion in a spherical wave may have the result that it is absorbed before discontinuities can be formed.

[7] This constant is not in general equal to r_1. The reason is that the argument of the logarithm has to be dimensionless, and therefore, when $r >> r_1$, we cannot simply neglect $\log r_1$ in (102.11). The determination of the coefficient of r in the large logarithm requires a more exact allowance for the original form of the profile.

[8] This is the asymptotic form of the profile for any periodic wave.

profile of a monochromatic spherical wave.

Solution. Writing the wave in the form $rv = A\cos(kr - \omega t)$, we can allow for the distortion, in the first approximation, by adding δr to r on the right-hand side of this equation, and expanding in powers of δr. This gives, by (102.11),

$$rv = A\cos(kr - \omega t) - \frac{\alpha k}{2c} A^2 \log\frac{r}{r_1} \sin 2(kr - \omega t);$$

here r_1 must be taken as a distance at which the wave can still be regarded, with sufficient accuracy, as strictly monochromatic. The second term in this formula is the second harmonic in the spectral resolution of the wave. Its total (time-averaged) intensity I_2 is

$$I_2 = \frac{\alpha^2 k^2}{8\pi c^3 \rho_0} \log^2\left(\frac{r}{r_1}\right) L_1^2,$$

where $I_1 = 2\pi c\rho A^2$ is the intensity of the first harmonic.

§103. Characteristics

The definition of characteristics, given in §82, as lines along which small disturbances are propagated (in the approximation of geometrical acoustics) has general validity, and is not restricted to the plane steady supersonic flow discussed in §82.

For one-dimensional non-steady flow, we can introduce the characteristics as lines in the *xt*-plane whose slope dx/dt is equal to the velocity of propagation of small disturbances relative to a fixed coordinate system. Disturbances propagated relative to the gas with the velocity of sound, in the positive or negative *x*-direction, move relative to the fixed coordinate system with velocity $v \pm c$. The differential equations of the two families of characteristics, which we shall call C_+ and C_-, are accordingly

$$\left(\frac{dx}{dt}\right)_+ = v + c, \quad \left(\frac{dx}{dt}\right)_- = v - c \tag{103.1}$$

Disturbances transmitted with the gas are propagated in the *xt*-plane along characteristics belonging to a third family C_0, for which

$$\left(\frac{dx}{dt}\right)_0 = v. \tag{103.2}$$

These are just the "**streamlines**" in the *xt*-plane; cf. the end of §82.[9] It should be emphasized that, for characteristics to exist, it is no longer necessary for the gas flow to be supersonic. The "**directional**" propagation of disturbances, as evidenced by the characteristics, is here simply due to the causal relation between the motions at successive instants.

As an example, let us consider the characteristics of a **simple wave**. For a wave propagated in the positive *x*-direction we have, by (101.5), $x = t(v + c) + f(v)$. Differentiating this relation, we have

$$dx = (v + c)dt + [t + tc'(v) + f'(v)]dv.$$

Along a characteristic C_+, we have $dx = (v + c)dt$; comparing the two equations, we find that along such a characteristic $[t + tc'(v) + f'(v)]dv = 0$. The expression in brackets cannot vanish identically, and therefore $dv = 0$, i.e., v = constant. Thus we conclude that, along any characteristic C_+, the velocity is constant, and therefore so are all other quantities. The same property holds for the characteristics C_- in a wave propagated to the left. We shall see in §104 that this is no accident, but is a mathematical consequence of the nature of simple waves.

From this property of the characteristics C_+ for a simple wave, we can in turn conclude

[9] The same equations (103.1) and (103.2) determine the characteristics for non-steady spherically symmetrical flow, if *x* is replaced by the radial coordinate *r* (the characteristics now being lines in the *rt*-plane).

that they are a family of straight lines in the xt-plane; the velocity is constant along the lines $x = t[v + c(v)] + f(v)$ (101.5). In particular, for a similarity rarefaction wave (a simple wave with $f(v) = 0$), these lines form a pencil through the origin in the xt-plane. For this reason, a similarity simple wave is said to be ***centred***.

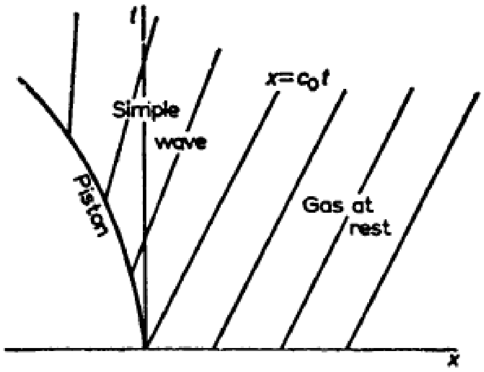

FIG. 86

Figure 86 shows the family of characteristics C_+ for the simple rarefaction wave formed when a piston moves out of a pipe with acceleration. It is a family of diverging straight lines, which begin from the curve $x = X(t)$ giving the motion of the piston. To the right of the characteristic $x = c_0 t$ lies a region of gas at rest, where the characteristics become parallel.

Figure 87 is a similar diagram for the simple compression wave formed when a piston moves into a pipe with acceleration. In this case the characteristics are converging straight lines, which eventually intersect. Since every characteristic has a constant value of v, their

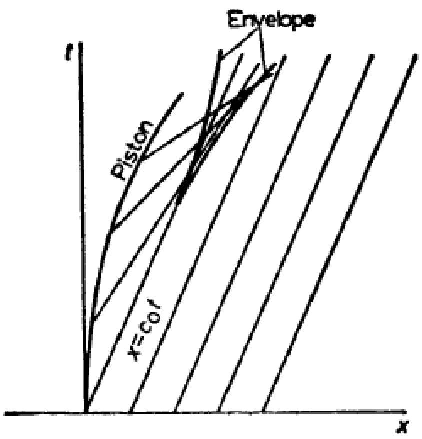

FIG. 87

intersection shows that the function $v(x, t)$ is many-valued, which is physically meaningless. This is the geometrical interpretation of the result obtained in §101: a simple compression wave cannot exist indefinitely, and a shock wave must be formed in it. The geometrical interpretation of the conditions (101.12), which determine the time and place of formation of the shock wave, is as follows. The intersecting family of rectilinear characteristics has an envelope, which, for a certain least value of t, has a cusp; this gives the instant at which many-valuedness first occurs. If the equations of the characteristics are given in the parametric form $x = x(v)$, $t = t(v)$, the position of the cusp is given by equations (101.12).[10]

[10] The whole of the region between the two branches of the envelope is occupied by three sets of characteristics, in accordance with the three-valuedness caused by the turn-over of the wave profile. The particular case where the shock wave occurs at the boundary of the gas at rest corresponds to that where one branch of the envelope is part of

We shall now indicate briefly how the physical definition, given above, of the characteristics as lines along which disturbances are propagated corresponds to the mathematical sense of the word in the theory of partial differential equations. Let us consider a partial differential equation having the form

$$A\frac{\partial^2\phi}{\partial x^2}+2B\frac{\partial^2\phi}{\partial x\partial t}+C\frac{\partial^2\phi}{\partial t^2}+D=0\,, \qquad (103.3)$$

which is linear in the second derivatives; the coefficients A, B, C, D can be any functions, both of the independent variables x, t and of the unknown function ϕ and its first derivatives.[11] Equation (103.3) is of the elliptic type if $B^2-AC<0$ everywhere, and of the hyperbolic type if $B^2-AC>0$. In the latter case, the equation

$$Adt^2-2Bdxdt+Cdx^2=0\,, \qquad (103.4)$$

or

$$\frac{dx}{dt}=\frac{B\pm\sqrt{B^2-AC}}{C}\,, \qquad (103.5)$$

determines two families of curves in the xt-plane, the ***characteristics*** (for a given solution $\phi(x,t)$ of equation (103.3)). We may point out that, if the coefficients A, B, C are functions only of x and t, then the characteristics are independent of the particular solution ϕ .

Let a given flow correspond to some solution $\phi=\phi_0(x,t)$ of equation (103.3), and let a small perturbation ϕ_1 be applied to it. We assume that this perturbation satisfies the conditions for geometrical acoustics to be valid: it does not greatly affect the flow (ϕ_1 and its first derivatives are small), but varies considerably over short distances (the second derivatives of ϕ_1 are relatively large). Putting in equation (103.3) $\phi=\phi_0+\phi_1$, we then obtain for ϕ_1, the equation

$$A\frac{\partial^2\phi_1}{\partial x^2}+2B\frac{\partial^2\phi_1}{\partial x\partial t}+C\frac{\partial^2\phi_1}{\partial t^2}=0\,,$$

with $\phi=\phi_0$ in the coefficients A, B, C. Following the method used in changing from wave optics to geometrical optics, we write $\phi_1=ae^{i\psi}$, where the function ψ (the ***eikonal***) is large, and obtain for ψ the equation

$$A\left(\frac{\partial\psi}{\partial x}\right)^2+2B\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}+C\left(\frac{\partial\psi}{\partial t}\right)^2=0\,. \qquad (103.6)$$

The equation of ray propagation in geometrical acoustics is obtained by equating $\frac{dx}{dt}$ to the group velocity: $\frac{dx}{dt}=\frac{d\omega}{dk}$, where $k=\frac{\partial\psi}{\partial x}$, $\omega=-\frac{\partial\psi}{\partial t}$. Differentiating the relation $Ak^2-2Bk\omega+C\omega^2=0$, we obtain $\frac{dx}{dt}=\frac{B\omega-Ak}{C\omega-Bk}$, and, eliminating $\frac{k}{\omega}$ by the same relation, we again arrive at equation (103.5).

PROBLEM

Find the equation of the second family of characteristics in a centred simple wave in a polytropic gas.

Solution. In a centred simple wave propagated into gas at rest to the right of it, we have

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the characteristic $x=c_0t$.

[11] The velocity potential satisfies an equation of this form in one-dimensional non-steady flow.

$\dfrac{x}{t} = v + c = c_0 + \dfrac{\gamma+1}{2} v$. The characteristics C_+ form the pencil $x = \text{constant} \times t$. The characteristics C_-, on the other hand, are determined by the equation

$$\frac{dx}{dt} = v - c = \frac{3-\gamma}{\gamma+1}\frac{x}{t} - \frac{4}{\gamma+1} c_0 .$$

Integrating, we find

$$x = -\frac{2}{\gamma-1} c_0 t + \frac{\gamma+1}{\gamma-1} c_0 t_0 \left(\frac{t}{t_0}\right)^{(3-\gamma)/(\gamma+1)} ,$$

where the constant of integration has been chosen so that the characteristic C_- passes through the point $x = c_0 t_0$, $t = t_0$ on the characteristic C_+ ($x = c_0 t$) which is the boundary between the simple wave and the region at rest.

The "streamlines" in the *xt*-plane are given by the equation

$$\frac{dx}{dt} = v = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right),$$

whence, for the characteristic C_0,

$$x = -\frac{2}{\gamma-1} c_0 t + \frac{\gamma+1}{\gamma-1} c_0 t_0 \left(\frac{t}{t_0}\right)^{2/(\gamma+1)} .$$

§104. Riemann invariants

An arbitrary small disturbance is in general propagated along all three characteristics (C_+, C_-, C_0) leaving a given point in the *xt*-plane. However, an arbitrary disturbance can be separated into parts each of which is propagated along only one characteristic.

Let us first consider isentropic gas flow. We write the equation of continuity and Euler's equation in the form

$$\boxed{\begin{gathered}\frac{\partial \rho}{\partial t} + div(\rho v) = 0 \\ \frac{1}{c^2}\frac{\partial p}{\partial t} + \rho\frac{\partial v}{\partial x} + v\frac{\partial \rho}{\partial x} = 0 \\ \frac{1}{c^2}\frac{\partial p}{\partial t} + \rho\frac{\partial v}{\partial x} + \frac{v}{c^2}\frac{\partial p}{\partial x} = 0\end{gathered}}$$

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial x} + \rho c^2 \frac{\partial v}{\partial x} = 0 ,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 ;$$

in the equation of continuity we have replaced the derivatives of the density by those of the pressure, using the formulae

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t}, \quad \frac{\partial \rho}{\partial x} = \frac{1}{c^2}\frac{\partial p}{\partial x} .$$

Dividing the first equation by $\pm\rho c$ and adding it to the second, we obtain

$$\frac{\partial v}{\partial t} \pm \frac{1}{\rho c}\frac{\partial p}{\partial t} + \left(\frac{\partial v}{\partial x} \pm \frac{1}{\rho c}\frac{\partial p}{\partial x}\right)(v \pm c) = 0 . \qquad (104.1)$$

We now introduce as new unknown functions

$$J_+ = v + \int \frac{dp}{\rho c}, \quad J_- = v - \int \frac{dp}{\rho c}, \qquad (104.2)$$

which are called ***Riemann invariants***. It should be remembered that, in isentropic flow, ρ and c are definite functions of *p*, and the integrals on the right-hand sides are therefore definite functions. For a polytropic gas

$$J_+ = v + \frac{2c}{\gamma-1}, \quad J_- = v - \frac{2c}{\gamma-1} \qquad (104.3)$$

In terms of these quantities, the equations of motion take the simple form

$$\left[\frac{\partial}{\partial t}+(v+c)\frac{\partial}{\partial x}\right]J_+=0\,,\quad \left[\frac{\partial}{\partial t}+(v-c)\frac{\partial}{\partial x}\right]J_-=0\,. \tag{104.4}$$

The differential operators acting on J_+ and J_- are just the operators of differentiation along the characteristics C_+ and C_- in the *xt*-plane. Thus we see that J_+ and J_- remain constant along each characteristic C_+ or C_-, respectively. We can also say that small perturbations of J_+ are propagated only along the characteristics C_+, and those of J_- only along C_-.

In the general case of anisentropic flow, the equations (104.1) cannot be written in the form (104.4), since $\dfrac{dp}{\rho c}$ is not a perfect differential. These equations, however, still permit the separation of perturbations propagated along characteristics of only one family. Such perturbations are those of the form $\delta v \pm \dfrac{\delta p}{\rho c}$, where δv and δp are arbitrary small perturbations of the velocity and pressure. Their propagation is described by the linearized equations

$$\left[\frac{\partial}{\partial t}+(v\pm c)\frac{\partial}{\partial x}\right]\left(\delta v\pm\frac{\delta p}{\rho c}\right)=0\,. \tag{104.5}$$

In order to obtain a complete system of equations of motion, these must be supplemented by the adiabatic equation

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right]\delta s=0\,, \tag{104.6}$$

which shows that perturbations δs are propagated along the characteristics C_0. An arbitrary small perturbation can always be separated into independent parts of the three kinds mentioned.

A comparison with formula (101.4) shows that the Riemann invariants (104.2) are the quantities which, in simple waves, are constant throughout the region of the flow at all times: J_- is constant in a simple wave propagated to the right, and J_+ in one travelling to the left. Mathematically, this is the fundamental property of simple waves, from which follows, in particular, the property mentioned in §103: one family of characteristics consists of straight lines. For example, let the wave be propagated to the right. Each characteristic C_+ has a constant value of J_+ and, furthermore, a constant value of J_-, which value is the same everywhere. Since both J_+ and J_- are constant, it follows that v and p are constant (and therefore so are all the other quantities), and we obtain the property of the characteristics C_+ deduced in §103, which in turn shows that they are straight lines.

If the flow in two adjoining regions of the *xt*-plane is described by two analytically different solutions of the equations of motion, then the boundary between the regions is a characteristic. For this boundary is a discontinuity in the derivatives of some quantity, i.e., it is a weak discontinuity, and therefore must necessarily coincide with some characteristic.

The following property of simple waves is of great importance in the theory of one-dimensional isentropic flow. The flow in a region adjoining a region of constant flow (in which v = constant, p = constant) must be a simple wave.

This statement is very easily proved. Let the region 1 in the *xt*-plane be bounded on the right by a region (2) of constant flow (Fig. 88). Both invariants J_+ and J_- are evidently constant in the latter region, and both families of characteristics are straight lines. The boundary between the two regions is a characteristic C_+, and the lines C_+ in one region do not enter the other region. The characteristics C_- pass continuously from one region to the other, and carry the constant value of J_- into region 1 from region 2. Thus J_- is constant throughout region 1 also, so that the flow in the latter is a simple wave.

The ability of characteristics to transmit constant values of certain quantities throws some light on the general problem of initial and boundary conditions for the equations of

fluid dynamics. In particular cases of physical interest, there is usually no doubt about the choice of these conditions, which is dictated by physical considerations. In more complex cases, however, mathematical considerations based on the general properties of characteristics may be useful.

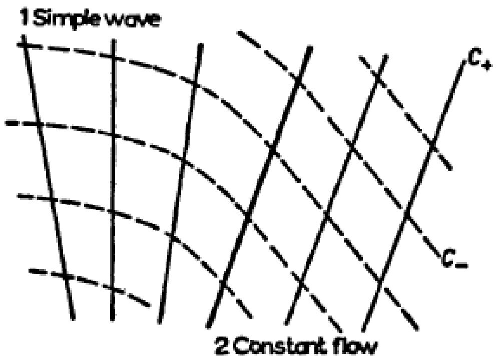

FIG. 88

We shall discuss specifically a one-dimensional isentropic gas flow. Mathematically, a problem of gas dynamics usually amounts to the determination of two unknown functions (for instance, v and p) in a region of the xt-plane lying between two given curves (OA and OB in Fig. 89a), on which the boundary conditions are known. The problem is to find how many quantities can take given values on these curves. In this respect it is very important to know how each curve is situated relative to the directions (shown by arrows in Fig. 89) of the two characteristics C_+ and C_- leaving[12] each point of it. Two cases can occur: either both characteristics lie on the same side of the curve, or they do not. In Fig. 89a, the curve OA belongs to the first case and the curve OB to the second. It is clear that, for a complete determination of the unknown functions in the region AOB, the values of two quantities must be given on the curve OA (e.g., the two invariants J_+ and J_-), and those of only one quantity on OB. For the values of the second quantity are transmitted to the curve OB from the curve OA by the characteristics of the corresponding family, and therefore cannot be given arbitrarily.[13] Similarly, Figs. 89b and c show cases where one and two quantities respectively are given on each bounding curve.

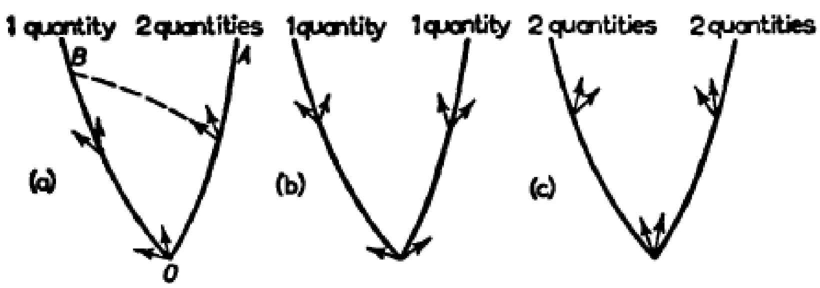

FIG. 89

It should also be mentioned that, if the bounding curve coincides with a characteristic, two independent quantities cannot be specified on it, since their values are related by the

[12] In the xt-plane, the characteristics leaving a given point are those which go in the direction of t increasing.

[13] An example of this case may be given as an illustration: the gas flow when a piston moves into or out of an infinite pipe. Here we are concerned with finding a solution of the equations of gas dynamics in the region of the xt-plane lying between two lines, the positive x-axis and the line $x = X(t)$ which gives the movement of the piston (Figs. 86, 87). On the first line the values of two quantities are given (the initial conditions $v = 0$, $p = p_0$ for $t = 0$), and on the second line those of one quantity ($v = u$, where $u(t)$ is the velocity of the piston).

condition that the corresponding Riemann invariant be constant.

The problem of specifying boundary conditions for the general case of anisentropic flow can be discussed in a similar manner.

We have everywhere above spoken of the characteristics of one-dimensional flow as lines in the *xt*-plane. The characteristics can, however, also be defined in the plane of any two variables describing the flow. For example, we can consider the characteristics in the *vc*-plane. For isentropic flow, the equations of these characteristics are given simply by J_+ = constant, J_- = constant, with various constants on the right; we call these characteristics Γ_+ and Γ_-. For a polytropic gas these are, by (104.3), two families of parallel lines (Fig. 90).

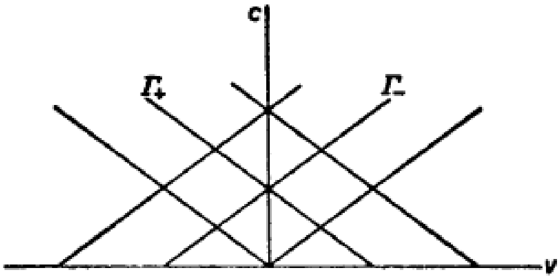

FIG. 90

It should be noted that these characteristics are entirely determined by the properties of the gas, and do not depend on any particular solution of the equations of motion. This is because the equation of isentropic flow in the variables *v*, *c* is (as we shall see in §105) a linear second-order partial differential equation with coefficients which depend only on the independent variables.

The characteristics in the *xt* and *vc* planes are transformations of one another involving the particular solution of the equations of motion. The transformation need not be one-to-one, however. In particular, only one characteristic in the *vc*-plane corresponds to a given simple wave, and all the characteristics in the *xt*-plane are transformed into it. For a wave travelling to the right (e.g.), it is one of the characteristics Γ_-; the characteristics C_- are transformed into the line Γ_-, and the characteristics C_+ into its various points.

§105. Arbitrary one-dimensional gas flow

Let us now consider the general problem of arbitrary one-dimensional isentropic gas flow (without shock waves). We shall first show that this problem can be reduced to the solution of a linear differential equation.

Any one-dimensional flow (i.e., a flow depending on only one spatial coordinate) must be a potential flow, since any function *v*(*x*, *t*) can be written as a derivative: $v(x,t)=\dfrac{\partial\phi(x,t)}{\partial x}$. We can therefore use, as a first integral of Euler's equation, Bernoulli's equation (9.3): $\dfrac{\partial\phi}{\partial t}+\dfrac{1}{2}v^2+w=0$. From this, we find the differential

$$\begin{aligned} d\phi &= \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial t}dt \\ &= vdx-\left(\frac{1}{2}v^2+w)\right)dt \end{aligned}.$$

Here the independent variables are *x* and *t*; we now change to the independent variables *v* and *w*. To do so, we use Legendre's transformation; putting

$$d\phi = d(xv) - xdv - d\left[t\left(w + \frac{1}{2}v^2\right)\right] + td\left(w + \frac{1}{2}v^2\right)$$

and replacing ϕ by a new auxiliary function

$$\chi = \phi - xv + t\left(w + \frac{1}{2}v^2\right),$$

we obtain

$$d\chi = -xdv + td\left(w + \frac{1}{2}v^2\right) = tdw + (vt - x)dv\ ,$$

where χ is regarded as a function of v and w. Comparing this relation with the equation $d\chi = \frac{\partial \chi}{\partial w}dw + \frac{\partial \chi}{\partial v}dv$, we have $t = \frac{\partial \chi}{\partial w}$, $vt - x = \frac{\partial \chi}{\partial v}$, or

$$t = \frac{\partial \chi}{\partial w},\quad x = v\frac{\partial \chi}{\partial w} - \frac{\partial \chi}{\partial v}. \tag{105.1}$$

If the function $\chi(v, w)$ is known, these formulae determine v and w as functions of the coordinate x and the time t.

We now derive an equation for χ. To do so, we start from the equation of continuity, which has not yet been used:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \equiv \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0\,.$$

We transform this equation to one in terms of the variables v, w. Writing the partial derivatives as Jacobians, we have

$$\frac{\partial(\rho, x)}{\partial(t, x)} + v\frac{\partial(t, \rho)}{\partial(t, x)} + \rho\frac{\partial(t, v)}{\partial(t, x)} = 0\ ,$$

or, multiplying by $\frac{\partial(t, x)}{\partial(w, v)}$,

$$\frac{\partial(\rho, x)}{\partial(w, v)} + v\frac{\partial(t, \rho)}{\partial(w, v)} + \rho\frac{\partial(t, v)}{\partial(w, v)} = 0\ .$$

To expand these Jacobians we must use the following result. According to the equation of state of the gas, ρ is a function of any two other independent thermodynamic quantities; for example, we may regard ρ as a function of w and s. If s = constant, we have simply $\rho = \rho(w)$, and the density is independent of v. Expanding the Jacobians, we therefore have

$$\frac{d\rho}{dw}\frac{\partial x}{\partial v} - v\frac{d\rho}{dw}\frac{\partial t}{\partial v} + \rho\frac{\partial t}{\partial w} = 0\ .$$

Substituting here the expressions (105.1) for t and x, we obtain

$$\frac{1}{\rho}\frac{d\rho}{dw}\left(\frac{\partial \chi}{dw} - \frac{\partial^2 \chi}{\partial v^2}\right) + \frac{\partial^2 \chi}{\partial w^2} = 0\,.$$

If s = constant, we have $dw = dp/\rho$, whence $\frac{d\rho}{dw} = \frac{d\rho}{dp}\frac{dp}{dw} = \frac{\rho}{c^2}$. We finally have for χ the equation

$$c^2\frac{\partial^2 \chi}{\partial w^2} - \frac{\partial^2 \chi}{\partial v^2} + \frac{\partial \chi}{\partial w} = 0\,;$$

here the velocity of sound c is to be regarded as a function of w. The problem of integrating the non-linear equations of motion has thus been reduced to that of solving a linear equation.

Let us apply this result to the case of a polytropic gas. We have $c^2 = (\gamma - 1)w$, and the fundamental equation (105.2) becomes

$$(\gamma-1)w\frac{\partial^2\chi}{\partial w^2}-\frac{\partial^2\chi}{\partial v^2}+\frac{\partial\chi}{\partial w}=0\,. \tag{105.3}$$

This equation has an elementary general integral if $\dfrac{3-\gamma}{\gamma-1}$ is an even integer:

$$\frac{3-\gamma}{\gamma-1}=2n\,,\ \text{or}\ \ \gamma=\frac{3+2n}{2n+1}\,,\ n=0,\,1,\,2,\,\ldots \tag{105.4}$$

This condition is satisfied by monatomic ($\gamma=5/3$, n = 1) and diatomic ($\gamma=7/5$, n = 2) gases. Expressing γ in terms of n, we can rewrite (105.3) as

$$\frac{2}{2n+1}w\frac{\partial^2\chi}{\partial w^2}-\frac{\partial^2\chi}{\partial v^2}+\frac{\partial\chi}{\partial w}=0\,. \tag{105.5}$$

We denote by χ_n a function which satisfies this equation for a given n. For the function χ_0 we have

$$2w\frac{\partial^2\chi_0}{\partial w^2}-\frac{\partial^2\chi_0}{\partial v^2}+\frac{\partial\chi_0}{\partial w}=0\,.$$

Introducing in place of w the variable $u=\sqrt{2w}$, we obtain

$$\frac{\partial^2\chi_0}{\partial u^2}-\frac{\partial^2\chi_0}{\partial v^2}=0\,.$$

This is just the ordinary wave equation, whose general solution is

$$\chi_0=f_1(u+v)+f_2(u+v)\,,$$

f_1 and f_2 being arbitrary functions. Thus

$$\chi_0=f_1[\sqrt{2w}+v]+f_2[\sqrt{2w}-v]\,. \tag{105.6}$$

We shall now show that, if the function χ_n is known, the function χ_{n+1} can be obtained by differentiation. For, differentiating equation (105.5) with respect to w, we easily find on rearrangement

$$\frac{2}{2n+1}w\frac{\partial^2}{\partial w^2}\left(\frac{\partial\chi_n}{\partial w}\right)+\frac{2n+3}{2n+1}\frac{\partial}{\partial w}\left(\frac{\partial\chi_n}{\partial w}\right)-\frac{\partial^2}{\partial v^2}\left(\frac{\partial\chi_n}{\partial w}\right)=0\,.$$

Putting $v=v'\sqrt{(2n+1)/(2n+3)}$, we have for $\dfrac{\partial\chi_n}{\partial w}$ the equation

$$\frac{2}{2n+1}w\frac{\partial^2}{\partial w^2}\left(\frac{\partial\chi_n}{\partial w}\right)+\frac{\partial}{\partial w}\left(\frac{\partial\chi_n}{\partial w}\right)-\frac{\partial^2}{\partial v'^2}\left(\frac{\partial\chi_n}{\partial w}\right)=0\,,$$

which is equation (105.5) for the function $\chi_{n+1}(w,v')$. Thus we conclude that

$$\chi_{n+1}(w,v')=\frac{\partial}{\partial w}\chi(w,v)=\frac{\partial}{\partial w}\chi_n\left(w,v'\sqrt{\frac{2n+1}{2n+3}}\right). \tag{105.7}$$

Using this formula n times and taking χ_0 from (105.6), we find that the general solution of equation (105.5) is

$$\chi=\frac{\partial^n}{\partial w^n}\left\{f_1\left[\sqrt{2(2n+1)w}+v\right]+f_2\left[\sqrt{2(2n+1)w}-v\right]\right\},$$

or

$$\chi=\frac{\partial^{n-1}}{\partial w^{n-1}}\left\{\frac{F_1\left[\sqrt{2(2n+1)w}+v\right]+F_2\left[\sqrt{2(2n+1)w}-v\right]}{\sqrt{w}}\right\}, \tag{105.8}$$

where F_1 and F_2 are again two arbitrary functions.

If we express w in terms of the velocity of sound by $w = \frac{c^2}{\gamma - 1} = \frac{1}{2}(2n+1)c^2$, the solution (105.8) becomes

$$\chi = \left(\frac{\partial}{c\partial c}\right)^{n-1} \left\{ \frac{1}{c} F_1\left(c + \frac{v}{2n+1}\right) + \frac{1}{c} F_1\left(c - \frac{v}{2n+1}\right) \right\}. \qquad (105.9)$$

The expressions $c \pm \frac{v}{2n+1} = c \pm \frac{\gamma - 1}{2} v$ which are the arguments of the arbitrary functions are just the **Riemann invariants** (104.3), which are constant along the characteristics.

In applications it is often necessary to calculate the values of the function $\chi(v, c)$ on a characteristic. The following formula[14] is useful for this purpose:

$$\left(\frac{\partial}{c\partial c}\right)^{n-1} \left\{ \frac{1}{c} F\left(c \pm \frac{v}{2n+1}\right) \right\} = \frac{1}{2^{n-1}} \left(\frac{\partial}{\partial c}\right)^{n-1} \frac{F(2c+a)}{c^n}, \qquad (105.10)$$

with $\pm \frac{v}{2n+1} = c + a$ (a being an arbitrary constant).

Let us now ascertain the relation between the general solution just found and the solution of the equations of gas dynamics which describes a simple wave. The latter is distinguished by the property that in it v is a definite function of w: $v = v(w)$, and therefore the Jacobian $\Delta = \frac{\partial(v, w)}{\partial(x, t)}$ vanishes identically. In transforming to the variables v and w, however, we divided the equation of motion by this Jacobian, and the solution for which $\Delta \equiv 0$ is therefore "lost". Thus a simple wave cannot be directly obtained from the general integral of the equations of motion, but is a special integral of these equations.

To understand the nature of this special integral, we must observe that it can be obtained from the general integral by a certain passage to a limit, which is closely related to the physical significance of the characteristics as the paths of propagation of small disturbances. Let us suppose that the region of the vw-plane in which the function $\chi(v, w)$ is not zero becomes a very narrow strip along a characteristic. The derivatives of χ in the direction transverse to the characteristic then take a very wide range of values, since χ diminishes very rapidly in that direction. Such solutions $\chi(v, w)$ of the equations of motion must exist. For, regarded as a perturbation in the vw-plane, they satisfy the conditions of geometrical acoustics, and are therefore non-zero along characteristics, as such perturbations must be.

It is clear from the foregoing that, for such a function χ, the time $t = \frac{\partial \chi}{\partial w}$ will take an arbitrarily large range of values. The derivative of χ along the characteristic, however, is finite. Along a characteristic (for instance, a characteristic Γ_-) we have

$$\frac{dJ_-}{dv} = 1 - \frac{1}{\rho c}\frac{dp}{dw}\frac{dw}{dv} = 1 - \frac{1}{c}\frac{dw}{dv} = 0\ .$$

The derivative of χ with respect to v along a characteristic, which we denote by $-f(v)$, is

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[14] It is most simply derived by using Cauchy's theorem in the theory of functions of a complex variable. For an arbitrary function $F(c+u)$ we have

$$\left(\frac{\partial}{c\partial c}\right)^{n-1} \frac{F(c+u)}{c} = 2^{n-1}\left(\frac{\partial}{\partial c^2}\right)^{n-1} \frac{F(c+u)}{c} = 2^{n-1}\frac{(n-1)!}{2\pi i}\oint \frac{F(\sqrt{z}+u)}{\sqrt{z}(z-c^2)^n}dz,$$

where the integral is taken along a contour in the complex z-plane which encloses the point $z = c^2$. Putting now $u = c + a$ and substituting in the integral $\sqrt{z} = 2\zeta - c$, we obtain

$$\frac{1}{2^{n-1}}\frac{(n-1)!}{2\pi i}\oint \frac{F(2\zeta + a)}{\zeta^n(\zeta - c)^n}d\zeta,$$

where the contour of integration encloses the point $\zeta = c$; again applying Cauchy's theorem, we have the result (105.10).

therefore

$$\frac{d\chi}{dv} = \frac{\partial\chi}{\partial v} + \frac{\partial\chi}{\partial w}\frac{\partial w}{\partial v} = \frac{\partial\chi}{\partial v} + c\frac{\partial\chi}{\partial w} = -f(v)\,.$$

Expressing the partial derivatives of χ in terms of x and t by (105.1), we obtain the relation $x = (v+c)t + f(v)$, i.e., the equation (101.5) for a simple wave. The relation (101.4), which gives the relation between v and c in a simple wave, is necessarily satisfied, since J_- is constant along a characteristic Γ_-.

We have shown in §104 that, if the solution of the equations of motion reduces to constant flow in some part of the *xt*-plane, then there must be a simple wave in the adjoining regions. The motion described by the general solution (105.8) must therefore be separated from a region of constant flow (in particular, a region of gas at rest) by a simple wave. The boundary between the simple wave and the general solution, like any boundary between two analytically different solutions, is a characteristic. In solving particular problems, the value of the function $\chi(w,v)$ on this boundary characteristic must be determined.

The joining condition at the boundary between the simple wave and the general solution is obtained by substituting the expressions (105.1) for x and t in the equation of the simple wave $x = (v \pm c)t + f(v)$; this gives

$$\frac{\partial\chi}{\partial v} \pm c\frac{\partial\chi}{\partial w} + f(v) = 0\,.$$

Moreover, in a simple wave (and therefore on the boundary characteristic), we have $dv = \pm\frac{dp}{\rho c} = \pm\frac{dw}{c}$, or $\pm c = \frac{dw}{dv}$. Substituting this in the above condition, we obtain

$$\frac{\partial\chi}{\partial v} + \frac{\partial\chi}{\partial w}\frac{dw}{dv} + f(v) = \frac{d\chi}{dv} + f(v) = 0\,,$$

or, finally,

$$\chi = -\int f(v)dv\,, \tag{105.11}$$

which determines the required boundary value of χ. In particular, if the simple wave has a centre at the origin, i.e., if $f(v) \equiv 0$, then χ = constant; since the function χ is defined only to within an additive constant, we can without loss of generality take χ = 0 on the boundary characteristic.

PROBLEMS

Problem 1. Determine the resulting flow when a centred rarefaction wave is reflected from a solid wall.

Solution. Let the rarefaction wave be formed at the point x = 0 at time t = 0, and propagated in the positive x-direction; it reaches the wall after a time $t = l/c_0$, where l is the distance to the wall. Figure 91 shows the characteristics for the reflection of the wave. In regions 1 and 1' the gas is at rest; in region 3 it moves with a constant velocity $v = -U$.[15] Region 2 is the incident rarefaction wave (with rectilinear characteristics C_+), and region 5 is the reflected wave (with rectilinear characteristics C_-). Region 4 is the "region of interaction", in which the solution is required; the linear characteristics become curved on entering this region. The solution is entirely determined by the boundary conditions on the segments *ab* and *ac*. On *ab* (i.e., on the wall) we must have v = 0 for x = l; by (105.1), we hence obtain the condition $\frac{\partial\chi}{\partial v} = -l$ for v = 0. The boundary *ac* with the rarefaction wave is part of a

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[15] If the rarefaction wave is due to a piston which begins to move out of a pipe at a constant velocity, then U is the velocity of the piston.

characteristic C_-, and we therefore have $c-\frac{\gamma-1}{2}v=c-\frac{v}{2n+1}=\text{constant}$; since, at the point *a*, v = 0 and $c=c_0$, the constant is c_0. On this boundary χ must be zero, so that

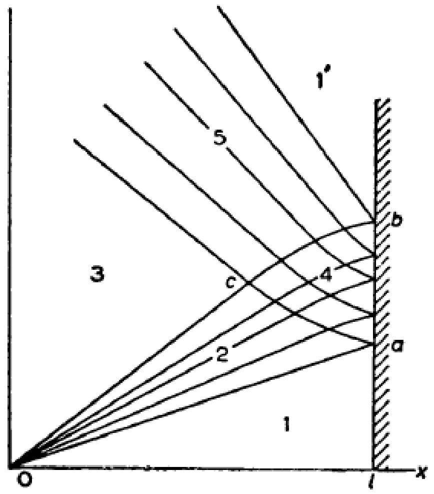

FIG. 91

we have the condition $\chi=0$ for $c-\frac{v}{2n+1}=c_0$. It is easily seen that a function of the form (105.9) which satisfies these conditions is

$$\chi=\frac{l(2n+1)}{2^n n!}\left(\frac{\partial}{c\partial c}\right)^{n-1}\left\{\frac{1}{c}\left[\left(c-\frac{v}{2n+1}\right)^2-c_0{}^2\right]^n\right\}, \qquad (1)$$

and this gives the required solution.

The equation of the characteristic *ac* is (see §103, Problem)

$$x=-(2n+1)c_0t+2(n+1)l\left(\frac{tc_0}{l}\right)^{(2n+1)/2((n+1)}$$

Its intersection with the characteristic *Oc*

$$\frac{x}{t}=c_0-\frac{\gamma+1}{2}U=c_0-\frac{2(n+1)}{2n+1}U$$

determines the time at which the incident wave disappears:

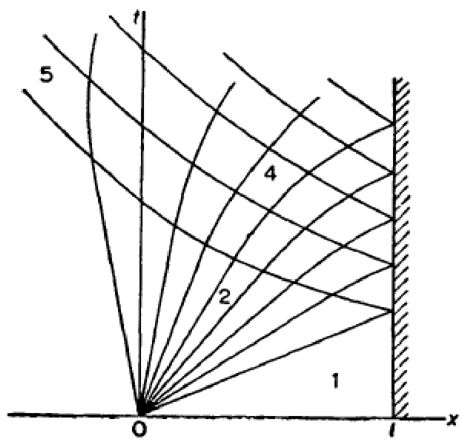

FIG. 92

$$t_c = \frac{l(2n+1)^{n+1} c_0{}^n}{[(2n+1)c_0 - U]^{n+1}}.$$

In Fig. 91 it is assumed that $U < \frac{2c_0}{\gamma + 1}$; in the opposite case, the characteristic Oc is in the negative x-direction (Fig. 92). The interaction of the incident and reflected waves then lasts for an infinite time (not, as in Fig. 91, for a finite time).

The function (1) also describes the interaction between two equal centred rarefaction waves which leave the points $x = 0$ and $x = 2l$ at time $t = 0$ and are propagated towards each other; this is evident from symmetry (Fig. 93).

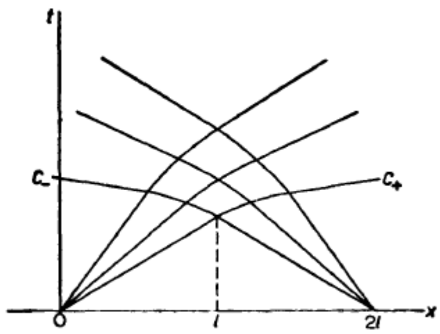

FIG. 93

Problem 2. Derive the equation analogous to (105.3) for one-dimensional isothermal flow of a perfect gas.

Solution. For isothermal flow, the heat function w in Bernoulli's equation is replaced by

$$\mu = \int dp/\rho = c_T{}^2 \int d\rho/\rho = c_T{}^2 \log \rho,$$

where $c_T{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_T$ is the square of the isothermal velocity of sound. For a perfect gas $c_T = \text{constant}$. Taking the quantity μ (instead of w) as an independent variable, we obtain, by the same method as in the text, the following linear equation with constant coefficients:

$$c_T{}^2 \frac{\partial^2 \chi}{\partial \mu^2} + \frac{\partial \chi}{\partial \mu} - \frac{\partial^2 \chi}{\partial v^2} = 0.$$