§117. Characteristics in steady two-dimensional flow

Some general properties of characteristics in steady (supersonic) two-dimensional flow have already been discussed in §82. We shall now derive the equations which give the characteristics in terms of a given solution of the equations of motion.

In steady two-dimensional supersonic flow there are, in general, three families of characteristics. All small disturbances, except those of entropy and vorticity, are propagated along two of these families (which we call the characteristics C_+ and C_-); disturbances of entropy and vorticity are propagated along characteristics (C_0) of the third family, which coincide with the streamlines. For a given flow, the streamlines are known, and the problem is to determine the characteristics belonging to the first two families.

The directions of the characteristics C_+ and C_- passing through each point in the plane lie on opposite sides of the streamline through that point, and make with it an angle equal to the local value of the Mach angle α (Fig. 51, §82). We denote by m_0 the slope of the streamline at a given point, and by m_+, m_- the slopes of the characteristics C_+, C_-. Then we have

$$\frac{m_+ - m_0}{1 + m_0 m_+} = \tan\alpha\,, \quad \frac{m_- - m_0}{1 + m_0 m_-} = -\tan\alpha\,,$$

whence

$$m_\pm = \frac{m_0 \pm \tan\alpha}{1 \pm m_0 \tan\alpha}\,;$$

the upper signs everywhere relate to C_+ and the lower to C_-. Substituting $m_0 = \dfrac{v_y}{v_x}$, $\tan\alpha = \dfrac{c}{\sqrt{v^2 - c^2}}$ and simplifying, we obtain the following expression for the slopes of the characteristics:

$$m_\pm \equiv \left(\frac{dy}{dx}\right)_\pm = \frac{v_x v_y \pm c\sqrt{v^2 - c^2}}{v_x{}^2 - c^2}\,. \tag{117.1}$$

If the velocity distribution is known, this is a differential equation which determines the characteristics C_+, and C_-.[1]

Besides the characteristics in the *xy*-plane, we may consider those in the hodograph plane, which are especially useful in the discussion of isentropic potential flow; we shall take this case in what follows. Mathematically, these are the characteristics of Chaplygin's equation (116.8), which is of hyperbolic type for $v > c$. Following the general method familiar in mathematical physics (see §103), we form from the coefficients the equation of the characteristics:

$$dv^2 + d\theta^2 \frac{v^2}{1 - v^2/c^2} = 0\,,$$

or

$$\left(\frac{d\theta}{dv}\right)_\pm = \pm\frac{1}{v}\sqrt{\frac{v^2}{c^2} - 1}\,. \tag{117.2}$$

[1] Equation (117.1) also determines the characteristics for steady axially symmetrical flow if v_y and *y* are replaced by v_r and *r*, where *r* is the cylindrical polar coordinate (the distance from the axis of symmetry, which is the *x*-axis); it is clear that the derivation is unchanged if we consider, instead of the *xy*-plane, an *xr*-plane through the axis of symmetry.

The characteristics given by this equation do not depend on the particular solution of Chaplygin's equation considered, because the coefficients in that equation are independent of Φ. The characteristics in the hodograph plane are a transformation of the characteristics C_+ and C_- in the physical plane, and we call them respectively the characteristics Γ_+ and Γ_-, in accordance with the signs in (117.2).

The integration of equation (117.2) gives relations of the form $J_+(v,\theta) = \text{constant}$, $J_-(v,\theta) = \text{constant}$. The functions J_+ and J_- are quantities which remain constant along the characteristics C_+ and C_- (i.e., Riemann invariants). For a polytropic gas, equation (117.2) can be integrated explicitly. There is, however, no need to go through the calculations, since the result can be seen from formulae (115.3) and (115.4). For, according to the general properties of simple waves (see §104), the dependence of v on θ for a simple wave is given by the condition that one of the Riemann invariants be constant in all space. The arbitrary constant in formulae (115.3) and (115.4) is ϕ_*; eliminating the parameter ϕ from these formulae, we obtain

$$J_\pm = \theta \pm \left\{ \arcsin\sqrt{\frac{\gamma+1}{2}\left(1-\frac{c_*{}^2}{v^2}\right)} - \sqrt{\frac{\gamma+1}{\gamma-1}}\arcsin\sqrt{\frac{\gamma-1}{2}\left(\frac{v^2}{c_*{}^2}-1\right)} \right\}. \qquad (117.3)$$

The characteristics in the hodograph plane are a family of **epicycloids**, occupying the space between two circles with radii $v = c_*$ and $v = \sqrt{\frac{\gamma+1}{\gamma-1}}c_*$ (Fig. 117).

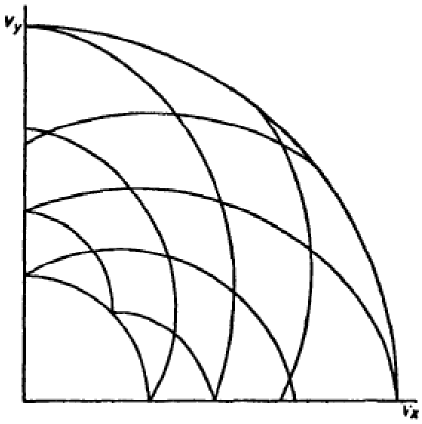

Fig. 117

For isentropic potential flow, the characteristics Γ_+, Γ_- have the following important property: the families Γ_+, Γ_- are orthogonal to the families C_-, C_+, respectively (it is assumed that the coordinate axes of x and y are mapped parallel to those of v_x and v_y).[2]

To prove this, we start from equation (114.3) for two-dimensional potential flow, which has the form

$$A\frac{\partial^2\phi}{\partial x^2} + 2B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} = 0, \qquad (117.4)$$

with no free term. The slopes $m_\pm$ of the characteristics $C_\pm$ are the roots of the quadratic

$$Am^2 - 2Bm + C = 0.$$

Let us consider the expression $dv_x{}^+dx^- + dv_y{}^+dy^-$, in which the velocity differentials are taken along the characteristics Γ_+, and the coordinate differentials along C_-. We have, identically,

[2] This does not apply to the characteristics of axially symmetrical flow in the xr-plane.

$$dv_x{}^+dx^- + dv_y{}^+dy^- = \frac{\partial^2\phi}{\partial x^2}dx^+dx^- + \frac{\partial^2\phi}{\partial x\partial y}(dx^+dy^- + dx^-dy^+) + \frac{\partial^2\phi}{\partial y^2}dy^+dy^- .$$

Dividing by dx^+dx^-, we obtain as the coefficients of $\frac{\partial^2\phi}{\partial x\partial y}$ and $\frac{\partial^2\phi}{\partial y^2}$, respectively $m_+ + m_- = \frac{2B}{A}$ and $m_+m_- = \frac{C}{A}$. It is then clear that the expression is zero, by (117.4). Thus

$$dv_x{}^+dx^- + dv_y{}^+dy^- = d\mathbf{v}^+ \cdot d\mathbf{r}^- = 0 .$$

Similarly, $d\mathbf{v}^- \cdot d\mathbf{r}^+ = 0$. These equations are equivalent to the result stated.

§118. The Euler-Tricomi equation. Transonic flow

The investigation of the properties resulting from the transition between subsonic and supersonic flow is of fundamental interest. Steady flows in which this transition occurs are called ***mixed*** or ***transonic*** flows, and the surface where the transition occurs is called the ***transitional*** or ***sonic*** surface.

Chaplygin's equation is particularly useful in investigating the flow near the transition, since it is much simplified there. At the boundary where the transition occurs $v = c = c_*$, and near it (in the transonic region) the differences v - c and $v - c_*$ are small; they are related by (114.8):

$$\frac{v}{c} - 1 = \alpha_*\left[\frac{v}{c_*} - 1\right].$$

Let us effect the corresponding simplification in Chaplygin's equation. The third term in equation (116.8) is small compared with the second, which contains $1 - \frac{v^2}{c^2}$ in the denominator. In the second term we put approximately

$$\frac{v^2}{1 - v^2/c^2} = \frac{c_*{}^2}{2(1 - v/c)} = \frac{c_*}{2\alpha_*(1 - v/c_*)} .$$

Finally, replacing the velocity v by a new variable

$$\eta = (2\alpha_*)^{1/3}\frac{v - c_*}{c_*}, \qquad (118.1)$$

we obtain the required equation in the form

$$\frac{\partial^2\Phi}{\partial\eta^2} - \eta\frac{\partial^2\Phi}{\partial\theta^2} = 0 . \qquad (118.2)$$

An equation of this form is called in mathematical physics the ***Euler-Tricomi equation***.[3] In the half-plane $\eta > 0$ it is hyperbolic, but in $\eta < 0$ it is elliptic. We shall discuss here some mathematical properties of this equation which are important in connection with various physical problems.

The characteristics of equation (118.2) are given by the equation $\eta d\eta^2 - d\theta^2 = 0$, which has the general integral

$$\theta \pm \frac{2}{3}\eta^{3/2} = C , \qquad (118.3)$$

where C is an arbitrary constant. This equation represents two families of curves in the $\eta\theta$-plane, which are branches of semi-cubical parabolae in the right half-plane with cusps on the θ-axis (Fig. 118)

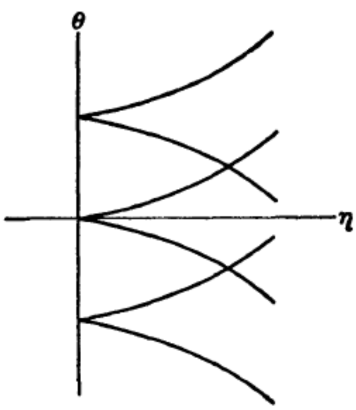

FIG. 118

[3] The application of this equation to the problem here considered is due to F. I. Frankl' (1945).

In investigating the flow in a small region[4] of space, where the direction of the gas velocity varies only slightly, we can always take the direction of the *x*-axis such that the angle θ measured from it is small throughout the region considered. The equations (116.6) which determine the coordinates *x*, *y* from the function $\Phi(\eta,\theta)$ are then much simplified also:[5] $x=(2\alpha_*)^{1/3}\dfrac{\partial\Phi}{\partial\eta}$, $y=\dfrac{\partial\Phi}{\partial\theta}$. In order to avoid the appearance of the factor $(2\alpha_*)^{1/3}$, we shall replace the coordinate *x* in §§118-121 by $x(2\alpha_*)^{-1/3}$ and call the latter quantity *x*. Then

$$x=\frac{\partial\Phi}{\partial\eta}, \quad y=\frac{\partial\Phi}{\partial\theta}. \tag{118.4}$$

It is useful to note that, since it is so simply related to Φ, the function $y(\eta,\theta)$ (but not $x(\eta,\theta)$) also satisfies the Euler-Tricomi equation. Using this fact, we can write the Jacobian of the transformation from the physical plane to the hodograph plane as

$$\Delta=\frac{\partial(x,y)}{\partial(\theta,\eta)}=\Phi_{\eta\theta}{}^2-\Phi_{\eta\eta}\Phi_{\theta\theta}=\left(\frac{\partial y}{\partial\eta}\right)^2-\eta\left(\frac{\partial y}{\partial\theta}\right)^2. \tag{118.5}$$

As has already been mentioned, the Euler-Tricomi equation has usually to be applied to investigate the properties of the solution near the origin in the $\eta\theta$-plane. In cases of physical interest, the origin is a singular point of the solution. For this reason especial significance attaches to the family of particular integrals of the Euler-Tricomi equation which possess certain properties of homogeneity. These solutions are homogeneous in the variables θ^2 and η^3; such solutions must exist, since the transformation $\theta^2\to a\theta^2$, $\eta^3\to a\eta^3$ leaves the equation (118.2) unchanged. We shall seek these solutions in the form $\Phi=\theta^{2k}f(\xi)$, $\xi=1-\dfrac{4\eta^3}{9\theta^2}$, where *k* is a constant, the degree of homogeneity of the function Φ with respect to the transformation mentioned. We have taken the variable ξ so that it vanishes on the characteristics which pass through the point $\eta=\theta=0$. Making the above substitution, we obtain for the function $f(\xi)$ the equation

$$\xi(1-\xi)f''+\left[\frac{5}{6}-2k-\xi\left(\frac{3}{2}-2k\right)\right]f'-k\left(k-\frac{1}{2}\right)f=0.$$

This is a hypergeometric equation. Using the well-known expressions for the two independent integrals of that equation, we find the required solution (for $2k + 1/6$ not integer):

$$\Phi_k=\theta^{2k}\left[AF\left(-k,-k+\frac{1}{2};-2k+\frac{5}{6};1-\frac{4\eta^3}{9\theta^2}\right)+B\left(1-\frac{4\eta^3}{9\theta^2}\right)^{2k+1/6}F\left(k+\frac{1}{6},k+\frac{2}{3};2k+\frac{7}{6};1-\frac{4\eta^3}{9\theta^2}\right)\right] \tag{118.6}$$

Using the relations between hypergeometric functions of arguments *z*, $1/z$, $1-z$, $1/(1-z)$ and $z/(1-z)$, we can also put this solution in five other forms, all of which are needed in various problems.[6] We shall give two of these:

$$\Phi_k=\theta^{2k}\left[AF\left(-k,-k+\frac{1}{2};\frac{2}{3};\frac{4\eta^3}{9\theta^2}\right)+B\frac{\eta}{\theta^{2/3}}F\left(-k+\frac{1}{3},-k+\frac{5}{6};\frac{4}{3};\frac{4\eta^3}{9\theta^2}\right)\right] \tag{118.7}$$

$$\Phi_k=\eta^{3k}\left[AF\left(-k,-k+\frac{1}{3};\frac{1}{2};\frac{9\theta h2}{4\eta^3}\right)+B\frac{\eta}{\theta^{3/2}}F\left(-k+\frac{1}{2},-k+\frac{5}{6};\frac{3}{2};\frac{9\theta^2}{4\eta^3}\right)\right]; \tag{118.8}$$

---

[4] This phrase must not be taken literally, of course. The region concerned may be the neighbourhood of the point at infinity, i.e., the region at large distances from the body.

[5] We omit a factor $1/c_*$ on the right-hand sides; this simply means that Φ is replaced by $c_*\Phi$, which does not affect equation (118.2) and is therefore always permissible.

[6] The relevant formulae are given, for example, in *QM*, Mathematical Appendices, e.

the constants A and B in formulae (118.6) - (118.8) are not the same, of course. These expressions yield at once the following important property of the functions Φ_k, which is not evident from (118.6): the lines $\eta = 0$ and $\theta = 0$ are not singular lines (it is seen from (118.7) that, near $\eta = 0$, Φ_k can be expanded in integral powers of η, and from (118.8) the same is true of θ). It is seen from the expression (118.6) that the characteristics, on the other hand, are singular lines of the general (i.e., containing the two constants A and B) homogeneous integral Φ_k of the Euler-Tricomi equation: if $2k + 1/6$ is not an integer, the factor $(9\theta^2 - 4\eta^3)^{2k+1/6}$ has branch points, while if $2k + 1/6$ is an integer, one term of (118.6) is meaningless[7] (or degenerates to the other term if $2k + 1/6 = 0$), and must be replaced by the second independent solution of the hypergeometric equation, which in this case has a logarithmic singularity.

The following relations hold between the integrals Φ_k with different values of k:

$$\Phi_k = \Phi_{-k-1/6}(9\theta^2 - 4\eta^3)^{2k+1/6}, \tag{118.9}$$

$$\Phi_{k-1/2} = \frac{\partial \Phi_k}{\partial \theta}. \tag{118.10}$$

The first of these follows immediately from (118.6), and the second from the fact that $\dfrac{\partial \Phi_k}{\partial \theta}$ satisfies the Euler-Tricomi equation, and its degree of homogeneity is that of $\Phi_{k-1/2}$. In these formulae Φ_k means, of course, the general expression, with two arbitrary constants.

In investigating the solution near the point $\eta = \theta = 0$, we have to follow its variation along a contour round this point. For example, let the function Φ_k (118.6) represent the solution at the point A near the characteristic $\theta = \frac{2}{3}\eta^{3/2}$ (Fig. 119), and suppose that we

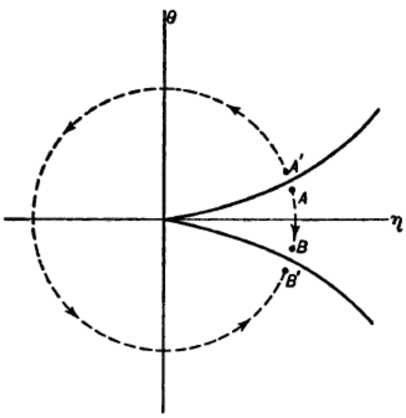

FIG. 119

require the form of the solution near the characteristic $\theta = -\frac{2}{3}\eta^{3/2}$ (at the point B). The passage from A to B involves crossing the axis of abscissae, and $\theta = 0$ is a singular line of the hypergeometric functions in the expression (118.6), so that their argument is infinite there. In order to go from A to B, therefore, it is necessary to transform the hypergeometric functions into functions of the reciprocal argument $\dfrac{9\theta^2}{9\theta^2 - 4\eta^3}$, for which $\theta = 0$ is not a singularity, and then change the sign of θ, finally returning to the original argument by repeating the transformation. In this way we obtain the following transformation formulae for the functions which appear in (118.6):

[7] We recall that the series $F(\alpha, \beta; \gamma; z)$ is meaningless for $\gamma = 0, -1, -2, \ldots$.

$$
\left.\begin{aligned}
F_1 &\to \frac{F_1}{2\sin(2k+1/6)\pi} + F_2 \cdot 2^{-4k-1/3}\,\frac{\Gamma(2k-1/6)\Gamma(-2k+5/6)}{\Gamma(-2k)\Gamma(-2k+2/3)} \\
F_2 &\to \frac{-F_2}{2\sin(2k+1/6)\pi} + F_1 \cdot 2^{4k+1/3}\,\frac{\Gamma(2k+1/6)\Gamma(2k+7/6)}{\Gamma(2k+1)\Gamma(2k+1/3)}
\end{aligned}\right\} \qquad (18.11)
$$

where F_1 and F_2 signify

$$
\left.\begin{aligned}
F_1 &= |\theta|^{2k}\, F\left(-k,-k+\frac{1}{2};-2k+\frac{5}{6};1-\frac{4\eta^3}{9\theta^2}\right) \\
F_2 &= |\theta|^{2k}\left|1-\frac{4\eta^3}{9\theta^2}\right|^{2k+1/6} F\left(k+\frac{1}{6},k+\frac{2}{3};2k+\frac{7}{6};1-\frac{4\eta^3}{9\theta^2}\right)
\end{aligned}\right\} \qquad (118.12)
$$

in which the moduli of θ and $1-\dfrac{4\eta^3}{9\theta^2}$ are taken in the coefficients of the hypergeometric functions.

We can similarly obtain transformation formulae for the passage from A' to B' (Fig. 119) round the origin in the opposite direction. The calculations are more involved, since we have to pass through three singularities of the hypergeometric function (one with $\theta = 0$ = 0 and two with $\eta = 0$; we recall that the singularities of a hypergeometric function with argument z are z = 1 and $z = \infty$). The final formulae are

$$
\left.\begin{aligned}
F_1 &\to -\frac{\sin(4k-1/6)\pi}{\sin(2k+1/6)\pi}F_1 + F_2 \cdot 2^{-4k+2/3}\cos(2k+1/6)\pi\,\frac{\Gamma(-2k-1/6)\Gamma(-2k+5/6)}{\Gamma(-2k)\Gamma(-2k+2/3)} \\
F_2 &\to \frac{\sin(4k-1/6)\pi}{\sin(2k+1/6)\pi}F_2 + F_1 \cdot 2^{4k+4/3}\cos(2k+1/6)\pi\,\frac{\Gamma(2k+1/6)\Gamma(2k+7/6)}{\Gamma(2k+1)\Gamma(2k+1/3)}
\end{aligned}\right\}
$$

(118.13)

As well as this family of homogeneous solutions there are, of course, other families of particular integrals of the Euler-Tricomi equation. We may mention here a family which results from a Fourier expansion in terms of θ. If we seek Φ in the form

$$
\Phi_\nu = g_\nu(\eta)e^{\pm i\nu\theta}, \qquad (118.14)
$$

where ν is an arbitrary constant, we obtain for the function g_ν the equation $g_\nu'' + \nu^2\eta g_\nu = 0$. This is the equation for the Airy function; its general integral is

$$
g_\nu(\eta) = \sqrt{\eta}\,Z_{1/3}\left(\frac{2}{3}\nu\eta^{3/2}\right), \qquad (118.15)
$$

where $Z_{1/3}$ is an arbitrary linear combination of Bessel functions of order 1/3.

Finally, it is useful to bear in mind that the general integral of the Euler-Tricomi equation may be written

$$
\Phi = \int_{C_z} f(\zeta)dz, \quad \zeta = z^3 - 3\eta z + 3\theta, \qquad (118.15)
$$

where $f(\zeta)$ is an arbitrary function and the integration in the complex z-plane is taken along any contour C_z at whose ends the derivative $f'(\zeta)$ has equal values. For a direct substitution of (118.16) in the Euler-Tricomi equation gives

$$
\frac{\partial^2\Phi}{\partial\eta^2} - \eta\frac{\partial^2\Phi}{\partial\theta^2} = 9\int_{C_z}(z^2-\eta^2)f''(\zeta)dz = 3\int_{C_\zeta} f''(\zeta)d\zeta = 3\left[f((\zeta)\right]_{C_\zeta} = 0,
$$

i.e., the equation is satisfied.

§119. Solutions of the Euler-Tricomi equation near non-singular points of the sonic surface

Let us now ascertain which solutions Φ_k correspond to cases where the gas flow has no physical singularities (weak discontinuities or shock waves) near the transition. To do this it is more convenient to start, not from the Euler-Tricomi equation itself, but from the equation for the velocity potential in the physical plane. This equation has been derived in §114; for a two-dimensional flow, equation (114.10) becomes, with the substitution $x \to x(2\alpha_*)^{1/3}$,

$$\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial y^2}. \tag{119.1}$$

We recall that the potential ϕ in this equation is defined so that its derivatives with respect to the coordinates give the velocity according to the equations

$$\frac{\partial\phi}{\partial x} = \eta, \quad \frac{\partial\phi}{\partial y} = \theta. \tag{119.2}$$

We may also note that the Euler-Tricomi equation can be obtained directly from equation (119.1) by changing to the independent variables θ,η by Legendre's transformation, with $\Phi = -\phi + x\eta + y\theta$, or

$$\phi = -\Phi + \eta\frac{\partial\Phi}{\partial\eta} + \theta\frac{\partial\Phi}{\partial\theta}. \tag{119.3}$$

Taking the origin in the *xy*-plane at the point on the transition or ***sonic line*** whose neighbourhood we are investigating, we expand ϕ in powers of *x* and *y*. In the general case, the first term of an expansion which satisfies equation (119.1) is

$$\phi = \frac{xy}{a}. \tag{119.4}$$

Here $\theta = \frac{x}{a}$, $\eta = \frac{y}{a}$, so that

$$\Phi = a\theta\eta. \tag{119.5}$$

It is clear from the degree of homogeneity of this function that it corresponds to one of the functions $\Phi_{5/6}$; this is the second term of the expression (118.7), in which the hypergeometric function with $k = 5/6$ reduces to 1 simply: $\eta\theta F\left(-\frac{1}{2},0;\frac{4}{3};\frac{4\eta^3}{9\theta^2}\right) = \eta\theta$.

If we wish to find the equation of the sonic line in the physical plane, the first term of the expansion does not suffice. The next term is of degree 1, i.e., it corresponds to one of the functions Φ_1, namely the first term in the expression (118.7), which reduces to a polynomial for $k = 1$:

$$\theta^2 F\left(-1,-\frac{1}{2};\frac{2}{3};\frac{4\eta^3}{9\theta^2}\right) = \theta^2 + \frac{1}{3}\eta^3.$$

Thus the first two terms of the expansion of Φ are

$$\Phi = a\eta\theta + b\left(\theta^2 + \frac{1}{3}\eta^3\right). \tag{119.6}$$

Hence

$$\left.\begin{aligned} x &= a\theta + b\eta^2 \\ y &= a\eta + 2b\theta \end{aligned}\right\}. \tag{119.7}$$

The sonic line ($\eta = 0$) is the straight line $y = 2bx/a$.

To find the equation of the characteristics in the physical plane we need only the first term

of the expansion. Substituting $\theta = \frac{x}{a}$, $\eta = \frac{y}{a}$ in the equation of the hodograph characteristics $\theta = \pm\frac{2}{3}\eta^{3/2}$, we obtain $x = \pm\frac{2}{3}\frac{y^{3/2}}{\sqrt{a}}$, i.e., again two branches of a semi-cubical parabola with a cusp on the sonic line (the thick line in Fig. 120). This property of the characteristics is evident also from the following simple argument. At points on the sonic line, the Mach angle is $\pi/2$. This means that the tangents to the characteristics of the two families coincide, so that there is a cusp (Fig. 120). The streamlines intersect the sonic line perpendicularly to the characteristics, and do not have singularities there.

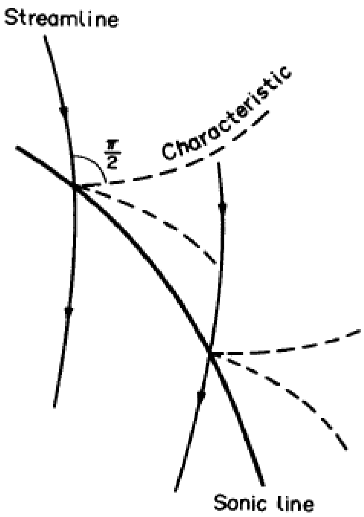

FIG. 120

The solution (119.6) is not applicable in the exceptional case where the streamline is perpendicular to the sonic line at the point considered.[8] Near such a point the flow is evidently symmetrical about the x-axis. This case requires special consideration, which has been given by F. I. Frankl' and S. V. Fal'kovich (1945).

The symmetry of the flow means that, when the sign of y is changed, the velocity v_y changes sign and v_x remains unchanged. That is, the potential ϕ must be an even function of y, and the potential Φ an even function of θ. The first terms in the expansion of in this case therefore have the form

$$\phi = \frac{1}{2}ax^2 + \frac{1}{2}a^2xy^2 + \frac{1}{24}a^3y^4; \qquad (119.8)$$

the relative order of smallness of x and y is not known *a priori*, so that all three terms may be of the same order. Hence we find the following formulae for the transformation from the physical plane to the hodograph plane:

$$\left.\begin{aligned} \eta &= ax + \frac{1}{2}a^2y^2 \\ \theta &= a^2xy + \frac{1}{6}a^3y^3 \end{aligned}\right\}. \qquad (119.9)$$

Without explicitly solving these equations for x and y, we can easily see that the degree of the function $y(\theta,\eta)$ is 1/6. Hence the corresponding function Φ has $k = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$, i.e., it is a particular case of the general integral $\Phi_{2/3}$.

Eliminating x from equations (119.9), we obtain a cubic equation for the function $y(\theta,\eta)$:

---

[8] This would correspond to the case $a = 0$ in (119.6); the solution then ceases to hold, because the Jacobian Δ vanishes on the line $\eta = 0$.

$$(ay)^3 - 3\eta ay + 3\theta = 0\,. \tag{119.10}$$

For $9\theta^2 - 4\eta^3 > 0$, i.e., throughout the region to the left of the hodograph characteristics which pass through the point $\eta = \theta = 0$ (including the whole of the subsonic region $\eta < 0$; Fig. 121), this equation has only one real root, which must be the function $y(\theta, \eta)$. In the region to the right of the characteristics, all three roots are real, and we must take the one which is the continuation of the real root in the region to the left.

The characteristics in the physical plane (which pass through the origin) are obtained by substituting the expressions (119.9) in the equation $4\eta^3 = 9\theta^2$. This gives two parabolae:

the characteristics 23 and 56: $x = -\frac{1}{4}ay^2$,

the characteristics 34 and 45: $x = \frac{1}{2}ay^2$. (119.11)

The numbers show which two regions in the physical plane are separated by the characteristic in question. The sonic line ($\eta = 0$ in the hodograph plane) is the parabola $x = -\frac{1}{2}ay^2$ in the physical plane (the thick line in Fig. 121). We may notice the following property of the point where the sonic line intersects the axis of symmetry: four branches of characteristics leave this point, whereas only two leave any other point on the sonic line.

Figure 121 shows by corresponding numbers the regions of the hodograph plane which correspond to the various regions of the physical plane. This correspondence is not one-to-one;[9] when we go completely round the origin in the physical plane, the region between the two characteristics in the hodograph plane is covered three times, as shown by the dashed line in Fig. 121, which is twice reflected from the characteristics.

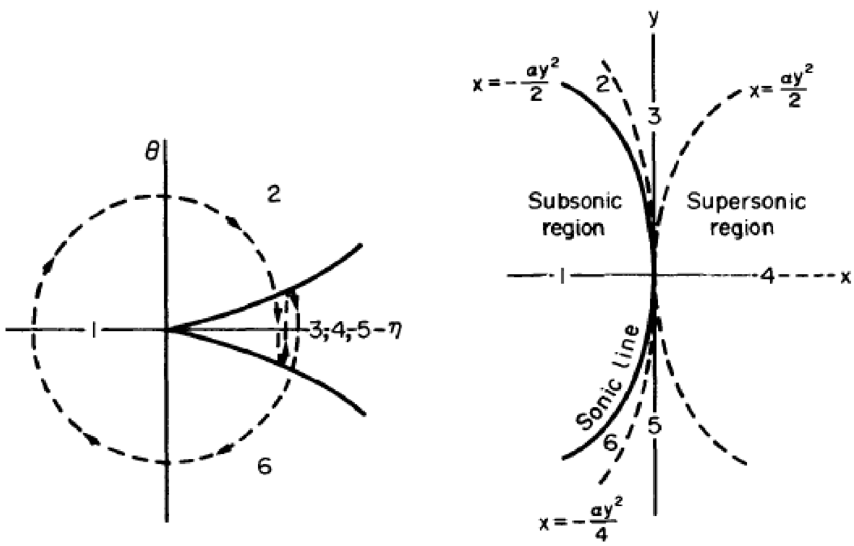

FIG. 121

Since the function $y(\theta, \eta)$ itself satisfies the Euler-Tricomi equation, it must be obtainable from the general integral $\Phi_{1/6}$. Near the characteristic 23 in the physical plane, it is

[9] In accordance with the fact that $\Delta = \infty$ on the characteristic $x = \frac{1}{2}ay^2$ in the physical plane; see the first footnote to §116.

$$y = \frac{1}{a}\left(\frac{3}{2}\theta\right)^{1/3} F\left(-\frac{1}{6}, \frac{1}{3}; \frac{1}{2}; 1 - \frac{4\eta^3}{9\theta^2}\right); \tag{119.12}$$

the first term in (118.6) has no singularity on this characteristic. Continuing this analytically to the neighbourhood of the characteristic 56 (by a path through the subsonic region 1, i.e., by means of formulae (118.13)), we obtain the same function there. Near the characteristics 34 and 45, however, $y(\theta, \eta)$ is given by linear combinations of that function and

$$\theta^{1/3}\sqrt{\frac{4\eta^3}{9\theta^2} - 1}F\left(\frac{1}{3}, \frac{5}{6}; \frac{3}{2}; 1 - \frac{4\eta^3}{9\theta^2}\right). \tag{119.13}$$

i.e., the second term of (118.6). These combinations are obtained by analytical continuation, using formulae (118.11); here it must be borne in mind that the square root in (119.13) changes sign at each reflection from a hodograph characteristic.

Mathematically, these results show that the functions $\Phi_{1/6}$ are linear combinations of the roots of the cubic equation

$$f^3 - 3\eta f + 3\theta = 0, \tag{119.14}$$

i.e., they are algebraic functions.[10] As well as $\Phi_{1/6}$, all the Φ_k with

$$k = \frac{1}{6} \pm \frac{1}{2}n, \quad n = 0, 1, 2, \ldots \tag{119.15}$$

reduce to algebraic functions; they are obtained from $\Phi_{1/6}$, according to formulae (118.9) and (118.10), by successive differentiation, a remark due to F. I. Frankl' (1947).

The functions Φ_k with

$$k = \pm\frac{1}{2}n, \quad k = \frac{1}{3} \pm \frac{1}{2}n, \tag{119.16}$$

in which the hypergeometric function reduces to a polynomial,[11] also reduce to algebraic functions; e.g., for $k = \frac{1}{2}n$ we have the first term of the expression (118.6), and for $k = -\frac{1}{2}n$ the second term.

These three families of algebraic functions Φ_k include, in particular, all the functions which can be potentials Φ corresponding to flows having no singularity in the physical plane. In such flows, all the terms in the expansion of Φ near an asymmetric point on the sonic line (the first two terms of which are given by formula (119.6)) must have either $k = \frac{5}{6} + \frac{1}{2}n$ or $k = 1 + \frac{1}{2}n$. The expansion of Φ near a symmetric point, however, which begins with a term with $k = \frac{2}{3}$, can also contain functions with $k = \frac{2}{3} + \frac{1}{2}n$.

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[10] It is not convenient in practice to use the explicit forms of these functions, which are obtained from (119.14) by Cardan's formula.

[11] Here it must be recalled that $F(\alpha, \beta; \gamma; z)$ reduces to a polynomial if α (or β) is such that $\alpha = -n$ or $\gamma - \alpha = -n$.