§111. The intersection of shock waves with a solid surface

An important part in the phenomenon of steady interaction of shock waves with the surface of a body is played by their intersection with the boundary layer. This interaction is very complex, and a detailed discussion is outside the scope of this book; we shall give here only some general results.[1]

The pressure is discontinuous in a shock wave, and increases in the direction of motion of the gas. Hence, if the shock wave intersects the surface, there must be a finite increment of pressure over a very short distance near the place of intersection, i.e., there must be a very large positive pressure gradient. We know, however, that such a rapid increase in pressure cannot occur near a solid wall (see the end of §40); it would cause separation, and the pattern of flow round the body is changed in such a way that the shock wave moves away to a sufficient distance from the surface. An exception occurs only when the shock wave is weak. It is clear from the proof given at the end of §40 that the impossibility of a positive pressure discontinuity at the boundary layer is a consequence of the assumption that this discontinuity is large: it must exceed a certain limit depending on the value of R, which diminishes when R increases.

Thus we reach the following important conclusions. The steady intersection of strong shock waves with a solid surface is impossible. A solid surface can intersect only weak shock waves, and the limiting intensity decreases with increasing R. The maximum permissible intensity of the shock wave also depends on whether the boundary layer is laminar or turbulent. If the boundary layer is turbulent, the onset of separation is retarded (§45). In a turbulent boundary layer, therefore, stronger shock waves can leave the surface of the body than in a laminar boundary layer.

It should be emphasized that these arguments rely on the fact that the boundary layer exists in front of the shock wave (i.e., upstream of it). The results obtained therefore do not relate to shock waves which leave the leading edge of the body; the latter can occur, for instance in flow past an acute-angled wedge, a case which is discussed in detail in §112. In the latter case the gas reaches the vertex of the angle from outside, i.e., from a region in which there is no boundary layer. It is therefore clear that the present arguments do not deny that shock waves can occur which leave the vertex of such an angle.

In subsonic flow, separation can occur only when the pressure in the main stream increases downstream along the surface. In supersonic flow, however, it is found that separation can occur even when the pressure decreases downstream. Such a phenomenon can occur by the combination of a weak shock wave with a separation, the pressure increase necessary for separation taking place in the shock wave; the pressure may either increase or decrease downstream in the region in front of the shock wave.

All the above discussion relates only to a steady intersection, with the shock wave and the body at relative rest. Let us now consider **non-steady** intersections, when a moving shock wave is incident on a solid body, so that the line of intersection moves on the surface. Such an intersection is accompanied by reflection of the shock wave: besides the incident wave, a reflected wave leaving the body is formed.

We shall examine the phenomenon in a system of coordinates which moves with the line of intersection; in this system the shock waves are steady. The simplest type of reflection occurs

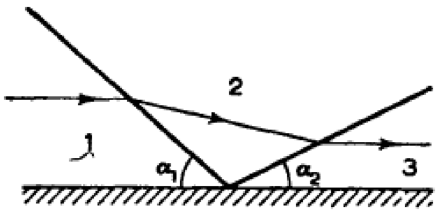

Fig. 105

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[1] The boundary layer necessarily contains a subsonic part adjoining the surface, into which the shock wave cannot penetrate. In speaking of the intersection, we ignore this fact, which does not affect the following discussion.

when the reflected wave leaves the line of intersection itself; this is called ***regular reflection*** (Fig. 105). If the angle of incidence a! and the intensity of the incident shock are given, the flow in region 2 is uniquely determined. The gas velocity in the reflected shock must be turned through an angle such that it is again parallel to the surface. When this angle is given, the position and intensity of the reflected shock are obtained from the equation of the shock polar. For a given angle, the shock polar determines two different shock waves, those of the **weak** and **strong** families (§92). Experimental results show that in fact the reflected shock always belongs to the weak family, and we shall assume this in what follows. It should be pointed out that, when the intensity of the incident shock tends to zero, the intensity of the reflected shock then tends to zero also, and the angle of reflection α_2 tends to the angle of incidence α_1, as we should expect in accordance with the acoustic approximation. In the limit $\alpha_1 \to 0$, the reflected shock of the weak family passes continuously into the shock obtained when a shock wave is incident "frontally" (§100, Problem 1).

The mathematical calculations for regular reflection (in a perfect gas) offer no difficulty in principle, but the algebra is extremely laborious. Here we shall give only some of the results.[2]

It is clear from the general properties of the shock polar that regular reflection is not possible for arbitrary values of the parameters of the incident shock (the angle of incidence α_1 and the ratio p_2 / p_1). For a given ratio p_2 / p_1 there is a maximum possible angle α_{1k}, and for $\alpha_1 > \alpha_{1k}$ regular reflection is impossible. As $p_2 / p_1 \to \infty$, the maximum angle tends to a value which depends on γ (= 40° for air). As $p_2 / p_1 \to 1$, α_{1k} tends to 90°, i.e., regular reflection is possible for any angle of incidence. Figure 106 shows α_{1k} as a function of p_1 / p_2 for $\gamma = 7/5$ and 5/3.

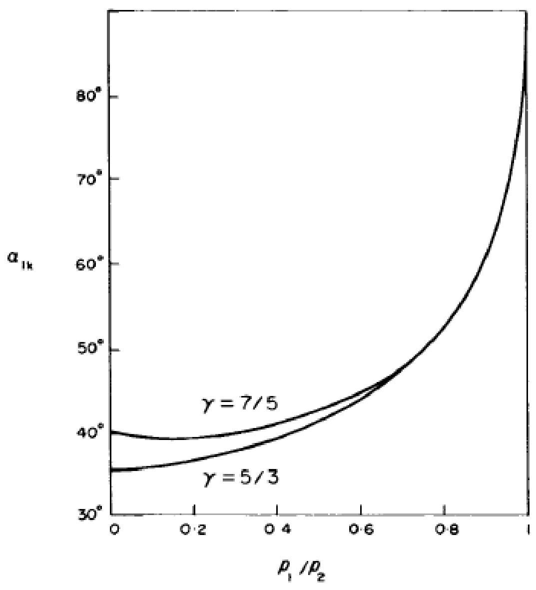

FIG. 106

The angle of reflection α_2 is not in general the same as the angle of incidence. There is a value α_* of the angle of incidence such that, if $\alpha_1 < \alpha_*$, the angle of reflection $\alpha_2 < \alpha_1$; if $\alpha_1 > \alpha_*$ on the other hand, $\alpha_2 > \alpha_1$. The value of α_* is $\frac{1}{2}\arccos\frac{\gamma - 1}{2}$ (= 39.2° for air); it does not depend on the intensity of the incident shock.

For $\alpha_1 > \alpha_{1k}$ regular reflection is impossible, and the incident shock wave must break up at a distance from the surface, so that we have the pattern shown in Fig. 107, with three shock

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[2] A more detailed account of the reflection of shock waves is given by R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, New York 1948, Chapter IV, by R. von Mises, *Mathematical Theory of Compressible Fluid Flow*, New York 1958, §23, and by W. Bleakney and A. H. Taub, *Reviews of Modern Physics*, **21**, 584, 1949.

waves, and a tangential discontinuity leaving the point where the incident shock wave divides. This is called ***Mach reflection***.

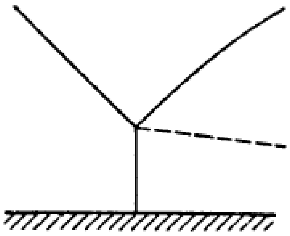

FIG. 107

§112. Supersonic flow round an angle

In investigating the flow near the vertex of an angle on the surface, it is again sufficient to consider small portions of the vertex and suppose it straight, the angle being formed by two intersecting planes. We shall speak of flow outside an angle if the angle is greater than π, and of flow inside an angle if it is less than π.

Subsonic flow past an angle is not essentially different from the flow of an incompressible fluid. Supersonic flow, however, is entirely different; an important property of it is the occurrence of discontinuities leaving the vertex of the angle.

Let us first consider the possible flow patterns when a supersonic gas stream reaches the vertex along one of the sides of the angle. In accordance with the general properties of supersonic flow, the stream remains uniform up to the vertex. The turning of the stream into the direction parallel to the other side of the angle occurs in a rarefaction wave leaving the vertex, and the flow pattern consists of three regions separated by weak discontinuities (*Oa* and *Ob* in Fig. 108): the uniform gas stream 1 moving along the side *AO* is turned in the rarefaction wave 2 and then moves, again with constant velocity, along the other side of the angle. It should be noticed that no turbulent region is formed; in a similar flow of an incompressible fluid, on the other hand, a turbulent region must be formed, with a line of separation at the vertex of the angle (Fig. 24, §36).

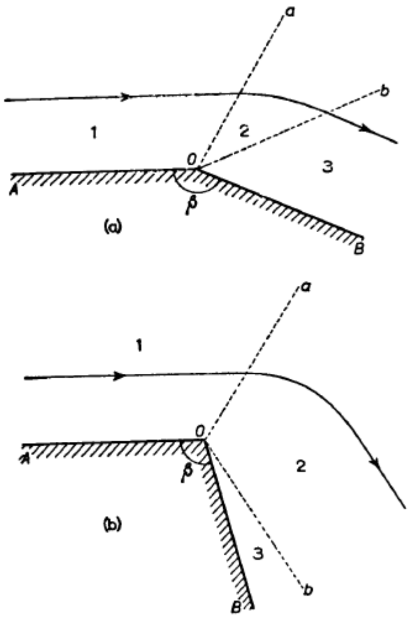

FIG. 108

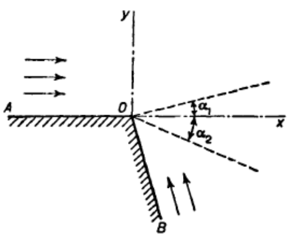

FIG. 24

Let v_1 be the velocity of the incident stream (1 in Fig. 108), and c_1 the velocity of sound in it. The position of the weak discontinuity *Oa* is determined immediately from the Mach number $M_1 = v_1 / c_1$ by the condition that it intersect the streamlines at the Mach angle. The changes in velocity and pressure in the rarefaction wave are determined by formulae (109.12) - (109.15); all that is needed is the direction from which the angle ϕ in these formulae is to be measured. The straight line $\phi = 0$ corresponds to $v = c = c_*$; for $M_1 > 1$, there is in fact no such line, since $v / c > 1$ everywhere. However, if the rarefaction wave is imagined to be formally extended into the region to the left of *Oa*, we can use formula (109.12), and we find that the discontinuity *Oa* must correspond to a value of ϕ given by

$$\phi_1 = \sqrt{\frac{\gamma+1}{\gamma-1}} \arccos \frac{c_1}{c_*},$$

and that ϕ must increase from *Oa* to *Ob*. The position of the discontinuity *Ob* is determined by the fact that the direction of the velocity becomes parallel to the side *OB* of the angle.

The angle through which the stream turns in the rarefaction wave cannot exceed the value $\chi_{\max}$ determined in §109, Problem 2. If the angle β round which the flow occurs is less than $\pi - \chi_{\max}$, the rarefaction wave cannot turn the stream through the necessary angle, and we have the flow pattern shown in Fig. 108b. The rarefaction in the wave 2 then proceeds to zero pressure (reached on the line *Ob*), so that the rarefaction wave is separated from the wall by a vacuum (region 3).

The flow pattern described above is not the only possible one, however. Figures 109 and 110 show patterns in which a region of gas at rest adjoins the second side of the angle, this region being separated from the moving gas by a tangential discontinuity; as usual, the latter becomes a turbulent region, so that the case considered corresponds to the presence of separation.[3] The stream is turned through a certain angle in a rarefaction wave (Fig. 109) or in a shock wave (Fig. 110). The latter case, however, is possible only if the shock wave is not too strong (in accordance with the general considerations given in §111).

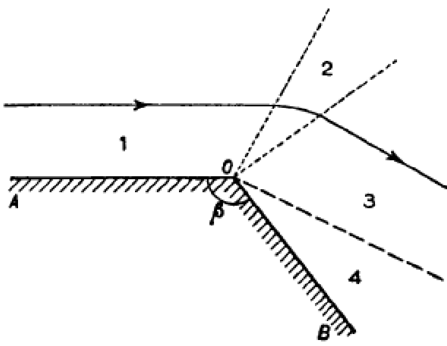

FIG. 109

Which of these flow patterns will occur in any particular case depends in general on the conditions far from the angle. For instance, when gas flows out of a nozzle (the vertex of the angle being here the edge of the outlet), the relation between the pressure p_1 of the outgoing gas and the pressure p_e of the external medium is of importance. If $p_e < p_1$, the flow is of the type shown in Fig. 109; the position and angle of the rarefaction wave are then determined by the condition that the pressure in regions 3 and 4 be equal to p_e. The smaller p_e, the greater the angle through which the stream must be turned. If, however, the angle β (Fig. 109) is large, the gas pressure cannot reach the required value p_e; the direction of the velocity becomes parallel to the side *OB* of the angle before the pressure falls to p_e. The flow near the edge of the outlet will then be as shown in Fig. 107. The pressure near the outer

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[3] According to experimental results, the compressibility of the gas somewhat diminishes the angle of the turbulent region resulting from the tangential discontinuity.

side OB of the outlet is entirely determined by the angle β, and does not depend on the pressure p_e; the final decrease of the pressure to p_e occurs only at a distance from the outlet.

If $p_e > p_1$, on the other hand, the flow round the edge of the outlet is of the type shown in Fig. 110, with a shock wave which leaves the edge and raises the pressure from p_1 to p_e. This is possible, however, only if the difference between p_e and p_1 is not too large, i.e., the shock wave is not too strong; otherwise there is separation at the inner surface of the nozzle, and the shock wave moves into the nozzle, in the manner described in §97.

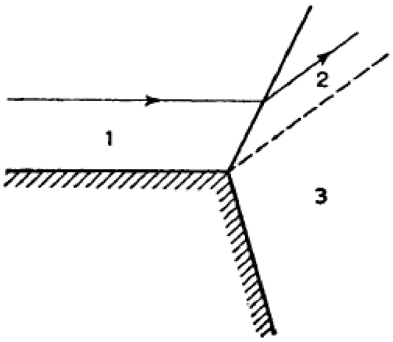

FIG. 110

Next, let us consider flow inside an angle. In the subsonic case such a flow is accompanied by separation at a point ahead of the vertex (see the end of §40). For a supersonic incident flow, however, the change in direction may be effected by a shock wave leaving the vertex (Fig. 111). Here it must again be mentioned that such a simple separationless flow pattern is possible only if the shock wave is not too strong. Its intensity increases with the angle χ through which the stream is turned, and we can therefore say that separationless flow is possible only when χ is not too large.

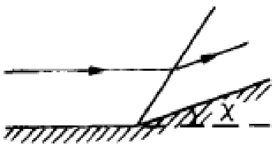

FIG. 111

Let us now consider the flow pattern which results when a free supersonic stream is incident on the vertex of an angle (Fig. 112). The stream is turned into directions parallel to the sides of the angle by shock waves leaving the vertex. As has been shown in §111, this is the exceptional case where a shock wave of arbitrary intensity can leave a solid surface.

If we know the velocities v_1 and c_1 in the incident stream 1, we can determine the positions of the shock waves and the gas flow in the regions behind them. The direction of the velocity v_2 must be parallel to the side OA of the angle: $\dfrac{v_{2y}}{v_{2x}} = \tan\chi$. Thus v_2 and the angle ϕ giving the position of the shock wave can be determined immediately from the shock polar, using a chord through the origin at the known angle χ to the axis of abscissae (Fig. 64), as explained in §92. We have seen that, for a given χ, the shock polar gives two different shock waves, with different values of ϕ. One of these (corresponding to the point B in Fig. 64) is the weaker, and in general leaves the flow supersonic; the other, stronger, shock renders the flow subsonic. In the present case of flow past an angle on a finite solid surface,

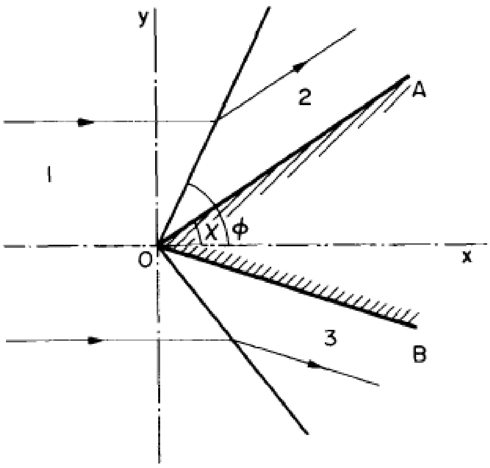

FIG. 112

we must always take the former, i.e., the weak shock. It should be borne in mind that this choice is really decided by the conditions of the flow far from the angle. In flow past a very acute angle (χ small), the resulting shock wave must obviously be very weak. It is natural to suppose that, as the angle increases, the intensity of the shock increases monotonically; this corresponds to a movement along the arc QC of the shock polar (Fig. 64), from Q towards C.[4]

We have also seen in §92 that the angle through which the velocity vector is turned in a shock wave cannot exceed a certain value $\chi_{\max}$, which depends on M_1. The flow pattern described above is therefore impossible if either of the sides of the angle makes an angle greater than $\chi_{\max}$ with the direction of the incident stream. In this case the gas flow near the angle must be subsonic; this is achieved by the appearance of a shock wave somewhere in front of the angle (see §122). Since $\chi_{\max}$ increases monotonically with M_1, we can also say that, for a given value of the angle χ, M_1 for the incident stream must be greater than a certain value $M_{1,\min}$.

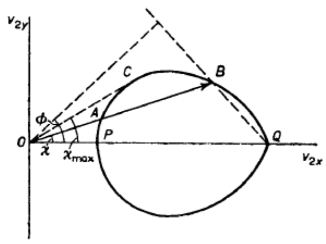

FIG. 64

Finally, it may be mentioned that, if the sides of the angle are situated, relative to the incident stream, as shown in Fig. 113, then a shock wave is of course formed on only one side of the angle; the stream is turned on the other side by a rarefaction wave.

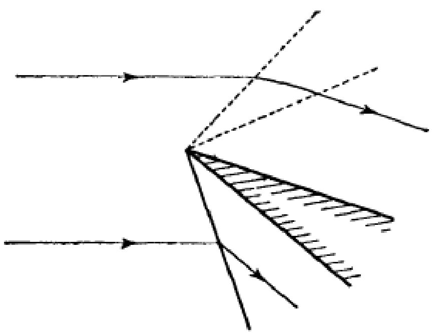

FIG. 113

PROBLEMS

Problem 1. Determine the position and intensity of the shock wave in flow past a very small

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[4] Cf., however, the first footnote to §113. The purely formal problem of flow past a wedge formed by the intersection of two infinite planes is of no physical interest.

angle ($\chi \ll 1$) for Mach numbers such that $M_1\chi \ll 1$.

Solution. When $\chi \ll 1$, the shock polar gives two values: close to $\pi/2$ (near *P* in Fig. 64) and close to the Mach angle α_1 (near *Q*). The relevant shock wave in the weak family corresponds to the latter. From (92.11), when $\chi \ll 1$,

$$M_1^2 \sin^2\phi - 1 \cong \chi \cdot \frac{\gamma+1}{2} M_1^2 \tan\alpha_1$$
$$= \chi \cdot \frac{\gamma+1}{2} \frac{M_1^2}{\sqrt{M_1^2-1}} .$$

Substitution of this in (92.9) gives

$$\frac{p_2 - p_1}{p_1} = \frac{\gamma M_1^2 \chi}{\sqrt{M_1^2-1}} .$$

The angle ϕ is sought in the form $\phi = \alpha_1 + \varepsilon$, where $\varepsilon \ll \alpha_1$; the same formula gives

$$\phi - \alpha_1 = \frac{\gamma+1}{4} \frac{M_1^2 \chi}{\sqrt{M_1^2-1}} .$$

When $M_1 \gg 1$, the angle $\alpha_1 \cong 1/M_1$, and for the above expressions to be valid we must have $M_1\chi \ll 1$.

Problem 2. The same as Problem 1, but for a Mach number so great that $M_1\chi \gg 1$.

Solution. In this case, ϕ and χ have the same order of magnitude. From (92.11), $\phi = \frac{\gamma+1}{2}\chi$. The pressure ratio is, from (92.9),

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1} M_1^2 \phi^2 = \frac{\gamma(\gamma+1)}{2} M_1^2 \chi^2 .$$

The value of M_2 behind the shock is, from (92.12),

$$M_2 = \frac{1}{\chi}\sqrt{\frac{2}{\gamma(\gamma-1)}} ,$$

and thus remains large in comparison with unity but not in comparison with $1/\chi$. In the same approximation,

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}, \quad \frac{v_2}{v_1} = 1 ;$$

the difference $v_1 - v_2 \sim v_1\chi^2$. The decrease in the Mach number is therefore actually due only to the increase in the velocity of sound: $\frac{M_2}{M_1} = \frac{c_1}{c_2}$.

§113. Flow pasta conical obstacle

The problem of steady supersonic flow near a pointed projection on the surface of a body is three-dimensional, and is very much more complicated than that of flow past an angle with a line vertex. A problem that has been completely solved is that of axially symmetrical flow past a projecting point, and we shall discuss this case.

Near its vertex, an axially symmetrical projection can be regarded as a right cone with circular cross-section, and so the problem consists in investigating the flow of a uniform stream past a cone whose axis is in the direction of incidence. The flow pattern is qualitatively as follows.

As in the analogous problem of flow past a two-dimensional angle, a shock wave must be formed (A. Busemann 1929), and it is evident from symmetry that this shock is a conical surface coaxial with the cone and having the same vertex (Fig. 114 shows the cross-section of

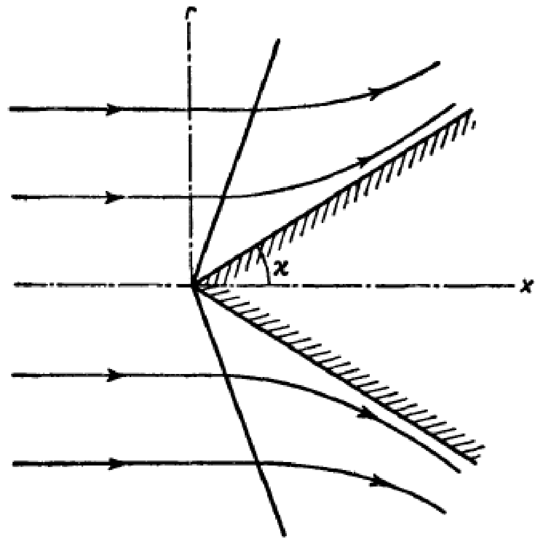

FIG. 114

the cone by a plane through its axis). Unlike what happens in the two-dimensional case, however, the shock wave does not turn the gas velocity through the whole angle χ necessary for the gas to flow along the surface of the cone (2χ being the vertical angle of the cone). After passing through the surface of discontinuity, the streamlines are curved, and asymptotically approach the generators of the cone. This curvature is accompanied by a continuous increase in density (besides the increase which occurs at the shock itself) and by a corresponding decrease in the velocity.

The change in the magnitude and direction of the velocity at the shock wave itself is determined by the shock polar; here again, the solution which occurs corresponds to the "**weak**" branch of the polar.[5] Accordingly, for each value of the incident stream Mach number $M_1 = v_1 / c_1$, there is a definite limiting value of the half-angle $\chi_{\max}$ of cone, beyond which such flow becomes impossible and the shock wave detaches itself from the cone vertex. Since there is a further rotation of the flow beyond the shock, the values of $\chi_{\max}$ for flow past a cone exceed (for the same M_1) those in the two-dimensional case of flow past a wedge. Immediately behind the shock wave, the gas flow is usually supersonic, but may be subsonic when χ is close to $\chi_{\max}$. The supersonic flow may become subsonic as the cone surface is approached, in which case the velocity passes through that of sound on a certain conical surface.

The conical shock wave intersects all streamlines in the incident flow at the same angle, and is therefore of constant intensity. Hence it follows (see §114) that we have **isentropic potential flow** behind the shock wave also.

From the symmetry of the problem and its similarity properties (there are no characteristic constant lengths in the conditions imposed), it is evident that the distribution of all quantities (velocity, pressure) in the flow behind the shock wave will depend only on the angle θ which the radius vector from the vertex of the cone to the point considered makes with the axis of the cone (the x-axis in Fig. 114). Accordingly, the equations of motion are ordinary differential equations; the boundary conditions on these equations at the shock wave are determined by the equation of the shock polar, while those at the surface of the cone are that the velocity should be parallel to the generators. These equations, however, cannot be integrated analytically, and have to be solved numerically. We refer the reader elsewhere[6] for

[5] This may not be so, however, for certain "exotic" shapes of the obstacle. For example, it appears that the "strong" family shock may occur in flow past a cone at the forward end of a broad blunt body.

[6] See G. I. Taylor and J. W. Maccoll, *Proceedings of the Royal Society* A139,278,1933; J. W. Maccoll, *ibid*. 159, 459, 1937; and N. E. Kochin, I. A. Kibel' and N. V. Roze, *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 2, §27, Moscow 1963.

the results of the calculations, and merely give the curve (Fig. 65, §92) which shows the maximum possible angle $\chi_{\max}$ as a function of M_1. We may also mention that, as $M_1 \to 1$, the angle $\chi_{\max}$ tends to zero:

$$\chi_{\max} = \text{constant} \times \sqrt{\frac{\mathrm{M}_1 - 1}{\gamma + 1}}, \tag{113.1}$$

as may be deduced from the general law of transonic similarity (126.11); the constant is independent both of M_1 and of the gas involved.

An analytical solution of the problem of flow past a cone is possible only in the limit of small vertical angles (T. von Karman and N. B. Moore 1932). It is evident that in this case the gas velocity nowhere differs greatly from the velocity v_1 of the incident stream. Denoting by $\boldsymbol{v}$ the small difference between the gas velocity at the point considered and v_1, and using its potential ϕ), we can apply the linearized equation (114.4); if we take cylindrical polar coordinates x, r, ω with the polar axis along the axis of the cone (ω being the polar angle), this equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \omega^2} - \beta^2 \frac{\partial^2 \phi}{\partial x^2} = 0, \tag{113.2}$$

or, for an axially symmetrical solution,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) - \beta^2 \frac{\partial^2 \phi}{\partial x^2} = 0, \tag{113.3}$$

where

$$\beta = \sqrt{M_1^{\,2} - 1}. \tag{113.4}$$

In order that the velocity distribution should be a function of θ only, the potential must be of the form $\phi = x f(\xi)$, where $\xi = r/x = \tan\theta$. Substituting this, we obtain for the function $f(\xi)$ the equation

$$\xi(1 - \beta^2 \xi^2) f'' + f' = 0,$$

of which the solution is elementary. The trivial solution f = constant corresponds to a uniform flow; the other solution is

$$f = \text{constant} \times \left[\sqrt{1 - \beta^2 \xi^2} - \cosh^{-1}\frac{1}{\beta\xi}\right].$$

The boundary condition on the surface of the cone (i.e., for $\xi = \tan\chi \cong \chi$) is

$$\frac{v_r}{v_1 + v_x} \cong \frac{1}{v_1}\frac{\partial \phi}{\partial r} = \chi \tag{113.5}$$

or $f' = v_1 \chi$. Hence the constant is $v_1 \chi^2$, and we have the following expression for the potential in the region $x > \beta r$:[7]

$$\phi = v_1 \chi^2 \left[\sqrt{x^2 - \beta^2 r^2} - x\cosh^{-1}\frac{x}{\beta r}\right]. \tag{113.6}$$

It should be noticed that ϕ has a logarithmic singularity for $r \to 0$.

We can now find the velocity components:

$$\left.\begin{aligned} v_x &= -v_1 \chi^2 \cosh^{-1}\frac{x}{\beta r} \\ v_r &= \frac{v_1 \chi^2}{r}\sqrt{x^2 - \beta^2 r^2} \end{aligned}\right\}. \tag{113.7}$$

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[7] In this approximation, the cone $x = \beta r$ is a surface of weak discontinuity. In the next approximation, a shock wave occurs whose strength (relative to the pressure discontinuity) is proportional to χ^4; the vertical semi-angle exceeds the Mach angle by an amount that is likewise proportional to χ^4.

The pressure on the surface of the cone is calculated from formula (114.5); since ϕ has a logarithmic singularity for $r \to 0$, the velocity v_r on the surface of the cone (i.e., for small r) is large compared with v_x, and therefore we need retain only the term in v_r^2 in the formula for the pressure. The result is

$$p - p_1 = \rho_1 v_1^2 \chi^2 \left[\log \frac{2}{\beta\chi} - \frac{1}{2} \right]. \qquad (113.8)$$

All these formulae, which have been derived by means of a linearized theory, cease to be valid for large M_1, comparable with $1/\chi$ (see §127).