CHAPTER XII
TWO-DIMENSIONAL GAS FLOW

§114. Potential flow of a gas

In what follows we shall meet with many important cases where the flow of a gas can be regarded as potential flow almost everywhere. Here we shall derive the general equations of potential flow and discuss the question of their validity.[1]

After passing through a shock wave, potential flow of a gas usually becomes rotational flow. An exception, however, is formed by cases where a steady potential flow passes through a shock wave whose intensity is constant over its area; such, for example, is the case where a uniform stream passes through a shock wave intersecting every streamline at the same angle.[2] The flow behind the shock wave is then potential flow also. To prove this, we use Euler's equation in the form

$$\frac{1}{2}\mathrm{grad}v^2 - \boldsymbol{v}\times\mathrm{curl}\boldsymbol{v} = -\frac{1}{\rho}\mathrm{grad}p$$

(cf. (2.10)), or

$$\mathrm{grad}(w+\frac{1}{2}v^2) - v\times\mathrm{curl}\boldsymbol{v} = T\mathrm{grad}s\,,$$

where we have used the thermodynamic relation $dw = Tds + dp/\rho$. In potential flow, however, $w+\frac{1}{2}v^2 = \text{constant}$ in front of the shock wave, and this quantity is continuous at the shock; it is therefore constant everywhere behind the shock wave, so that

$$\boldsymbol{v}\times\mathrm{curl}\boldsymbol{v} = -T\mathrm{grad}s\,. \qquad (114.1)$$

The potential flow in front of the shock wave is isentropic. In the general case of an arbitrary shock wave, for which the discontinuity of entropy varies over its surface, $\mathrm{grad}s \neq 0$ in the region behind the shock, and $\mathrm{curl}\boldsymbol{v}$ is therefore also not zero. If, however, the shock wave is of constant intensity, then the discontinuity of entropy in it is constant, so that the flow behind the shock is also isentropic, i.e., $\mathrm{grad}s = 0$. From this it follows that either $\mathrm{curl}\boldsymbol{v} = 0$ or the vectors $\boldsymbol{v}$ and $\mathrm{curl}\boldsymbol{v}$ are everywhere parallel. The latter, however, is impossible; at the shock wave, $\boldsymbol{v}$ always has a non-zero normal component, but the normal component of $\mathrm{curl}\boldsymbol{v}$ is always zero (since it is given by the tangential derivatives of the tangential velocity components, which are continuous).

Another important case where potential flow continues despite the shock wave is that of a **weak shock**. We have seen (§86) that in such a shock wave the discontinuity of entropy is of the third order relative to the discontinuity of pressure or velocity. We therefore see from (114.1) that $\mathrm{curl}\boldsymbol{v}$ behind the shock is also of the third order. This enables us to assume that we have potential flow behind the shock wave, the error being of a higher order of smallness.

We shall now derive the general equation for the velocity potential in an arbitrary steady potential flow of a gas. To do so, we eliminate the density from the equation of continuity $\mathrm{div}(\rho\boldsymbol{v}) \equiv \rho\mathrm{div}\boldsymbol{v} + \boldsymbol{v}\cdot\mathrm{grad}\rho = 0$, using Euler's equation

$$(\boldsymbol{v}\cdot\mathrm{grad})\boldsymbol{v} = -\frac{1}{\rho}\mathrm{grad}p = -\frac{c^2}{\rho}\mathrm{grad}\rho$$

and obtaining

$$c^2\mathrm{div}\boldsymbol{v} - \boldsymbol{v}\cdot(\boldsymbol{v}\cdot\mathrm{grad})\boldsymbol{v} = 0\,.$$

Introducing the velocity potential by $\boldsymbol{v} = \mathrm{grad}\phi$ and expanding in components, we obtain the equation

$$(c^2-\phi_x{}^2)\phi_{xx} + (c^2-\phi_y{}^2)\phi_{yy} + (c^2-\phi_z{}^2)\phi_{zz} - 2(\phi_x\phi_y\phi_{xy} + \phi_y\phi_z\phi_{yz} + \phi_z\phi_y\phi_y) = 0\,, \qquad (1\ 14.2)$$

where the suffixes here denote partial derivatives. In particular, for two-dimensional flow we

[1] In §114, the flow is not yet assumed to be two-dimensional.

[2] We have already met with this situation in connection with supersonic flow past a wedge or cone (§§112, 113).

have

$$(c^2-\phi_x{}^2)\phi_{xx}+(c^2-\phi_y{}^2)\phi_{yy}-2\phi_x\phi_y\phi_{xy}=0\,. \qquad (1\ 14.3)$$

In these equations, the velocity of sound must itself be expressed in terms of the velocity; this can in principle be done by means of Bernoulli's equation, $w+\frac{1}{2}v^2=\text{constant}$, and the isentropic equation, s = constant. For a polytropic gas, c as a function of v is given by formula (83.18).

Equation (114.2) is much simplified if the gas velocity nowhere differs greatly in magnitude or direction from that of the stream incident from infinity.[3] This implies that the shock waves (if any) are **weak**, and so the potential flow is not destroyed.

We denote by $\boldsymbol{v}'$ the small difference between the gas velocity $\boldsymbol{v}$ at a given point and that of the main stream, The potential ϕ is replaced by that of the velocity $\boldsymbol{v}'$: $\boldsymbol{v}'$ = grad ϕ'. The equation for this potential is obtained from (114.2) by substituting $\phi=\phi'+xv_1$; we take the x-axis in the direction of the vector $\boldsymbol{v}_1$. We then regard ϕ' as a small quantity, and omit all terms of order higher than the first, obtaining the following linear equation:

$$(1-M_1{}^2)\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial y^2}+\frac{\partial^2\phi'}{\partial z^2}=0\,, \qquad (114.4)$$

where $M_1=\frac{v_1}{c_1}$; the velocity of sound is, of course, given its value at infinity.

The pressure at any point is determined in terms of the velocity in the same approximation, by a formula which can be obtained as follows. We regard p as a function of w (for given s), and use the fact that $\left(\frac{\partial w}{\partial p}\right)_s=\frac{1}{\rho}$, writing $p-p_1\cong\left(\frac{\partial p}{\partial w}\right)_s(w-w_1)=\rho_1(w-w_1)$. From Bernoulli's equation we have

$$w-w_1=-\frac{1}{2}\left[(\boldsymbol{v}_1+\boldsymbol{v})^2-\boldsymbol{v}_1{}^2\right]\cong-\frac{1}{2}(v_y{}^2+v_z{}^2)-v_1v_x\,,$$

so that

$$p-p_1=-\rho_1v_1v_x-\frac{1}{2}\rho_1(v_y{}^2+v_z{}^2)\,. \qquad (114.5)$$

In this expression the term in the squared transverse velocity must in general be retained, since, in the region near the x-axis (and, in particular, on the surface of the body itself), the derivatives $\frac{\partial\phi'}{\partial y}$, $\frac{\partial\phi'}{\partial z}$ may be large compared with $\frac{\partial\phi'}{\partial x}$.

Equation (114.4), however, is not valid if the number M_1 is very close to unity (transonic flow), so that the coefficient of the first term is small. It is clear that, in this case, terms of higher order in the x-derivatives of ϕ must be retained. To derive the corresponding equation, we return to the original equation (114.2); when the terms which are certainly small are neglected, this becomes

$$\left(1-\frac{\phi_x{}^2}{c^2}\right)\phi_{xx}+\phi_{yy}+\phi_{zz}=0\,. \qquad (114.6)$$

In the present case, the velocity $v_x\cong v$, and the velocity of sound c is close to the critical velocity c_*. We can therefore put $c-c_*=(v-c_*)\left(\frac{dc}{dv}\right)_{v=c_*}$, or

[3] One such case was discussed in §113 (flow past a narrow cone), and others will be found in connection with flow past arbitrary thin bodies.

$c-v=(c_*-v)\left[1-\left(\frac{dc}{dv}\right)_{v=c_*}\right]$. Using the fact that, for $v=c=c_*$, we have by (83.4) $\frac{d\rho}{dv}=-\frac{\rho}{c}$, we put (for $v=c_*$)

$$\frac{dc}{dv}=\frac{dc}{d\rho}\frac{d\rho}{dv}=-\frac{\rho}{c}\frac{dc}{d\rho},$$

so that

$$c-v=\frac{c_*-v}{c}\frac{d(\rho c)}{d\rho}=\alpha_*(c_*-v). \tag{114.7}$$

We have here used the expression (99.9) for the derivative $\frac{d(\rho c)}{d\rho}$, while α_* denotes the value of α (102.2) for $v=c_*$; for a polytropic gas, α is constant, so that $\alpha_*=\alpha=\frac{\gamma+1}{2}$. To the same accuracy, this equation can be written as

$$\frac{v}{c}-1=\alpha_*\left(\frac{v}{c_*}-1\right). \tag{114.8}$$

This gives the general relation between the Mach numbers M and M_* in transonic flow.

Using this formula, we can put

$$1-\frac{{v_x}^2}{c^2}\cong 1-\frac{v^2}{c^2}\cong 2\left(1-\frac{v}{c}\right)\cong 2\alpha_*\left(1-\frac{v}{c_*}\right).$$

Finally, we introduce a potential by the substitution $\phi\to c_*(x+\phi)$, so that

$$\frac{\partial\phi}{\partial x}=\frac{v_x}{c_*}-1,\ \frac{\partial\phi}{\partial y}=\frac{v_y}{c_*},\ \frac{\partial\phi}{\partial z}=\frac{v_z}{c_*}. \tag{114.9}$$

Substituting these formulae in (114.6), we obtain the following final equation for the **velocity potential in a transonic flow** (with the velocity everywhere almost parallel to the *x*-axis):

$$2\alpha_*\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}. \tag{11410}$$

The properties of the gas appear here only through the constant α_*. We shall see later that this constant governs the entire dependence of the properties of transonic flow on the nature of the gas.

The linearized equation (114.4) becomes invalid also in another limiting case, that of very large values of M_1; however, the appearance of strong shock waves has the result that potential flow cannot actually occur for such values of M_1 (see §127).

§115. Steady simple waves

Let us determine the general form of those solutions of the equations of steady two-dimensional supersonic gas flow which describe flows in which there is a uniform plane-parallel stream at infinity, which then turns through an angle as it flows round a curved profile. We have already met a particular case of such a solution in discussing the flow near an angle; the flow considered was essentially a plane-parallel one along one side of the angle, which turned at the vertex of the angle. In this particular solution all quantities (the two velocity components, the pressure and the density) were functions of only one variable, the angle ϕ. Each of these quantities could therefore be expressed as a function of any other. Since this solution must be a particular case of the required general solution, it is natural to seek the latter on the assumption that each of the quantities p, ρ, v_x, v_y (the plane of the motion being taken as the *xy*-plane) can be expressed as a function of any other. This assumption is, of course, a very considerable restriction on the solution of the equations of motion, and the solution thus obtained is not the general integral of those equations. In the

general case, each of the quantities p, ρ, v_x, v_y which are functions of the two coordinates x, y, can be expressed as a function of any two of them.

Since we have a uniform stream at infinity, in which all quantities, and in particular the entropy s, are constants, and since in steady flow of an ideal fluid the entropy is constant along the streamlines, it is clear that s = constant in all space if there are no shock waves in the gas, as we shall assume.

Euler's equations and the equation of continuity are

$$\begin{cases} v_x \dfrac{\partial v_x}{\partial x} + v_y \dfrac{\partial v_x}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ v_x \dfrac{\partial v_y}{\partial x} + v_y \dfrac{\partial v_y}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \\ \dfrac{\partial}{\partial x}(\rho v_x) + \dfrac{\partial}{\partial y}(\rho v_y) = 0 \end{cases}.$$

Writing the partial derivatives as Jacobians, we can convert these equations to the form

$$\begin{cases} v_x \dfrac{\partial(v_x, y)}{\partial(x,y)} - v_y \dfrac{\partial(v_x, x)}{\partial(x,y)} = -\dfrac{1}{\rho}\dfrac{\partial(p,x)}{\partial(x,y)} \\ v_x \dfrac{\partial(v_y, y)}{\partial(x,y)} - v_y \dfrac{\partial(v_y, x)}{\partial(x,y)} = \dfrac{1}{\rho}\dfrac{\partial(p,x)}{\partial(x,y)} \\ \dfrac{\partial(\rho v_x, y)}{\partial(x,y)} - \dfrac{\partial(\rho v_y, x)}{\partial(x,y)} = 0 \end{cases}.$$

We now take x and p as independent variables. In order to effect this transformation, we need only multiply the above equations by $\dfrac{\partial(x,y)}{\partial(x,p)}$, obtaining the same equations except that $\partial(x,p)$ replaces $\partial(x,y)$ in the denominator of each Jacobian. We now expand the Jacobians, bearing in mind that all the quantities ρ, v_x, v_y are assumed to be functions of p but not of x, so that their partial derivatives with respect to x are zero. We then obtain

$$\begin{cases} \left(v_y - v_x \dfrac{\partial y}{\partial x}\right)\dfrac{dv_x}{dp} = \dfrac{1}{\rho}\dfrac{\partial y}{\partial x} \\ \left(v_y - v_x \dfrac{\partial y}{\partial x}\right)\dfrac{dv_y}{dp} = -\dfrac{1}{\rho} \\ \left(v_y - v_x \dfrac{\partial y}{\partial x}\right)\dfrac{d\rho}{dp} + \rho\left(\dfrac{dv_y}{dp} - \dfrac{\partial y}{\partial x}\dfrac{dv_x}{dp}\right) = 0 \end{cases}.$$

Here $\dfrac{\partial y}{\partial x}$ denotes $\left(\dfrac{\partial y}{\partial x}\right)_p$. All the quantities in these equations except $\dfrac{\partial y}{\partial x}$ are functions of p only, by hypothesis, and x does not appear explicitly. We can therefore conclude, first of all, that $\dfrac{\partial y}{\partial x}$ also is a function of p only: $\left(\dfrac{\partial y}{\partial x}\right)_p = f_1(p)$, whence

$$y = xf_1(p) + f_2(p), \qquad (115.1)$$

where $f_2(p)$ is an arbitrary function of the pressure.

No further calculations are necessary if we use the particular solution, already known, for a rarefaction wave in flow past an angle (§§109, 112). It will be recalled that, in this solution, all quantities (including the pressure) are constants along any straight line (characteristic) through the vertex of the angle. This particular solution evidently corresponds to the case where the arbitrary function $f_2(p)$ in the general expression (115.1) is identically zero. The function $f_1(p)$ is determined by the formulae obtained in §109.

Equation (115.1) for various constant p gives a family of straight lines in the xy-plane. These lines intersect the streamlines at every point at the Mach angle. This is seen

immediately from the fact that the lines $y = xf_1(p)$ in the particular solution with $f_2 \equiv 0$ have this property. Thus one of the families of characteristics (those leaving the surface of the body) consists, in the general case, of straight lines along which all quantities remain constant; these lines, however, are no longer concurrent.

The properties of the flow described above are, mathematically, entirely analogous to those of **one-dimensional simple waves**, in which one family of characteristics is a family of straight lines in the *xt*-plane (see §§101, 103, 104). Hence the class of flows under consideration occupies the same place in the theory of steady (supersonic) two-dimensional flow as do simple waves in non-steady one-dimensional flow. On account of this analogy, such flows are also called ***simple waves***; in particular, the rarefaction wave which corresponds to the case $f_2 \equiv 0$ is called a ***centred simple wave***.

As in the non-steady case, one of the most important properties of steady simple waves is that the flow in any region of the *xy*-plane bounded by a region of uniform flow is a simple wave (cf. §104).

We shall now show how the simple wave corresponding to flow round a given profile can be constructed. Figure 115 shows the profile in question; to the left of the point *O* it is straight, but to the right it begins to curve. In supersonic flow the effect of the curvature is, of course, propagated only downstream of the characteristic *OA* which leaves the point *O*. Hence the flow to the left of this characteristic is uniform; we denote by the suffix 1 quantities pertaining to this region. All the characteristics there are parallel and at an angle to the *x*-axis which is equal to the Mach angle $\alpha_1 = \arcsin\dfrac{c_1}{v_1}$.

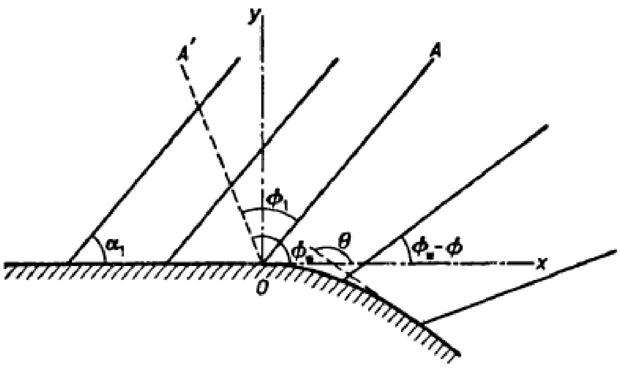

FIG. 115

In formulae (109.12) - (109.15), the angle ϕ of the characteristics is measured from the line on which $v = c = c_*$. This means (cf. §112) that the characteristic *OA* must have a value of ϕ given by

$$\phi_1 = \sqrt{\frac{\gamma+1}{\gamma-1}} \arccos\frac{c_1}{c_*},$$

and the angle ϕ is to be measured from *OA*' (Fig. 115). The angle between the characteristics and the *x*-axis is then $\phi_* - \phi$, where $\phi_* = \alpha_1 + \phi_1$. According to formulae (109.12) - (109.15), the velocity and pressure are given in terms of ϕ by

$$v_x = v\cos\theta\,,\quad v_y = v\sin\theta\,, \tag{115.2}$$

$$v^2 = {c_*}^2\left[1 + \frac{2}{\gamma-1}\sin^2\sqrt{\frac{\gamma-1}{\gamma+1}}\phi\right], \tag{115.3}$$

$$\theta = \phi_* - \phi - \arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\cot\sqrt{\frac{\gamma-1}{\gamma+1}}\phi\right), \tag{115.4}$$

$$p = p_* \cos^{2\gamma/(\gamma-1)} \sqrt{\frac{\gamma-1}{\gamma+1}}\phi . \qquad (115.5)$$

The **equation of the characteristics** can be written

$$y = x \tan(\phi_* - \phi) + F(\phi) . \qquad (115.6)$$

The arbitrary function $F(\phi)$ is determined as follows when the form of the profile is given. Let the latter be $Y = Y(X)$, where X and Y are the coordinates of points on it. At the surface, the gas velocity is tangential, i.e.,

$$\tan\theta = \frac{dY}{dX} . \qquad (115.7)$$

The equation of the line through the point (X, Y) at an angle $\phi_* - \phi$ to the x-axis is

$$y - Y = (x - X)\tan(\phi_* - \phi) .$$

This equation is the same as (115.6) if we put

$$F(\phi) = Y - X \tan(\phi_* - \phi) . \qquad (115.8)$$

Starting from the given equation $Y = Y(X)$ and equation (115.7), we express the form of the profile in parametric equations $X = X(\theta)$, $Y = Y(\theta)$, the parameter being the inclination θ of the tangent. Substituting θ in terms of ϕ from (115.4), we obtain X and Y as functions of ϕ; finally, substituting these in (115.8), we obtain the required function $F(\phi)$.

In flow past a convex surface, the angle θ between the velocity vector and the x-axis decreases downstream (Fig. 115), and the angle $\phi_* - \phi$ between the characteristic and the x-axis therefore decreases monotonically also (we always mean the characteristic leaving the surface). For this reason, the characteristics do not intersect (in the region of flow, that is). Thus, in the region downstream of the characteristic OA (which is a weak discontinuity), we have a continuous (no shock waves) and increasingly rarefied flow.

The situation is different in flow past a **concave** profile. Here the inclination θ of the tangent increases downstream, and therefore so does the inclination of the characteristics. Consequently, the characteristics intersect in the region of flow. On different non-parallel characteristics, however, all quantities (velocity, pressure, etc.) have different values. Thus all these quantities become many-valued at points where characteristics intersect, which is physically impossible. We have already met a similar phenomenon in connection with a non-steady one-dimensional simple compression wave (§101). As in that case, it signifies that in reality a **shock wave** is formed. The position of the discontinuity cannot be completely determined from the solution under consideration, since this was derived on the assumption that there are no discontinuities. The only result that can be obtained is the place where the shock wave begins (the point O in Fig. 116, where the shock is shown by the continuous line OB). It is the point of intersection of characteristics whose streamline lies nearest to the surface of the body. On streamlines passing below O (i.e., nearer to the surface) the solution is everywhere single-valued; its many-valuedness begins at O. The equations for the coordinates x_0, y_0 of this point can be obtained in the same way as the corresponding equations which

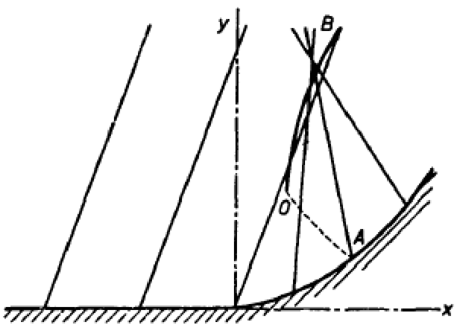

FIG. 116

determine the time and place of formation of the discontinuity in a one-dimensional non-steady simple wave. If we regard the inclination of the characteristics as a function of the coordinates (x, y) of points through which they pass, then this function becomes many-valued when x and y exceed certain values x_0, y_0. In §101 the situation was the same in relation to the function $v(x, t)$, and so we need not repeat the arguments used there, but can write down immediately the equations

$$\left(\frac{\partial y}{\partial \phi}\right)_x = 0\,, \quad \left(\frac{\partial^2 y}{\partial \phi^2}\right)_x = 0\,, \qquad (115.9)$$

which now determine the place of formation of the shock wave. Mathematically, this point is a **cusp** on the envelope of the family of straight characteristics (cf. §103).

In flow past a concave profile, the simple wave exists along streamlines passing above O as far as the points where these lines intersect the shock wave. The streamlines passing below O do not intersect the shock wave at all, but we cannot conclude from this that the solution in question is valid at all points on these streamlines. The reason is that the shock wave has a perturbing effect even on the gas which flows along these streamlines, and so alters the flow from what it would be in the absence of the shock wave. By a property of supersonic flow, however, these perturbations reach only the gas downstream of the characteristic OA (of the second family) which leaves the point where the shock wave begins. Thus the solution under consideration is valid everywhere to the left of AOB. The line OA itself is a weak discontinuity. We see that there cannot be a continuous (no shock waves) simple compression wave everywhere in flow past a concave surface, which would correspond to the simple rarefaction wave in flow past a convex surface.

The shock wave formed in flow past a concave profile is an example of a shock which "begins" at a point inside the stream, away from the solid walls. The point where the shock begins has some general properties, which may be noted here. At the point itself the intensity of the shock wave is zero, and near the point it is small. In a weak shock wave, however, the discontinuities of entropy and vorticity are of the third order of smallness, and so the change in the flow on passing through the shock differs from a continuous potential isentropic change only by quantities of the third order. Hence it follows that, in the weak discontinuities which leave the point where the shock wave begins, only the third derivatives of the various quantities can be discontinuous. There will in general be two such discontinuities: a **weak discontinuity coinciding with the characteristic**, and a **weak tangential discontinuity coinciding with the streamline** (see the end of §96).

§116. Chaplygin's equation: the general problem of steady two-dimensional gas flow

Having dealt with steady simple waves, let us now consider the general problem of an arbitrary steady two-dimensional potential flow. We assume that the flow is isentropic and contains no shock waves.

It is possible to reduce this problem to the solution of a single linear partial differential equation (S. A. Chaplygin 1902). This is achieved by means of a transformation to new independent variables, the velocity components v_x, v_y; this transformation is often called the ***hodograph transformation***, the $v_x v_y$-plane being called the ***hodograph plane*** and the xy-plane the ***physical plane***.

For potential flow we can replace Euler's equations by their first integral, Bernoulli's equation:

$$w + \frac{1}{2}v^2 = w_0\,. \qquad (116.1)$$

The equation of continuity is

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) = 0\,. \qquad (116.2)$$

For the differential of the velocity potential ϕ we have $d\phi = v_x dx + v_y dy$. We transform from the independent variables x, y to the new variables v_x, v_y by Legendre's transformation, putting

$$d\phi = d(xv_x) - xdv_x + d(yv_y) - ydv_y \,,$$

introducing the function

$$\Phi = -\phi + xv_x + yv_x \,, \qquad (116.3)$$

and obtaining

$$d\Phi = xdv_x + ydv_y \,,$$

where Φ is regarded as a function of v_x and v_y. Hence

$$x = \frac{\partial \Phi}{\partial v_x} \,, \quad y = \frac{\partial \Phi}{\partial v_y} \,. \qquad (116.4)$$

It is more convenient, however, to use, not the Cartesian components of the velocity, but its magnitude v and the angle θ which it makes with the x-axis:

$$v_x = v\cos\theta \,, \quad v_y = v\sin\theta \,. \qquad (116.5)$$

The appropriate transformation of the derivatives gives, instead of (116.4),

$$x = \cos\theta \frac{\partial \Phi}{\partial v} - \frac{\sin\theta}{v}\frac{\partial \Phi}{\partial \theta} \,, \quad y = \sin\theta \frac{\partial \Phi}{\partial v} + \frac{\cos\theta}{v}\frac{\partial \Phi}{\partial \theta} \,. \qquad (116.6)$$

The relation between the potential ϕ and the function Φ is given by the simple formula

$$\phi = -\Phi + v\frac{\partial \Phi}{\partial v} \,. \qquad (116.7)$$

Finally, in order to obtain the equation which determines the function $\Phi(v,\theta)$, we must transform the equation of continuity (116.2) to the new variables. Writing the derivatives as Jacobians:

$$\frac{\partial(\rho v_x, y)}{\partial(x, y)} - \frac{\partial(\rho v_y, x)}{\partial(x, y)} = 0 \,,$$

multiplying by $\dfrac{\partial(x, y)}{\partial(v, \theta)}$ and substituting (116.5), we have

$$\frac{\partial(\rho v\cos\theta, y)}{\partial(v, \theta)} - \frac{\partial(\rho v\sin\theta, x)}{\partial(v, \theta)} = 0 \,.$$

To expand these Jacobians, we must substitute (116.6) for x and y. Furthermore, since the entropy s is a given constant, if we express the density as a function of s and w and substitute $w = w_0 - \frac{1}{2}v^2$ we find that the density can be written as a function of v only: $\rho = \rho(v)$. We therefore obtain, after a simple calculation, the equation

$$\frac{d(\rho v)}{dv}\left(\frac{\partial \Phi}{\partial v} + \frac{1}{v}\frac{\partial^2 \Phi}{\partial \theta^2}\right) + \rho v \frac{\partial^2 \Phi}{\partial v^2} = 0 \,.$$

According to (83.5),

$$\frac{d(\rho v)}{dv} = \rho\left(1 - \frac{v^2}{c^2}\right),$$

and so we have finally ***Chaplygin's equation*** for the function $\Phi(v,\theta)$:

$$\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{v^2}{1 - v^2/c^2}\frac{\partial^2 \Phi}{\partial v^2} + v\frac{\partial \Phi}{\partial v} = 0 \,. \qquad (116.8)$$

Here the velocity of sound is a known function $c(v)$, determined by the equation of state of the gas together with Bernoulli's equation.

The equation (116.8), together with the relations (116.6), is equivalent to the equations of motion. Thus the problem of solving the non-linear equations of motion is reduced to the solution of a linear equation for the function $\Phi(v,\theta)$. It is true that the boundary conditions on this equation are non-linear. These conditions are as follows. At the surface of the body, the gas velocity must be tangential. Expressing the equation of the surface in the parametric form $X = X(\theta)$, $Y = Y(\theta)$ (as in §115), and substituting X and Y in place of x and y in (116.6), we

obtain two equations, which must be satisfied for all values of θ; this is not possible for every function $\Phi(v,\theta)$. The boundary condition is, in fact, that these two equations are compatible for all θ, i.e., one of them must be deducible from the other.

The satisfying of the boundary conditions, however, does not ensure that the resulting solution of Chaplygin's equation determines a flow that is actually possible everywhere in the physical plane. The following condition must also be met: the Jacobian $\Delta \equiv \dfrac{\partial(x,y)}{\partial(\theta,v)}$ must nowhere be zero, except in the trivial case when all its four component derivatives vanish. It is easy to see that, unless this condition holds, the solution becomes complex when we pass through the line (called the ***limiting line***) in the *xy*-plane given by the equation $\Delta = 0$.[4] For, let $\Delta = 0$ on the line $v = v_0(\theta)$, and suppose that $\left(\dfrac{\partial y}{\partial \theta}\right)_\theta \neq 0$. Then we have

$$-\Delta\left(\frac{\partial \theta}{\partial y}\right)_v = \frac{\partial(x,y)}{\partial(v,\theta)}\frac{\partial(v,\theta)}{\partial(v,y)} = \frac{\partial(x,y)}{\partial(v,y)} = \left(\frac{\partial x}{\partial v}\right)_y = 0.$$

Hence we see that, near the limiting line, *v* is determined as a function of *x* (for given *y*) by

$$x - x_0 = \frac{1}{2}\left(\frac{\partial^2 x}{\partial v^2}\right)_y (v - v_0)^2,$$

and *v* becomes complex on one side or the other of the limiting line.[5]

It is easy to see that a limiting line can occur only in regions of supersonic flow. A direct calculation, using the relations (116.6) and equation (116.8), gives

$$\Delta = \frac{1}{v}\left[\left(\frac{\partial^2 \Phi}{\partial\theta\partial v} - \frac{1}{v}\frac{\partial \Phi}{\partial \theta}\right)^2 + \frac{v^2}{1 - v^2/c^2}\left(\frac{\partial^2 \Phi}{\partial v^2}\right)^2\right]. \qquad (116.9)$$

It is clear that, for $v \le c$, $\Delta > 0$, and Δ can become zero only if $v > c$.

The appearance of limiting lines in the solution of Chaplygin's equation indicates that, under the given conditions, a continuous flow throughout the region is impossible, and shock waves must occur. It should be emphasized, however, that the position of these shocks is not the same as that of the limiting lines.

In §115 we discussed the particular case of steady two-dimensional supersonic flow (a simple wave), which is characterized by the fact that the velocity in it is a function only of its direction: $v = v(\theta)$. This solution cannot be obtained from Chaplygin's equation, since $1/\Delta \equiv 0$, and the solution is lost when the equation of continuity is multiplied by the Jacobian Δ in the transformation to the hodograph plane. The situation is exactly analogous to that found in the theory of non-steady one-dimensional flow. The remarks made in §105 concerning the relation between the simple wave and the general integral of equation (105.2) are wholly applicable to the relation between the steady simple wave and the general integral of Chaplygin's equation.

The fact that the Jacobian Δ is positive in subsonic flow gives a rule for finding the direction in which the velocity vector turns along the flow (A. A. Nikol'skii and G. I. Taganov 1946). We have identically

$$\frac{1}{\Delta} \equiv \frac{\partial(\theta,v)}{\partial(x,y)} = \frac{\partial(\theta,v)}{\partial(x,v)}\frac{\partial(x,v)}{\partial(x,y)},$$

or

[4] There is no objection to a passage through points where Δ becomes infinite. If $1/\Delta = 0$ on some line, this merely means that the correspondence between the *xy* and $v\theta$ planes is no longer one-to-one: in going round the *xy*-plane, we cover some part of the $v\theta$-plane two or three times.

[5] This result clearly remains valid even if $\left(\dfrac{\partial^2 x}{\partial v^2}\right)_y$ vanishes with Δ but $\left(\dfrac{\partial x}{\partial v}\right)_y$ again changes sign for $v = v_0$, i.e., the difference $x - x_0$ is proportional to a higher even power of $v - v_0$.

$$\frac{1}{\Delta} = \left(\frac{\partial\theta}{\partial x}\right)_v \left(\frac{\partial v}{\partial y}\right)_x . \qquad (116.10)$$

In a subsonic flow $\Delta > 0$, and we see that the derivatives $\left(\frac{\partial\theta}{\partial x}\right)_v$ and $\left(\frac{\partial v}{\partial y}\right)_x$ have the same sign. This has a simple geometrical significance: if we move along a line $v = \text{constant} \equiv \mathrm{v}_0$, with the region $v < v_0$ to the right, the angle θ increases monotonically, i.e., the velocity vector turns always counterclockwise. This result holds, in particular, for the line of transition between subsonic and supersonic flow, on which $v = c = c_*$.

In conclusion, we may give Chaplygin's equation for a polytropic gas, writing *c* explicitly in terms of *v*:

$$\frac{\partial^2\Phi}{\partial\theta^2} + v^2 \frac{1 - \frac{\gamma-1}{\gamma+1}\frac{v^2}{c_*^{\,2}}}{1 - \frac{v^2}{c_*^{\,2}}} \frac{\partial^2\Phi}{\partial v^2} + v\frac{\partial\Phi}{\partial v} = 0 . \qquad (116.11)$$

This equation has a family of particular integrals expressible in terms of hypergeometric functions.[6]

[6] See, for instance, L. I. Sedov, *Two-dimensional problems in Hydrodynamics and Aerodynamics*, Chapter X, New York 1965; R. von Mises, *Mathematical Theory of Compressible Fluid Flow*, § 20, New York 1958.