§106. A strong explosion

Let us consider the propagation of a strong spherical shock wave resulting from a strong explosion, that is, the instantaneous release of a large amount of energy *E* in a small volume. The gas in which the wave is propagated will be assumed to be polytropic.[1]

We shall consider the wave at not too great distances from the source, where it is still strong. These distances are nevertheless large compared with the dimensions of the source: this enables us to assume that the energy *E* is released at one point, the origin.

The shock wave is strong, and the pressure discontinuity in it is therefore very large. We shall suppose that the pressure p_2 behind the shock is so much larger than the pressure p_1 of the undisturbed gas in front of it that

$$\frac{p_2}{p_1} >> \frac{\gamma+1}{\gamma-1}.$$

This enables us to neglect p_1 everywhere in comparison with p_2; the density ratio ρ_2/ρ_1 has its limiting value $\frac{\gamma+1}{\gamma-1}$ (see §89).

Thus the gas flow pattern is determined by only two parameters: the initial gas density ρ_1, and the amount of energy *E* released in the explosion. From these parameters and the two independent variables (the time *t* and the radial coordinate *r*), we can form only one dimensionless combination, which we write as $r\left(\frac{\rho_1}{Et^2}\right)^{1/5}$. Consequently, we have a certain type of similarity flow.

We can say, first of all, that the position of the shock wave itself at every instant must correspond to a certain constant value of this dimensionless combination. This gives at once the manner in which the shock wave moves with time; denoting by *R* the distance of the shock from the origin, we have

$$R = \beta\left(\frac{Et^2}{\rho_1}\right)^{1/5}, \tag{106.1}$$

where β is a numerical constant (depending on γ), which is itself determined by solving the equations of motion. The velocity of the shock wave relative to the undisturbed gas, i.e., relative to a fixed coordinate system, is

$$u_1 = \frac{dR}{dt} = \frac{2R}{5t} = \frac{2\beta E^{1/5}}{5\rho_1^{1/5}t^{3/5}}. \tag{106.2}$$

Thus in this problem the movement of the shock wave can be determined (to within a constant factor) by simple dimensional arguments.

The gas pressure p_2, the density ρ_2 and the velocity $v_2 = u_2 - u_1$ (relative to a fixed coordinate system) at the back of the shock can be expressed in terms of u_1 by means of the formulae derived in §89. According to (89.10) and (89.11),[2]

$$v_2 = \frac{2u_1}{\gamma+1},\quad \rho_2 = \rho_1\frac{\gamma+1}{\gamma-1},\quad p_2 = \frac{2\rho_1 u_1^2}{\gamma+1}. \tag{106.3}$$

The density is constant in time, while v_2 and p_2 decrease as $t^{-3/5}$ and $t^{-6/5}$, respectively. We may also note that the pressure p_2 due to the shock increases with the total energy of the explosion as $E^{2/5}$.

Let us next determine the gas flow throughout the region behind the shock. Instead of the gas velocity *v*, the density ρ and the squared velocity of sound $c^2 = \gamma p/\rho$ (which

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[1] The solution given below was found independently by L. I. Sedov (1946) and J. von Neumann (1947). The problem was treated less fully by G. I. Taylor (1941, published 1950), who did not derive an analytical solution.

[2] We here denote by u_1 and u_2 the velocities of the shock wave, relative to the gas, given by formulae (89.11).

replaces the pressure p as a variable), we introduce dimensionless variables V, G, Z defined by[3]

$$v = \frac{2rV}{5t}, \quad \rho = \rho_1 G, \quad c^2 = \frac{4r^2 Z}{25t^2}. \tag{106.4}$$

They can be functions only of a single independent dimensionless "**similarity**" variable, which we define as

$$\xi = \frac{r}{Rt} = \frac{r}{\beta}\left(\frac{\rho_1}{Et^2}\right)^{1/5} \tag{106.5}$$

In accordance with (106.3), their values at the discontinuity surface ($\xi = 1$) must be

$$V(1) = \frac{2}{\gamma+1}, \quad G(1) = \frac{\gamma+1}{\gamma-1}, \quad Z(1) = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \tag{106.6}$$

The equations of centrally-symmetrical adiabatic gas flow are

$$\left.\begin{aligned} &\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \\ &\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial r} + \frac{2\rho v}{r} = 0 \\ &\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\log\frac{p}{\rho^\gamma} = 0 \end{aligned}\right\} \tag{106.7}$$

The last equation is the equation of conservation of entropy, with the expression (83.12) for the entropy of a polytropic gas substituted. After substituting (106.4), we obtain a set of ordinary differential equations for the functions V, G and Z. The integration of these equations is facilitated by the fact that one integral can be obtained immediately, using the following arguments.

The fact that we have neglected the pressure p_1 of the undisturbed gas means that we neglect the original energy of the gas in comparison with the energy E which it acquires as a result of the explosion. It is therefore clear that the total energy of the gas within the sphere bounded by the shock wave is constant and equal to E. Furthermore, since we have a similarity flow, it is evident that the energy of the gas inside any sphere of a smaller radius, which increases with time in such a way that $\xi =$ any constant (not only 1), must remain constant; the radial velocity of points on this sphere is $v_n = \frac{2r}{5t}$ (cf. (106.2)).

It is easy to write down the equation which expresses the constancy of this energy. On the one hand, an amount of energy $dt \cdot 4\pi r^2 \rho v(w + \frac{1}{2}v^2)$ leaves the sphere (whose area is $4\pi r^2$) in time dt. On the other hand, the volume of the sphere is increased in that time by $dt \cdot v_n \cdot 4\pi r^2$, and the energy of the gas in this extra volume is $dt \cdot 4\pi r^2 \rho v_n(\varepsilon + \frac{1}{2}v^2)$. Equating the two expressions, substituting ε and w from (83.10) and (83.11), and introducing the dimensionless functions by (106.4), we obtain

$$Z = \frac{\gamma(\gamma-1)(1-V)V^2}{2(\gamma V - 1)}, \tag{106.8}$$

which is the required integral. It automatically satisfies the boundary conditions (106.6).

When the integral (106.8) is known, the integration of the equations is elementary though laborious. The second and third equations (106.7) give

[3] The symbol V in §§106 and 107 should not be confused with the specific volume used elsewhere.

$$\left.\begin{aligned} \frac{dV}{d\log\xi} - (1-V)\frac{d\log G}{d\log\xi} &= -3V \\ \frac{d\log Z}{d\log\xi} - (\gamma-1)\frac{d\log G}{d\log\xi} &= -\frac{5-2V}{1-V} \end{aligned}\right\}. \tag{106.9}$$

From these two equations we can express the derivatives $\frac{dV}{d\log\xi}$ and $\frac{d\log G}{dV}$, by means of (106.8), as functions of V only, and then an integration with the boundary conditions (106.6) gives

$$\xi^5 = \left[\frac{\gamma+1}{2}V\right]^{-2}\left\{\frac{\gamma+1}{7-\gamma}[5-(3\gamma-1)V]\right\}^{\nu_1}\left[\frac{\gamma+1}{\gamma-1}(\gamma V-1)\right]^{\nu_2},$$

$$G = \frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+}{\gamma-1}(\gamma V-1)\right]^{\nu_3}\left\{\frac{\gamma+1}{7-\gamma}[5-(3\gamma-1)V\right\}^{\nu_4}\left[\frac{\gamma+1}{\gamma-1}(1-V\right]^{\nu_5},$$

$$\nu_1 = -\frac{13\gamma^2-7\gamma+12}{(3\gamma-1)(2\gamma+1)}, \qquad \nu_2 = \frac{5(\gamma-1)}{2\gamma+1},$$

$$\nu_3 = \frac{3}{2\gamma+1}, \qquad \nu_4 = -\frac{\nu_1}{2-\gamma},$$

$$\nu_5 = -\frac{2}{2-\gamma}. \tag{106.10}$$

Formulae (106.8) and (106.10) give the complete solution of the problem. The constant β in the definition of the independent variable ξ is determined by the condition

$$E = \int_0^R \rho\left[\frac{1}{2}v^2 + \frac{c^2}{\gamma(\gamma-1)}\right]4\pi r^2 dr,$$

which states that the total energy of the gas is equal to the energy E of the explosion. In terms of the dimensionless quantities, this condition becomes

$$\beta^5\frac{16\pi}{25}\int_0^1 G\left[\frac{1}{2}V^2 + \frac{Z}{\gamma(\gamma-1)}\right]\xi^4 d\xi = 1. \tag{106.11}$$

For air ($\gamma = 7/5$), $\beta = 1.033$.

It is easily seen from (106.10) that, as $\xi \to \infty$, V tends to a constant limit and G to zero:

$$V - \frac{1}{\gamma} \propto \xi^{5/\nu_2}, \quad G \propto \xi^{5\nu_3/\nu_2}.$$

Hence it follows that v/v_2 and ρ/ρ_2 as functions of $r/R = \xi$ tend to zero as $\xi \to 0$:

$$\frac{v}{v_2} \propto \frac{r}{R}, \quad \frac{\rho}{\rho_2} \propto \left(\frac{r}{R}\right)^{3/(\gamma-1)}; \tag{106.12}$$

the pressure ratio p/p_2 tends to a constant limit, and the temperature ratio accordingly becomes infinite.[4]

Figure 94 shows the quantities v/v_2, p/p_2 and ρ/ρ_2 as functions of r/R for air

[4] These statements relate to values $\gamma < 7$, the function $V(\xi)$ varying from $V(1) = 2/(\gamma+1)$ to $V(0) = 1/\gamma$. For actual gases whose thermodynamic functions could be approximated by the expressions for a polytropic gas, the inequality is certainly satisfied; the upper limit of γ in practice is 5/3 for a monatomic gas. For formal completeness, however, it may be noted that when $\gamma > 7$ the function $V(\xi)$ varies from $2/(\gamma+1)$ for $\xi = 1$ to the limit of unity reached at a value $\xi_0 < 1$ which depends on γ; at this point, G is zero, and an expanding spherical vacuum is formed.

($\gamma = 1.4$). The very rapid decrease of the density into the sphere is noticeable: almost all the gas is in a relatively thin layer behind the shock wave. This is, of course, due to the fact that the gas on the surface of greatest radius (R) has a density six times the normal density.[5]

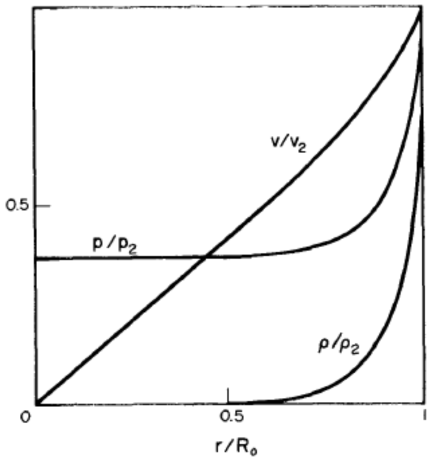

FIG. 94

§107. An imploding spherical shock wave

There are a number of instructive features in the problem of a strong shock wave converging to a centre.[6] We shall not be concerned with the specific mechanism whereby such a shock wave is formed; it is sufficient to suppose it produced by some kind of "**spherical piston**" which gives the gas an initial impetus, and the shock becomes stronger as the centre is approached.

We will consider the gas flow at the stage when the radius R of the spherical surface of discontinuity is already much less than its initial value, the radius R_0 of the "piston". At this stage the flow is largely (to an extent described below) independent of the specific initial conditions. The shock wave will be assumed so strong that the pressure of the gas in front of it is (as in §106) negligible in comparison with the pressure p_2 behind it. The total energy of the gas in the (variable) region $r \sim R \ll R_0$ considered is by no means constant; it will be shown later to decrease in the course of time.

The spatial scale of the flow in question can only be determined by the time-dependent radius $R(t)$ of the shock wave, and the scale of the velocity by the derivative dR/dt. Under these conditions, it is reasonable to suppose that there is similarity flow with the independent variable $\xi = r / R(t)$. The function $R(t)$, however, cannot be determined by dimensional arguments alone.

Let the time when the shock wave is focused (i.e., when $R = 0$) be taken as $t = 0$. Then the times before focusing correspond to $t < 0$. We shall seek the function $R(t)$ in the form

$$R(t) = A(-t)^{\alpha}, \tag{107.1}$$

with an initially unknown ***similarity index*** α. It is found that this index is determined by the condition for a solution of the equations of motion to exist (in the region $r \ll R_0$) with the necessary boundary conditions. This also determines the dimensions of the constant parameter A, whose value, however, remains indeterminate and can in principle be found only by solving the gas flow problem as a whole, that is, by joining the similarity solution to the solution at distances $r \sim R_0$, which depends on the specific initial conditions. This

[5] The results of calculations for other values of γ are given by L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, Chapter IV, §11, London, 1959. The corresponding problem with cylindrical symmetry is also discussed.

[6] This was discussed independently by G. Guderley (1942) and by L. D. Landau and K. P. Stanyukovich (1944, published 1955).

parameter alone governs the flow for $R << R_0$ in relation to the way in which the shock wave is initially formed.

We shall now show how the problem thus formulated is solved. As in §106, we use dimensionless unknown functions defined by

$$v = \frac{\alpha r}{t} V(\xi), \quad \rho = \rho_1 G(\xi), \quad c^2 = \frac{\alpha^2 r^2}{t^2} Z(\xi), \tag{107.2}$$

where

$$\xi = \frac{r}{R(t)} = \frac{r}{A(-t)^\alpha}; \tag{107.3}$$

when $\alpha = 2/5$, these definitions are the same as (106.4). Here, v is the radial velocity of the gas, relative to fixed coordinates in which the gas is at rest in a sphere with radius $r = R_0$; the gas moves with the shock wave towards the centre, corresponding to $v < 0$, so that $V(\xi) > 0$.

In fact, the desired solution of the equations of motion relates only to the region $r \sim R$ behind the shock wave and to sufficiently short times t, for which $R << R_0$. But formally the solution obtained covers all space $r >> R$, from the surface of discontinuity to infinity, and all times $t \le 0$; the variable ξ, then takes all values from 1 to ∞. Accordingly, the boundary conditions for the functions G, V and Z must be specified for $\xi = 1$ and $\xi = \infty$.

The value $\xi = 1$ corresponds to the surface of the shock wave; the boundary conditions there are the same as (106.6).

To establish the conditions at infinity (for ξ), we note that when $t = 0$ (the shock is focused) all the quantities v, ρ and c^2 must remain finite at any finite distance from the centre. When $t = 0$ and $r \ne 0$, we have $\xi = \infty$. If the functions $v(r, t)$ and $c^2(r,t)$ then remain finite, it follows that $V(\xi)$ and $Z(\xi)$ must tend to zero:

$$V(\infty) = 0, \quad Z(\infty) = 0. \tag{107.4}$$

Substitution of (107.2) and (107.3) brings the equations (106.7) to the form

$$\left.\begin{aligned}
(1-V)\frac{dV}{d\log\xi} - \frac{Z}{\gamma}\frac{d\log G}{d\log\xi} - \frac{1}{\gamma}\frac{dZ}{d\log\xi} &= \frac{2}{\gamma}Z - V\left(\frac{1}{\alpha} - V\right) \\
\frac{dV}{d\log\xi} - (1-V)\frac{d\log G}{d\log\xi} &= -3V \\
(\gamma-1)Z\frac{d\log G}{d\log\xi} - \frac{dZ}{d\log\xi} &= \frac{2Z(1/\alpha - V)}{1-V}
\end{aligned}\right\} \tag{107.5}$$

compare (106.9) for the last two equations. The independent variable ξ appears here only as the differential $d\log\xi$; the constant $\log A$ disappears from the equations entirely, as stated above.

The coefficients of the derivatives in equations (107.5) and the right-hand sides involve only V and Z, not G.[7] By solving these equations for the derivatives, we can express the latter in terms of the two functions V and Z. This gives

$$\frac{d\log\xi}{dV} = -\frac{Z - (1-V)^2}{(3V-\kappa)Z - V(1-V)(1/\alpha - V)}, \tag{107.6}$$

$$(1-V)\frac{d\log G}{d\xi} = 3V - \frac{(3V-\kappa)Z - V(1-V)(1/\alpha - V)}{Z - (1-V)^2}, \tag{107.7}$$

where $\kappa = \dfrac{2(1-\alpha)}{\alpha\gamma}$. As a third equation, we write the result of dividing $\dfrac{dZ}{d\log\xi}$ by

[7] This is the advantage of using v, ρ and c^2 as the fundamental variables instead of v, ρ and p.

$\frac{dV}{d\log\xi}$:

$$\frac{dZ}{dV}=\frac{Z}{1-V}\left\{\frac{[Z-(1-V)^2][2/\alpha-(3\gamma-1)V]}{(3V-\kappa)Z-V(1-V)(1/\alpha-V)}+\gamma-1\right\}. \tag{107.8}$$

If the required solution of (107.8) has been found, i.e., the functional $Z(V)$, the solution of equations (107.6) and (107.7) to find $\xi(V)$ and then $G(\xi)$ is reduced to quadratures.

The whole problem thus reduces to first solving equation (107.8). The integral curve in the VZ-plane must start from a point Y with coordinates $V(1)$, $Z(1)$, the image of the shock wave in the VZ-plane. This point specifies the solution of (107.8) for a given a: the integral curve of a first-order equation is uniquely defined by one (non-singular) point on it. Let us next ascertain the condition which establishes the value of a giving the "correct" integral curve.

This condition follows from an obvious physical requirement, that the dependence of all quantities on ξ must be single-valued: to each value of ξ there must correspond unique values of V, G and Z. This means that throughout the range of variation of ξ ($1\le\xi\le\infty$, i.e., $0\le\log\xi\le\infty$) the functions $\xi(V)$, $\xi(G)$ and $\xi(Z)$ cannot have extrema. Thus the derivatives $\frac{d\log\xi}{dV}$ etc. can nowhere be zero. In Fig. 95, curve 1 is the parabola

$$Z=(1-V)^2. \tag{107.9}$$

It is easily seen that Y lies above this parabola.[8] The integral curve corresponding to the solution of the problem stated must reach the origin in accordance with the limiting condition (107.4), and must therefore cross the parabola (107.9). According to (107.6)—(107.8), all the derivatives mentioned are expressed by fractions with $Z-(1-V)^2$ in the numerator. If these do not become zero, then at the point where the integral curve intersects the parabola (107.9) we must also have

$$(3V-\kappa)Z=V(1-V)(1/\alpha-V). \tag{107.10}$$

That is, the integral curve must pass through the point where the parabola (107.9) meets the curve (107.10) (curve 2 in Fig. 95); this point is a singularity of (107.8), since $\frac{dZ}{dV}=\frac{0}{0}$. The same condition determines the value of the similarity index α; two values given by numerical calculation are

$$\alpha=0.6884 \quad \text{for} \quad \gamma=\frac{5}{3},$$

$$\alpha=0.7172 \quad \text{for} \quad \gamma=\frac{7}{5}. \tag{107.11}$$

After passing through the singular point, the integral curve goes to the origin O corresponding to the limiting values (107.4). To elucidate the mathematical situation, we will briefly describe the distribution of the integral curves of equation (107.8) in the

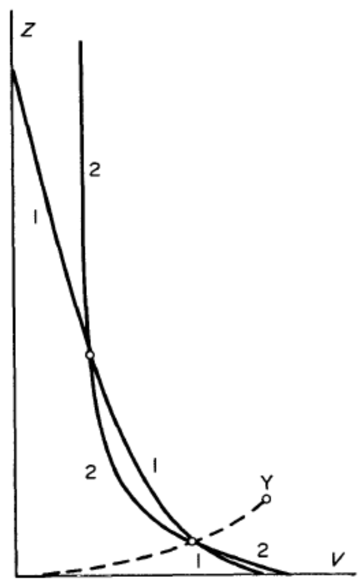

FIG. 95

[8] A result which simply expresses the fact that the gas velocity at the back of the discontinuity surface is less than the velocity of sound in the gas.

VZ-plane (for the "correct" value of α), without going through the calculations.[9]

The curves (107.9) and (107.10) intersect, in general, at two points as shown by the circles in Fig. 95 (in addition to the unimportant point $V = 1$, $Z = 0$ on the abscissa axis). The equation also has a singular point c where the curve (107.10) intersects the straight line $(3\gamma - 1)V = 2/\alpha$ (on which the second factor in the numerator of (107.8) is zero). The point a through which the "correct" integral curve passes is a saddle point; b and c are nodes. The origin is also a node singularity. Near it, equation (107.8) becomes

$$\frac{dZ}{dV} = \frac{2Z}{V + \kappa Z}.$$

An elementary integration of this homogeneous equation shows that, as $V \to \infty$, $Z(V)$ tends to zero faster than V:

$$Z \cong \text{constant} \times V^2. \qquad (107.12)$$

There is thus an infinity of integral curves leaving the origin, with different values of the constant in (107.12). All of them go to either b or c, except one which goes to the saddle point a (one of the two separatrices, the only integral curves through the saddle point).[10]

The origin corresponds to $\xi = \infty$, i.e. the time when the shock wave is focused at the centre. Let us determine the limiting distributions of all quantities with respect to radial distances at this time. With (107.12), we find from (107.6) - (107.7) that

$$V = \text{constant} \times \xi^{-1/\alpha}, \quad Z = cons\tan t \times \xi^{-2/\alpha}, \quad G = \text{constant} \quad \text{as} \quad \xi \to \infty; \qquad (107.13)$$

the values of the constant coefficients can be found only by a numerical determination of the integral curve over its whole extent. Substituting these expressions in the definitions (107.2), we find[11]

$$|v| \propto c \propto r^{-(1/\alpha - 1)}, \quad \rho = \text{constant}, \quad p \propto r^{-2(1/\alpha - 1)}. \qquad (107.14)$$

These relations could also be derived directly from dimensional arguments (when the dimensions of A are known). We have two available parameters, ρ_1 and A, and one variable, r; from these, only one combination with the dimensions of velocity can be formed, $A^{1/\alpha} r^{1-1/\alpha}$. The quantity with the dimensions of density must be ρ_1 itself.

Let us also find the time dependence of the total energy of the gas in the region of similarity flow. The radial size of this region is of the order of the radius R of the shock wave, and decreases with R. Let us arbitrarily take as the boundary of the similarity region a particular value $r/R = \xi_1$. The total energy of the gas in a spherical shell between radii R and $\xi_1 R$ is given, in dimensionless variables, by

$$E_{sim} = \frac{\alpha^2 \rho_1 R^5}{t^2} \int_1^{\xi_1} G\left[\frac{1}{2}V^2 + \frac{Z}{\gamma(\gamma - 1)}\right] \cdot 4\pi\xi^2 d\xi;$$

cf. (106.11). The integral is a constant.[12] Hence we find

$$E_{sim} \propto R^{5-2/\alpha} \propto (-t)^{5\alpha - 2}. \qquad (107.15)$$

For any actual value of γ, the exponent is positive. Although the shock wave itself becomes stronger as it approaches the centre, the volume of the region of similarity flow decreases, and this reduces the total energy contained in it.

After focusing at the centre, a "reflected" shock wave is formed, which (when $t > 0$)

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[9] The procedure is to use the general methods of qualitative differential equation theory. The types of singular points for a first-order equation are classified as described by V. V. Stepanov, *Differential Equations* (*Kurs differentsial'nykh uravnenii*), Chpater II; G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, 2nd ed., Waltham (Mass.), 1969.

[10] The picture described is found to be valid only for $\gamma < \gamma_1 = 1.87\ldots$. When $\gamma = \gamma_1$ and α has the "correct" value, the points a and b coincide; when $\gamma > \gamma_1$, the distribution of integral curves changes and a fuller investigation is needed. However, in actual cases, $\gamma \le 5/3$, see the penultimate footnote to §106.

[11] The limiting value of ρ/ρ_1 at the time of focusing is 20.1 for $\gamma = 7/5$ and 9.55 for $\gamma = 5/3$.

[12] The integral diverges as $\xi_1 \to \infty$. This is because the similarity regime does not apply at distances $r >> R$.

expands to meet the gas moving towards the centre. This is again a similarity flow, with the same index α, so that the expansion occurs according to $R \propto t^{\alpha}$. We shall not give here a more detailed analysis of this stage.[13]

The problem thus provides an example of similarity flow, but one in which the similarity index (i.e., the form of the similarity variable ξ) cannot be determined from dimensional arguments; it is found only by solving the equations of motion themselves, using the conditions imposed by the physical formulation of the problem. Mathematically, it is characteristic that these conditions are formulated as requiring that the integral curve of a first-order differential equation should pass through a singular point of the equation. The similarity index is in general irrational.[14]

§108. Shallow-water theory

There is a remarkable analogy between gas flow and the flow in a gravitational field of an incompressible fluid with a free surface, when the depth of the fluid is small (compared with the characteristic dimensions of the problem, such as the dimensions of the irregularities on the bottom of the vessel). In this case the vertical component of the fluid velocity may be neglected in comparison with its velocity parallel to the surface, and the latter may be regarded as constant throughout the depth of the fluid. In this (***hydraulic***) approximation, the fluid can be regarded as a "two-dimensional" medium having a definite velocity $\boldsymbol{v}$ at each point and also characterized at each point by a quantity h, the depth of the fluid.

The corresponding general equations of motion differ from those obtained in §12 only in that the changes in quantities during the motion need not be assumed small, as they were in §12 in discussing long gravity waves with small amplitude. Consequently, the second-order velocity terms in Euler's equation must be retained. In particular, for one-dimensional flow in a channel, depending only on one coordinate x (and on the time), the equations are

$$\left.\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} = 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x} \end{aligned}\right\}, \qquad (108.1)$$

the depth h is here assumed constant across the channel.

Long gravity waves are, in a general sense, small perturbations of the flow now under consideration. The results of §12 show that such perturbations are propagated relative to the fluid with a finite velocity, namely

$$c = \sqrt{gh}\ . \qquad (108.2)$$

This velocity here plays the part of the velocity of sound in gas dynamics. Just as in §82, we can conclude that, if the fluid moves with velocities $v < c$ (***streaming flow***), the effect of the perturbations is propagated both upstream and downstream. If the fluid moves with velocities $v > c$ (***shooting flow***), however, the effect of the perturbations is propagated only into certain regions downstream.

The pressure p (reckoned from the atmospheric pressure at the free surface) varies with depth in the fluid according to the hydrostatic law $p = \rho g(h - z)$, where z is the height above the bottom. It is useful to note that, if we introduce the quantities

$$\bar{\rho} = ph\,,\quad \bar{p} = \int_0^h p dz = \frac{1}{2}\rho g h^2 = \frac{g\bar{\rho}^2}{2\rho}\,, \qquad (108.3)$$

then equations (108.1) become

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[13] But simply mention that the reflection of the shock wave is accompanied by a further compression of the gas, reaching 145 for $\gamma = 7/5$ and 32.7 for $\gamma = 5/3$.

[14] Another example of this kind of similarity flow is the propagation of a shock wave formed by a short sharp impact on a gas-filled half-space (Ya. B. Zel'dovich, *Soviet Physics Acoustics* 2, 25, 1956). The problem is also described in chapter XII of the book by Zel'dovich and Raizer cited in §86, and in G. I. Barenblatt's *Similarity, Self-Similarity, and Intermediate Asymptotics*, New York 1979

$$
\left.\begin{aligned}
\frac{\partial \bar{\rho}}{\partial t}+\frac{\partial (v\bar{\rho})}{\partial x}=0 \\
\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}
\end{aligned}\right\} \qquad (108.4)
$$

which are formally identical with the equations of adiabatic flow of a polytropic gas with $\gamma = 2$ ($\bar{p} \propto \bar{\rho}^2$). This enables us to apply immediately to shallow-water theory all the results of gas dynamics for flow in the absence of shock waves. If shock waves are present, however, the results of shallow-water theory differ from those of perfect-gas dynamics.

A "**shock wave**" in a fluid in a channel is a discontinuity in the fluid height h, and therefore in the fluid velocity v (what is called a ***hydraulic jump***). The relations between the values of the quantities on the two sides of the discontinuity can be obtained from the conditions of continuity of the fluxes of mass and momentum. The mass flux density (per unit width of the channel) is $j = \rho v h$. The momentum flux density is obtained by integrating $p + \rho v^2$ over the depth of the fluid, and is

$$\int_0^h (p + \rho v^2)dz = \frac{1}{2}\rho g h^2 + \rho v^2 h .$$

The conditions of continuity therefore give two equations:

$$
\left.\begin{aligned}
v_1 h_1 = v_2 h_2 \\
v_1^2 h_1 + \frac{1}{2} g h_1^2 = v_2^2 h_2 + \frac{1}{2} g h_2^2
\end{aligned}\right\}. \qquad (108.5)
$$

These give the relations between the four quantities v_1, v_2, h_1, h_2, two of which can be specified arbitrarily. Expressing the velocities v_1 and v_2 in terms of the heights h_1 and h_2, we obtain

$$
\left.\begin{aligned}
v_1^2 = \frac{g h_2 (h_1 + h_2)}{2h_1} \\
v_2^2 = \frac{g h_1 (h_1 + h_2)}{2h_2}
\end{aligned}\right\}. \qquad (108.6)
$$

The energy fluxes on the two sides of the discontinuity are not the same, and their difference is the amount of energy dissipated in the discontinuity per unit time. The energy flux density in the channel is

$$q = \int_0^h \left(\frac{p}{\rho} + \frac{1}{2}v^2\right)\rho v dz = \frac{1}{2} j (gh + v^2) .$$

Using (108.6), we find the difference to be

$$q_1 - q_2 = \frac{gj(h_1^2 + h_2^2)(h_2 - h_1)}{4h_1 h_2} .$$

Let the fluid move through the discontinuity from side 1 to side 2. Then the fact that energy is dissipated means that $q_1 - q_2 > 0$, and we conclude that

$$h_2 > h_1 , \qquad (108.7)$$

i.e., the fluid moves from the smaller to the greater height. We then can deduce from (108.6) that

$$v_1 > c_1 = \sqrt{g h_1} , \quad v_2 < c_2 = \sqrt{g h_2} , \qquad (108.8)$$

is complete analogy to the results for shock waves in gas dynamics. The inequalities (108.8) could also be derived as the necessary conditions for the discontinuity to be stable, as in §88.

PROBLEM

Find the stability condition for a tangential discontinuity in shallow water, i.e., a line such

that the liquid on either side is moving with different velocities (S. V. Bezdenkov and O. P. Pogutse 1983).

Solution. Because of the analogy mentioned above between the hydrodynamics of shallow water and polytropic gas dynamics, the problem is equivalent to that of the stability of a tangential discontinuity in a gas (§84, Problem 1). There is a difference, however, because in the shallow-water case we have to consider perturbations depending only on the coordinates in the plane of the liquid layer (parallel and perpendicular to the velocity *v*), not on the depth coordinate [15] *z*; the shallow-water approximation corresponds to perturbations with wavelength $\lambda >> h$. The velocity v_k found in §84, Problem 1, is therefore now the limit of instability: the discontinuity is stable for $v > v_k$, where *v* is the velocity change there. Since the density and the depth are the same on either side of the discontinuity, the velocity of sound is the same on either side, $c_1 = c_2 = \sqrt{gh}$, and the discontinuity is therefore stable if $v > 2\sqrt{2gh}$.

[15] Corresponding to *y* in §84, Problem 1.